STUDIA MATHEMATICA 226 (2) (2015)

Approximate identities in Banach function algebras

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Abstract. In this paper, we shall study contractive and pointwise contractive Banach function algebras, in which each maximal modular ideal has a contractive or pointwise contractive approximate identity, respectively, and we shall seek to characterize these algebras. We shall give many examples, including uniform algebras, that distinguish between contractive and pointwise contractive Banach function algebras. We shall describe a contractive Banach function algebra which is not equivalent to a uniform algebra. We shall also obtain results about Banach sequence algebras and Banach function algebras that are ideals in their second duals.

1. Introduction. Let A be a Banach function algebra. A *contractive approximate identity* for A is a bounded approximate identity of bound 1; A is *contractive* if A and all its maximal modular ideals have a contractive approximate identity. The uniform algebra $C_0(K)$ on a locally compact space K is an example of a contractive Banach function algebra. The first main question that we shall address in this paper is how to characterize contractive Banach function algebras. Are there any examples other than those of the form $C_0(K)$? We shall prove that a natural uniform algebra is contractive if and only if it is a Cole algebra, so that every point of its character space is a p-point; in a future paper [13], we shall discuss the (wide) class of Banach function algebras that 'have a BSE norm' and prove more generally that every contractive Banach function algebra in this class is necessarily equivalent to a Cole algebra. However, we shall show in Example 5.2 that there are contractive Banach function algebras whose norm is not equivalent to the uniform norm.

A net (f_α) in a Banach function algebra $(A, \|\cdot\|)$ is a *contractive pointwise approximate identity* if $\|f_\alpha\| \le 1$ for all α and $\lim_\alpha \varphi(f_\alpha) = 1$ for each character φ on A; A is *pointwise contractive* if A and all its maximal modular ideals have contractive pointwise approximate identities, so that every con-

2010 *Mathematics Subject Classification*: Primary 46B15; Secondary 46B28, 46B42, 47L10.
Key words and phrases: Banach function algebra, approximate identity, uniform algebra, Banach sequence algebra, Fourier algebra.

DOI: 10.4064/sm226-2-3

tractive Banach function algebra is pointwise contractive. The second main question that we shall address is whether there are Banach function algebras in various classes that are pointwise contractive, but not contractive. We shall give several examples that distinguish between these two properties. For example, in Example 5.1, we shall describe a pointwise contractive Banach function algebra without any approximate identity, and, in Examples 4.8, (vi) and (viii), we shall give a pointwise contractive uniform algebra without a bounded approximate identity and a uniform algebra with a contractive pointwise approximate identity, but no approximate identity. We shall prove that a uniform algebra is pointwise contractive if and only if each point of its character space is a one-point Gleason part; this characterization will be extended to Banach function algebras with a BSE norm in [13].

We now summarize the main results of this paper.

In §2, we shall recall some notation and preliminary results about Banach function algebras. Our examples include the Fourier, Fourier–Stieltjes, and Figà-Talamanca–Herz algebras on locally compact groups, and various Segal algebras defined with respect to the Fourier algebra. In §3 we shall give a series of examples illustrating and distinguishing between various notions of approximate identity. Also we shall show in Theorem 3.5 that a natural, pointwise contractive Banach sequence algebra on a set S is equivalent to $c_0(S)$. In Example 3.15, we shall show that the minimum bound of a bounded pointwise approximate identity in a maximal modular ideal of a Fourier algebra $A(\Gamma)$ is 2 whenever Γ is amenable as a discrete group. In §4, we shall prove that a uniform algebra on a locally compact space K is contractive if and only if it is a Cole algebra, as defined in the text, and that it is pointwise contractive if and only if each Gleason part in K is a singleton; we shall give many examples of uniform algebras delineating several properties.

In §5, we shall construct two unital Banach function algebras on closed intervals. The first gives a unital, pointwise contractive Ditkin algebra such that one of its maximal ideals does not have a bounded approximate identity. The second exhibits a unital, contractive Banach function algebra that is not equivalent to a uniform algebra.

2. Preliminaries.

In this section we shall recall some notation, definitions, and standard results about Banach function algebras.

2.1. Banach function algebras.

We set: $\mathbb{I} = [0,1]$, the closed unit interval; $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, the open unit disc in the plane; and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, the unit circle. For $n \in \mathbb{N}$, we set $\mathbb{N}_n = \{1, \ldots, n\}$.

Let E be a linear space, always taken to be over the complex field $\mathbb{C}$. Then $\operatorname{lin} F$ denotes the linear span of a subset F of E; the set of extreme points of a convex set S in E is denoted by $\operatorname{ex} S$; the convex hull of a subset S of E is denoted by $\langle S\rangle$.

Let E be a Banach space. Then we denote the dual space of E by E', with the duality specified by $(x,\lambda)\mapsto\langle x,\lambda\rangle$, $E\times E'\to\mathbb{C}$; the second dual space of E is E''. The closed ball of radius $r\geq 0$ in E is $E_{[r]}$, and the weak-$*$ topology on E' is $\sigma=\sigma(E',E)$.

Let A be a (complex, associative) algebra. The set of characters on A is denoted by Φ_A. We set

$$A^{[2]}=\{ab : a,b\in A\}\quad\text{and}\quad A^2=\operatorname{lin}A^{[2]};$$

the algebra A *factors* if $A=A^{[2]}$.

Let A be an algebra that is also a Banach space with respect to a norm $\|\cdot\|$. Then A is a *Banach algebra* if $\|ab\|\leq\|a\|\,\|b\|$ $(a,b\in A)$; in the case where A has an identity e_A, we also require that $\|e_A\|=1$. The Banach algebra of all bounded linear operators on a Banach space E is denoted by $\mathcal{B}(E)$. For a monograph on Banach algebras, see [6]; we shall usually follow the notation of this book.

