Kernels and point processes associated with Whittaker functions

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Abstract. This article considers Whittaker's function $W_{\kappa,\mu}$ where κ is real and μ is real or purely imaginary. Then $\varphi(x) = x^{-\mu-1/2}W_{\kappa,\mu}(x)$ arises as the scattering function of a continuous time linear system with state space $L^2(1/2,\infty)$ and input and output spaces $\mathbf{C}$. The Hankel operator Γ_φ on $L^2(0,\infty)$ is expressed as a matrix with respect to the Laguerre basis and gives the Hankel matrix of moments of a Jacobi weight w. The operation of translating φ is equivalent to multiplying w by an exponential factor to give w_ε. The determinant of the Hankel matrix of moments of w_ε satisfies the σ form of Painlevé's transcendental differential equation PV. It is shown that Γ_φ gives rise to the Whittaker kernel from random matrix theory, as studied by Borodin and Olshanski (Comm. Math. Phys. 211 (2000), 335–358).

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1. Introduction

The Whittaker function $W_{\kappa,\mu}$ is the solution of the second order linear differential equation

$$y'' + \Big(\frac{-1}{4} + \frac{\kappa}{x} + \frac{1/4-\mu^2}{x^2}\Big)y = 0 \tag{1.1}$$

that is asymptotic to $x^\kappa e^{-x/2}$ as $x \to \infty$ through real values, and possibly has a logarithmic singularity at $x = 0$. The general solution of (1.1) is given by linear combinations of the Whittaker functions $M_{\kappa,\pm\mu}$. See [14, 15, 36 p. 343] for basic definitions and properties, such as $W_{\kappa,\mu} = W_{\kappa,-\mu}$. We consider the case in which κ is real, and μ is either real or purely imaginary; hence $W_{\kappa,\mu}(x)$ is real for all $x > 0$. In random matrix theory [4, 5, 6, 23], kernels such as

$$K(x,y) = \big((\kappa-1/2)^2-\mu^2\big)\sqrt{xy}\,\frac{W_{\kappa-1,\mu}(x)W_{\kappa,\mu}(y) - W_{\kappa,\mu}(x)W_{\kappa-1,\mu}(y)}{x-y}, \tag{1.2}$$

provide integral operators on $L^2((0,\infty);\mathbf{C})$, and are associated with determinantal random point fields. The purpose of this note is to provide some transparent proofs of some basic properties of these kernels and their associated determinantal random point fields. We show that the Whittaker kernels are closely related to systems of orthogonal polynomials for a Pollaczek–Jacobi type weight.

The classical Jacobi weight function [30] can be translated onto $[0,1]$ to become

$$w(x) = x^b(1-x)^a \qquad (0<x<1), \tag{1.3}$$

where $a,b>-1$. We will consider deformations of this weight which arise from multiplying by a positive factor, for instance $e^{-t/x}$ where t is the deformation parameter.

Definition With $a,b>-1$ and $t\geq 0$, we introduce

$$\mathcal{D}_N(t;a,b) = \det\Bigl[\int_0^1 x^{j+k}x^b(1-x)^a e^{-t/x}\,dx\Bigr]_{j,k=0}^{N-1} \tag{1.4}$$

or equivalently

$$\mathcal{D}_N(t;a,b) = \frac{1}{N!}\int_{[0,1]^N}\prod_{1\leq j<k\leq N}(x_j-x_k)^2\prod_{\ell=1}^{N} x_\ell^b(1-x_\ell)^a e^{-t/x_\ell}\,dx_\ell. \tag{1.5}$$

In section two, we introduce the notion of a linear system and scattering function. In section three, we show how the determinants $\mathcal{D}_N(t;a,b)$ are related to Whittaker functions and state the Painlevé transcendental differential equations [16, 20, 24] that they satisfy. In section four, we introduce the Whittaker kernels and obtain factorization theorems in the style of Tracy and Widom [32, 33, 34], which express Fredholm determinants of the kernels as determinants of Hankel integral operators on $L^2(0,\infty)$. These determinants do not involve N directly. The case $\kappa=a+1/2$ and $\mu=a$ is of particular interest, as in [25], and we consider this in sections 5 and 6.

A stochastic point process is a probability measure on the space of point configurations. See section 5.4 of [11] for the general definition of finite point processes. The process is said to be determinantal when the correlation functions are given by Fredholm determinants, as follows.

Definition (Determinantal point process) Let S be a continuous kernel on $(0,\infty)$ such that

(i) $S(x,y)=\overline{S(y,x)}$ for all $x,y\in(0,\infty)$;

(ii) the integral operator with kernel S satisfies the operator inequality $0\leq S\leq I$ as self-adjoint operators on $L^2(0,\infty)$;

(iii) for all $0 < u < v < \infty$, the integral operator with kernel $\mathbf{I}_{[u,v]}S(x,y)\mathbf{I}_{[u,v]}$ is trace class, where $\mathbf{I}_{[u,v]}$ denotes the indicator function of $[u,v]$.

Then S gives rise to a determinantal point process on (u,v) in Soshnikov's sense [7, 29]. Let T satisfy $I+T=(I-S)^{-1}$ as operators on $L^2(u,v)$. Then the probability that there are exactly N points in the realization, one in each subset dx_j for $j=1,\dots,N$ and none elsewhere is equal to

$$\det(I+T)^{-1}\det[T(x_j,x_k)]_{j,k=1}^{N}dx_1\dots dx_N, \tag{1.6}$$

where the $x_j\in[0,1]$.

2. Linear systems for the Whittaker functions

Definition (Linear systems) Let H be a complex separable Hilbert space known as the state space, and H_0 a finite dimensional complex Hilbert space which serves as the input and output space. Let $\mathcal{L}(H)$ be the space of bounded linear operators on H with the operator norm. A linear system $(-A,B,C)$ consists of:

(i) $-A$, the generator of a C_0 (strongly continuous) semigroup $(e^{-tA})_{t\geq 0}$ of bounded linear operators on H such that $\|e^{-tA}\|\leq Me^{-\omega_0 t}$ for all $t\geq 0$ and some $M,\omega_0\geq 0$;

(ii) $B: H_0\rightarrow H$ a bounded linear operator;

(iii) $C: H\rightarrow H_0$ a bounded linear operator.

