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Citation for published version

Assa, H. and Gospodinov, N. (2018) Market consistent valuations with financial imperfection. Decisions in Economics and Finance, 41 (1). pp. 65-90. ISSN 1593-8883.

DOI

https://doi.org/10.1007/s10203-018-0207-2

Link to record in KAR

https://kar.kent.ac.uk/87561/

Document Version

Publisher pdf

Decisions Econ Finan (2018) 41:65–90
https://doi.org/10.1007/s10203-018-0207-2

Market consistent valuations with financial imperfection

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Received: 4 March 2017 / Accepted: 2 April 2018 / Published online: 7 April 2018

Abstract In this paper, we study market consistent valuations in imperfect markets. In the first part of the paper, we observe that in an imperfect market one needs to distinguish two type of market consistencies, namely types I and II. We show that while market consistency of type I holds without very strong conditions, market consistency of type II (which in the literature is known as the usual definition of market consistency) is only well defined in perfect markets. This is important since the existing literature on market consistency considers perfect markets where the two market consistencies are equivalent. In the second part of the paper, by introducing a *best estimator* we find strong connections between hedging and market consistency of either type. We show under very general conditions, the type I and the type II market consistent evaluators are best estimators, and establish a two-step representation for the market consistent risk evaluators. In the third part of the paper, we present several families of market consistent evaluators in imperfect markets.

Keywords Imperfect financial valuation · Risk evaluation · Hedging · Market consistent valuation

JEL Classification G11 · G13 · C22 · E44

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1 Introduction

The main assumption of a market consistent valuation is that the fully hedged portfolios cannot improve the actuarial valuation; see, for example, Wüthrich et al. (2010), Wüthrich and Merz (2013), Pelsser and Stadje (2014), Happ et al. (2015) and Dhaene et al. (2017). But given that a perfect approach to hedging does not always exist, this assumption needs to be examined more carefully. Indeed, this assumption postulates that liquidly traded assets and payoffs replicable by them do not carry any risk as they can be converted to cash at any time. In this paper, we show that this assumption necessitates the hedging strategy to be perfect, i.e., the market pricing rule is cone linear over the cone of fully hedged portfolios. However, the possibility of a perfect hedging can be challenged in practice along many dimensions. For instance, nonzero ask–bid spreads, costly dynamic hedging or model risk are among the reasons that hedging strategies do not need to be perfect. We will discuss these particular reasons using a few examples in Sect. 2.3.

This paper considers a financial market where hedging is not necessarily perfect. In the first part, we argue that with an imperfect hedging strategy, we have to distinguish between two different (type I and II) market consistencies. Market consistency of type I only asserts that the valuation of a fully hedged position is the same as its market price, whereas market consistency of type II assumes further that hedging with hedgeable strategies cannot improve the valuation of the risky positions. The existing literature uses the type II market consistency as the usual definition. We characterize market consistent evaluators of either type and prove that a market consistent valuation of type II is well defined only in perfect markets.

In the second part of the paper, we observe that market consistent valuations are strongly related to hedging. Interestingly, this connection will help us characterize market consistent valuation by introducing a *best estimator*. We show that if some market principle conditions hold (e.g., compatibility[1]), the type I and II market consistent evaluators are best estimators. In addition, we demonstrate how the best estimator characterization of market consistent evaluators can facilitate obtaining a two-step representation.

Finally, we introduce and discuss practical ways for constructing market consistent evaluators. First, we introduce a family of two-step market consistent evaluators. Second, inspired by super-hedging pricing methods, we introduce the family of super-evaluators.

The rest of the paper is organized as follows. Section 2 introduces the notation, provides some preliminary definitions and states the main problem. Section 3 discusses the concepts and studies the properties of market consistent evaluators. Section 4 develops a general framework for hedging in an imperfect market and shows its relation to the market consistencies of either type. In Sect. 5, we provide several examples of market consistent valuations. Section 6 concludes.

[1] It can be shown in many cases this is equivalent to the No Good Deal assumption, see Remark 9.

2 Preliminaries and analytical setup

We consider a probability triple $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is a set of scenarios, $\mathcal{F}$ is a set of events and $\mathbb{P}$ is a probability measure. We denote the expectation by E and for any other probability triple $(\Omega, \mathcal{F}, \mathbb{Q})$, the associated expectation is denoted by $E^{\mathbb{Q}}$. For $p \in [1, \infty]$, let L^p be the Banach space defined as:

$$L^p = \left\{x : \Omega \to \mathbb{R} | \mathcal{F}\text{-measurable random variables and } E\left(|x|^p\right) < \infty\right\}.$$

L^p is endowed with the norm $\|x\|_{L^p} = E\left(|x|^p\right)$. Consider L^p and L^q where $1/p + 1/q = 1$. There is a duality relation between L^p and L^q defined as

$$(y, y') \mapsto E(yy'), \forall (y, y') \in L^p \times L^q.$$

The smallest topology induced by L^q on L^p is denoted by $\sigma(L^p, L^q)$. Similarly, one can introduce the topology $\sigma(L^q, L^p)$. For any sub-sigma-field $\mathcal{G} \subseteq \mathcal{F}$, $L^p(\mathcal{G})$ and $L^q(\mathcal{G})$ represent the same spaces as described above just for $\mathcal{G}$ measurable random variable. Note that $L^p = L^p(\mathcal{F})$ and $L^q = L^q(\mathcal{F})$. For any set of random variables $x_1, \ldots, x_n$, the smallest sigma-field generated by them is denoted by $\varsigma(x_1, \ldots, x_n)$.

For a time interval $[0, T]$, we consider two right-continuous filtration $\left\{\mathcal{F}_t^A\right\}_{0 \le t \le T}$ and $\left\{\mathcal{F}_t^S\right\}_{0 \le t \le T}$, representing the flows of information for insurance and financial markets, respectively. We assume that $\mathcal{F}_0^A = \mathcal{F}_0^S = \{\varnothing, \Omega\}$, and $\mathcal{F} = \mathcal{F}_T^A \vee \mathcal{F}_T^S$ (the smallest sigma-field containing $\mathcal{F}_T^A$ and $\mathcal{F}_T^S$). We also assume that $\mathcal{F}_T^S$ contains all measure zero sets of $\mathcal{F}$.

Remark 1 As it is common in the literature, we have to specify whether we are working with loss/profit or deficit/surplus variables. However, in this paper, we do not need to do this. Indeed, our framework is flexible to cover both approaches; i.e., one can decide if the set of random variables represents losses (or deficit)—which is of interest to actuaries, or if it represents profits (surplus)—which is of interest to financial modelers. We will come back to this point later after introducing risk evaluators and pricing rules.

To fix the terminology and the notation, it is useful to briefly review some concepts from convex analysis. We assume that all mappings $f : L^p \to (-\infty, \infty]$ in this paper are $\sigma(L^p, L^q)$-lower semi-continuous, i.e., for any number $a \in \mathbb{R}$, the set $\{x \in L^p | f(x) \le a\}$ is $\sigma(L^p, L^q)$-closed in L^p. This assumption implies the existence of a dual representation (see Proposition 4.1, Ekeland and Témam 1999) for a lower semi-continuous convex mapping $f : L^p \to (-\infty, +\infty]$ as follows,

$$f(x) = \sup_{z \in L^q} \left\{E(zx) - f^*(z)\right\},$$

where $f^* : L^q \to (-\infty, \infty]$ is the dual of f defined as

$$f^*(z) = \sup_{x \in L^p} \{E(zx) - f(x)\}.$$

It can be easily seen that for any positive homogeneous (i.e., $f(\lambda x) = \lambda f(x), \forall \lambda \geq 0, x \in L^p$) and convex function f, f^* is 0 on a closed convex set Δ_f, and infinity otherwise. Therefore, the dual representation of a positive homogeneous and convex function f has the form

$$f(x) = \sup_{z \in \Delta_f} E(zx).$$

Then, it is straightforward to see that $\Delta_f = \{z \in L^q | E(zx) \leq f(x), \forall x \in L^p\}$.

2.1 Risk evaluator

A *risk evaluator* Π is a mapping from L^p to the set of real numbers $\mathbb{R}$ which maps each random variable in L^p to a real number representing its risk with the additional property $\Pi(0) = 0$. Each risk evaluator can have one or more of the following properties:

(P1) $\Pi(\lambda x) = \lambda \Pi(x)$, for all $\lambda \geq 0$ and $x \in L^p$ (positive homogeneity);
(P2) $\Pi(x + y) \leq \Pi(x) + \Pi(y)$, for all $x, y \in L^p$ (sub-additivity);
(P3) $\Pi(\lambda x + (1 - \lambda)y) \leq \lambda \Pi(x) + (1 - \lambda)\Pi(y)$, for all $x, y \in L^p$ and $\lambda \in [0, 1]$ (convexity).

There are several families of risk evaluators introduced and studied in different areas: for instance, coherent risk measure, convex risk measures and expectation bounded risks. Coherent and convex risk measures are introduced by Artzner et al. (1999) and Föllmer and Schied (2002), respectively, while expectation bounded risks are first defined in Rockafellar et al. (2006).

The following examples are risk evaluators when L^p represents the loss variables (or deficit). A mean-variance risk evaluator on L^2, is defined as

$$MV_\delta(x) = \delta\sigma(x) + E(x),$$

where $\sigma(x)$ is the standard deviation of x and δ is a nonnegative number representing the level of risk aversion. One can replace σ by semi pth moment to have the following family of risk evaluators on L^p

$$MV_\delta^p(x) = \delta E\left(\max\{x - E(x), 0\}^p\right) + E(x).$$

One popular risk evaluator on L^p, (even on the set of all random variables) is the value at risk defined as

$$\text{VaR}_\alpha(x) = \inf\{a \in \mathbb{R} | \mathbb{P}[x > a] \leq \alpha\}.$$

Here $\alpha \in (0, 1)$ is the risk aversion parameter. Therefore, one can introduce the following risk evaluator on L^p,

$$\Pi_\delta^{\text{VaR}_\alpha}(x) = \delta \text{VaR}_\alpha(x - E(x)) + E(x).$$

In contrast, the conditional value at risk (CVaR), expressed as the sum over all VaR above $1-\alpha$ percent

$$\mathrm{CVaR}_\alpha(x) = \frac{1}{1-\alpha}\int_{1-\alpha}^{1} \mathrm{VaR}_\beta(x)\mathrm{d}\beta, \tag{1}$$

is a coherent risk evaluator on L^p. Accordingly, one can introduce the following risk evaluator based on CVaR on L^p,

$$\Pi_\delta^{\mathrm{CVaR}_\alpha}(x) = \delta\mathrm{CVaR}_\alpha(x - E(x)) + E(x).$$

In this paper, we use the following two categories of risk evaluators in our statements.