Let $(A,\|\cdot\|)$ be a Banach algebra. Then each $\varphi\in\Phi_A$ is a continuous linear functional on A with $\|\varphi\|\leq 1$, and so we may regard Φ_A as a subset of $A'_{[1]}$. The space Φ_A is locally compact with respect to the weak-$*$ topology; it is compact in the case where A has an identity, and is non-empty when A is commutative and has an identity. In the case where A is unital, we define

$$K_A=\{\lambda\in A' : \|\lambda\|=\langle e_A,\lambda\rangle=1\},$$

the *state space* of A. The space K_A is a non-empty, convex, and weak-$*$ compact subset of $A'_{[1]}$, and hence $\overline{\langle\operatorname{ex}K_A\rangle}^{\sigma}=K_A$ by the Kreĭn–Mil'man theorem. Clearly $\Phi_A\subset K_A$.

Let A be a Banach algebra. The spaces A' and A'' are Banach A-bimodules, as in [6]. We denote the module operations on A' and A'' by $\cdot$, and set

$$A\cdot A'=\{a\cdot\lambda : a\in A,\ \lambda\in A'\}\quad\text{and}\quad AA'=\operatorname{lin}A\cdot A'.$$

There are two *Arens products*, denoted by $\Box$ and $\Diamond$, respectively, on the second dual space A'', each extending the module operations on A''; now $(A'',\Box)$ and $(A'',\Diamond)$ are Banach algebras each containing A as a closed subalgebra. The algebra A is *Arens regular* if $\mathrm{M}\,\Box\,\mathrm{N}=\mathrm{M}\,\Diamond\,\mathrm{N}$ $(\mathrm{M},\mathrm{N}\in A'')$. An element $\lambda\in A'$ is *weakly almost periodic* if the map

$$R_\lambda : a\mapsto a\cdot\lambda,\quad A\to A',\tag{2.1}$$

is weakly compact; the Banach space of all weakly almost periodic functionals on A is denoted by $WAP(A)$. For details of the above, see [6, §2.6], and also [10, 11, 30].

The function constantly equal to 1 on a non-empty set S is written as 1_S or just 1.

Let K be a non-empty, locally compact (Hausdorff) space. We write $C_0(K)$ and $C^b(K)$ for the spaces of all complex-valued, continuous functions on K which vanish at infinity and which are bounded on K, respectively; we write $C(K)$ for $C_0(K)$ when K is compact. Then $C^b(K)$ is a unital algebra with respect to the pointwise algebraic operations; the function 1_K is the identity of $C^b(K)$. We define

$$|f|_K = \sup\{|f(x)| : x \in K\} \quad (f \in C^b(K)),$$

so that $|\cdot|_K$ is the *uniform norm* on K and $(C^b(K), |\cdot|_K)$ is a commutative, unital Banach algebra; $C_0(K)$ is a closed ideal in $C^b(K)$.

The support of a function $f \in C_0(K)$ is denoted by $\operatorname{supp} f$.

Let K be a non-empty, locally compact space. A *function algebra* on K is a subalgebra A of $C^b(K)$ that separates strongly the points of K, in the sense that, for each $x, y \in K$ with $x \neq y$, there exists $f \in A$ with $f(x) \neq f(y)$, and, for each $x \in K$, there exists $f \in A$ with $f(x) \neq 0$. A function algebra A is *self-adjoint* if $\overline{f} \in A$ whenever $f \in A$; here $\overline{f}(x) = \overline{f(x)}$ $(x \in K)$. A *Banach function algebra* on K is a function algebra A on K with a norm $\|\cdot\|$ such that $(A, \|\cdot\|)$ is a Banach algebra; a Banach function algebra $(A, \|\cdot\|)$ is *equivalent to a uniform algebra* if $\|\cdot\|$ and $|\cdot|_K$ are equivalent norms on A, so that A is closed in $(C^b(K), |\cdot|_K)$; $(A, \|\cdot\|)$ is a *uniform algebra* if $\|\cdot\| = |\cdot|_K$.

Let A be a function algebra on K. For each $x \in K$, define

$$\varepsilon_x(f) = f(x) \quad (f \in A).$$

Then each ε_x is a character on A, called the *evaluation character* at x, and so we may regard K as a subset of Φ_A by identifying $x \in K$ with $\varepsilon_x \in \Phi_A$. A Banach function algebra A on K is *natural* if $K = \Phi_A$ with this identification, and then $A \subset C_0(K)$ and $\|f\| \geq |f|_K$ $(f \in A)$; clearly, $C_0(K)$ is natural and the uniform closure of A in $C_0(K)$ is natural whenever A is natural.

Let $(A, \|\cdot\|)$ be a natural Banach function algebra on a non-empty, compact space K. It follows from Šilov's idempotent theorem [6, Corollary 2.4.35] that $1_K \in A$, and so A is unital; our definition requires that $\|1_K\| = 1$.

Let A be a commutative Banach algebra. For $\varphi \in \Phi_A$, set $M_\varphi = \ker \varphi$. Then M_φ is a maximal modular ideal in A, and each maximal modular ideal has this form. In the case where $\Phi_A \neq \emptyset$, define $\widehat{a} \in C_0(\Phi_A)$ for $a \in A$ by $\widehat{a}(\varphi) = \varphi(a)$ $(\varphi \in \Phi_A)$. Then the Gel'fand transformation

$$\mathcal{G} : a \mapsto \widehat{a}, \quad A \to C_0(\Phi_A),$$

is a norm-decreasing homomorphism from A onto a function algebra $\widehat{A}$: $\widehat{A}$ is a natural Banach function algebra on Φ_A with respect to the quotient norm from A. The map $\mathcal{G}$ is injective whenever A is semi-simple, and so we can regard each commutative, semi-simple Banach algebra as a natural Banach function algebra on Φ_A.

