FINITE GENERATION IN C^*-ALGEBRAS AND HILBERT C^*-MODULES

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Abstract. We characterize C^*-algebras and C^*-modules such that every maximal right ideal (resp. right submodule) is algebraically finitely generated. In particular, C^*-algebras satisfy the Dales–Żelazko conjecture.

1. Introduction

Magajna's paper [11] characterizing C^*-modules consisting of compact operators has been much emulated, as is revealed by a cursory search in a citation index. Here we prove a complementary characterization, inspired by the recent Dales–Żelazko conjecture that if A is a unital Banach algebra all of whose maximal right ideals are algebraically finitely generated as right modules over A, then A is finite dimensional [8]. Indeed the instigation of this paper was a question Dales asked independently of both authors, and which both authors answered around August 2012, as to whether this conjecture was true for C^*-algebras. (He was able to answer this for special classes of C^*-algebras.) One ingredient of the solution is a characterization of algebraically finitely generated one-sided ideals in C^*-algebras. Although this is well known to experts (the algebraically finitely generated projective modules over a C^*-algebra constitute one of the common ways to picture its K-theory, and hence are well understood), we could not find it in the literature. Thus we include a direct proof due to Rørdam, as well as a very short C^*-module proof. We then use this to characterize C^*-algebras and C^*-modules such that every maximal right ideal (resp. right submodule) is algebraically finitely generated.

Turning to notation and background, we denote by A^1 the unitization of the C^*-algebra A. By 'projection' in this paper we mean a self-adjoint idempotent e in A. Then e is a *minimal projection* in A if eAe is one dimensional (which if A is a von Neumann algebra, is equivalent to e having no non-trivial proper subprojections). For convenience we usually work with right modules in this paper. It is well known that all C^*-algebras have an abundant supply of maximal right ideals. This is equivalent to saying that the bidual of A, A'', which is a von Neumann algebra, has an abundant supply of non-zero minimal projections (see 3.13.6 in [13], or the paragraph before Lemma 2.2 below, for the correspondence between minimal projections and maximal right ideals). Indeed, every

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right ideal is an intersection of maximal right ideals (see 3.13.5 in [13]). We will not really use the facts in the present paragraph though, except for those which we prove below.

Before we proceed we require another piece of terminology. Let H and K be Hilbert spaces. A closed subspace Z of $\mathscr{B}(K,H)$ is called a *ternary ring of operators* (or a *TRO*, for short) if it is closed under the *ternary product*, that is, $ZZ^*Z \subseteq Z$. Every Hilbert C^*-module Z may be viewed as a TRO by identifying it with the (1-2)-corner of its linking algebra (see *e.g.* 8.1.19 and 8.2.8 in [2]). Thus we will write z^*w in place of $\langle z, w\rangle$ for elements in a right C^*-module Z. Also, the so-called compact operators $\mathbb{K}(Z)$ may be written as ZZ^* (here and below for sets X, Y we write XY for the closure of the span of products xy for $x \in X, y \in Y$). To say that two TRO's are isomorphic as TRO's means that there is a linear isomorphism between them which is a ternary morphism (that is, $T(xy^*z) = T(x)T(y)^*T(z)$). Hamana showed that this is equivalent to inducing a corner-preserving $*$-isomorphism between the Morita linking C^*-algebras of the TRO's; and it is also equivalent to being completely isometric as operator spaces (a result also contributed to by Harris, Kaup, Kirchberg, Ruan, and no doubt others; see *e.g.* [2] for references and self-contained proofs).

2. Finitely generated ideals

The following lemma is well known to experts, although we could not find a reference for it. We shall present a direct self-contained proof that we are grateful to Mikael Rørdam for having communicated to us. Of course there are many other proofs, including the one in the next Remark.

Lemma 2.1. *Every algebraically finitely generated closed left (resp. right) ideal of a C^*-algebra A is actually singly generated, and equals Ap (resp. pA) for a projection $p \in A$.*

Proof. We consider first the case where A is unital. Let $J \subseteq A$ be a closed left ideal. We shall use the following general fact: for each positive element $a \in J$ and each continuous function $f\colon [0,\infty) \to \mathbb{R}$ with $f(0) = 0$, we have $f(a) \in J$. (This follows by approximating f uniformly on the spectrum of a by real polynomials vanishing at 0.)