We define the scattering function $\varphi\in C((0,\infty);\mathcal{L}(H_0))$ by $\varphi(t)=Ce^{-tA}B$, and the Hankel operator with scattering function φ by

$$\Gamma_\varphi f(x)=\int_0^\infty \varphi(x+y)f(y)\,dy \qquad (f\in L^2((0,\infty);H_0)), \tag{2.1}$$

as in [26]. Note that if the integral $\int_0^\infty t\|\varphi(t)\|^2_{\mathcal{L}(H_0)}dt$ converges, then Γ_φ defines a Hilbert–Schmidt operator. In particular, this holds if $\omega_0>0$.

We also introduce $R_\varepsilon: H\rightarrow H$ by

$$R_\varepsilon=\int_0^\infty e^{-tA}B_\varepsilon C_\varepsilon e^{-tA}\,dt, \tag{2.2}$$

and this integral plainly converges whenever $\omega_0>0$. See [3] for some related results.

For Whittaker functions, the basic linear system is the following. Let $\varepsilon>0$, $H_0=\mathbf{C}$ and $H=L^2((1/2,\infty);\mathbf{C})$ and $\mathcal{D}(A)=\{f(s)\in H: sf(s)\in H\}$. Then

$$\begin{aligned} A:&f(s)\mapsto sf(s), \qquad (f\in\mathcal{D}(A)); \qquad (2.3)\\ B_\varepsilon:&b\mapsto e^{-\varepsilon s}(s+1/2)^{(\kappa+\mu-1/2)/2}(s-1/2)^{(-\kappa+\mu-1/2)/2}b, \qquad (b\in\mathbf{C});\\ C_\varepsilon:&f(s)\mapsto\int_{1/2}^\infty e^{-\varepsilon s}\frac{(s+1/2)^{(\kappa+\mu-1/2)/2}(s-1/2)^{(-\kappa+\mu-1/2)/2}}{\Gamma(\mu-\kappa+1/2)}f(s)ds, \qquad (f\in\mathcal{D}(A)).\end{aligned}$$

2.1 Lemma *(i) Then the scattering function is*

$$\varphi_{(\varepsilon)}(x) = \frac{W_{\kappa,\mu}(x+2\varepsilon)}{(x+2\varepsilon)^{\mu+1/2}}. \tag{2.4}$$

(ii) Suppose that $\kappa, \mu \in \mathbf{R}$ and $-\kappa + \mu + 1/2 > 0$. Then R_ε is self-adjoint and nonnegative.

(iii) Suppose that $\Re\mu > \kappa - 1/2$, and either $\varepsilon > 0$, or $\Re\mu < 1/2$ and $\varepsilon = 0$. Then R_ε is trace class and

$$\det(I - \lambda R_\varepsilon) = \det(I - \lambda \Gamma_{\varphi_{(\varepsilon)}}) \qquad (\lambda \in \mathbf{C}). \tag{2.5}$$

Proof. (i) This is by direct computation, in which one uses a representation formula

$$\frac{W_{\kappa,\mu}(x)}{x^{\mu+1/2}} = \int_{1/2}^\infty e^{-sx}(s+1/2)^{(\kappa+\mu-1/2)}(s-1/2)^{(-\kappa+\mu-1/2)}\frac{ds}{\Gamma(-\kappa+\mu+1/2)} \tag{2.6}$$

for the Whittaker function from [13, 15].

(ii) The operator R_ε on $L^2(0,\infty)$ is represented by the kernel

$$\frac{e^{-\varepsilon s}(s+1/2)^{(\kappa+\mu-1/2)/2}}{(s-1/2)^{(\kappa-\mu+1/2)/2}}\frac{e^{-\varepsilon t}(t+1/2)^{(\kappa+\mu-1/2)/2}}{(t-1/2)^{(\kappa-\mu+1/2)/2}}\frac{1}{(s+t)\Gamma(-\kappa+\mu+1/2)}. \tag{2.7}$$

The first two factors are multiplication operators, while the final factor in Carleman's operator Γ on $L^2(0,\infty)$ with kernel $1/(x+y)$, as discussed in [26, p.440]. Now Γ is non negative as an operator, hence R_ε is also non negative by (2.7). See [17] for more analysis of operators of this form.

(iii) One can introduce Hilbert–Schmidt operators $\Xi, \Theta : L^2(0,\infty) \to L^2(1/2,\infty)$ by $\Theta f = \int_0^\infty e^{-tA^\dagger}C_\varepsilon^\dagger f(s)\,ds$ and $\Xi f = \int_0^\infty e^{-tA}B_\varepsilon f(s)\,ds$ such that $\Gamma_{\varphi_{(\varepsilon)}} = \Theta^\dagger \Xi$ and $R_\varepsilon = \Xi\Theta^\dagger$; see [3] for more details. The kernel of Ξ is

$$e^{-\varepsilon s - st}(s+1/2)^{(\kappa+\mu-1/2)/2}(s-1/2)^{(-\kappa+\mu-1/2)/2} \qquad (s > 1/2, t > 0), \tag{2.8}$$

which is Hilbert–Schmidt, hence R_ε and likewise $\Gamma_{\varphi_{(\varepsilon)}}$ are trace class with

$$\det(I - \lambda R_\varepsilon) = \det(I - \lambda \Xi\Theta^\dagger) = \det(I - \lambda\Theta^\dagger\Xi) = \det(I - \lambda\Gamma_{\varphi_{(\varepsilon)}}). \tag{2.9}$$

For $\alpha > 0$, let $L_n^{(\alpha)}(x)$ be the monic generalized Laguerre polynomial of degree n, defined by

$$L_n^{(\alpha)}(x) = (-1)^n \frac{e^x}{x^\alpha}\frac{d^n}{dx^n}\big(x^{n+\alpha}e^{-x}\big); \tag{2.10}$$

then the n^{th} generalized Laguerre function is $\phi_n^{(\alpha)}(x) = e^{-x/2}x^\alpha L_n^{(\alpha)}(x)/\Gamma(n+1)$.