Let A be a Banach function algebra on a locally compact space K, and take $x \in K$. We write M_x for M_{ε_x} and set $M_\infty = A$. More generally, for a closed subset E of K, we set

$$I(E) = \{f \in A : f|E = 0\},$$

so that $I(E)$ is a closed ideal in A. Further, we write J_∞ or $J_\infty(A)$ for the set of functions in A of compact support, so that J_∞ is an ideal in A, and we set

$$J_x = J_x(A) = \{f \in J_\infty : x \notin \operatorname{supp} f\} \quad (x \in K),$$

so that each J_x is also an ideal in A with $J_x \subset M_x$. We write A_0 for the closure of J_∞ in A; the algebra A is *Tauberian* if $A_0 = A$, and so A_0 is a Tauberian Banach function algebra whenever J_∞ separates strongly the points of K. A natural Banach function algebra A is *strongly regular* if J_x is dense in M_x for each $x \in K \cup \{\infty\}$, and A is *regular* if, for each non-empty, closed subset F of K and each $x \in K \setminus F$, there exists $f \in A$ with $f(x) = 1$ and $f(y) = 0$ $(y \in F)$. A strongly regular algebra is regular. The algebra A is a *Ditkin algebra* if $f \in \overline{fJ_x}$ $(f \in M_x)$ for each $x \in K \cup \{\infty\}$, so that a Ditkin algebra is strongly regular.

Let A be a natural Banach function algebra on a locally compact, non-compact space K, and take $K_\infty = K \cup \{\infty\}$ to be the compact space that is the one-point compactification of K. We regard A as a subalgebra of $C(K_\infty)$ by taking $f \in A$ to have the value 0 at the point ∞; the constant function on K_∞ is 1, and the *unitization* of A is defined to be the subalgebra $A^\sharp = \{f + z1 : f \in A,\, z \in \mathbb{C}\}$ of $C(K_\infty)$. We define a norm $\|\cdot\|$ on $A^\sharp$ by setting

$$\|f + z1\| = \|f\| + |z| \quad (f + z1 \in A^\sharp).$$

It is clear that $(A^\sharp, \|\cdot\|)$ is a natural Banach function algebra on K_∞ and that $A^\sharp$ contains A as a maximal ideal. Further, $A^\sharp$ is equivalent to a uniform algebra on K_∞ if and only if A is equivalent to a uniform algebra on K.

For further details of the above theory, see [6], especially §§2.5, 4.1.

2.2. Approximate identities and units. Let A be a Banach algebra. A net (e_α) in A is an *approximate identity* for A if

$$\lim_\alpha ae_\alpha = \lim_\alpha e_\alpha a = a \quad (a \in A);$$

an approximate identity is *sequential* if it is a sequence in A indexed by $\mathbb{N}$; an approximate identity (e_α) is *bounded* if $\sup_\alpha \|e_\alpha\| < \infty$, and then $\sup_\alpha \|e_\alpha\|$ is the *bound* of the approximate identity; an approximate identity is *contractive* if it has bound 1. We refer to a BAI and a CAI, respectively, in these two cases. For the theory of approximate identities in Banach algebras, see [6, §2.9].

Definition 2.1. Let $(A, \|\cdot\|_A)$ be a natural Banach function algebra on a non-empty, locally compact space K. A Banach function algebra $(B, \|\cdot\|_B)$ is an *abstract Segal algebra* (with respect to A) if B is an ideal in A and there is a net in B that is an approximate identity for both $(A, \|\cdot\|_A)$ and $(B, \|\cdot\|_B)$.

Thus, in the above situation, B is dense in A, B is natural, and we may suppose that

$$\|f\|_A \le \|f\|_B \quad (f \in B) \quad \text{and} \quad \|fg\|_B \le \|f\|_A \|g\|_B \quad (f \in A,\, g \in B).$$

A natural Banach function algebra A on a non-empty, locally compact space K is a *strong Ditkin algebra* if M_x has a BAI contained in J_x for each $x \in K \cup \{\infty\}$; see [6, Definition 4.1.31]. Clearly, a strong Ditkin algebra is a Ditkin algebra.

A commutative Banach algebra $(A, \|\cdot\|)$ has *bounded approximate units of bound* m if, for each $a \in A$ and $\varepsilon > 0$, there exists $u \in A_{[m]}$ with $\|a - ua\| < \varepsilon$. By [6, Proposition 2.9.14(ii)], A has bounded approximate units of bound m if and only if A has a BAI of bound m. Suppose that A has bounded approximate units of bound $m + \varepsilon$ for each $\varepsilon > 0$. Then A has bounded approximate units of bound m.

Let A be a Banach function algebra on a non-empty, locally compact space K. Then A has *bounded relative approximate units* (BRAUs) *of bound* m if, for each non-empty, compact subset L of Φ_A and each $\varepsilon > 0$, there exists $f \in A_{[m]}$ with $|1 - f(y)| < \varepsilon$ $(y \in L)$. It is easy to see that A has BRAUs of bound m whenever A has a BAI of bound m; Example 2.9, to be given below, will show that the converse need not hold.

Theorem 2.2. *Let A be a Banach algebra with a* BAI. *Then:*

(i) *$A = A^{[2]}$ and $\overline{AA'} = A \cdot A'$;*
(ii) *$WAP(A) \subset A \cdot A'$;*
(iii) *in the case where A is Arens regular, $A' = A \cdot A'$.*

Proof. (i) This is a form of *Cohen's factorization theorem*; considerably stronger forms of this theorem are given in [6, Theorem 2.9.24].

(ii) This is part of [9, Proposition 3.12].

(iii) By a result of [33] (also contained in [11]), $WAP(A) = A'$ if and only if A is Arens regular, and so this follows from (ii). ∎

Definition 2.3. Let A be a natural Banach function algebra on a non-empty, locally compact space K. Then A is *contractive* if M_x has a CAI for each $x \in K \cup \{\infty\}$.