Suppose that J is generated as a left ideal by the elements $a_1, \ldots, a_n$ for some $n \in \mathbb{N}$, and set $b = a_1^*a_1 + \cdots + a_n^*a_n \in J$. Then $b^{1/4}$ belongs to J by the above fact, so that $b^{1/4} = c_1a_1 + \cdots + c_na_n$ for some $c_1, \ldots, c_n \in A$. We may suppose that J is non-zero, which implies that b is non-zero, and consequently $c_1, \ldots, c_n$ are not all zero. Since $x^*x + y^*y - (x^*y + y^*x) = (x-y)^*(x-y) \geqslant 0$ for any $x, y \in A$, we deduce that

$$a_j^*c_j^*c_ka_k + a_k^*c_k^*c_ja_j \leqslant a_j^*c_j^*c_ja_j + a_k^*c_k^*c_ka_k$$

for $j \neq k$. Hence

$$b^{1/2} = (b^{1/4})^*b^{1/4} = \sum_{j,k=1}^{n} a_j^*c_j^*c_ka_k \leqslant n\sum_{j=1}^{n} a_j^*c_j^*c_ja_j \leqslant nK\sum_{j=1}^{n} a_j^*a_j = nKb,$$

where $K = \max_{1\leqslant j\leqslant n} \|c_j\|^2 > 0$. By elementary spectral calculus, this implies that the spectrum of b is contained in the set $\{0\} \cup [(nK)^{-2}, \infty)$, so that we can take a continuous function $f\colon [0,\infty) \to [0,1]$ such that $f(0) = 0$ and $f(t) = 1$ for each $t \geqslant (nK)^{-2}$. Then $p = f(b)$ is a projection such that $pb = bp = b$, and p belongs to J by the fact stated above. In particular we have

$$0 = (1-p)b(1-p) = \sum_{j=1}^{n} \big(a_j(1-p)\big)^* a_j(1-p),$$

which implies that $a_j = a_j p \in Ap$ for each $j \in \{1, \ldots, n\}$. Hence $J = Ap$, and the result follows.

Let us now consider the case where A is non-unital. Let J be a closed, finitely generated left ideal of A. Then J is finitely generated when regarded as a left ideal of A^1. Let p be a projection in A^1 such that $J = A^1 p = \{x \in A^1 : xp = x\}$. Then $p = 1p \in J \subset A$, so

$$J = \{x \in A : xp = x\} = Ap.$$

The right-ideal case is similar or follows by symmetry by considering the opposite C^*-algebra. □

Remark. Lemma 2.1 also follows from a well-known C^*-module 'generalization' of it, which is a basic result in the theory of Hilbert C^*-modules (see *e.g.* p. 255–257 in [16] or the proof of 8.1.27 in [2]). Namely, a right C^*-module Z over A is algebraically finitely generated over A iff there are finitely many $z_k \in Z$ with $z = \sum_k z_k z_k^* z$ for all $z \in Z$. Note that this immediately implies Lemma 2.1 by taking Z to be the right ideal of A in Lemma 2.1: in this case if $e = \sum_k z_k z_k^*$, which is in Z, then $e^2 = e$ and $e \geqslant 0$. So e is a projection in the right ideal, and now it is easy to see that this right ideal equals eA.

If K is a maximal right ideal of A then e, the complement of the support projection of K, is a minimal projection in A''. This is well known (see 3.13.6 in [13]), but here is a simple argument for this. We recall that the support projection of K is the smallest projection $p \in A''$ with $px = x$ for all $x \in K$. Thus e is the largest projection in A'' with $ex = 0$ for all $x \in K$. We will assume for simplicity that $e \in A$, which will be the case for us in Corollary 2.3 below, but the general case is very similar (but uses modifications of some steps below using Cohen factorization and 'second dual techniques' valid in any Arens regular Banach algebra, and one should replace eAe and eA below by $\{a \in A\colon a = eae\}$ and $\{a \in A\colon a = ea\}$). We will use only the well-known fact that every non-trivial C^*-algebra has a proper non-zero closed right ideal, *e.g.* the right kernel of any non-faithful state. If e is not minimal, that is if $A_e = eAe$ is not one dimensional, then A_e has a proper closed non-zero right ideal I, and $I = IA_e$ as usual. Then $W = IA$ is a closed right ideal of A. Note that $W \neq eA$ since $\{w \in W : we = w\} \subset I \neq A_e$. On the other hand, $K + W = A$ by maximality of K (note $K \cap W \subset (1-e)A \cap eA = \{0\}$). Thus $eA = e(K+W) = W$. This contradiction shows that e is a minimal projection.

