# A SHORT PROOF OF THE FACT THAT THE MATRIX TRACE IS THE EXPECTATION OF THE NUMERICAL VALUES

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ABSTRACT. Using the fact that the normalised matrix trace is the unique linear functional $f$ on the algebra of $n \times n$ matrices which satisfies $f(I) = 1$ and $f(AB) = f(BA)$ for all $n \times n$ matrices $A$ and $B$, we derive a well-known formula expressing the normalised trace of a complex matrix $A$ as the expectation of the numerical values of $A$; that is the function $\langle Ax, x \rangle$, where $x$ ranges the unit sphere of $\mathbb{C}^n$.

Let $A = [a_{ij}]$ be an $n \times n$ complex matrix. The aim of this note is to give an easy proof of the fact that the normalised trace of $A$, $\operatorname{tr} A = \frac{1}{n}(a_{11} + a_{22} + \ldots + a_{nn})$, can be thought of as the expectation of the numerical values of $A$; that is the function $x \mapsto \langle Ax, x \rangle$ defined on the Euclidean unit sphere in $\mathbb{C}^n$, endowed with the normalised Lebesgue surface measure $\mu$. More precisely,

$$(1) \qquad \operatorname{tr} A = \int\limits_{\|x\|=1} \langle Ax, x \rangle \, \mu(\mathrm{d}x).$$

The above formula is a particular version of a more general identity for symmetric 2-tensors on Riemannian manifolds (consult *e.g.* [2]; see also [1] for the proof[1]). We offer here an elementary proof of Equation (1) relying on two folklore facts from linear algebra.

**Lemma 1.** *The matrix trace is unique in the sense that it is the unique linear functional* $f \colon M_n(\mathbb{C}) \to \mathbb{C}$ *satisfying the following properties:*
   (i) $f(I) = 1$, *where* $I$ *denotes the* $n \times n$ *identity matrix,*
   (ii) $f(AB) = f(BA)$ *for all* $A, B \in M_n(\mathbb{C})$.

It is evident that the standard normalised trace tr on $M_n(\mathbb{C})$ satisfies conditions (i)-(ii), so in order to prove the above lemma, it is enough to show that a functional $f$ enjoying (i)-(ii) agrees with tr on the standard matrix units $e_{ij} = [\delta_{i,j}]$ $(1 \leqslant i, j \leqslant n)$; that is, $f(e_{ij}) = \frac{1}{n}$ whenever $i = j$ and $f(e_{ij}) = 0$ otherwise. We leave this as an exercise for the reader.

We shall require also the following easy and well-known fact. (See also [3, Lemma 3.2.21].)

**Lemma 2.** *Every complex* $n \times n$ *matrix is a linear combination of unitary matrices.*

*Proof.* Every matrix $A \in M_n(\mathbb{C})$ can be written as a linear combination of two self-adjoint matrices, so without loss of generality it is enough to show that each self-adjoint matrix

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[1]Bennett Chow blames one of his students for a neat proof he posted at [1].

$A$ with $\|A\| \leqslant 1$ can be written as a linear combination of unitaries. To this end, set $U = A - i(I - A^2)^{\frac{1}{2}}$ and note that $U$ is unitary. Clearly $A = \frac{1}{2}U + \frac{1}{2}U^*$. $\hfill\square$

We are now in a position to prove Equation (1).

*Proof.* Let $f(A)$ denote the right hand side of Equation (1). It is enough to verify that $f$ meets conditions (i)-(ii) of Lemma 1. Evidently, $f$ is linear, and $f(I) = 1$ because $\mu$ is a probability measure. It remains to show that (ii) holds.

Let $A, B \in M_n(\mathbb{C})$ and let us write $B$ as a linear combination of some unitary matrices $U_1, \ldots, U_m$, that is $B = \sum_{k=1}^m a_k U_k$ for some scalars $a_1, \ldots, a_m$. We may assume additionally that each matrix $U_k$ $(k \leqslant n)$ has determinant 1, as we can always write $B = \sum_{k=1}^m (a_k \det U_k)\frac{U_k}{\det U_k}$. We have $f(AB) = f(BA)$ as soon as $f(AU_k) = f(U_k A)$ for all $k \leqslant m$, so that without lost of generality we may suppose that $B$ is unitary and $\det B = 1$. Making the substitution $x = B^*z$ and taking into account that the determinant of $B$ is equal to 1 (hence also the Jacobian of $B$, regarded as a map from the real $(2n-1)$-sphere to itself, is equal to 1), we arrive at the conclusion that

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\begin{aligned}
f(AB) &= \int_{\|x\|=1} \langle ABx, x\rangle \, \mu(\mathrm{d}x) \\
&= \int_{\|x\|=1} \langle Az, B^*z\rangle \, \mu(\mathrm{d}z) \\
&= \int_{\|x\|=1} \langle BAz, z\rangle \, \mu(\mathrm{d}z) \\
&= f(BA),
\end{aligned}
$$

which completes the proof. $\hfill\square$

## References

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   http://mathoverflow.net/questions/127692/geometric-interpretation-of-tracing/150260#150260.
2. S. Gallot, D. Hulin, J. Lafontaine, *Riemannian geometry.* Universitext, Springer-Verlag, Berlin, Heidelberg, New York, 1987.
3. G. K. Pedersen, *Analysis Now*, Springer-Verlag, New York, 1989.

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