2.2 Lemma *Let φ be the scattering function $\varphi(x) = W_{\kappa,\mu}(x)x^{-\mu-1/2}$, and let w be the translated Jacobi weight*

$$w(\xi) = \xi^{\mu-\kappa-1/2}(1-\xi)^{2\alpha-2\mu+1} \qquad (0 < \xi < 1). \tag{2.11}$$

Then the operation of Γ_φ on the generalized Laguerre basis is represented by a matrix of moments for the weight w, so

$$\big\langle \Gamma_\varphi \phi_\ell^{(\alpha)}, \phi_n^{(\alpha)} \big\rangle_{L^2(0,\infty)} = \frac{\Gamma(n+1+\alpha)}{\Gamma(n+1)} \frac{\Gamma(\ell+1+\alpha)}{\Gamma(\ell+1)} \int_0^1 \xi^{\ell+n} w(\xi)d\xi. \tag{2.12}$$

Proof. This is suggested by [27]. The Laplace transform of $\phi_n^{(\alpha)}$ satisfies

$$\hat{\phi}_n^{(\alpha)}(s) = \frac{1}{\Gamma(\alpha+n)} \frac{(s-1/2)^n}{(s+1/2)^{n+1+\alpha}}, \tag{2.13}$$

as one checks by repeatedly integrating by parts. Using the representation formula for φ as in (2.6), one can express $\langle \Gamma_\varphi \phi_n^{(\alpha)}, \phi_\ell^{(\alpha)} \rangle$ as an integral with respect to s over $(1/2, \infty)$. By changing variables to $\xi = (s-1/2)/(s+1/2)$, one obtains the integral of moments with respect to the weight w.

We now show how the leading minors of the Hankel operator Γ_φ are related to the Jacobi unitary ensemble. The joint probability density function of the Jacobi unitary ensemble on $[0,1]^N$ as in [23, 34] is

$$\frac{1}{N!} \frac{1}{\Gamma(\kappa+\mu+1)^N} \prod_{j=0}^{N-1} \frac{\Gamma(j+1+\alpha)}{\Gamma(j+1)} \prod_{0\leq j<k\leq N-1} (x_j - x_k)^2 \prod_{j=0}^{N-1} w(x_j). \tag{2.14}$$

Let $\Delta_N(t)$ be the multiple integral

$$\Delta_N(t) = \frac{1}{N!} \frac{1}{\Gamma(\kappa+\mu+1)^N} \prod_{j=0}^{N-1} \frac{\Gamma(j+1+\alpha)}{\Gamma(j+1)} \int_{[t,1]^N} \prod_{0\leq j<k\leq N-1} (x_j - x_k)^2 \prod_{j=0}^{N-1} w(x_j)dx_j, \tag{2.15}$$

as in Chen and Zhang [10].

2.3 Proposition *(i) The leading minors of the determinant of Γ_φ satisfy*

$$\det\Big[\big\langle \Gamma_\varphi \phi_n^{(\alpha)}, \phi_\ell^{(\alpha)} \big\rangle\Big]_{\ell,n=0}^{N-1} = \Delta_N(0). \tag{2.16}$$

(ii) Let $x_0, \dots, x_{N-1}$ be a sample of N points from the Jacobi unitary ensemble. Then the probability of the event $[x_j \geq t$ for all $j]$ equals $\Delta_N(t)/\Delta_N(0)$.

Proof. (i) This identity follows from Lemma 2.2 and the Heine–Andreief identity from [30].

(ii) Let $P_n^{(a,b)}(x)$ be the monic Jacobi polynomial of degree n for the weight $w(x) = x^b(1-x)^a$ on $[0,1]$, where we choose $b = \mu + \kappa - 1/2$ and $a = 2\alpha - 2\mu + 1$. See [30, page 58]. Then with the constants $\gamma_j = \int P_j^{(a,b)}(x)^2 w(x)dx$, the kernel

$$J_N(x,y) = \sum_{j=0}^{N-1} \frac{P_j^{(a,b)}(x)P_j^{(a,b)}(y)}{\gamma_j} \tag{2.17}$$

defines a self-adjoint operator on $L^2(w,[0,1])$ such that $0 \leq J_N \leq I$. Hence

$$\frac{\Delta_N(t)}{\Delta_N(0)} = \det\bigl(I - J_N \mathbf{I}_{(0,t)}\bigr). \tag{2.18}$$

2.4 Proposition *Let*

$$H_N(t) = t(1-t)\frac{d}{dt}\log \Delta_N(t). \tag{2.19}$$

Then $\sigma(t) = H_N(t) - d_1 - td_2$ satisfies the σ form of Painlevé's transcendental differential equation PVI, so

$$\sigma'\bigl[t(t-1)\sigma''\bigr]^2 + \bigl[2\sigma'(t\sigma'-\sigma) - (\sigma')^2 - \nu_1\nu_2\nu_3\nu_4\bigr]^2 = (\sigma'+\nu_1^2)(\sigma'+\nu_2^2)(\sigma'+\nu_3^2)(\sigma'+\nu_4^2) \tag{2.20}$$

where $\nu_1 = (a+b)/2$, $\nu_2 = (b-a)/2$, $\nu_3 = \nu_4 = (2N+a+b)/2$ with initial conditions $\sigma(0) = d_2$ and $\sigma'(0) = d_1$.

Proof. See [10], and [20].

3. Determinant formulas for the Pollaczek–Jacobi type weight

In this section, we consider the Pollaczek–Jacobi type weights, and show that translating the scattering function φ has the same effect as deforming the weights through multiplication by $e^{-2t/x}$. The m^{th} moment of the Pollaczek–Jacobi weight is defined to be

$$\mu_m(t;a,b) = \int_0^1 x^m x^b (1-x)^a e^{-t/x}\, dx \qquad (m = 0, 1, \dots). \tag{3.1}$$

3.1 Proposition *The moments satisfy*

$$\mu_m(t;a,b) = \Gamma(a+1)e^{-t/2}t^{(b+m)/2}W_{-(2a+b+m+2)/2,-(b+m+1)/2}(t). \tag{3.2}$$

Proof. Making the change of variable $1/x = s + 1/2$ in (2.6) we have

$$\mu_m(t;a,b) = e^{-t/2}\int_{1/2}^{\infty} e^{-st}(s-1/2)^{-\kappa+\mu-1/2}(s+1/2)^{\kappa+\mu-1/2}\, ds \tag{3.3}$$

with $\kappa = -(2a+b+m+2)/2$ and $\mu = -(b+m+1)/2$.