Definition 1 Let Π be a risk evaluator.

1. Π is sub-linear if it satisfies P1 and P2.
2. Π is convex if it satisfies P3.

As discussed earlier, it is clear that every sub-linear evaluator can be represented as

$$\Pi(x) = \sup_{z\in\Delta_\Pi} E(zx), \quad \forall x \in L^p, \tag{2}$$

for a closed convex set Δ_Π of L^q.

Remark 2 Following Remark 1, a few remarks regarding the cash invariance and monotonicity, as they are commonly used in the literature, seem warranted. For those properties, we need to specify whether L^p represents loss variables (deficit) or profit (surplus). More specifically, if we suppose random variables model the losses (profits) then for all $x, y \in L^p$, $x \le y$ a.s. and $\forall c \in \mathbb{R}$, cash invariance and monotonicity are $\Pi(x+c) = \Pi(x)+c$ and $\Pi(x) \le \Pi(y)$, ($\Pi(x+c) = \Pi(x)-c$ and $\Pi(x) \ge \Pi(y)$), respectively. But, our approach in this paper does not need to specify if we are working with loss/deficit or profit/surplus variables, since our theory is not dependent on cash invariance or monotonicity property.

2.2 Pricing rule

We now turn our attention to pricing rules. First, we need to fix a set of assets that are fully hedged. Let $\mathcal{X}$ be a closed subset of $L^p\left(\mathcal{F}_T^S\right)$ which contains the origin. In the subsequent discussions, we will assume that $\mathcal{X}$ possesses one or several properties from the following list:

(S1) *Positive homogeneity* $\lambda\mathcal{X} \subseteq \mathcal{X}$, for all $\lambda \ge 0$;
(S2) *Sub-additivity* $\mathcal{X} + \mathcal{X} \subseteq \mathcal{X}$;
(S3) *Convexity* $\lambda\mathcal{X} + (1-\lambda)\mathcal{X} \subseteq \mathcal{X}$ for all $\lambda \in (0, 1)$.

If $\mathcal{X}$ has properties S1 and S2, it is called a convex cone and if S3 it is simply called convex.

A *pricing rule* $\pi : \mathcal{X} \to \mathbb{R}$ is a mapping from $\mathcal{X}$ to the set of real numbers $\mathbb{R}$ which maps each random variable in $\mathcal{X}$ to a real number representing its price, with an additional property $\pi(0) = 0$. Just realize that in principle the main difference between the definition of π and Π is the domain of these two mappings. If π satisfies properties P1, P2, or P3, $\mathcal{X}$ has to satisfy properties S1, S2, or S3, respectively. Jouini and Kallal (1995a, b, 1999) argue that for a wide range of market imperfections the pricing rule is sub-linear, i.e., π has P1 and P2. That is why in this paper we develop our theoretical framework for sub-linear pricing rules.

Remark 3 Like in Remarks 1 and 2, to emphasize the generality of our framework, note that we do not need to specifically assume that if π is cash invariant or monotone. Just a further attention needed to be paid when considering cash invariance or monotonicity: if π is cash invariance we have to consider $\mathbb{R} \subseteq \mathcal{X}$ and if it is non-decreasing (non-increasing), $\forall x \in \mathcal{X}$ and $y \geq x$, $(y \leq x)$, $y \in \mathcal{X}$.

Next, we introduce more rigorously a perfect market:

Definition 2 A pricing rule π on a cone $\mathcal{X}$ is perfect if $\pi(x + \lambda y) = \pi(x) + \lambda\pi(y), \forall x, y \in \mathcal{X}, \lambda > 0$. When π is perfect we say the market and the hedging strategy are perfect.

It is clear that a perfect pricing rule is sub-linear.

2.3 Examples of pricing rules

Using several examples, we show how all different types of markets and pricing rules exist. For simplicity, in all examples, we assume that there is a variable (loss or profit) $h \in L^1$ for the insurance company. We assume that the insurance information is given by $\mathcal{F}_t^A = \{\varnothing, \Omega\}$ for $0 \leq t < T$, and $\mathcal{F}_T^A = \varsigma(h)$. Therefore, we only need to focus our attention on introducing the financial part.

The first four examples use a standard mathematical finance setup to model portfolios with Brownian motions (e.g., see Karoui and Quenez 1995 for more details). Let $\left(\overrightarrow{W}_t\right)_{0\leq t\leq T} = \left(\left(W_{1,t}, \ldots, W_{d,t}\right)\right)_{0\leq t\leq T}$ be a standard Brownian motion, where all components are independent. For a natural number $N \geq d$, let $\overrightarrow{\mu} = (\mu_1, \ldots \mu_N)$ be a vector of real numbers (representing the drifts) when also $\mu_i > r, i = 1, \ldots, d$, and $r > 0$ is the interest rate. Let $\overrightarrow{\sigma}(t) = \left(\sigma_{i,j}(t)\right)_{0\leq t\leq T}$ be an N by d matrix of previsible volatility processes. Let $\mathcal{F}_t^S = \varsigma(\overrightarrow{W}_s, 0 \leq s \leq t)$. Introduce the discounted value of the N assets $x_{i,t}$, $i = 1, \ldots, N$ by

$$\mathrm{d}x_{i,t} = x_{i,t}\left((\mu_i - r)\,\mathrm{d}t + \sum_{j=1}^{d}\sigma_{i,j}(t)\,\mathrm{d}W_{j,t}\right), \quad 0 \leq t \leq T, \quad 1 \leq i \leq N.$$

It is known that for any previsible process $\overrightarrow{\theta} = \left(\theta_{1,t}, \ldots, \theta_{d,t}\right)$ that solves $\mu_i - r = \sum_{j=1}^{d}\sigma_{i,j}(t)\,\theta_{j,t}$ and $\int_0^T \theta_s^2 \mathrm{d}s < \infty$ a.s., there is an equivalent martingale measure associated with $\overrightarrow{\theta}$ whose Radon–Nikodym derivative is given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T \sum_{j=1}^{d} \theta_{j,t} dW_{j,t} - \frac{1}{2}\int_0^T \sum_{j=1}^{d} \theta_{j,t}^2 dt\right).$$

Such process $\overrightarrow{\theta} = \left(\theta_{1,t}, \ldots, \theta_{d,t}\right)$ is known as the market price of risk. It is known that if $N = d$, and if $\left(\sigma_{i,j}(t)\right)$ is a full rank matrix for every t, then the market is complete and otherwise, incomplete. Let us also introduce the following set:

$$\mathcal{A} = \left\{ c + \int_0^T \sum_{i=1}^{N} h_{i,t} dx_{i,t} \;\middle|\; \begin{array}{c} c \in \mathbb{R},\, h_t \text{ is a previsible \&} \\ \exists a \in \mathbb{R},\, \int_0^t \sum_{i=1}^{N} h_{i,t} dx_{i,t} \geq a \text{ a.s.}, \forall 0 \leq t \leq T \end{array} \right\}.$$

It is clear that $\mathcal{A}$ is a convex cone.

Example 1 (A Perfect Complete Market) Consider the following one-dimensional ($N = 1$) model for the discounted values of an asset

$$\frac{dx_t}{x_t} = (\mu - r)\, dt + \sigma dW_t, 0 \leq t \leq T,$$

for real numbers $\mu > r$ and $\sigma > 0$. As mentioned above the Radon–Nikodym derivative of the unique equivalent martingale measure of this model is given as

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\theta W_T - \frac{1}{2}\theta^2 T\right),$$

where $\theta = \frac{\mu - r}{\sigma}$. Let $\mathcal{X}$ be the L^2 closer of $\mathcal{A} \cap L^2\left(F_T^S\right)$. Note that since $\mathbb{R} \subseteq \mathcal{A}$, then $\mathcal{X} \neq \varnothing$. Since $L^2\left(F_T^S\right)$ is closed in L^2, so is $\mathcal{X}$. In fact, one can easily see that any complete market is perfect, since there is a unique stochastic discount factor that gives the pricing rule.