We see quickly that uniform algebras of the form $C_0(K)$ are contractive, but it is not immediately obvious that there are any other contractive Banach function algebras; examples will be given in §4 and §5.

Let $(A, \|\cdot\|_A)$ and $(B, \|\cdot\|_B)$ be natural Banach function algebras on a non-empty, locally compact space, and suppose that A is a dense subalgebra of B and that $\|f\|_B \le \|f\|_A$ $(f \in A)$. Then B is contractive whenever A is contractive.

Next we recall two classical Banach function algebras, and note that they are not contractive.

EXAMPLE 2.4. As in [6, Example 2.1.13(ii)], we denote by $A(\overline{\mathbb{D}})$ the *disc algebra* of all functions f in $C(\overline{\mathbb{D}})$ such that $f|\mathbb{D}$ is analytic. Thus $A(\overline{\mathbb{D}})$ is a natural uniform algebra on $\overline{\mathbb{D}}$. The maximal ideal M_z at $z \in \overline{\mathbb{D}}$ has a CAI if $z \in \mathbb{T}$, and so $M_z = M_z^{[2]}$, but

$$M_z^{[2]} = \overline{M_z^2} = \{f \in M_z : f'(z) = 0\} \subsetneq M_z$$

when $z \in \mathbb{D}$, and so M_z has no approximate identity when $z \in \mathbb{D}$. Thus $A(\overline{\mathbb{D}})$ is not contractive.

Let A be a uniform algebra on a compact space K, and take $f \in A_{[1]}$ and $F \in A(\overline{\mathbb{D}})_{[1]}$. Then we note that $F \circ f \in A_{[1]}$. In particular, for $a \in \mathbb{D}$, set $\psi_a(z) = (z-a)/(1-\overline{a}z)$ $(z \in \overline{\mathbb{D}})$, so that ψ_a is a Möbius transformation with $\psi_a(a) = 0$. Then $\psi_a \in A(\overline{\mathbb{D}})_{[1]}$, and so $\psi_a \circ f \in A_{[1]}$.

In the following result and later we shall use some specific functions that are defined on $\mathbb{I}$.

DEFINITION 2.5. Take $t \in \mathbb{I}$ and $n \in \mathbb{N}$, and define g_n to be the restriction to $\mathbb{I}$ of the function which is 0 on the interval $[t-1/n, t+1/n]$, which is equal to 1 outside the interval $[t-2/n, t+2/n]$, and which is linear on $[t-2/n, t-1/n]$ and $[t+1/n, t+2/n]$.

Then (g_n) is a sequence in $C(\mathbb{I})$, and each g_n is zero on a neighbourhood of t.

EXAMPLE 2.6. Consider the algebra $A = BVC(\mathbb{I})$ of continuous functions of bounded variation on $\mathbb{I}$; this algebra is discussed in [6, Theorem 4.4.35]. Fix $\gamma > 0$. Here we note that A is a natural, unital Banach function algebra with respect to the norm $\|\cdot\|_\gamma$ defined by

$$\|f\|_\gamma = |f|_{\mathbb{I}} + \gamma \operatorname{var}_{\mathbb{I}}(f) \qquad (f \in BVC(\mathbb{I})),$$

where $\operatorname{var}_{\mathbb{I}}(f)$ is the variation of f over $\mathbb{I}$; the algebra A is regular, but it is not a uniform algebra.

Take $t \in \mathbb{I}$. Clearly the above sequence (g_n) is contained in $J_t(A)$ and is a BAI of bound $1+2\gamma$ for $M_t(A)$, and so $(BVC(\mathbb{I}), \|\cdot\|_\gamma)$ is a strong Ditkin algebra. However, this algebra is not contractive.

This example shows that we cannot replace 'CAI in each maximal ideal' by 'BAI of bound c in each maximal ideal' for any $c > 1$ when seeking contractive Banach function algebras.

2.3. Banach sequence algebras. Let S be a non-empty set. The algebra of all functions on S of finite support is denoted by $c_{00}(S)$; the characteristic function of the singleton $\{s\}$ for $s \in S$ is denoted by δ_s, so that $\delta_s \in c_{00}(S)$ $(s \in S)$. The following definition is given in [6, §4.1].

DEFINITION 2.7. Let S be a non-empty set. A *Banach sequence algebra* on S is a Banach function algebra A on S such that $c_{00}(S) \subset A$.

Let A be a Banach sequence algebra on a set S. Then $J_\infty(A) = c_{00}(S)$ and A_0 is the closure of $c_{00}(S)$ in A. Thus a Banach sequence algebra is Tauberian if and only if $c_{00}(S)$ is dense in A. A natural Banach sequence algebra is always regular, and it is strongly regular if and only if it is Tauberian.

A form of converse to the following result will be given in Proposition 3.1.

PROPOSITION 2.8. *Let A be a Tauberian Banach sequence algebra on a non-empty set S. Then A is natural and A is an ideal in A''.*

Proof. That A is natural is [6, Proposition 4.1.35(i)]. To see that A is an ideal in A'', set $R_f(g) = gf$ $(g \in A)$ for each $f \in A$, so that $R_f \in \mathcal{B}(A)$. For each $f \in c_{00}(S)$, the operator R_f has finite-dimensional range, and so is compact. Since $c_{00}(S)$ is dense in A, R_f is compact, and hence weakly compact, for each $f \in A$. By [30, Proposition 1.4.13], A is an ideal in A''. ∎