See [15, 3.4712, page 367]. With the change of integration variable $x \mapsto 1/\xi$,

$$\mu_m(t;a,b) = \int_1^\infty \xi^{-2-a-b-m}(\xi-1)^a e^{-t\xi}d\xi.$$

The translation operation $\varphi(t) \mapsto \varphi_{(\varepsilon)}(t) = \varphi(t+2\varepsilon)$ replaces w by

$$w_\varepsilon(x) = w(x)\exp\Big(-2\varepsilon\Big(\frac{1}{x}-\frac{1}{2}\Big)\Big). \tag{3.4}$$

For this problem, we shall be concerned with

$$\begin{aligned} &D_N(\varepsilon) \\ &= \prod_{j=0}^{N-1}\frac{\Gamma(j+\alpha+1)}{\Gamma(j+1)}[\Gamma(\kappa-\mu+(1/2))]^{-N}\frac{1}{N!}\int_{[0,1]^N}\prod_{0\le j<k\le N-1}(x_k-x_j)^2\prod_{\ell=1}^N w_\varepsilon(x_\ell)dx_\ell. \end{aligned} \tag{3.5}$$

Hence a straightforward change of variables gives

$$D_N(\varepsilon) = C_N(\varepsilon)\mathcal{D}_N(2\varepsilon;a,b) \tag{3.6}$$

where $\mathcal{D}_N(t;a,b)$ is defined in (1.4) and

$$C_N(\varepsilon) = e^{\varepsilon N}[\Gamma(\kappa-\mu+1/2)]^{-N}\prod_{j=0}^{N-1}\frac{\Gamma(j+\alpha+1)}{\Gamma(j+1)}. \tag{3.7}$$

As in Lemma 2.2 and Proposition 2.3, we have

$$D_N(\varepsilon) = \det\Big[\big\langle\Gamma_{\varphi_{(\varepsilon)}}\phi_n^{(\alpha)},\phi_\ell^{(\alpha)}\big\rangle\Big]_{\ell,n=0}^{N-1}. \tag{3.8}$$

Let the quantity $\tilde{H}_N(t)$ be defined as follows

$$\tilde{H}_N(t) = t\frac{d}{dt}\log\mathcal{D}_N(t) - N(N+b+a). \tag{3.9}$$

3.2 Theorem *Then $\tilde{H}_N(t)$ satisfies the Jimbo–Miwa–Okamoto σ form of Painlevé's PV for a special choice of parameters, so*

$$\begin{aligned} (t\tilde{H}_N'')^2 &= -4t(\tilde{H}_N')^3 + (\tilde{H}_N')^2\big(4\tilde{H}_N + (b+2a+t)^2 + 4N(N+a+b) - 4a(b+a)\big) \\ &\quad + 2\tilde{H}_N'\big(-(b+2a+t)\tilde{H}_N - 2Na(N+b+a)\big) + (\tilde{H}_N)^2. \end{aligned} \tag{3.10}$$

Proof. This was found in Chen and Dai [9, p. 2161].

3.3 Remarks (i) Chen and Dai [9, Theorem 6.1] also how that $(\tilde{H}_N)_{N=1}^{\infty}$ satisfies a second order nonlinear difference equation.

(ii) $\mathcal{D}_N(t;a,b)$ is the Wronskian determinant of

$$\{\mu_{2N-1}(t;a,b), \mu'_{2N-1}(t;a,b), \dots, \mu_{2N-1}^{(N-1)}(t;a,b)\}; \tag{3.11}$$

thus $\mu_{2N-1}(t;a,b)$ determines $\mathcal{D}_N(t;a,b)$.

(iii) The Painlevé V equation has previously appeared in various ensembles in random matrix theory. Tracy and Widom obtained PV in [34] from the Laguerre ensemble, and also from the Bessel ensemble from [32] which is associated with hard edge distributions. In [34], they also considered the ensembles U_N of $N \times N$ complex unitary matrices U with Haar measure, and obtained PV from the distribution of trace(U). Remarkably, this is related to the uniform measure on the symmetric group S_N of permutations as $N \to \infty$. Borodin and Olshanski considered the pseudo-Jacobi ensemble and obtained PV from a Fredholm determinant associated with the Whittaker functions $M_{\kappa,\mu}(1/x)$ with κ and μ real as in [6]. Their results have applications to the infinite dimensional unitary group U_∞. Lisovyy considered how the determinant for the hypergeometric kernel degenerates to the determinant for the Whittaker kernel, and realised PV as a limiting case of PVI; see [22, section 10].

The modified Bessel function as in (6.1) satisfies $K_\mu(z) = (2z/\pi)^{-1/2} W_{0,\mu}(2z)$ as in [28]; so in view of these results, it is fitting that PV should emerge from the Whittaker kernel.

4. The matrix Whittaker kernel

Borodin and Olshanski [5, 6] have considered kernels of the form (1.2). We can factorize the kernel in terms of products of Hankel operators, by analogy to the results of [31, 34]. In section 5, we see that the case $\kappa = a + 1/2$ and $\mu = a$ is of particular interest.

4.1 Proposition *The kernel satisfies*

$$\sqrt{zw}\frac{W_{\kappa,\mu}(z)W_{\kappa-1,\mu}(w) - W_{\kappa-1,\mu}(z)W_{\kappa,\mu}(w)}{(w-z)}$$
$$= \int_1^\infty \Big(\sqrt{z}W_{\kappa,\mu}(sz)\sqrt{w}W_{\kappa-1,\mu}(sw) + \sqrt{z}W_{\kappa-1,\mu}(sz)\sqrt{w}W_{\kappa,\mu}(sw)\Big)\frac{ds}{2s}. \tag{4.1}$$

Proof. Note that the left-hand side is a continuously differentiable function of $(z,w) \in (0,\infty)^2$ and that the left-hand side converges to zero as $z \to \infty$ or $w \to \infty$.