Example 2 (A Perfect Incomplete Market with Good Deal Pricing) Consider the following model for a discounted asset value

$$\frac{dx_t}{x_t} = (\mu - r)\, dt + \sigma_1 dW_{1,t} + \sigma_2 dW_{2,t}, 0 \leq t \leq T.$$

It is clear that this is a model for an incomplete market. Let $\{(\theta_{1,t}, \theta_{2,t})\}_{0 \leq t \leq T}$ be a previsible process that can solve $\sigma_1\theta_{1,t} + \sigma_2\theta_{2,t} = \mu - r$, and $\int_0^T \theta_{i,t}^2 dt < \infty$. Then, probability measure $\mathbb{Q}$, whose Radon–Nikodym derivative is given as follows, is an equivalent martingale measure:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T \theta_{1,t} dW_{1,t} - \int_0^T \theta_{2,t} dW_{2,t} - \frac{1}{2}\int_0^T \theta_{1,t}^2 dt - \frac{1}{2}\int_0^T \theta_{2,t}^2 dt\right). \tag{3}$$

By changing $\mathbb{P}$ to $\mathbb{Q}$, we are eventually replacing $\left(W_{i,t}\right)_{0\leq t\leq T}$ by $\mathbb{Q}$-Brownian motions $\tilde{W}_{i,t} = W_{i,t} + \int_0^t \theta_{i,s}\mathrm{d}s$, $i = 1, 2$. Let us assume that it is believed that the market prices of risk cannot be greater than a particular number $M > 0$. This is justifiable as ruling out Good Deals from the market in the same way considered in Cochrane and Saa-Requejo (2000). Let us denote the Radon–Nikodym derivative of all martingale measures satisfying $|\theta_{i,t}| \leq M, 0 \leq t \leq T, i = 1, 2$, by $\mathcal{P}_M$. It is clear that $\mathcal{P}_M \subseteq L^2\left(F_T^S\right)$ and for any $x \in L^2\left(F_T^S\right)$, we have $\sup_{\mathbb{Q}\in\mathcal{P}_M} E\left(x\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right) \leq \exp\left(M^2T\right)\|x\|_{L^2}$. To establish this, we use repeatedly the Cauchy–Schwarz inequality:

$$
\begin{aligned}
\left\|\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right\|_{L^2} &= \left\|\exp\left(-\int_0^T \theta_{1,t}\mathrm{d}W_{1,t} - \int_0^T \theta_{2,t}\mathrm{d}W_{2,t} - \frac{1}{2}\int_0^T \theta_{1,t}^2\mathrm{d}t - \frac{1}{2}\int_0^T \theta_{2,t}^2\mathrm{d}t\right)\right\|_{L^2} \\
&= \sqrt{E\left(\exp\left(-2\int_0^T \theta_{1,t}\mathrm{d}W_{1,t} - 2\int_0^T \theta_{2,t}\mathrm{d}W_{2,t} - \int_0^T \theta_{1,t}^2\mathrm{d}t - \int_0^T \theta_{2,t}^2\mathrm{d}t\right)\right)} \\
&= \sqrt{\exp\left(-\int_0^T \theta_{1,t}^2\mathrm{d}t - \int_0^T \theta_{2,t}^2\mathrm{d}t\right)\left(\exp\left(2\int_0^T \theta_{1,t}^2\mathrm{d}t + 2\int_0^T \theta_{2,t}^2\mathrm{d}t\right)\right)} \\
&= \sqrt{\exp\left(\int_0^T \theta_{1,t}^2\mathrm{d}t + \int_0^T \theta_{2,t}^2\mathrm{d}t\right)} \leq \sqrt{\exp\left(\int_0^T M^2\mathrm{d}t + \int_0^T M^2\mathrm{d}t\right)} \\
&= \exp\left(M^2T\right).
\end{aligned}
$$

Let Δ be the L^2 closed convex hull of $\left\{\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}, \mathbb{Q} \in \mathcal{P}_M\right\}$ and define $\pi(x) = \sup_{z\in\Delta} E(zx)$. This is the upper Good Deal bound. It is clear that π is Lipschitz continuous, i.e., $|\pi(x) - \pi(y)| \leq |\pi(x-y)| \leq L\|x-y\|_{L^2}$, where $L = \exp(M^2T)$. Therefore, π is well defined on L^2. Let $\mathcal{X}$ be the L^2 closer of $\mathcal{A} \cap L^2\left(F_T^S\right)$. Note that since $\mathbb{R} \subseteq \mathcal{X}$, $\mathcal{X} \neq \varnothing$. Observe that π is linear over $\mathcal{A}$, since for any martingale measure $\mathbb{Q}$ we have:

$$
E^{\mathbb{Q}}\left(c + \int_0^T h_t\mathrm{d}x_t + \lambda\left(c' + \int_0^T h'_t\mathrm{d}x_t\right)\right) = c + \lambda c'.
$$

Since π is Lipschitz, it is linear over $\mathcal{X}$, so, the pricing rule is perfect.

Example 3 (An Imperfect Complete Market with Model Uncertainty) Let us consider the complete market model in Example 1 for volatility $\sigma_0 > 0$:

$$
\frac{\mathrm{d}x_t}{x_t} = (\mu - r)\,\mathrm{d}t + \sigma_0\mathrm{d}W_t, 0 \leq t \leq T.
$$

The value of the asset is given by $x_t = \exp\left(\left((\mu - r) - \frac{1}{2}\sigma_0^2\right)t + \sigma_0 W_t\right)$. Suppose that there is uncertainty over the value of σ_0. More precisely, let $0 < \sigma_1 < \sigma_2$, we assume that the true value σ_0 belongs to $[\sigma_1, \sigma_2]$. In that case, any of the following random variables can be a candidate for the Radon–Nikodym derivative of the true equivalent martingale measure:

$$\frac{\mathrm{d}\mathbb{Q}_\theta}{\mathrm{d}\mathbb{P}} = \exp\left(-\theta W_T - \frac{1}{2}\theta^2 T\right) \text{ where } \theta \in \left[\frac{\mu - r}{\sigma_2}, \frac{\mu - r}{\sigma_1}\right].$$

A robust approach to pricing takes the maximum price for all the possible values of θ, i.e., the pricing rule is given by $\pi(x) = \sup_{z\in\Delta} E(zx)$, where Δ is the L^2 closed convex hull of $\left\{\frac{\mathrm{d}\mathbb{Q}_\theta}{\mathrm{d}\mathbb{P}}\middle|\theta \in \left[\frac{\mu-r}{\sigma_2}, \frac{\mu-r}{\sigma_1}\right]\right\}$. Similar to the previous example, one can show that for all $x \in L^2$, $\pi(x) \le L\|x\|_{L^2}$, where $L = \exp\left(\frac{\mu-r}{\sigma_1}T\right)$. Let $\mathcal{X}$ be the same set as in Example 1 when $\sigma = \sigma_0$. Since, for the true model there is a unique martingale measure given by $\mathbb{Q}_{\theta_0}$, then for a given $\theta \neq \theta_0$, there must be $\tau \in [0, T)$ so that $\mathbb{P}\left\{E^{\mathbb{Q}_\theta}(x_T|\mathcal{F}_\tau) \neq x_\tau\right\} > 0$. Without loss of generality suppose $\mathbb{P}\left(E^{\mathbb{Q}_\theta}(x_T|\mathcal{F}_\tau) > x_\tau\right) > 0$. Let us introduce $h_t(\omega) = \mathbb{I}_{\{(\omega,t)\in A\times(\tau,T]\}}$ where $A = \left\{E^{\mathbb{Q}_\theta}(x_T|\mathcal{F}_\tau) > x_\tau\right\}$ and $\mathbb{I}$ is the indicator function. It is clear that $(h_t)_{0\le t\le T}$ is previsible. Let $u_t = \int_0^t h_s \mathrm{d}x_s$. It is clear that for $t \le \tau$, $u_t = 0$, and for $t > \tau$, $u_t = (x_t - x_\tau)\mathbb{I}_A$. Observe that $E^{\mathbb{Q}_\theta}(u_T) = E^{\mathbb{Q}_\theta}\left(E^{\mathbb{Q}_\theta}(u_T|\mathcal{F}_\tau)\right) = E^{\mathbb{Q}_\theta}\left(1_A\left(E^{\mathbb{Q}_\theta}(x_T|\mathcal{F}_\tau) - x_\tau\right)\right) > E^{\mathbb{Q}_{\theta_0}}(0) = 0$. This yields $\pi(u_T) \ge E^{\mathbb{Q}_\theta}(u_T) > 0$. On the other hand, let $w_t = -u_t = -\int_0^t h_s \mathrm{d}x_s$. Since, $E^{\mathbb{Q}_{\theta_0}}(w_T) = 0$, then $\pi(w_T) \ge 0$. Now, let

$$u_T^c = \begin{cases} c & u_T \ge c \\ u_T & |u_T| < c \\ -c & u_T \le -c \end{cases},$$

and $u_t^c = E^{\mathbb{Q}_{\theta_0}}\left(u_T^c|\mathcal{F}_t\right), 0 \le t < T$. It is clear that $\left(u_t^c\right)_{0\le t\le T}$ is a $\mathbb{Q}_{\theta_0}$ martingale. By martingale representation and the Girsanov theorem, there is a previsible process h_t^c so that $du_t^c = \theta_0 h_t^c \mathrm{d}t + h_t^c \mathrm{d}W_t$. From the true model, putting $\frac{\mathrm{d}x_t}{\sigma_0 x_t} - \theta_0 \mathrm{d}t$ in lieu of $\mathrm{d}W_t$ we get

$$du_t^c = \theta_0 h_t^c \mathrm{d}t + h_t^c\left(\frac{\mathrm{d}x_t}{\sigma_0 x_t} - \theta_0 \mathrm{d}t\right) = \frac{h_t^c}{\sigma_0 x_t}\mathrm{d}x_t = H_t^c \mathrm{d}x_t,$$

where $H_t := \frac{h_t^c}{\sigma_0 x_t}, 0 \le t \le T$. Note that, since $\left(h_t^c\right)_{0\le t\le T}$ is previsible and $(x_t)_{0\le t\le T}$ is continuous and nonzero then, $(H_t)_{0\le t\le T}$ is again previsible. Given this, and that u_t^c is bounded by c, we get $u_T^c \in \mathcal{A}$. Similarly, one gets $w_T^c = -u_T^c \in \mathcal{A}$. If we choose c large enough, we get $\left|E^{\mathbb{Q}_\theta}(u_T) - E^{\mathbb{Q}_\theta}\left(u_T^c\right)\right| < \frac{E^{\mathbb{Q}_\theta}(u_T)}{3}$ and $\left|E^{\mathbb{Q}_{\theta_0}}\left(w_T^c\right)\right| = \left|E^{\mathbb{Q}_{\theta_0}}(w_T) - E^{\mathbb{Q}_{\theta_0}}\left(w_T^c\right)\right| < \frac{E^{\mathbb{Q}_\theta}(u_T)}{3}$. Therefore, we have

$$\pi\left(u_T^c\right) \ge E^{\mathbb{Q}_\theta}\left(u_T^c\right) \ge E^{\mathbb{Q}_\theta}(u_T) - \frac{E^{\mathbb{Q}_\theta}(u_T)}{3} = \frac{2E^{\mathbb{Q}_\theta}(u_T)}{3},$$

and

$$\pi\left(w_T^c\right) \geq E^{\mathbb{Q}_{\theta_0}}\left(w_T^c\right) \geq -\frac{E^{\mathbb{Q}_\theta}(u_T)}{3}.$$

Summing these two inequalities, we get

$$\pi\left(w_T^c\right) + \pi\left(u_T^c\right) \geq \frac{E^{\mathbb{Q}_\theta}(u_T)}{3} > 0 = \pi(0) = \pi\left(w_T^c + u_T^c\right).$$

This shows that the pricing rule is not perfect.