EXAMPLE 2.9. For $\alpha = (\alpha_k) \in \mathbb{C}^{\mathbb{N}}$, set

$$p_n(\alpha) = \frac{1}{n}\sum_{k=1}^{n} k|\alpha_{k+1} - \alpha_k| \quad (n \in \mathbb{N}), \qquad p(\alpha) = \sup\{p_n(\alpha) : n \in \mathbb{N}\},$$

and define A to be $\{\alpha \in c_0 : p(\alpha) < \infty\}$, so that A is a self-adjoint Banach sequence algebra on $\mathbb{N}$ for the norm given by

$$\|\alpha\| = |\alpha|_{\mathbb{N}} + p(\alpha) \quad (\alpha \in A).$$

The details of this example, which is due to Feinstein, are given in [6, Example 4.1.46]. It is shown that A is natural, and that, for each $m \in \mathbb{N}$ and each compact subset K of $\mathbb{N}$ with $m \notin K$, there exists $\alpha \in A$ with $\alpha(m) = 0$, with $\alpha(j) = 1$ $(j \in K)$, and with $\|\alpha\| \le 4$. Thus each maximal modular ideal of A has BRAUs of bound 4. It is also shown that $A^2 = A_0^2 = A_0$, that A_0 is separable, and that A is non-separable, and so A^2 is a closed subspace of infinite codimension in A. Thus A is not Tauberian and A does not have any approximate identity.

The following remarkable example of Blecher and Read from [2] was the first to exhibit a natural Banach sequence algebra that has a BAI, but which is not Tauberian.

EXAMPLE 2.10. There is a natural Banach sequence algebra A on $\mathbb{N}$ with all of the following properties:

(i) A is self-adjoint, and so dense in c_0;
(ii) A has a CAI, and so $A^{[2]} = A$;
(iii) there exists $g \in A$ such that the singly-generated subalgebra $\mathrm{lin}\{g^n : n \in \mathbb{N}\}$ is dense in A, and so A is separable;
(iv) A is non-Tauberian, and A/A_0 is an infinite-dimensional space;
(v) the closed ideal A_0 also has a CAI;
(vi) each maximal modular ideal in A has a BAI (but there is no upper bound to the bounds of these BAIs);
(vii) the CAI for A_0 is contained in c_{00}, and so A_0 is a strong Ditkin algebra;
(viii) A is Arens regular, but A is not an ideal in A''.

2.4. Pointwise approximate identities. There is a related notion concerning approximate identities in Banach function algebras. The following definition originates with Jones and Lahr [27]; see also [26, 29, 37]. However, our terminology is different.

DEFINITION 2.11. Let A be a natural Banach function algebra on a non-empty, locally compact space K. A net (e_α) in A is a *pointwise approximate identity* (PAI) if

$$\lim_\alpha e_\alpha(x) = 1 \quad (x \in K);$$

the PAI is *bounded*, with *bound* $m > 0$, if $\sup_\alpha \|e_\alpha\| \leq m$, and then (e_α) is a *bounded pointwise approximate identity* (BPAI); a bounded pointwise approximate identity of bound 1 is a *contractive pointwise approximate identity* (CPAI). The algebra A is *pointwise contractive* if M_x has a CPAI for each $x \in K \cup \{\infty\}$.

Thus a natural Banach function algebra A on K is pointwise contractive if and only if, for each $x \in K \cup \{\infty\}$, each non-empty, finite subset F of K with $x \notin F$, and each $\varepsilon > 0$, there exists $f \in M_x$ with $\|f\| \leq 1$ and $|1 - f(y)| < \varepsilon$ $(y \in F)$.

Clearly each Banach function algebra has a PAI. A Banach function algebra with BRAUs has a BPAI, with the same bound. Example 2.9 exhibited a Banach sequence algebra with a BPAI, but no approximate identity. In Examples 3.13 and 3.14, it will be shown that there are natural Banach function algebras A that have a CPAI, but no BAI. Example 4.8(vi) will give a natural, pointwise contractive uniform algebra that is not contractive, and Example 4.8(viii) will give a natural uniform algebra with a CPAI, but no approximate identity. In Example 5.1, we shall exhibit a natural, pointwise contractive Banach function algebra that does not have any approximate identity.

As we stated at the beginning of this paper, our aim is to understand the structure of contractive and pointwise contractive Banach function algebras.

PROPOSITION 2.12. *Let A be a natural, Banach function algebra on a non-empty, locally compact space K, and take $x \in K$. Suppose that there exists $n > 0$ such that, for each neighbourhood U of x, there exists $g \in A_{[n]}$ with $g(x) = 1$ and $\operatorname{supp} g \subset U$.*

(i) *Suppose that $\overline{J_x} = M_x$ and A has a* BAI *of bound m. Then M_x has a* BAI *of bound $m(1+n)$.*
(ii) *Suppose that A has a* BPAI *of bound m. Then M_x has a* BPAI *of bound $m(1+n)$.*

Proof. (i) Take $h \in M_x$ and $\varepsilon > 0$. Then there exists $h_1 \in J_x$ with $\|h - h_1\| < \varepsilon$. Since A has a BAI of bound m, there exists $f \in A_{[m]}$ with $\|h_1 - fh_1\| < \varepsilon$. There is a neighbourhood U of x with $U \cap \operatorname{supp} h_1 = \emptyset$, and so, by hypothesis, there exists $g \in A_{[n]}$ with $g(x) = 1$ and $\operatorname{supp} g \subset U$, so that $h_1 g = 0$. Now we have $f - fg \in (M_x)_{[m(n+1)]}$ and

$$\begin{aligned}\|h - h(f - fg)\| &\le \|h - h_1\| + \|h_1 - fh_1\| + \|h - h_1\|\,\|f - fg\| \\ &< \varepsilon + \varepsilon + \varepsilon m(1+n).\end{aligned}$$

Thus M_x has bounded approximate units of bound $m(n+1)$, and hence a BAI of bound $m(1+n)$.