In the following proof we use that matrices

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \tilde{J} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{4.2}$$

Combining the differential equation (1.1) with the recurrence relation

$$z\frac{d}{dx}W_{\kappa,\mu}(z) = (\kappa - z/2)W_{\kappa,\mu}(z) - \big(\mu^2 - (\kappa - 1/2)^2\big)W_{\kappa-1,\mu}(z), \tag{4.3}$$

we obtain the matrix differential equation

$$z\frac{d}{dz}W = \big((1/2)I + \Omega(z)\big)W, \tag{4.4}$$

$$W(z) = \begin{bmatrix} W_{\kappa,\mu}(z) \\ W_{\kappa-1,\mu}(z) \end{bmatrix}, \quad \Omega(z) = \begin{bmatrix} \kappa - 1/2 - z/2 & (\kappa - 1/2)^2 - \mu^2 \\ -1 & 1/2 - \kappa + z/2 \end{bmatrix} \tag{4.5}$$

in which $\mathrm{trace}(\Omega) = 0$. The eigenvalues of $(1/2)I + \Omega(0)$ are $(1/2) \pm \mu$, so the eigenvalues differ by an integer if and only if 2μ is an integer; this characterizes the exceptional case for Birkhoff factorization into canonical form [35].

The system (4.4) resembles the system of matrix differential equation considered by Tracy and Widom [34], although the trace of the coefficient matrix is non-zero, so we use a variant on their methods as in [1]. We compute

$$\begin{aligned}\Big(z\frac{d}{dz} + w\frac{d}{dw}\Big)\frac{\langle JW(z), W(w)\rangle}{z-w} &= \frac{\langle JzW'(z), W(w)\rangle}{z-w} + \frac{\langle JW(z), wW'(w)\rangle}{z-w} \\ &\quad - \frac{\langle JW(z), W(w)\rangle}{z-w} \\ &= \frac{\langle J\Omega(z)W(z), W(w)\rangle}{z-w} + \frac{\langle JW(z), \Omega(w)W(w)\rangle}{z-w},\end{aligned} \tag{4.6}$$

where $J\Omega(z) + \Omega(w)^t J = -(1/2)(z-w)\tilde{J}$, so

$$\Big(z\frac{d}{dz} + w\frac{d}{dw}\Big)\frac{\langle JW(z), W(w)\rangle}{z-w} = -(1/2)\langle \tilde{J}W(z), W(w)\rangle. \tag{4.7}$$

For comparison, we have

$$\begin{aligned}&\Big(z\frac{d}{dz} + w\frac{d}{dw}\Big)\frac{1}{2}\int_1^\infty \langle \tilde{J}W(sz), W(sw)\rangle \frac{ds}{s} \\ &\qquad = \frac{1}{2}\int_1^\infty \Big(\langle \tilde{J}zW'(sz), W(sw)\rangle + \langle JW(sz), wW'(sw)\rangle\Big)\,ds \\ &\qquad = \frac{1}{2}\int_1^\infty \frac{d}{ds}\langle \tilde{J}W(sz), W(sw)\rangle ds \\ &\qquad = -\frac{1}{2}\langle \tilde{J}W(z), W(w)\rangle.\end{aligned} \tag{4.8}$$

Hence the sum

$$\frac{\langle JW(z), W(w)\rangle}{z-w} - \frac{1}{2}\int_1^\infty \langle \tilde{J}W(sz), W(sw)\rangle \frac{ds}{s} \tag{4.9}$$

is a function of z/w, which converges to zero as $z \to \infty$ or $w \to 0$ in any way whatever, so this sum is zero. To obtain the stated result, we multiply by $\sqrt{zw}$ and rearrange the terms.

Let K be the integral operator on $L^2((a_1, a_2); dx)$ with kernel

$$K(x,y) = \frac{\langle JW(x), W(y)\rangle}{x-y}. \tag{4.10}$$

For suitable a_1 and a_2, we can suppose that $\|K\| < 1$ as an operator on $L^2((a_1, a_2); dx)$ and let $S = K(I-K)^{-1}$; so that $(I+S)(I-K) = I$ and $S : L^2((a_1, a_2); dx) \to L^2((a_1, a_2); dx)$ has a kernel

$$S(x,y) = \frac{Q(x)P(y) - Q(y)P(x)}{x-y}, \tag{4.11}$$

and we take $S(x,x) = Q'(x)P(x) - Q(x)P'(x)$ on the diagonal.

Let $D(a_1, a_2; K)$ be the Fredholm determinant of K, regarded as a function of the endpoints a_1 and a_2.

4.2 Proposition *(i) The restrictions of the kernels of the operators S and S^2 to the diagonal satisfy*

$$x\frac{d}{dx}S(x,x) = -P(x)Q(x) + (S^2)(x,x). \tag{4.12}$$

(ii) The exterior derivative with respect to the endpoints satisfies

$$d\log D(a_1, a_2; K) = S(a_1, a_1)da_1 - S(a_2, a_2)da_2. \tag{4.13}$$

Proof. (i) Let M be the operator of multiplication by the independent variable x and D the operator of differentiation with respect to x. For a integral operator T let $\delta T = [MD, T] - T$, so that δT has kernel $(x\partial/\partial x + y\partial/\partial y)T(x,y)$; thus δ is a pointwise derivation on the kernels, while $T \mapsto [MD, T]$ is a derivation on operator composition. We use $\doteq$ to mean that an operator corresponds to a certain kernel. Then

$$[M, K] \doteq W_{\kappa,\mu}(x)W_{\kappa-1,\mu}(y) - W_{\kappa,\mu}(y)W_{\kappa-1,\mu}(x) \tag{4.14}$$

and (4.7) shows that

$$\delta K \doteq -2^{-1}\langle \tilde{J}W(x), W(y)\rangle.$$

Then $[M,S](I-K)=(I+S)[M,K]$, so the kernels have

$$[M,S] \doteq Q(x)P(y)-Q(y)P(x) \tag{4.15}$$

where $Q=(I-K)^{-1}W_{\kappa,\mu}$ and $P=(I-K)^{-1}W_{\kappa-1,\mu}$ are differentiable functions. Hence S is also an integral operator with kernel of the form (4.11). Also, by some straightforward manipulations, we have

$$\delta S=(I-K)^{-1}(\delta K)(I-K)^{-1}+S^2, \tag{4.16}$$

where

$$(I-K)^{-1}(\delta K)(I-K)^{-1} \doteq -(1/2)(P(x)Q(y)+P(y)Q(x)) \tag{4.17}$$

and a short calculation shows that $(\delta S)(x,x)=x(d/dx)S(x,x)$. Hence the result.

(ii) This is a standard consequence of the results from [19].