Example 4 (An Imperfect Incomplete Market with Static Hedging) The standard delta hedging method is not only time-consuming but also costly, as it needs continuous re-balancing. Therefore, the practitioners use alternative ways of hedging. Static hedging is a method in which the hedging portfolio needs only be re-balanced a finite number of times or only once by using the most liquid assets in the market. We consider a static hedging framework in an incomplete market. Let us consider the same model in Example 2 where $\sigma_1 = \sigma_2 = \sigma$, $\mu > r$ and M is a number larger than $\theta = \frac{\mu - r}{\sigma}$. Let us consider two assets whose value processes are given by $\frac{\mathrm{d}x_{i,t}}{x_{i,t}} = \theta \mathrm{d}W_{i,t}, 0 \leq t \leq T, i = 1, 2$. This simply gives that

$$x_{i,t} = \exp\left(-\frac{1}{2}\theta^2 t + \theta W_{i,t}\right), \quad i = 1, 2. \tag{4}$$

Let us consider two binary swaps introduced by $u = \mathbb{I}_{\{x_{1,T} > x_{2,T}\}}$ and $w = \mathbb{I}_{\{x_{1,T} < x_{2,T}\}} = 1 - u$, a.s. We assume that u and w are among the family of most liquid derivatives in the market. Note also that, $u, w \in L^2\left(\mathcal{F}_T^S\right)$. Let $\mathcal{X}$ be the L^2 closer of $\{a_1 u + a_2 w + v | a_1, a_2 \in [0, \infty], v \in \mathcal{A}\} \cap L^2\left(\mathcal{F}_T^S\right)$. Like Example 2, let Δ be the L^2 closed convex hull of $\left\{\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}, \mathbb{Q} \in \mathcal{P}_M\right\}$ and define $\pi(x) = \sup_{z \in \Delta} E(zx)$.

We want to prove the pricing rule is not perfect. Let $\mathbb{Q}_1, \mathbb{Q}_2$ be martingale measures associated with pairs $(\theta_1 = \frac{\mu - r}{\sigma}, \theta_2 = 0), (\theta_1 = 0, \theta_2 = \frac{\mu - r}{\sigma})$, respectively. Note that

$$\frac{\mathrm{d}\mathbb{Q}_i}{\mathrm{d}\mathbb{P}} = \exp\left(-\theta W_{i,T} - \frac{\theta^2}{2}T\right), \quad i = 1, 2. \tag{5}$$

Furthermore, note that by assumption since $\frac{\mu - r}{\sigma} < M$, then $\frac{\mathrm{d}\mathbb{Q}_2}{\mathrm{d}\mathbb{P}}, \frac{\mathrm{d}\mathbb{Q}_1}{\mathrm{d}\mathbb{P}} \in \Delta$.

First, we claim that $E^{\mathbb{Q}_2}(u) > \frac{1}{2}$. To prove this, note since $W_{1,T}$ and $W_{2,T}$ are *i.i.d*, so are $\frac{\mathrm{d}\mathbb{Q}_2}{\mathrm{d}\mathbb{P}}, \frac{\mathrm{d}\mathbb{Q}_1}{\mathrm{d}\mathbb{P}}$, and as a result we have that $E\left(\frac{\mathrm{d}\mathbb{Q}_1}{\mathrm{d}\mathbb{P}} \mathbb{I}_{\left\{\frac{d\mathbb{Q}_1}{\mathrm{d}\mathbb{P}} > \frac{\mathrm{d}\mathbb{Q}_2}{\mathrm{d}\mathbb{P}}\right\}}\right) = E\left(\frac{\mathrm{d}\mathbb{Q}_2}{\mathrm{d}\mathbb{P}} \mathbb{I}_{\left\{\frac{\mathrm{d}\mathbb{Q}_2}{\mathrm{d}\mathbb{P}} > \frac{\mathrm{d}\mathbb{Q}_1}{\mathrm{d}\mathbb{P}}\right\}}\right)$. Using this, and since $\frac{\mathrm{d}\mathbb{Q}_2}{\mathrm{d}\mathbb{P}}, \frac{d\mathbb{Q}_1}{\mathrm{d}\mathbb{P}}$ are continuous, we get,

$$
\begin{aligned}
1 = E\left(\frac{\mathrm{d}\mathbb{Q}_1}{\mathrm{d}\mathbb{P}}\right) &= E\left(\frac{\mathrm{d}\mathbb{Q}_1}{\mathrm{d}\mathbb{P}}\mathbb{I}_{\left\{\frac{d\mathbb{Q}_1}{\mathrm{d}\mathbb{P}}<\frac{\mathrm{d}\mathbb{Q}_2}{\mathrm{d}\mathbb{P}}\right\}}\right) + E\left(\frac{\mathrm{d}\mathbb{Q}_1}{\mathrm{d}\mathbb{P}}\mathbb{I}_{\left\{\frac{\mathrm{d}\mathbb{Q}_1}{\mathrm{d}\mathbb{P}}>\frac{\mathrm{d}\mathbb{Q}_2}{\mathrm{d}\mathbb{P}}\right\}}\right) \\
&= E\left(\frac{\mathrm{d}\mathbb{Q}_1}{\mathrm{d}\mathbb{P}}\mathbb{I}_{\left\{\frac{d\mathbb{Q}_1}{\mathrm{d}\mathbb{P}}<\frac{\mathrm{d}\mathbb{Q}_2}{\mathrm{d}\mathbb{P}}\right\}}\right) + E\left(\frac{\mathrm{d}\mathbb{Q}_2}{\mathrm{d}\mathbb{P}}\mathbb{I}_{\left\{\frac{\mathrm{d}\mathbb{Q}_2}{\mathrm{d}\mathbb{P}}>\frac{\mathrm{d}\mathbb{Q}_1}{\mathrm{d}\mathbb{P}}\right\}}\right) \\
&< E\left(\frac{\mathrm{d}\mathbb{Q}_2}{\mathrm{d}\mathbb{P}}\mathbb{I}_{\left\{\frac{d\mathbb{Q}_1}{\mathrm{d}\mathbb{P}}<\frac{\mathrm{d}\mathbb{Q}_2}{\mathrm{d}\mathbb{P}}\right\}}\right) + E\left(\frac{\mathrm{d}\mathbb{Q}_2}{\mathrm{d}\mathbb{P}}\mathbb{I}_{\left\{\frac{\mathrm{d}\mathbb{Q}_2}{\mathrm{d}\mathbb{P}}>\frac{\mathrm{d}\mathbb{Q}_1}{\mathrm{d}\mathbb{P}}\right\}}\right) = 2E^{\mathbb{Q}_2}(u).
\end{aligned}
$$

Note that by (4) and (5), in the last equality above we used $\left\{\frac{\mathrm{d}\mathbb{Q}_2}{\mathrm{d}\mathbb{P}} > \frac{\mathrm{d}\mathbb{Q}_1}{\mathrm{d}\mathbb{P}}\right\} = \{W_{1,T} > W_{2,1}\} = \{x_{1,T} > x_{2,T}\}$.

So this inequality gives $\pi(u) \geq E^{\mathbb{Q}_2}(u) > \frac{1}{2}$. Similarly, one can show that $E^{\mathbb{Q}_1}(w) > \frac{1}{2}$, and as a result $\pi(w) > \frac{1}{2}$. So, we have

$$
\pi(u) + \pi(w) > \frac{1}{2} + \frac{1}{2} = 1 = \pi(1) = \pi(u + w). \tag{6}
$$

The inequality in (6) shows that the pricing rule is not perfect.

Example 5 (A Financial Approach) This example is based on an approach that is popular in the financial literature and is strongly related to factor models. It is also referred to as a nonparametric approach due to the lack of explicit assumptions about the asset models. In this approach, we need a set of test assets $x_0, x_1, \ldots, x_N$ with associated prices $p_0, p_1, \ldots, p_N$, which are assumed to be the most liquid assets in the market. We assume all pricing rules are able to correctly price the test assets. This implies that for any stochastic discount factor $z \in L^q$, we have $E(zx_i) = p_i, \forall i = 0, 1, \ldots, N$. Also note that if we need to impose a no-arbitrage condition, then we have to assume $z \geq 0$. In this approach, one can consider any closed cone $\mathcal{X}$ that is a sub-cone of all portfolios $\{\sum_{i=0}^{N} a_i x_i \,|a_i \in \mathbb{R}, i = 0, 1, \ldots, N\}$, containing the origin. If we denote the set of all stochastic discount factors by SDF, then any sub-linear pricing rule π can be expressed as (2), where $\Delta_\pi \subseteq$ SDF. In the financial literature, the typical set of test assets consists of excess returns[2] on a Fama–French portfolio (either the FF25, FF 50 or FF 100), plus the risk free asset. The associated prices are $p_0 = 1$, for the risk free and $p_1 = \cdots = p_N = 0$, for the other assets in the portfolio. One important implication of this approach is that for any sub-linear pricing rule π, it is linear on the set $\mathcal{X}$. Here, one can introduce a simple filtration $\mathcal{F}_t^S = \{\varnothing, \Omega\}$, for $t < T$ and $\mathcal{F}_T^S = \varsigma(x_0, x_1, \ldots, x_N)$. It is important to observe that the pricing rules are linear on the set of all hedgeable positions $\mathcal{X}$. Although this appears to be a credible assumption for the test assets which are liquidly traded in the market, in reality the ask and bid prices are different even for the most liquid assets. The implication of the bid–ask price spread is that the prices cannot be linear for hedgeable assets. Therefore, markets are usually imperfect and pricing rules are sub-linear on $\mathcal{X}$.