(ii) Take a non-empty, finite subset F of K with $x \notin F$, and take $\varepsilon > 0$. Then there exists $f \in A_{[m]}$ with $|1 - f(y)| < \varepsilon$ $(y \in F)$. Take a neighbourhood U of x with $U \cap F = \emptyset$, and then take $g \in A_{[n]}$ with $g(x) = 1$ and $\operatorname{supp} g \subset U$. Set $h = f - fg \in (M_x)_{[m(n+1)]}$. We see immediately that $|1 - h(y)| < \varepsilon$ $(y \in F)$, and so M_x has a BPAI of bound $m(1+n)$. ∎

The next examples exhibit some Banach function algebras whose maximal ideals do not have a BPAI.

EXAMPLE 2.13. The maximal ideals of the natural Banach function algebras $C^{(n)}(\mathbb{I})$ (for $n \in \mathbb{N}$) and $(\mathrm{Lip}_\alpha(\mathbb{I}), \|\cdot\|_\alpha)$ (for $0 < \alpha \le 1$), which are defined in [6, §4.4], do not have a BPAI. For example, for each $n \in \mathbb{N}$ and $f \in \mathrm{Lip}_\alpha(\mathbb{I})$ with $f(0) = 0$ and $f(1/n) > 1/2$, necessarily $\|f\|_\alpha \ge n^\alpha/2$.

EXAMPLE 2.14. Let A be the space $\ell^p = \ell^p(\mathbb{N})$, where $p \ge 1$. Then A is a natural, self-adjoint, Tauberian Banach sequence algebra on $\mathbb{N}$, and so A is an ideal in A''; in the case where $p > 1$, A is reflexive. Clearly A and each maximal modular ideal in A have approximate identities, but A does not have a BPAI. Here $A^{[2]} = A^2 = \ell^{p/2}$.

EXAMPLE 2.15. Let $\omega : \mathbb{N} \to [1, \infty)$ be a function, and set

$$B_\omega = \Big\{\alpha \in c_0 : p_\omega(\alpha) := \sum_{i=1}^{\infty} \omega(i)|\alpha_{i+1} - \alpha_i| < \infty\Big\};$$

for $\alpha \in B_\omega$, set $\|\alpha\|_\omega = |\alpha|_{\mathbb{N}} + p_\omega(\alpha)$, as in [12, p. 33]. Then B_ω is a natural, Tauberian Banach sequence algebra on $\mathbb{N}$. A slight extension of [12, Theorem 3.10.1] shows that the following are equivalent:

(a) B_ω has a BAI;
(b) B_ω has BRAUs;
(c) B_ω has a BPAI;
(d) $\liminf_{n\to\infty} \omega(n) < \infty$.

Various other equivalent properties involving amenability are given in the quoted theorem. Further remarks about this example will be given in [13].

2.5. Peaking properties. Let A be a function algebra on a non-empty, locally compact space K. A closed subset F of K is a *peak set* if there exists a function $f \in A$ with $f(x) = 1$ $(x \in F)$ and $|f(y)| < 1$ $(y \in K \setminus F)$; in this case, f *peaks* on F; a point $x \in K$ is a *peak point* if $\{x\}$ is a peak set, and a p*-point* if $\{x\}$ is an intersection of peak sets. The set of p-points of A is denoted by $\Gamma_0(A)$; it is sometimes called the *Choquet boundary* of A. In the case where A is a Banach function algebra, a countable intersection of peak sets is always a peak set, and so, when K is metrizable, $\Gamma_0(A)$ is the set of peak points of A. (However, even a uniform algebra may have p-points which are not peak points.) A closed subset L of K is a *closed boundary* for A if $|f|_L = |f|_K$ $(f \in A)$; the intersection of all the closed boundaries for A is called the *Šilov boundary*, $\Gamma(A)$ [6, Definition 4.3.1(iv)]. Suppose that K is compact and that A is a natural uniform algebra on K. Then, by [6, Corollary 4.3.7(i)], $\Gamma(A) = \overline{\Gamma_0(A)}$ and $\Gamma(A)$ is a closed boundary. Suppose that K is compact and metrizable and that A is a natural Banach function algebra on K. Then, by [5] (see also [6, Corollary 4.3.7(ii)]), the set of peak points is dense in $\Gamma(A)$.

The following theorem is proved by a small modification of the proof of [6, Theorem 4.3.5].

Theorem 2.16. *Let A be a Banach function algebra on a non-empty, compact space K, and take $x \in K$. Suppose that M_x has* BRAU*s. Then x is a p-point for A.* ∎

We now obtain a necessary condition for a Banach function algebra A to be contractive.

Theorem 2.17. *Let A be a natural Banach function algebra on a non-empty, locally compact space K such that each maximal modular ideal of A has a* BAI. *Then $\Gamma_0(A) = K$.*

Proof. Take $x \in K$. Then M_x has a BAI, and hence M_x has BRAUs. By Theorem 2.16, x is a p-point for A. ∎

Corollary 2.18. *Let A be a natural, contractive Banach function algebra on a non-empty, locally compact space K. Then $\Gamma_0(A) = K$.* ■

We shall see in Example 4.8(vi) and Theorem 4.9 that we cannot replace 'contractive' by 'pointwise contractive' in the above corollary, even in the case where A is a uniform algebra.

3. Relations among approximate identities. In this section we shall give some results and examples showing the relationships between various notions of approximate identity for Banach function algebras.

3.1. Examples. The first result, which is a (weak) converse to Proposition 2.8, is essentially given in [29, Theorem 3.1]; a similar result is contained in [2, Corollary 1.3].