Let $\phi_{\kappa-1,\mu}(t)=W_{\kappa-1,\mu}(e^t)$ and $\phi_{\kappa-1,\mu}(t)=W_{\kappa,\mu}(e^t)$ and form the matrix

$$\Phi(t)=\begin{bmatrix} 0 & 0 & \phi_{\kappa,\mu}(t) & 0\\ 0 & 0 & \phi_{\kappa-1,\mu}(t) & 0\\ \phi_{\kappa-1,\mu}(t) & \phi_{\kappa,\mu}(t) & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \tag{4.18}$$

and the integral operator on $L^2((0,\infty);\mathbf{C})$ with kernel

$$T \doteq 2\frac{\phi_{\kappa,\mu}(t)\phi_{\kappa-1,\mu}(u)-\phi_{\kappa,\mu}(u)\phi_{\kappa-1,\mu}(t)}{e^t-e^u}. \tag{4.19}$$

4.3 Corollary *The Hankel operator with scattering function* Φ *operates on* $L^2((0,\infty);\mathbf{C}^4)$ *and satisfies*

$$\det(I-T)=\det(I+\Gamma_\Phi). \tag{4.20}$$

Proof. By Proposition 4.1, with the change of variable $s=e^u$, we see that the integral operator satisfies $T=\Gamma_{\phi_{\kappa-1,\mu}}\Gamma_{\phi_{\kappa,\mu}}+\Gamma_{\phi_{\kappa,\mu}}\Gamma_{\phi_{\kappa-1,\mu}}$ where Γ_ϕ is in the standard form of a Hankel operator on $L^2(0,\infty)$ as in (2.1). By some elementary determinant manipulations, the Hankel with matrix valued scattering function satisfies $\det(I+\Gamma_\Phi)=\det(I-\Gamma_{\phi_{\kappa-1,\mu}}\Gamma_{\phi_{\kappa,\mu}}-\Gamma_{\phi_{\kappa,\mu}}\Gamma_{\phi_{\kappa-1,\mu}})$, which produces the stated result.

4.4 Remark By [15, 9.237], Whittaker's differential equation generalizes the differential equation that the associated Laguerre functions satisfy. The column vector $Y(z)=\sqrt{z}W(z)$ satisfies $z(d/dz)Y(z)=\Omega(z)Y(z)$, which resembles the differential equation for generalized Laguerre functions on [33, page 60]. In Remark 5.2 of [2] we obtained a factorization theorem for certain Whittaker kernels which expressly excluded the case of generalized Laguerre functions.

5. A special case of the Whittaker kernel

For $-1/2 < a < 1/2$, let R_a be the integral operator on $L^2(0,\infty)$ that has kernel

$$R_a \doteq \frac{(y/x)^a e^{-(x+y)/2}}{x+y} \tag{5.1}$$

and R be the operator on $L^2((0,\infty);\mathbf{C}^2)$ given by the block matrix

$$R = \begin{bmatrix} 0 & R_a \\ -R_{-a} & 0 \end{bmatrix} \tag{5.2}$$

and let Ψ be the matrix-valued scattering function

$$\Psi(t) = \begin{bmatrix} 0 & \Gamma(1-2a)/(1+t)^{1-2a} \\ -\Gamma(1+2a)/(1+t)^{1+2a} & 0 \end{bmatrix} \qquad (t>0). \tag{5.3}$$

5.1 Proposition *(i) The operators $R_{\pm a}$ are bounded on $L^2(0,\infty)$ for $|a|<1/2$.*

(ii) For all $0<a_1<b_1<\infty$, the operators R and Γ_Ψ are trace class on $L^2((a_1,b_1);\mathbf{C}^2)$ and satisfy

$$\det(I-\lambda R) = \det(I-\lambda\Gamma_\Psi) \qquad (\lambda\in\mathbf{C}). \tag{5.4}$$

(iii) The operator R is skew self-adjoint, and $I+R$ is invertible with block matrix form

$$R(I+R)^{-1} = \begin{bmatrix} R_aR_{-a}(I+R_aR_{-a})^{-1} & -R_a(I+R_{-a}R_a)^{-1} \\ -R_{-a}(I+R_aR_{-a})^{-1} & R_{-a}R_a(I+R_{-a}R_a)^{-1} \end{bmatrix}. \tag{5.5}$$

(iv) The integral operator R_aR_{-a} is represented by the kernel

$$\Gamma(1-2a)\frac{W_{a+1/2,a}(x)W_{a-1/2,a}(y)-W_{a-1/2,a}(x)W_{a+1/2,a}(y)}{(x-y)\sqrt{xy}}. \tag{5.6}$$

Proof. (i) This was proved by Olshanski [25] using a Fourier argument. For completeness we give an equivalent proof involving Mellin transforms. First we check that the operators $R_{\pm a}$ are bounded on $L^2(0,\infty)$. The expression

$$\int_0^\infty \frac{(x/y)^a}{(x/y)+1}\frac{f(y)dy}{y} = \frac{1}{\sqrt{x}}\int_0^\infty \frac{(x/y)^a}{\sqrt{(x/y)}+\sqrt{(y/x)}}\frac{\sqrt{y}f(y)\,dy}{y} \tag{5.7}$$

is a Mellin convolution as in [28], and one shows by a standard contour integration argument that

$$\int_0^\infty \frac{z^{a+i\sigma-1/2}dz}{z+1} = \pi\operatorname{sech}\pi(a+i\sigma) \tag{5.8}$$

is bounded for all $\sigma \in \mathbf{R}$ for all $|a| < 1/2$.