Remark 4 Even though the above examples provide a convincing argument for the generality and the practical relevance of the theoretical framework we considered in

[2] Excess return is the return of the discounted assets.

this paper, there are other important approaches to market consistent valuation that do not fit our framework. For instance, using a utility indifference pricing, in general, does not induce a sub-linear pricing rule.

3 Market consistent valuation

Let us now proceed with the definition of the market consistency.

Definition 3 Let Π be a risk evaluator and π be a pricing rule on $\mathcal{X}$. Π is market consistent of type I if

$$\Pi(x) = \pi(x), \forall x \in \mathcal{X}. \tag{7}$$

We say Π is market consistent of type II if

$$\Pi(x + y) = \pi(x) + \Pi(y), \forall x \in \mathcal{X} \text{ and } \forall y \in L^p. \tag{8}$$

Type II consistency states that hedging strategies cannot have an effect on the evaluation of the economic risks, i.e., it makes it neither better nor worse. Type I consistency does not have such an implication and only implies that for hedgeable positions, market and risk evaluators have similar valuation of risk. We will see that while the type II consistency holds only in perfect markets, the type I consistency can hold under very general conditions.

We have the following immediate result from the definition of market consistencies.

Proposition 1 *Market consistency of type II implies market consistency of type I.*

However, the opposite is not true as it is shown in the following example.

Example 6 Consider a risk evaluator Π with properties P3 which is not linear. Consider the pricing rule $\pi = \Pi$. If $\mathcal{X} = L^p$ then the market consistency of type I holds, while II does not!

One can prove the following theorem by following discussions in Pelsser and Stadje (2014).

Theorem 1 *Let π be a linear pricing rule on the space $\mathcal{X} = L^\infty(\mathcal{G})$ for a sigma-field $\mathcal{G} \subseteq \mathcal{F}$. If $\Pi : L^\infty \to \mathbb{R}$ is convex, then in a perfect market the following conditions are equivalent*

1. *Consistency of type I holds.*
2. *Consistency of type II holds.*
3. *Π can be presented as follows,*

$$\Pi(x) = \sup_{\{z \in L^1 | E_{\mathcal{G}}(z)=1\}} \{E(zx) - c(z)\},$$

for a penalty function $c : \{z \in L^1 \, | E_{\mathcal{G}}(z) = 1\} \to [0, \infty]$.

Even though one can easily introduce market consistent evaluators of type I, the same is not true for market consistent evaluators of type II in an imperfect market. Indeed, we will see in Theorem 3 that under general conditions, unless the market is perfect we cannot introduce a market consistent valuation of type II. For that, we need to introduce further propositions and theorem in the following.

Consider a sub-linear pricing rule π on a cone $\mathcal{X}$. In that case, we extend the range of π to $(-\infty, \infty]$

$$\bar{\pi}(x) = \begin{cases} \pi(x), & x \in \mathcal{X} \\ +\infty, & \text{otherwise.} \end{cases}$$

This extension allows us to use the dual representation of sub-linear pricing rules as

$$\bar{\pi}(x) = \sup_{z \in \Delta_{\bar{\pi}}} E(zx), \forall x \in L^p. \tag{9}$$

In order to obtain the dual representation for $\bar{\pi}$, we need to introduce the dual polar of a scalar cone of random payoffs. If $\mathcal{X}$ is a cone, the dual polar of the set $\mathcal{X}$ is given by

$$\mathcal{X}^\circ := \left\{ z \in L^q | E(zx) \leq 0 \, \forall x \in \mathcal{X} \right\}.$$

Note that $\mathcal{X}^\circ$ is a closed convex cone in L^q. We then have the following proposition in convex analysis.

Proposition 2 *For any function $\pi(x) := \sup_{z \in \Delta_\pi} E(zx)$, for some closed convex set Δ_π which is defined on a positive cone $\mathcal{X}$, we have that*

$$\bar{\pi}(x) = \sup_{z \in \Delta_\pi + \mathcal{X}^\circ} E(zx).$$

In other words, $\Delta_{\bar{\pi}} = \Delta_\pi + \mathcal{X}^\circ$.

Now, let us begin with the following theorem, which looks very similar to the results in Pelsser and Stadje (2014).

Theorem 2 *If Π is convex and π is sub-linear, the following are equivalent*

1. *$\Pi(x) \leq \pi(x)$ for all $x \in \mathcal{X}$.*
2. *$\Pi(x + y) \leq \pi(x) + \Pi(y)$ for all $x \in \mathcal{X}$ and $y \in L^p$.*
3. *Π accepts the following representation*

$$\Pi(x) = \sup_{z \in \Delta_{\bar{\pi}}} \{E(zx) - c(z)\}, \tag{10}$$

   for a penalty function $c \geq 0$.

4. *$\{\Pi^* < \infty\} \subseteq \Delta_{\bar{\pi}} = \Delta_\pi + \mathcal{X}^\circ$.*

Furthermore, if Π is also sub-linear then all are equivalent to $\Delta_\Pi + \mathcal{X}^\circ \subseteq \Delta_{\bar{\pi}} = \Delta_\pi + \mathcal{X}^\circ$.

Before we prove the theorem we need to introduce the inf-convolution and state some related propositions which prove to be very useful in the continuation. Let f_1 and f_2 be two convex functions defined from L^p to $(-\infty, \infty]$. Then, the inf-convolution of f_1 and f_2 is defined as

$$f_1 \Box f_2(y) = \inf_{x \in L^p} \{f_1(x) + f_2(y - x)\}.$$

The following proposition, which is a standard result in the literature of convex analysis, presents the necessary and sufficient conditions under which solution to the hedging problem exists (see Rockafellar 1997 for instance).

Proposition 3 *Let $x, y \in L^p$ be two random variables such that $f_1 \Box f_2(y) = f_1(x) + f_2(y - x)$. If f_1 is finite on a neighborhood around x, then $f_1 \Box f_2$ is proper, i.e., $\forall x \in L^p$, $f_1 \Box f_2(x) > -\infty$ and $\exists x \in L^p$, $f_1 \Box f_2(x) < \infty$.*

Proposition 4 *Let $f_1, f_2 : L^p \to (-\infty, \infty]$ be two convex functions. Then, the following two equalities hold*

$$(f_1 \Box f_2)^* = f_1^* + f_2^*,$$

and

$$(f_1 + f_2)^* = f_1^* \Box f_2^*,$$

with the convention that $\sup(\varnothing) = -\infty$.

Now, we can prove Theorem 2.

Proof of Theorem 2 We begin by proving $(1 \Rightarrow 3)$. First, observe that the statement 1 is equivalent to $\Pi(x) \le \bar{\pi}(x), \forall x \in L^p$; implying $\Pi^* \ge \bar{\pi}^*$. By Proposition 2 this means $\Pi^*(z) = +\infty$ if z does not belong to $\Delta_{\bar{\pi}}$. Using this fact, we have the following equalities

$$\begin{aligned}\Pi(x) &= \sup_{z \in L^q} \left\{E(zx) - \Pi^*(z)\right\} \\ &= \sup_{z \in \Delta_{\bar{\pi}}} \left\{E(zx) - \Pi^*(z)\right\}.\end{aligned}$$

Now, let us introduce the penalty function $c : \Delta_{\bar{\pi}} \to \mathbb{R}$ as $c(z) = \Pi^*(z)$. Note that $c \ge \bar{\pi}^* \ge 0$.

$(3 \Rightarrow 2)$. For all $x \in \mathcal{X}$ and $y \in L^p$ we easily have

$$\begin{aligned}\Pi(x + y) &= \sup_{z \in \Delta_{\bar{\pi}}} \{E(z(x + y)) - c(z)\} \\ &= \sup_{z \in \Delta_{\bar{\pi}}} \{E(zx) + E(zy) - c(z)\} \\ &\le \sup_{z \in \Delta_{\bar{\pi}}} E(zx) + \sup_{z \in \Delta_{\bar{\pi}}} \{E(zy) - c(z)\}\end{aligned}$$

$$= \sup_{z \in \Delta_\pi} E(zx) + \Pi(y)$$
$$= \pi(x) + \Pi(y).$$

$(2 \Rightarrow 1)$. It is enough to put $y = 0$.

$(2 \Leftrightarrow 4)$. It is clear that the statement 2 is equivalent to $\Pi(y) \leq \bar{\pi}(x) + \Pi(y - x), \forall x, y \in L^p$. This on its own is equivalent to $\Pi(y) \leq \inf_{x \in L^p} \{\bar{\pi}(x) + \Pi(y - x)\}$, $\forall y \in L^p$. But the right hand side is the inf-convolution between $\bar{\pi}$ and Π, so, $\Pi \leq \bar{\pi} \Box \Pi$. Therefore, the last inequality is equivalent to $\Pi^* \geq (\bar{\pi} \Box \Pi)^* = \bar{\pi}^* + \Pi^*$. This is also equivalent to $\{\Pi^* < \infty\} \subseteq \Delta_{\bar{\pi}}$. In particular, if Π is sub-linear then $\{\Pi^* < \infty\} = \Delta_\Pi$.

An immediate corollary is the following.

Corollary 1 *Assume Π is convex and π is sub-linear. If Π is market consistent of either type, then Π can be represented by* (10).

Remark 5 As one can see, if we instead of consistency of type I and II accept conditions 1 and 2 in Theorem 2, respectively, which can be interpreted as "sub-consistency," then we can prove the equivalence of the "sub-consistencies". However, as it will be made clear, we cannot always go further than this. That is why we will study consistency of types I and II separately.

Theorem *If Π is convex and π is sub-linear, the following are equivalent*

1. *Π is market consistent of type I*
2. *The following three statements hold together*
   (a) *$\Pi(x) \leq \pi(x), \forall x \in \mathcal{X}$;*
   (b) *Π is sub-linear on $\mathcal{X}$;*
   (c) *$\pi(x) = \sup_{\{\Pi^* < \infty\}} E(zx), \forall x \in \mathcal{X}$.*

Furthermore, if Π is also sub-linear then, both 1 and 2 are equivalent to $\Delta_\Pi + \mathcal{X}^\circ = \Delta_{\bar{\pi}}$.