Lemma 2.2. *A C^*-algebra A is unital if even one maximal right ideal is algebraically finitely generated over A.*

Proof. As we said above, a maximal right ideal of A has a support projection whose complement is a minimal projection $q \in A''$. On the other hand, if J is an algebraically finitely generated right ideal then by Lemma 2.1 the support projection of J is in J. Thus if J is an algebraically finitely generated right ideal which is a maximal right ideal, then $1-q \in J \subset A$ for a non-zero minimal projection q in A''. Hence $q = 1-(1-q)$ belongs to $M(A)$, the multiplier algebra of A, and of course $qAq \neq \{0\}$ since $q \neq 0$. Therefore $\{0\} \neq qAq = \mathbb{C}\, q \subset A$, and so q and $1 = (1-q)+q$ are in A. So A is unital. □

Corollary 2.3. *A C^*-algebra A is finite dimensional iff every maximal right ideal is algebraically finitely generated over A.*

Proof. For the non-obvious direction, by Lemma 2.2 we may suppose that A is unital. Let J be the right ideal generated by all the minimal projections in A. If $J \neq A$ let K be a maximal (proper) right ideal of A containing J. The support projection of K is in A by Lemma 2.1, hence its complement e is in A too. As we proved above Lemma 2.2, e is a minimal projection, and we obtain the contradiction $e \in J \subseteq K = (1-e)A$. So $A = J$, and therefore $1 = \sum_{k=1}^n e_k a_k = \sum_{k=1}^n a_k^* e_k$ for minimal projections e_k, and some $a_k \in A$. It is well known from pure algebra that $\dim(eAf) \leqslant 1$ for minimal $e, f \in A$ (a quick proof in our case where these are projections: if $v = eaf \neq 0$ then v^*v is a positive scalar multiple of f, so that left multiplication by v^* is an isomorphism $eAf \cong fAf$). From these facts it is clear that $A = \sum_{j,k=1}^n e_j A e_k$ is finite dimensional. □

Remark. Although we have chosen to give a selfcontained C^*-algebraic argument in the last proof, there are more algebraic arguments available that even allow one to generalize some of the above. For example, note that the hypothesis in the last result together with a result of the type of Lemma 2.1, implies that every maximal right ideal is a (module) direct summand. But the latter implies finite dimensionality. Indeed the elementary argument in the lines after Proposition 5.10 in [14] (which is a slight variant of our argument in the last proof) shows that any ring A whose maximal right ideals are (module) direct summands, equals its socle. Hence A is semisimple in the ring-theoretic sense, and one can apply the Wedderburn-Artin theorem. We thank Manuel Reyes for the last reference. So if A is in addition a Banach algebra over $\mathbb{C}$ it is now clear that it is finite dimensional. Similarly, one obtains the well known fact that a unital Banach algebra with dense socle (and hence equals its socle) is finite dimensional. This is also related to the theory of *modular annihilator algebras* (see *e.g.* 8.4.14 in [12], and its proof).

Corollary 2.4. *A unital C^*-algebra A is finite dimensional iff A contains all minimal projections in A''.*

Proof. If A contains all such projections and J is a maximal right ideal of A, then the support projection p of J is in A (since its complement is a minimal projection). So $J = pA$. The result now follows from Corollary 2.3. □

Remark. One might ask which of the results above extend to the class of not necessarily self-adjoint algebras of operators on a Hilbert space (resp. to classes of Banach algebras). In [4, 3] there are variants of one or two of the facts above for closed right ideals with a contractive (resp. 'real-positive' left approximate identity). For example, comparing with Lemma 2.1, such right ideals which are algebraically finitely generated as right modules over the algebra A, are precisely the right ideals of the form eA for a projection e (resp. a 'real-positive' idempotent) in the algebra (see [4, Corollary 2.13] and [3, Corollary 4.7]; in the latter reference it is also assumed that A has a contractive approximate identity but probably this is not necessary). Comparing with Lemma 2.2, and following its proof, one sees that A is a unital operator algebra say, which possesses even one such ideal which is algebraically finitely generated over A, and which is maximal in the sense that the complement e of its support projection is minimal in the sense that $eA''e$ is one dimensional. However even if A is unital, it need not have any right ideals of this type at all. Thus our techniques above towards the Dales–Żelazko conjecture break down in this case, although our method suggests that the way to proceed may be via the socle of A.

3. A C^*-module generalization

We now show that C^*-modules of the form $\bigoplus_{k=1}^m \mathscr{B}(\mathbb{C}^{n_k}, H_k)$ (that is, direct sums of rectangular matrix blocks with the length of the rows in each block allowed to be infinite), are the 'only' right C^*-modules Z such that every maximal right submodule of Z is algebraically finitely generated.