(ii) This follows as in Lemma 2.1. Letting M_2 be the 2×2 complex matrices with Hilbert–Schmidt norm, we introduce the state space $H = L^2((\omega_0, \infty); M_2)$ and the input and output space $H_0 = \mathbf{C}^2$, then

$$\begin{aligned} A : \mathcal{D}(A) \rightarrow H : &\quad f(x) \mapsto x f(x); \\ B_{\omega_0} : H_0 \rightarrow H : &\quad b \mapsto \begin{bmatrix} 0 & e^{-x/2} x^{-a} \\ -e^{-x/2} x^a & 0 \end{bmatrix} b; \\ C_{\omega_0} : H \rightarrow H_0 : &\quad g(y) \mapsto \int_{\omega_0}^\infty \begin{bmatrix} e^{-y/2} y^{-a} & 0 \\ 0 & e^{-y/2} y^a \end{bmatrix} g(y)\, dy. \end{aligned} \tag{5.9}$$

Then $\|e^{-tA}\| \leq e^{-t\omega_0}$ for all $t > 0$, and the scattering function is

$$C_{\omega_0} e^{-tA} B_{\omega_0} = \int_{\omega_0}^\infty \begin{bmatrix} 0 & e^{-ty-y} y^{-2a} \\ -e^{-ty-y} y^{2a} & 0 \end{bmatrix} dy; \tag{5.10}$$

and we note that

$$C_{\omega_0} e^{-tA} B_{\omega_0} \rightarrow \Psi(t) \tag{5.11}$$

as $\omega_0 \rightarrow 0+$, the corresponding R operator has kernel

$$R = \int_0^\infty e^{-tA} B_{\omega_0} C_{\omega_0} e^{-tA} dt; \tag{5.12}$$

so as $\omega_0 \rightarrow 0$, we obtain

$$\begin{aligned} R &\doteq \int_0^\infty \begin{bmatrix} 0 & e^{-t(x+y)-(x+y)/2} (y/x)^a \\ -(x/y)^a e^{-t(x+y)-(x+y)/2} & 0 \end{bmatrix} dt \\ &\doteq \begin{bmatrix} 0 & (y/x)^a e^{-(x+y)/2}/(x+y) \\ -(x/y)^a e^{-(x+y)/2}/(x+y) & 0 \end{bmatrix}. \end{aligned} \tag{5.13}$$

(iii) Note that $R_a^* = R_{-a}$ hence $R^* = -R$ and $I + R_{-a} R_a$ is invertible. By elementary row operations one checks that

$$\begin{bmatrix} I & R_a \\ -R_{-a} & I \end{bmatrix}^{-1} = \begin{bmatrix} (I + R_a R_{-a})^{-1} & -R_a (I + R_{-a} R_a)^{-1} \\ R_{-a} (I + R_a R_{-a})^{-1} & (I + R_{-a} R_a)^{-1} \end{bmatrix}, \tag{5.14}$$

in which we observe that the indices a and $-a$ alternate.

(iv) We have

$$\begin{aligned} R_a R_{-a} &\doteq \int_0^\infty \frac{(x/y)^a e^{-(x+y)/2}}{x+y} \frac{(y/z)^{-a} e^{-(y+z)/2}}{y+z} dy \\ &\doteq \frac{(xz)^a e^{-(x+z)/2}}{z-x} \int_0^\infty \Big(\frac{1}{x+y} - \frac{1}{y+z} \Big) y^{-2a} e^{-y}\, dy \end{aligned} \tag{5.15}$$

where [14, 14.2 (17) , and page 431] gives the final integral in terms of an incomplete Gamma function which reduces to Whittaker's function

$$\int_0^\infty \frac{y^{-2a}e^{-y}}{x+y}dy = \Gamma(1-2a)x^{-a-1/2}e^{x/2}W_{a-1/2,a}(x). \tag{5.16}$$

We also have the identity from [14, p 432]

$$x^a e^{-x/2} = x^{-1/2}W_{a+1/2,a}(x). \tag{5.17}$$

The result follows on substituting (5.16) and (5.17) into (5.15).

6. Diagonalizing the Whittaker kernel

The Whittaker's functions are related to Bessel's functions and Laguerre's functions. Let $K_\nu(x)$ be the modified Bessel function of the second kind given by

$$K_\nu(x) = \int_0^\infty \cosh(\nu t)e^{-x\cosh t}\, dt, \tag{6.1}$$

which is holomorphic on the open right half plane $\{x : \Re x > 0\}$ and decays rapidly as $x \to \infty$ through real values. When $\nu = im$ is purely imaginary, one refers to McDonald's function [28].

Let L_a be the differential operator

$$L_a f = -\frac{d}{dx}\Big(x^2\frac{df}{dx}\Big) + \big(x/2+a\big)^2 f(x). \tag{6.2}$$

Then $L_{-\kappa}$ has eigenfunctions

$$f_{\kappa,m}(x) = x^{-1}W_{\kappa,im}(x) \qquad (m \geq 0), \tag{6.3}$$

so that

$$L_{-\kappa}f_{\kappa,m}(x) = \big(1/4+m^2+\kappa^2\big)f_{\kappa,m}(x). \tag{6.4}$$

6.1 Proposition *(i) The differential operator L_{-a} commutes with R_aR_{-a} on $C_c^\infty((0,\infty);\mathbf{C})$, and*

$$L_{-a}R_a = R_aL_a. \tag{6.5}$$

(ii) R can be expressed as a diagonal operator with respect to $(f_{\pm a,m})_{m\geq 0}$.

Proof. (i) Suppose that f is smooth and has compact support inside $(0,\infty)$. Then we can repeatedly integrate by parts the integral

$$\int_0^\infty \frac{(y/x)^a e^{-(x+y)/2}}{(x+y)}\Big(-\frac{d}{dy}\Big(y^2\frac{df}{dy}\Big) + (y/2+a)^2 f(y)\Big)dy \tag{6.6}$$

without introducing boundary terms, to obtain

$$\int_0^\infty \frac{(x/y)^a e^{-(x+y)/2}}{x+y}\Big(\frac{y^2}{4}+a^2+\frac{y^2}{(x+y)^2}+ay + \frac{y^2}{x+y}+\frac{2ay}{x+y}-a-y+\frac{y^2}{(x+y)^2}-\frac{2y}{x+y}\Big) f(y)\,dy. \tag{6.7}$$

After a little reduction, one shows that this coincides with

$$\Big(-x^2\frac{d^2}{dx^2}-2x\frac{d}{dx}+(x/2-a)^2\Big)\int_0^\infty \frac{(y/x)^a e^{-(x+y)/2}}{x+y} f(y)\,dy. \tag{6.8}$$

Likewise, we have $L_aR_{-a}=R_{-a}L_{-a}$; so L_{-a} commutes with R_aR_{-a}.