Proposition 3.1. *Let A be a Banach function algebra such that A is an ideal in A'' and A has a* BPAI. *Then A also has a* BAI*, with the same bound. In the case where the* BPAI *is contained in A_0, the algebra A is Tauberian.*

Proof. Let (e_α) be a BPAI for A with bound m. Then (e_α) has a weak-$*$ accumulation point in $(A'')_{[m]}$, and we may suppose, by passing to a subnet, that weak-$*$-$\lim_\alpha e_\alpha = e \in (A'')_{[m]}$. For each $f \in A$, we know that $e \cdot f$ belongs to A because A is an ideal in A''. Further, for each $\varphi \in \Phi_A$, we have

$$\varphi(f) = \langle f, \varepsilon_\varphi\rangle = \lim_\alpha \langle e_\alpha f, \varepsilon_\varphi\rangle = \langle e, f \cdot \varepsilon_\varphi\rangle = \langle e \cdot f, \varepsilon_\varphi\rangle = \varphi(e \cdot f),$$

and so $e \cdot f = f$. Thus $e_\alpha f \to f$ in (A, σ), where $\sigma = \sigma(A, A')$. By [6, Proposition 2.9.14(iii)], A has a BAI with bound m.

Now suppose that (e_α) is contained in A_0, and again take $f \in A$. Then the net $(e_\alpha f)$ is in A_0, and so $f \in \overline{A_0}^{\sigma}$. By Mazur's theorem, $\overline{A_0}^{\sigma} = A_0$, and so $f \in A_0$. Thus A is Tauberian. ■

We shall show in Example 3.9 that a Banach sequence algebra A can be an ideal in A'' without being Tauberian, but we do not know, even in the case of Banach sequence algebras, whether a Banach function algebra A that is an ideal in A'' and has a BPAI is necessarily Tauberian.

Proposition 3.2. *Let A be a natural Banach function algebra on a non-empty, locally compact space K such that A is reflexive as a Banach space. Take $x \in K$, and suppose that M_x has a* BPAI. *Then M_x has an identity, x is an isolated point of K, and K is compact.*

Proof. Since A is reflexive, the closed balls of M_x are weakly compact, and so the BPAI in M_x has a weakly convergent subnet, with weak limit f, say. Clearly $f(y) = 1$ $(y \in K \setminus \{x\})$, and so f is the identity of M_x. Since $f \in C_0(K)$, the point x is an isolated point of K and K is compact. ■

We now give examples of unital Banach function algebras which are reflexive as Banach spaces (and so are ideals in their second duals) and have connected character spaces, so that they are not Banach sequence algebras. Their maximal ideals do not have a BPAI. We are grateful to David Blecher for reminding us of these examples.

EXAMPLE 3.3. Let S be either $\mathbb{Z}$ or $\mathbb{Z}^+$. Then $\omega = (\omega_n : n \in S)$ is a *weight sequence* on S if $\omega_n > 0$ $(n \in S)$, $\omega_0 = 1$, and

$$\omega_{m+n} \le \omega_m \omega_n \quad (m, n \in S).$$

Fix $p > 1$ and a weight sequence ω on S, and consider the Banach space

$$\ell^p(S, \omega) = \Big\{\alpha = (\alpha_n : n \in S) : \|\alpha\|_{p,\omega} = \Big(\sum_{n\in S} |\alpha_n|^p \omega_n^p\Big)^{1/p} < \infty\Big\}.$$

It is well-known that $\ell^p(S, \omega)$ is a Banach algebra for convolution multiplication, $\star$, provided that there exists a constant $C > 0$ such that

$$\sum_{m+n=k} \frac{\omega_k^q}{\omega_m^q \omega_n^q} \le C \quad (k \in S), \tag{3.1}$$

where q is the conjugate index to p. This follows from Hölder's inequality; the first explicit reference that we have found is [23, Theorem 6.7(D)]. Hence $\alpha \star \beta \in \ell^p(\mathbb{Z}^+, \omega)$ with $\|\alpha \star \beta\|_{p,\omega} \le C^{1/q} \|\alpha\|_{p,\omega} \|\beta\|_{p,\omega}$. To obtain a Banach function algebra that satisfies our precise definition, we must replace the given norm by an equivalent one. The interesting fact about these examples is that, as Banach spaces, they are reflexive (this is not affected by the re-norming), and so certainly they are Arens regular and ideals in their second duals.

Now choose $\omega_n = (1 + |n|)^r$ $(n \in S)$, where $r > 1/q$. It is easy to see that $\omega = (\omega_n : n \in S)$ is a weight sequence on S that satisfies the inequality (3.1), and so $\ell^p(\mathbb{Z}^+, \omega)$ and $\ell^p(\mathbb{Z}, \omega)$ are Banach algebras. Again by Hölder's inequality, $\ell^p(\mathbb{Z}, \omega) \subset \ell^1(\mathbb{Z})$. Both these algebras are semi-simple and they are natural, unital Banach function algebras on $\mathbb{T}$ and $\overline{\mathbb{D}}$, respectively. Let M_z be a maximal ideal of either of these algebras. Since z is not an isolated point of the character space of the algebra, it follows from Proposition 3.2 that this maximal ideal does not have a BPAI.

PROPOSITION 3.4. *Let A be a pointwise contractive Banach function algebra on a non-empty, locally compact space K. Suppose that F and G are disjoint, non-empty, finite subsets of K, and take $\varepsilon > 0$. Then there exists $f \in A_{[1]}$ such that $|1 - f(x)| < \varepsilon$ $(x \in F)$ and $f(x) = 0$ $(x \in G)$.*

Proof. Set $k = |G|$, and choose $\eta \in (0, \varepsilon/k)$. For each $y \in G$, there exists $f_y \in A_{[1]}$ with $f_y(y) = 0$ and $|1 - f_y(x)| < \eta$ $(x \in F)$. Now define

$$f = \prod\{f_y : y \in G\}.$$

Then clearly $f \in A_{[1]}$ and $f(y) = 0$ $(y \in G)$. For each $x \in F$, we have

$$|1 - f(x)| \le \sum \{|1 - f_y(x)| : y \in G\} < k\eta < \varepsilon,$$

as required. ∎

The following result shows that the only natural, pointwise contractive Banach sequence algebra on a set S is $c_0(S)$.