Theorem 3.1. *Let Z be a right C^*-module. Then every maximal right submodule of Z is algebraically finitely generated iff there are positive integers $m, n_1, \cdots, n_m$, and Hilbert spaces H_k, such that $Z \cong \bigoplus_{k=1}^m \mathscr{B}(\mathbb{C}^{n_k}, H_k)$ as TRO's.*

Proof. ($\Rightarrow$) Suppose that Z is a right C^*-module over a C^*-algebra B, and that every maximal right submodule of Z is algebraically finitely generated over B. Then every maximal right submodule W of Z is algebraically finitely generated over Z^*Z (since W is a non-degenerate Z^*Z-module and hence any $w \in W$ may be written as $w = w'c$ for $w' \in W, c \in Z^*Z$ by Cohen's factorization theorem. Hence $wb = w'(cb)$ with $cb \in Z^*Z$, for $b \in B$). So we may assume that $B = Z^*Z$.

We will be using the simple relationship between right submodules of Z and right ideals of ZZ^* perhaps first noticed by Brown [6]. If J is a maximal right ideal of $A = \mathbb{K}(Z) = ZZ^*$, then JZ is a right submodule of Z. If $JZ = Z$ then

$$J = JA = JZZ^* = ZZ^* = A,$$

a contradiction. So JZ is a proper right submodule of Z. If W is a proper closed right submodule of Z containing JZ, then WZ^* is a right ideal of $\mathbb{K}(Z)$ and it contains $JA = J$. If $WZ^* = A$, then $W = WZ^*Z = AZ = Z$, a contradiction. Hence $WZ^* = J$, so that $W = WZ^*Z = JZ$. Thus JZ is a maximal right submodule of Z, and hence JZ is finitely generated over Z^*Z. By the well-known argument/fact in the remark after Lemma 2.1 above, JZ has generators $z_1, \ldots, z_n$ with

$$\sum_{k=1}^{n} z_k z_k^* a z = az$$

for all $a \in J, z \in Z$. Hence $ea = a$ for all $a \in J$ where $e = \sum_{k=1}^n z_k z_k^* \in J$. Clearly $J = eZZ^*$. By Lemma 2.2 we see that ZZ^* is unital, and by Corollary 2.3 we have that ZZ^* is a finite dimensional C^*-algebra, hence $ZZ^* \cong \bigoplus_{k=1}^m M_{n_k}$ $*$-isomorphically. Now we are in well-known territory, indeed Hilbert C^*-modules over C^*-algebras of compact operators are completely understood. For example, by basic Morita equivalence (as in *e.g.* the proof on pp. 851–852 in [11], or p. 2125 of [15]) we have $Z^*Z \cong \bigoplus_{k=1}^m \mathbb{K}(H_k)$, and $Z \cong \bigoplus_{k=1}^m \mathscr{B}(\mathbb{C}^{n_k}, H_k)$, for Hilbert spaces H_k. (The cited papers do not explicitly use the term 'ternary morphism', but it is clear that their morphisms are such.)

($\Leftarrow$) This is the easy direction. Indeed, if $Z = \bigoplus_{k=1}^m \mathscr{B}(\mathbb{C}^{n_k}, H_k)$ then every right Z^*Z-submodule W is finitely generated over Z^*Z (since WW^* is finite dimensional, hence unital). And clearly this property is preserved by ternary isomorphisms. □

Remark. All right C^*-modules (which are not Hilbert spaces) have an abundant supply of maximal right submodules (one can see this for example from the paragraph before Lemma 2.2, and the correspondence in the proof of Theorem 3.1). Indeed, every right submodule is an intersection of maximal right submodules.

Closing remark. Let κ be a cardinal number. We will say that a right module V over A is *algebraically κ-generated* if there is a set $\{v_\alpha \colon \alpha < \kappa\}$ in V with cardinality κ such that every element in A is a finite sum $\sum_{k=1}^n v_{\alpha_k} a_k$ for some $a_k \in A^1$ and $\alpha_1, \ldots, \alpha_n < \kappa$. We call algebraically $\aleph_0$-generated modules *algebraically countably generated.* One might ask if 'algebraically finitely generated' could be replaced by 'algebraically countably generated' or 'algebraically κ-generated' for some uncountable cardinal κ in all of the results in our paper. In fact this is automatic in the countable case: It is proved in [5] that a right ideal of a Banach algebra is closed if its closure is algebraically countably generated in this sense. The proof in [5] works for modules too; thus a right submodule of a Banach module over A is closed if its closure is algebraically countably generated. Then as in [8, Corollary 1.6], closed algebraically countably generated right submodules of a Banach module over A are finitely generated. One can even go one step further using some set theory related to Martin's axiom. We shall use the so-called *pseudo-intersection number* $\mathfrak{p}$, a certain cardinal. That is, $\mathfrak{p}$ is the minimal cardinality of a family $(U_\alpha)_{\alpha<\lambda}$ of open dense subsets of $\mathbb{R}$ such that $\bigcap_{\alpha<\lambda} U_\alpha$ is not dense in $\mathbb{R}$.