(ii) Erdelyi [14, 14.3 (53)] quotes the following formula for the Stieltjes transform

$$\int_0^\infty \frac{y^{-1-a}e^{-y/2}}{x+y}W_{-a,im}(y)dy=\Gamma((1/2)-a+im)\Gamma((1/2)-a-im)x^{-a-1}e^{x/2}W_{a,im}(x), \tag{6.9}$$

so $f_{-a,m}(x)=x^{-1}W_{-a,im}(x)$ satisfies

$$R_af_{-a,m}(x)=\Gamma((1/2)-a+im)\Gamma((1/2)-a-im)f_{a,m}(x). \tag{6.10}$$

It follows that $f_{-a,m}$ is an eigenvector for $R_{-a}R_a$. Wimp [37] considered a Fourier–Plancherel formula which decomposes $L^2((0,\infty);\mathbf{C})$ as a direct integral with respect to $f_{a,m}(x)$, where $f_{a,m}(x)$ are eigenfunctions of L_{-a}, so there is a transform pair

$$\begin{aligned} g(m) &= \Gamma(1/2-\kappa-im)\Gamma(1/2-\kappa+im)\int_0^\infty f(t)f_{\kappa,m}(t)\,dt,\\ f(x) &= \frac{1}{\pi^2}\int_0^\infty t\sinh(2\pi t)f_{\kappa,t}(x)g(t)\,dt. \end{aligned} \tag{6.11}$$

Hence we can take $(f_{a,m})_{m\geq 0}$ to be a basis of $L^2((0,\infty);\mathbf{C})$, and $(f_{-a,m})_{m\geq 0}$ to be a basis of another copy of $L^2((0,\infty);\mathbf{C})$ and diagonalize R with respect to the combined basis $(f_{\pm a,m})_{m\geq 0}$ of $L^2((0,\infty);\mathbf{C}^2)$.

6.2 Corollary *For $-1/2<a<1/2$, the kernel*

$$K \doteq \frac{\Gamma(1-2a)\cos^2\pi a}{\pi^2}\frac{W_{a+1/2,a}(x)W_{a-1/2,a}(y)-W_{a-1/2,a}(x)W_{a+1/2,a}(y)}{\sqrt{xy}(x-y)} \tag{6.12}$$

gives a determinantal point process on $[0,\infty)$ such that random points λ_j, ordered by $0\leq\lambda_1\leq\lambda_2\leq\dots$, satisfy

$$\Pr[\lambda_1\geq s]=\det(I-K\mathbf{I}_{[0,s]}). \tag{6.13}$$

Proof. Note that

$$\Gamma(a+1/2+im)\Gamma(a+1/2-im)\Gamma(-a+1/2+im)\Gamma(-a+1/2-im) = 2\pi^2/(\cos 2\pi a+\cosh 2\pi m)$$

so

$$\|R_aR_{-a}\|_{\mathcal{L}(L^2(0,\infty))} = \pi^2\sec^2\pi a. \tag{6.14}$$

Hence we have $0 \leq K \leq I$ as operators on $L^2(0,\infty)$, and $K(x,y)$ has a continuous kernel. Hence by Mercer's theorem, K restricts to a trace class operator on $L^2(0,b)$ for all $0<b<\infty$ with

$$\mathrm{trace}\,(K) = \int_0^b K(x,x)\,dx. \tag{6.15}$$

Note that the right-hand side is finite for all $b>0$, but diverges to ∞ as $b\to\infty$, hence $\det(I-K\mathbf{I}_{[0,b]})\to 0$ as $b\to\infty$.

By Soshnikov's theorem [29], there is a point process on $[0,b]$ with K as the generating kernel. The point process distributes random x in $[0,b]$ such that only finitely many x can lie in each Borel subset of $[0,b]$, and the joint distribution of the points is specified as follows. Let B_j $(j=1,\dots,m)$ be disjoint Borel subsets of $[0,b]$ and $n_j=\sharp\{x\in B_j\}$ the number of random points that lie in B_j. Then the joint probability generating function of the random variables n_j is

$$\mathbf{E}\prod_{j=1}^m z_j^{n_j} = \det\Bigl(I+\mathbf{I}_{[0,b]}\sum_{j=1}^m (z_j-1)K\mathbf{I}_{B_j}\Bigr) \qquad (z_j\in\mathbf{C}). \tag{6.16}$$

Equivalently, as in [7, p. 597] we compress K to $\mathbf{I}_{[0,b]}K\mathbf{I}_{[0,b]}$ on $L^2([0,b])$ and introduce $T_{[0,b]}=\mathbf{I}_{[0,b]}K\mathbf{I}_{[0,b]}(I-\mathbf{I}_{[0,b]}K\mathbf{I}_{[0,b]})^{-1}$ so that

$$\det(I+T_{[0,b]}) = \bigl(\det(I-\mathbf{I}_{[0,b]}K\mathbf{I}_{[0,b]})\bigr)^{-1}. \tag{6.17}$$

For each integer $\ell\geq 0$, the ℓ-point correlation function is defined to be the positive symmetric function

$$\rho_\ell(x_1,\dots,x_\ell) = \det(I+T_{[0,b]})^{-1}\det\bigl[T_B(x_j,x_k)\bigr]_{j,k=1}^{\ell} \qquad (x_j\in[0,b]), \tag{6.18}$$

such that

$$\Pr(\text{there are exactly} \quad \ell \quad \text{particles in} \quad B) = \frac{1}{\ell!}\int_{B^\ell}\rho_\ell(x_1,\dots,x_\ell)dx_1\dots dx_\ell \tag{6.20}$$

for all Borel subsets B of $(0,b)$.

Remarks 6.3 (i) In particular [28, (7.4.25)] we can write

$$f_{0,m}(x) = \frac{1}{\sqrt{\pi x}} K_{im}(x/2) = \frac{1}{\sqrt{2}} \int_1^\infty P_{-1/2+im}(s) e^{-sx/2} ds \tag{6.21}$$

in terms of the MacDonald and associated Legendre functions. The function $W_{0,im}$ also occurs in the spectral decomposition of the Laplace operator over the fundamental domain that arises from the action of $SL(2, \mathbf{Z})$ on the upper half plane; see [21, p 318] for a discussion of Maass cusp forms. In [2], we considered the Hankel operators that commute with second order differential operators, and found the case L_0 and R_0 as Q(vii).

(ii) Another case of Q(vii) from [2] is $L_{-\kappa}$ commuting with the Hankel operator with scattering function $x^{-1}W_{\kappa,1/2}(x)$ on $C_c^\infty(0, \infty)$. If $a \neq 0$, then R_a is not a Hankel operator but an operator of Howland's type as in [17], and L_a does not commute with R_{-a}.

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