Proof Let us first prove $(1 \Rightarrow 2)$. If Π is market consistent of type I then clearly a) and b) hold.

On the other hand, condition 1 is equivalent to $\bar{\Pi} = \bar{\pi}$, implying that $\bar{\Pi}$ is positive homogeneous. That means $\bar{\pi}(x) = \bar{\Pi}(x) = \sup_{\{\bar{\Pi}^* = 0\}} E(zx)$. Now, observe that since $\Pi \leq \bar{\pi}$, we have that $\Pi^* \geq \bar{\pi}^* \geq 0$. In addition, $\Pi \leq \bar{\Pi} \Rightarrow \Pi^* \geq \bar{\Pi}^* \Rightarrow \{\Pi^* < \infty\} \subseteq \left\{\bar{\Pi}^* < \infty\right\} = \left\{\bar{\Pi}^* = 0\right\}$. Therefore, for $x \in \mathcal{X}$, we get

$$\pi(x) = \Pi(x) = \sup_{\{\Pi^* < \infty\}} \left\{E(zx) - \Pi^*(z)\right\} \leq \sup_{\{\Pi^* < \infty\}} E(zx) \leq \sup_{\{\bar{\Pi}^* = 0\}} E(zx)$$
$$= \bar{\Pi}(x) = \pi(x).$$

This completes the proof of the first implication.

Now, let us prove $(2 \Rightarrow 1)$. Since *a*) holds then we only need to prove $\pi \leq \Pi$ on $\mathcal{X}$. It is clear that by *b*), $\bar{\Pi}$ is sub-linear. Therefore, $\bar{\Pi}(x) = \sup_{\{\bar{\Pi}^* < \infty\}} E(zx)$. But since

$\Pi \leq \bar{\Pi}$, we get $\bar{\Pi}^* \leq \Pi^*$, and consequently $\{\Pi^* < \infty\} \subseteq \{\bar{\Pi}^* < \infty\}$. Combining this and c) with the last relation we get $\forall x \in \mathcal{X}, \bar{\Pi}(x) = \sup_{\{\bar{\Pi}^*<\infty\}} E(zx) \geq \sup_{\{\Pi^*<\infty\}} E(zx) = \pi(x)$. This completes the proof of the second implication. □

Remark 6 This theorem has a very interesting implication: if Π is market consistent of type I, the set of all true stochastic discount factors is equal to $\{\Pi^* < \infty\}$.

Now, we have the following theorem for consistency of type II.

Theorem 3 *If Π is convex and π is sub-linear, the following are equivalent*

1. *Π is market consistent of type II*
2. *$\forall z \in \{\Pi^* < \infty\}$ and $x \in \mathcal{X}, \pi(x) = E(zx)$.*

Furthermore, if Π is also sub-linear then all conditions are equivalent to $\forall (z, x) \in \Delta_\Pi \times \mathcal{X}, \pi(x) = E(zx)$.

Proof Let us assume that 1 holds. For a fixed $x \in \mathcal{X}$, introduce the following two mappings $\Pi_1(y) := \pi(x) + \Pi(y)$, and $\Pi_2(y) := \Pi(x + y)$. It is easy to see that $\Pi_1^*(z) = -\pi(x) + \Pi^*(z)$ and $\Pi_2^*(z) = -E(xz) + \Pi^*(z)$. Since $\Pi_1 = \Pi_2$, and therefore $\Pi_1^* = \Pi_2^*$, we get that $\forall z \in \{\Pi^* < \infty\}$ and $x \in \mathcal{X}, \pi(x) = E(zx)$. This completes the proof of $(1 \Rightarrow 2)$.

Now, let us assume that 2 holds. For any $z \in \{\Pi^* < \infty\}$ and $x \in \mathcal{X}$ we have that

$$\begin{aligned}\Pi(x+y) &= \sup_{z_1 \in L^q} \left\{E(z_1(x+y)) - \Pi^*(z_1)\right\} \\ &= \sup_{z_1 \in \{\Pi^*<\infty\}} \left\{E(z_1(x+y)) - \Pi^*(z_1)\right\} \\ &\geq E(z(x+y)) - \Pi^*(z) \\ &= \pi(x) + E(zy) - \Pi^*(z).\end{aligned}$$

Since $z \in \{\Pi^* < \infty\}$ is chosen arbitrarily, then by taking supremum from both sides of the last inequality we get $\Pi(x + y) \geq \pi(x) + \Pi(y)$. On the other hand, let us fix $z \in \{\Pi^* < \infty\}$. Then, according to 2 we have $\bar{\pi} = \bar{E}_z$ where $\bar{E}_z(x) = E(zx)$ if $x \in \mathcal{X}$ and $+\infty$, otherwise. Then, it is clear that $\bar{\pi}^* = \bar{E}_z^*$, which means $\Delta_{\bar{\pi}} = z + \mathcal{X}^\circ$. This implies that $z \in \Delta_{\bar{\pi}}$, which according to Theorem 2 implies $\Pi(x + y) \leq \pi(x) + \Pi(y)$. □

Remark 7 One can see that for a fixed $z \in \{\Pi^* < \infty\}$ we have that $\pi(x) = E(zx)$. This has two important consequences. First, in order to have the market consistency of type II, it is necessary to accept that the market is perfect. The second implication is that all members of $\{\Pi^* < \infty\}$ are true stochastic discount factors as mentioned in Remark 6.

Finally, the following corollary is very useful:

Corollary 2 *If Π is convex and π is sub-linear, then Π is market consistent of type II if and only if it is market consistent of type I and the market is perfect.*

This means that once a market consistent valuation of type II exists, it has to be also market consistent of type I.

4 Compatibility and market consistency

In this section, we present a general hedging framework for pricing financial positions that cannot be perfectly hedged in an incomplete market. The hedging strategy, introduced below, is based on the concept of a *best estimate* for actuarial evaluation of an insurance position. We will show that under reasonable conditions, market consistency of either type is enough to guarantee that the risk estimator is a best estimator. We will also see how the best estimator representation of a market consistent evaluator can help us to obtain a two-step representation of market consistent evaluators.

4.1 Best estimator and hedging

In the following discussion, we demonstrate the strong relation between a market consistent valuation, of either type, and hedging strategies as used in the literature on pricing (e.g., see Jaschke and Küchler 2001; Staum 2004; Xu 2006; Assa and Balbás 2011; Balbás et al. 2009a, b, 2010; Arai and Fukasawa 2014). We assume that the value of a variable is equal to the sum of a *best estimate* and a *risk margin*. For that, we assume any non-hedgeable position (i.e., $L^p \backslash \mathcal{X}$) can be decomposed into two parts: one which is fully hedged (associated with the best estimate) and a part which left and produces some risk (associated with the risk margin). However, for reasons that will be discussed below, we will extend the concept of a best estimate in a new direction.

More specifically, let us introduce the hedging strategy by considering a position y in an incomplete market whose risk has to be evaluated consistent with the market. To achieve this, we find a variable, among all variables in the set $\mathcal{X}$, that mimics y most closely. In other words, we want to project y on the set $\mathcal{X}$ as its best estimation (associated with the best estimate). Suppose for a moment that we know the best estimation and denote it by $x \in \mathcal{X}$. Hence, y can be decomposed into two parts: a best estimation x an unhedged part $y - x$, which is associated with the risk margin. The cost of the best estimation part is given by $\pi(x)$, and the risk generated by unhedged part, which cannot be diversified by any member of $\mathcal{X}$, is measured by $\Pi(y - x)$. We call $\pi(x)$ the best estimate and $\Pi(y - x)$ the risk margin. The idea is to minimize the aggregate cost of the hedging given as $\pi(x) + \Pi(y - x)$. Therefore, one can state the problem as follows,

$$\Pi_\pi(y) := \inf_{x \in \mathcal{X}} \{\pi(x) + \Pi(y - x)\}. \tag{11}$$

In this case, the market imperfections are reflected by the (nonlinear) pricing rule π and the risk evaluator Π which capture the market incompleteness, respectively. From an insurance point of view, the minimum hedging cost can be considered as a normal practice if we see the pair $(x, y - x)$ as a capital restructuring that can mitigate the risk of the insurance company. It is clear that all insurance companies will restructure their capital to achieve the minimum risk, justifying the infimum in (11). Hence, we introduce the following concept:

Definition 4 For a risk evaluator Π and a pricing rule π, the best estimator Π_π is introduced by (11).

Remark 8 If Π and π are sub-linear, $\Pi_\pi(x)$ is the Good Deal upper bound introduced in Staum (2004). We obtain the Good Deal upper bound within a general competitive pricing and hedging framework.

Now we move toward addressing if Π_π is a well-defined evaluator. Let us first state the following result for Π_π defined in (11) (for a proof see Barrieu and Karoui 2005).

Proposition 5 *Let*

$$Dom\,(\Pi_\pi) := \left\{ y \in L^p | \Pi_\pi(y) \in \mathbb{R} \right\}.$$

Then, the following statements hold:

1. Π_π *and* $Dom\,(\Pi_\pi)$ *are positive homogeneous if* Π *and* π *are.*
2. Π_π *and* $Dom\,(\Pi_\pi)$ *are sub-additive if* Π *and* π *are.*
3. Π_π *and* $Dom\,(\Pi_\pi)$ *are convex if* Π *and* π *are.*
4. Π_π *and* $Dom\,(\Pi_\pi)$ *are translation-invariant if* Π *and* π *are.*
5. Π_π *is monotone if* Π *and* π *are monotone.*

First, note that Proposition 5 does not say if Dom (Π_π) is equal to L^p. Second, the proposition also does not say under which conditions $\Pi_\pi(0) = 0$. Actually if these two conditions hold then Π_π is a risk evaluator. Interestingly, it turns out that Dom $(\Pi_\pi) = L^p$ and $\Pi_\pi(0) = 0$ hold under very general conditions, which will be discussed shortly.