Corollary 3.2. *A closed algebraically countably generated right submodule of a Banach module over A is finitely generated. Moreover, if a closed algebraically κ-generated right submodule of a Banach module is separable, where $\kappa < \mathfrak{p}$, then it is finitely generated.*

Proof. The 'countably generated' case is just as in the proof of [8, Corollary 1.6], but using the module version of Boudi's result discussed above. In the other case, let G be a set of algebraic generators for a closed submodule I, with $|G| < \mathfrak{p}$. Then the family of all finite subsets of G has the same cardinality. For cardinals $< \mathfrak{p}$, there is a generalization of Baire's category theorem valid in separable metric spaces; see *e.g.* [9, Corollary 22C]. We proceed similarly to the proof of [8, Corollary 1.6], but apply this generalized Baire principle to the union of the closed submodules generated by finite subsets of G, to see that one such submodule equals I. Finally, apply the module version of Boudi's result discussed above. □

Let us note that the separability assumption in Corollary 3.2 cannot be dropped. Indeed, let $A = C[0, \omega_1]$, that is, A is the commutative C^*-algebra of all continuous functions on the ordinal interval $[0, \omega_1]$. Let I be the ideal of A consisting of functions which vanish at ω_1. (As a Banach space, I is clearly non-separable.) Each function f in I has countable support $\operatorname{supp} f$, since continuous functions on $[0, \omega_1]$ are eventually constant. Let $f \in I$. We can then write $f = f \cdot \mathbf{1}_{[0,\alpha]}$, where $\mathbf{1}_{[0,\alpha]}$ is the characteristic function of the ordinal interval $[0, \alpha]$ and $\alpha = \sup \operatorname{supp} f$. Since α is countable, we have $\mathbf{1}_{[0,\alpha]} \in I$. Thus I is not finitely generated, but is algebraically $\aleph_1$-generated (regardless of whether $\aleph_1 < \mathfrak{p}$ or not).

References

[1] C. A. Akemann, Left ideal structure of C^*-algebras. *J. Funct. Anal.* **6** (1970), 305–317.
[2] D. P. Blecher and C. Le Merdy, *Operator algebras and their modules—an operator space approach,* Oxford Univ. Press, Oxford (2004).
[3] D. P. Blecher and N. Ozawa, Real positivity and approximate identities in Banach algebras, Preprint 2014. arXiv:1405.5551
[4] D. P. Blecher and C. J. R. Read, Order theory and interpolation in operator algebras, Preprint 2014. arXiv:1407.1356
[5] N. Boudi, Banach algebras in which every left ideal is countably generated, *Irish Math. Soc. Bull.* **48** (2002), 17–24.
[6] L. G. Brown, Close hereditary C^*-subalgebras and the structure of quasi-multipliers, MSRI preprint # 11211–85, 1985.
[7] H. G. Dales, *Banach algebras and automatic continuity*, London Mathematical Society Monographs. New Series, 24, Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 2000.
[8] H. G. Dales and W. Żelazko, Generators of maximal left ideals in Banach algebras, *Studia Math.* **212** (2012), 173–193.
[9] D. Fremlin, *Consequences of Martin's axiom,* Cambridge Tracts in Mathematics, 84. Cambridge University Press, Cambridge, 1984.
[10] R. V. Kadison and J. R. Ringrose, *Fundamentals of the theory of operator algebras,* Vol. 1, Graduate Studies in Mathematics, Amer. Math. Soc. Providence, RI, 1997.
[11] B. Magajna, Hilbert C^*-modules in which all closed submodules are complemented, *Proc. Amer. Math. Soc.* **125** (1997), 849–852.
[12] T. W. Palmer, *Banach algebras and the general theory of $*$-algebras, Vol. I. Algebras and Banach algebras,* Encyclopedia of Math. and its Appl., 49, Cambridge University Press, Cambridge, 1994.
[13] G. K. Pedersen, *C^*-algebras and their automorphism groups,* Academic Press, London (1979).
[14] M. L. Reyes, A one-sided prime ideal principle for noncommutative rings, *J. of Algebra and Applns.* **9** (2010), 877–919.
[15] J. Schweizer, A description of Hilbert C^*-modules in which all closed submodules are orthogonally closed, *Proc. Amer. Math. Soc.* **127** (1999), 2123–2125.
[16] N. E. Wegge-Olsen, *K-theory and C^*-algebras*, Oxford University Press, New York, 1993.

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