Definition 5 For a risk evaluator Π and a pricing rule π, compatibility holds if Π_π is a risk evaluator, i.e., if Dom $(\Pi_\pi) = L^p$ and $\Pi_\pi(0) = 0$. In the sequel, we denote compatibility by (C).

Theorem 4 *Let* Π *be convex and* π *be sub-linear. Then, the following conditions are equivalent:*

1. $\Pi_\pi(0) = 0$.
2. $Dom\,(\Pi_\pi) = L^p$.
3. $\Pi(-x) + \pi(x) \geq 0, \forall x \in \mathcal{X}$.
4. (C) *holds.*

In the case that Π *is convex and* π *is sub-linear, conditions above are equivalent to*

$$\{\Pi^* < \infty\} \cap \Delta_{\bar{\pi}} \neq \varnothing. \tag{12}$$

Proof First, let us prove $(1 \Leftrightarrow 3)$, i.e., 1 is equivalent to

$$\Pi(-x) + \pi(x) \geq 0, \quad \forall x \in \mathcal{X}. \tag{13}$$

It is easy to see that by construction, $\Pi_\pi(0) = 0$ implies (13). On the other hand, if (13) holds, it is easy to see that $\Pi_\pi(0) \geq 0$. In addition, by setting $x = 0$ in (11), it follows that $\Pi_\pi(0) = 0$.

Now we prove $(1 \Leftrightarrow 2)$. Given that Π is finite on L^p, in particular around a neighborhood of 0, and given that $\Pi(0) + \pi(0) = 0$, by Proposition 3 $\Pi_\pi(0) = 0$ is equivalent to Dom $(\Pi_\pi) = L^p$.

The implication (1 or 2 ⇔ 4) is easy to prove; indeed, form the definition it is clear that since 1 is equivalent to 2, then both are equivalent to compatibility.

Finally, we prove the last statement. Let $f_1(x) = \bar{\pi}(x)$ and $f_2(x) = \Pi(x)$. We have

$$(f_1^* + f_2^*)(z) = (\chi_{\Delta_{\bar{\pi}}} + \Pi^*)(z).$$

One can see that $(\bar{\pi}\Box\Pi)^*$ is proper if and only if $\{\Pi^* < \infty\} \cap \Delta_{\bar{\pi}} \neq \varnothing$. □

Corollary 3 *If Π and π are sub-linear and if $\Delta_\Pi \cap \Delta_{\bar{\pi}} \neq \varnothing$ then*

$$\Pi_\pi(y) = \sup_{z\in\Delta_\Pi\cap\Delta_{\bar{\pi}}} E(zy), \forall y \in L^p.$$

One important question is to establish the conditions under which a market consistent risk evaluator is also a best estimator. For that, we first state the following obvious proposition without proof.

Proposition 6 *If for pricing rule π and risk evaluator Π we have $\Pi(x + y) \leq \pi(x) + \Pi(y), \forall x \in \mathcal{X}, y \in L^p$, then (C) holds and $\Pi_\pi = \Pi$.*

Combining Proposition 6 with Theorem 2, we get the following theorem.

Theorem 5 *If π is sub-linear, Π is a convex market consistent risk evaluator of either type then $\Pi_\pi = \Pi$.*

Theorem 5 has two important implications: under the theorem's conditions, first Π is a best estimator, second, Π_π is market consistent of the same type as Π.

Remark 9 If Π is a coherent risk measure (i.e., monotone, cash invariant and sub-linear) and if π is sub-linear then condition 3 in Theorem 4 is equivalent to the No Good Deal assumption introduced and studied in Assa and Balbás (2011).

In general, it is not always true that Π_π is market consistent. Here, we illustrate with two examples that we cannot easily relax the assumptions in the previous theorem.

Example 7 Consider, $z_1, z_2 \in L^q$, $\pi(x) = E(z_1 x)$ and $\Pi(x) = \max\{E(z_1 x), E(z_2 x)\}$, and the cone $\mathcal{X} = \{x \in L^p | E(z_1 x) \geq E(z_2 x)\}$. First, observe that Π is market consistent of either type. One can easily see that $\mathcal{X}^\circ = \{z_2 - z_1\}$, and therefore, according to Theorem 4, $\Pi_\pi(x) = E(z_2 x)$. Again on the cone $\{x \in L^p | E(z_1 x) > E(z_2 x)\}$ the market consistency for Π_π, of either type, does not hold.

Example 8 Consider two different members z_1 and z_2 in L^q. Let $\pi(x) = \max\{E(z_1 x), E(z_2 x)\}$ and $\Pi(x) = E(z_2 x)$ and let $\mathcal{X} = L^p$. Then, according to Theorem 4 we have $\Pi_\pi(x) = E(z_2 x)$. This simply implies that on the cone $\{x \in L^p | E(z_1 x) > E(z_2 x)\}$, market consistency for Π_π, of either type, does not hold.

4.2 Best estimator in a perfect markets

Without knowing anything about the consistency of Π, it is difficult to prove whether the best estimator is market consistent. However, in a perfect market, we can answer this question by showing that Π_π is always market consistent of type I.

Theorem 6 *Assume that π is super-linear (i.e., P1 and $\pi(x+y) \geq \pi(x) + \pi(y), \forall x, y \in \mathcal{X}$) and that $\mathcal{X}$ is a vector space. If (C) holds, then Π_π is market consistent of type I.*

Proof One can easily see that (C) always implies $\Pi_\pi(0) = 0$. Now, we show $\Pi_\pi(0) = 0$ implies consistency of type I. Since $\mathcal{X}$ is a vector space, for a given $x \in \mathcal{X}$, we have that, $\mathcal{X} - x = \mathcal{X}$. Therefore, $\forall y \in \mathcal{X}$ by construction,

$$\Pi(y-x) + \pi(x-y) \geq \Pi_\pi(0) = 0,$$

and by super-linearity of π,

$$\Pi(y-x) + \pi(x) - \pi(y) \geq \Pi(y-x) + \pi(y-x) \geq 0$$

which implies that $\Pi(y-x) + \pi(x) \geq \pi(y)$. Therefore, we get $\Pi_\pi(y) \geq \pi(y)$.

On the other hand, if we let $x = y$, then we get $\pi(y) \geq \inf_{x \in \mathcal{X}} \Pi(y-x) + \pi(x) = \Pi_\pi(y)$. □

Corollary 4 *Assume that the market is perfect. If (C) holds, then Π_π is market consistent of type I.*

This corollary has a wide range of applications, since it shows how in a perfect market one can construct market consistent valuations.

4.3 Two-step evaluation and hedging

Pelsser and Stadje (2014) establish that if $\pi(x) = E(zx)$, for a unique stochastic discount factor z, then—under appropriate conditions—all market consistent risk evaluators can be represented within a two-step procedure

$$\Pi(x) = \pi\left(\Pi_{\mathcal{G}}(x)\right) = E\left(z\Pi_{\mathcal{G}}(x)\right),$$

for a particular mapping $\Pi_{\mathcal{G}} : L^\infty \to L^\infty(\mathcal{G})$, where at least $\Pi_{\mathcal{G}}(x) = x, \forall x \in L^\infty(\mathcal{G})$.

In order to have a two-step representation for market consistent risk evaluators in our setting, we generalize the concept of a two-step risk evaluator. First, for any $x \in \mathcal{X}$, let us introduce the following equivalent class $EQ_\pi(x)$

$$EQ_\pi(x) = \{y \in \mathcal{X} | \pi(y) = \pi(x)\}.$$

Moreover, define the class of all equivalent classes as follows,

$$EQ_\pi = \{B \subseteq L^p | \exists x \in \mathcal{X}; B \subseteq EQ_\pi(x)\}.$$

We have the following definition for a two-step evaluator.

Definition 6 An evaluation Π is a two-step evaluation if there exists a mapping $\Lambda : L^p \to EQ_\pi$ with $0 \in \Lambda(0)$, so that

$$\Pi = \pi \circ \Lambda. \tag{14}$$

It is clear that Π is a well-defined function.

First, using the hedging approach, we show market consistent evaluators can be represented, under particular conditions, as a two-step evaluation. Let Π be a risk evaluator and π a pricing rule. For any $y \in L^p$ let

$$S_{\pi,\Pi}(y) = \{x \in \mathcal{X} | x \text{ is a solution to the hedging problem}\}.$$

Then, we have the following theorem whose proof is straightforward and, hence, omitted.

Theorem 7 *Let $\pi : \mathcal{X} \to \mathbb{R}$ be a cash invariant pricing rule (i.e., $\pi(x + c) = \pi(x) + c, \forall, x \in \mathcal{X}, c \in \mathbb{R}$) and $\Pi : L^p \to \mathbb{R}$ be a risk evaluator. If for any $y \in L^p$, $S_{\pi,\Pi}(y) \neq \emptyset$ then Π_π can be represented as a two-step evaluator in (14) where $\Lambda(y) = \{x + \Pi(y - x) | x \in S_{\pi,\Pi}(y)\}$.*

Next, we combine this theorem with Theorem 5 to obtain the following representation.

Theorem 8 *If Π is convex, π is sub-linear and cash invariant (i.e., $\pi(x + c) = \pi(x) + c, \forall, x \in \mathcal{X}, c \in \mathbb{R}$) and Π is market consistent of either type and if for any $y \in L^p$, $S_{\pi,\Pi}(y) \neq \emptyset$ then Π has a two-step representation.*

5 Market consistent risk evaluators

So far, we have studied the conditions under which a risk estimator is market consistent. In the following sections we will introduce some families of market consistent evaluators and using discussions in Example 5 we show how one can construct them. Note that based on Corollary 2, in the following examples once we construct a convex market consistent risk evaluator Π, where its pricing rule is linear on $\mathcal{X}$, Π is automatically market consistent of type II. This is the main reason why we are mainly concerned with constructing market consistent valuations of type I.

5.1 A family of two-step estimators

Even though Theorem 8 assures that, under mild conditions, any market consistent evaluator can be represented as a two-step evaluator, it is not very helpful for constructing two-step evaluators in practice. For this reason, we opt to take a different path. Let us consider a mapping $\Pi_{\mathcal{X}} : L^p \to \{B : B \subseteq \mathcal{X}\}$, where $\Pi_{\mathcal{X}}(x) = \{x\}, \forall x \in \mathcal{X}$. It is clear that for any pricing rule π,

$$\Pi(y) = \min_{x \in \Pi_{\mathcal{X}}(y)} \pi(x)$$

is type I market consistent. Furthermore, if π is cone linear on $\mathcal{X}$ and Π is convex, then it is also type II market consistent. Therefore, the problem can be reduced to choosing an appropriate $\Pi_{\mathcal{X}}$. We propose the following strategy.

The first one is motivated by Pelsser and Stadje (2014), as explained in Sect. 4.3. Let $\mathcal{X} = L^p(\mathcal{G})$, for a sub-sigma-algebra $\mathcal{G} \subseteq \mathcal{F}$, and $\pi : L^p(\mathcal{G}) \to \mathbb{R}$ be any pricing rule. Let us also introduce $\Pi_{\mathcal{X}} = E_{\mathcal{G}}$, $\mathrm{VaR}_\alpha^{\mathcal{G}}$ or $\mathrm{CVaR}_\alpha^{\mathcal{G}}$, where they are expectation, VaR and CVaR conditioned on $\mathcal{G}$, respectively (for more details on this see Pelsser and Stadje 2014).The following examples are market consistent evaluators:

- $\Pi(x) = \pi\left(\delta E_{\mathcal{G}}(\max\{x - E_{\mathcal{G}}(x), 0\}^p) + E_{\mathcal{G}}(x)\right)$,
- $\Pi(x) = \pi\left(\delta \mathrm{VaR}_\alpha^{\mathcal{G}}(x - E_{\mathcal{G}}(x)) + E_{\mathcal{G}}(x)\right)$,
- $\Pi(x) = \pi\left(\delta \mathrm{CVaR}_\alpha^{\mathcal{G}}(x - E_{\mathcal{G}}(x)) + E_{\mathcal{G}}(x)\right)$.

However, a larger family of two-step evaluators can be constructed by using loss functions. Let $L : L^p \to [0, \infty]$ be a function so that $L(x) = 0$ if and only if $x = 0$. Let $\Pi_L(y) = \mathrm{argmin}_{x \in \mathcal{X}} L(y - x)$. For instance, one can consider $L(y) = \|y\|_{L^2}$. Therefore, the following market consistent evaluator can be constructed

$$\Pi^L(y) = \min_{\left\{x \in L^p \middle| L(y-x) = \min\limits_{x' \in \mathcal{X}} L(y-x')\right\}} \pi(x).$$

Note that, in many cases the minima x in $\left\{x \in L^p \,\middle|\, L(y-x) = \min_{x' \in \mathcal{X}} L(y-x')\right\}$ is unique; for instance, if we take $L(y) = \|y\|_{L^2}$ and $\mathcal{X}$ is a subspace in L^2. In order to develop more practical examples of market consistent risk evaluators, let us combine this idea with the approach we developed in Example 5. Indeed, let us consider $N + 1$ test assets $x_0, x_1, \ldots, x_N$, and associated prices $p_0, p_1, \ldots., p_N$, and consider $\mathcal{X} = \left\{\sum_{i=0}^N a_i x_i \,|a_i \in \mathbb{R}, i = 0, 1, \ldots, N\right\}$.

First, let us take $L(u) = \|u\|_{L^2}$. Then, for any $y \in L^2$, we have to solve the problem

$$\min_{(a_0, a_1, \ldots, a_N) \in \mathbb{R}^{N+1}} \sigma\left(y - \sum_{i=0}^N a_i x_i\right),$$

where $\sigma(.)$, denoted the standard deviation. The solution to this minimization problem is the OLS estimator of y on $x_0, x_1, \ldots, x_N$, which we denote by $\hat{a}$. Recall that since the pricing rule is linear over $\mathcal{X}$ for any $z \in \mathrm{SDF}$, the market consistent evaluator is

$$\Pi^{\mathrm{OLS}}(y) = E(z(\hat{a} \cdot X)) = \hat{a} \cdot p = p'\hat{a},$$

where $X = (x_0, x_1, \ldots, x_N)$ and $p = (p_0, p_1, \ldots, p_N)$.

If we denote the time series observations of each test asset x_i and the position y by $x_i = \left((x_{i,t})_{t=0}^T\right)'$ and $y = \left((y_t)_{t=0}^T\right)'$, then we know that the OLS estimator is obtained as $\hat{a} = (X'X)^{-1}y'X$, and therefore, the market consistent evaluation is given by

$$\Pi^{\mathrm{OLS}}(y) = p'(X'X)^{-1}y'X.$$

Note that if we use the Fama–French portfolios, then we also know that $p = (1, 0, 0, \ldots, 0)$.

Now let us consider another loss function to replace OLS with the quantile regression of y on$x_0, x_1, \ldots, x_N$. For that we have to assume

$$L(u) = \rho_{1-\alpha}(u) = u\left[(1-\alpha)\mathbb{I}_{\{u>0\}} - \alpha\mathbb{I}_{\{u\leq 0\}}\right],$$

and $\mathbb{I}\{\cdot\}$ denotes the indicator function. For a given tolerance level α, we have to solve the following problem

$$\min_{(a_0,a_1,\ldots,a_N)\in\mathbb{R}^{N+1}} \frac{1}{T}\sum_{t=1}^{T} \rho_\alpha\left(y_t - \sum a_i x_{i,t}\right). \tag{15}$$

It is interesting that one can also solve the quantile regression via solving the following problem

$$\mathrm{VaR}_\alpha\left(y - \sum_i a_i x_i\right) = 0.$$

In this case, a hedging strategy removes all the risk as measured by VaR_α. Therefore, the market consistent valuation is again

$$\Pi^{\mathrm{Q}}(y) = p'\hat{a},$$

where $\mathrm{VaR}_\alpha\left(y - \sum_i \hat{a}_i x_i\right) = 0$.

5.2 Super-evaluators

It is clear that for every type I market consistent evaluator Π, we have $\Pi \leq \bar{\pi}$. Indeed, if we assume for a moment that $+\infty$ belongs to the range of a risk evaluator, we can say that $\bar{\pi}$ is the largest type I market consistent risk evaluator. But the question is whether we can find the smallest type I market consistent evaluator.

In practice, pricing rules are non-decreasing since they have to be consistent with no-arbitrage condition. Thus, let us assume that π is non-decreasing. Then, motivated by the super-hedging strategy (e.g., see Karoui and Quenez 1995) for pricing, define the super estimator $\tilde{\Pi}$ as follows,

$$\tilde{\Pi}(x) = \inf_{\{y \in \mathcal{X}, y \geq x\}} \pi(x). \tag{16}$$

If we consider $\mathcal{X} = L^{\infty}$, it is clear that π is non-decreasing and as a result $\pi(-\|y\|_{\infty}) \leq \tilde{\Pi}(y) \leq \pi(\|y\|_{\infty})$. Therefore, $\tilde{\Pi}$ is well defined. On the other hand, by construction $\tilde{\Pi}(x) = \pi(x), x \in \mathcal{X}$, so $\tilde{\Pi}$ is type I market consistent.

Note that from a mathematical point of view, the super estimator is a best estimator when $\Pi(y) = \chi_{\{y \geq 0\}} = \begin{cases} 0, & y \geq 0 \\ +\infty, & \text{otherwise} \end{cases}$. We also have the following proposition.

Proposition 7 *Let us assume that π is non-decreasing. The, the following results hold:*

1. *$\tilde{\Pi}$ is positive homogeneous if π is.*
2. *$\tilde{\Pi}$ is sub-additive if π is.*
3. *$\tilde{\Pi}$ is convex if π is.*
4. *$\tilde{\Pi}$ is translation-invariant if π is.*
5. *$\tilde{\Pi}$ is non-decreasing.*

Consider again the approach we developed in Example 5. More specifically, we have to solve the following problem:

$$\begin{aligned} &\inf p'a \\ &\text{s.t. } y \leq \textstyle\sum_i a_i x_i \end{aligned} \tag{17}$$

In order to find the value of this linear programming problem, we solve the dual problem:

$$\begin{aligned} &\max E(\lambda y) \\ &E(x_i \lambda) = p_i, \quad i = 0, \ldots, N, \\ &\lambda \geq 0, \end{aligned} \tag{18}$$

where λ is a Lagrangian multiplier. If we assume the risk free asset is equal to one, i.e., $x_0 = 1$, then this corresponds to the so-called super-hedging price and λs are the members of the set of all SDF's.

6 Conclusion

To the best of our knowledge, this is the first paper that considers market consistency in imperfect markets. We presented several examples that justify the necessity of studying market consistent valuation in imperfect markets, including both complete and incomplete markets. In the first part of the paper, we distinguished between market consistency of two types, namely, types I and II. The type I consistency bears the very meaning of "consistency" by assuming that the market and risk evaluator are equal on hedgeable positions, whereas type II consistency further ensures that hedging strategies cannot improve the valuation of risky positions. While market consistency of type II implies the type I consistency, the opposite only can happen in perfect markets. Indeed, we demonstrated that the market consistency of type II only exists if the hedging strategy is perfect. This means that once a market consistent valuation of type II exists, it has to be also market consistent of type I. In the existing literature with perfect markets, the two definitions are equivalent. In the second part of the paper, motivated by the literature on pricing and hedging in incomplete markets, we introduced a best estimator and a risk margin. We showed that if the compatibility holds (e.g., Good Deals are ruled out), then a market consistent valuation is equal to its best estimator. We also used this to demonstrate how market consistent valuations can be represented in a two-step manner. Finally, we showed how to construct market consistent valuations as two-step estimators and super-evaluators.

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