THEOREM 3.5. *Let A be a natural Banach sequence algebra on a non-empty set S. Suppose that A is pointwise contractive. Then A is equivalent to the uniform algebra $c_0(S)$.*

Proof. Take F and G to be disjoint, closed, non-empty subsets of S_∞.

First, suppose that F and G are contained in S, and hence finite. By Proposition 3.4, there exists $f \in A_{[1]}$ with $|1 - f(x)| < 1/2$ $(x \in F)$ and $f|G = 0$. Second, suppose that $\infty \in G$. Then F is finite. Since A is pointwise contractive, there exists $f \in A_{[1]}$ with $|1 - f(x)| < 1/4$ $(x \in F)$. Set

$$H = \{y \in G : |f(y)| \ge 1/4\},$$

so that H is a compact, and hence finite, subset of S and $F \cap H = \emptyset$. By Proposition 3.4, there exists $g \in A_{[1]}$ with $|1 - g(x)| < 1/2$ $(x \in F)$ and $g|H = 0$. Set $h = fg$, so that $h \in A_{[1]}$. Then $|1 - h(x)| < 1/2$ $(x \in F)$ and $|h(y)| < 1/4$ $(y \in G)$. Thus the hypotheses of [6, Theorem 4.1.19] are satisfied (with $m = 1$), and so, by that theorem, $A^\sharp$ is equivalent to $C(S_\infty)$.

It follows that A is equivalent to $c_0(S)$. ∎

EXAMPLE 3.6. There is a norm $\|\cdot\|$ on c_0 such that $(c_0, \|\cdot\|)$ is a contractive Banach function algebra that is equivalent to a uniform algebra, but is not a uniform algebra.

Indeed, let B be the closed unit ball of $(c_0, |\cdot|_{\mathbb{N}})$, and set

$$C = \{(x_n) \in B : |x_1 - x_2| \le 1\}.$$

Then it is easily checked that C is absolutely convex and closed and that $(1/2)B \subset C \subset B$; further, $xy \in C$ whenever $x, y \in C$. Thus C is the closed unit ball of a norm, say $\|\cdot\|$, on c_0 such that $(c_0, \|\cdot\|)$ is a Banach function algebra that is equivalent to a uniform algebra; it is not a uniform algebra because $\|(-1, 1, 0, 0, 0, \dots)\| = 2$. Finally, we check that $(c_0, \|\cdot\|)$ is contractive. For example, the sequence $(\sum_{j=2}^{n} \delta_j)$ is a CAI for the maximal modular ideal $\{(x_n) \in c_0 : x_1 = 0\}$.

We now consider when the existence of a CPAI for a Banach function algebra A implies the existence of a BAI or even just an approximate identity for A. We have already shown in Proposition 3.1 that, in the case where A is an ideal in A'', the existence of a CPAI implies that of a CAI.

The first counter-example is the original one of Jones and Lahr [27]; it shows that a Banach function algebra can have a CPAI without having any approximate identity. We shall give further (some easier) related examples in Examples 3.13, 3.14, 3.17, and 5.1; we shall also give examples of pointwise contractive uniform algebras without any BAI in Example 4.8(vi) and Theorem 4.9 and an example of a uniform algebra with a CPAI, but no approximate identity, in Example 4.8(vii).

EXAMPLE 3.7. Let $S = (\mathbb{Q}^{+\bullet}, +)$ be the semi-group of strictly positive rational numbers. The semi-group algebra $A = (\ell^1(S), \star, \|\cdot\|_1)$ is a commutative, semi-simple Banach algebra. Then it is shown in [27] that there is a strictly increasing sequence (n_d) in $\mathbb{N}$ such that the sequence (δ_{1/n_d}) is a CPAI for A. However, A does not have any approximate identity. Indeed, for each $x \in S$, we have $\|\delta_x - \delta_x \star f\|_1 \geq 1$ $(f \in A)$.

Some future examples of natural Banach function algebras which are pointwise contractive, but which have no approximate identity, will be based on the following proposition.

PROPOSITION 3.8. *Let $(A, \|\cdot\|)$ be a natural Banach function algebra on a non-empty, locally compact space K. Suppose that $f_0 \in C_0(K) \setminus A$ is such that $ff_0 \in A$ and $\|ff_0\| \leq \|f\|$ for each $f \in A \cup \{f_0\}$. Set $B = A \oplus \mathbb{C}f_0$, with*

$$\|f + zf_0\| = \|f\| + |z| \quad (f \in A,\ z \in \mathbb{C}).$$

Then B is a natural Banach function algebra on K containing A as a proper closed ideal. Further, $B^2 \subset A$, and so B does not have an approximate identity.

Suppose that A has a CPAI *or is pointwise contractive. Then B has a* CPAI *or is pointwise contractive, respectively.*

Proof. It is clear that B is a Banach function algebra on K and that B contains A as a proper closed ideal.

Take $\varphi \in \Phi_B$. Then $\varphi | A \in \Phi_A$, and so there exists $x \in K$ such that $\varphi(f) = f(x)$ $(f \in A)$. Now take $f \in A$ such that $\varphi(f) = f(x) = 1$. Then $\varphi(f_0) = \varphi(ff_0) = (ff_0)(x) = f_0(x)$, and hence $\varphi(g) = g(x)$ $(g \in B)$. Thus B is natural on K.

Clearly $B^2 \subset A \subsetneq B$, and so B does not have an approximate identity.

Let $M_x(A)$ be a maximal ideal of A. Then a CPAI in $M_x(A)$ is also a CPAI in $M_x(B)$, the corresponding maximal ideal of B. ∎

The following example is a first concrete realization of the above proposition; others will be given in Examples 3.17 and 5.1.

EXAMPLE 3.9. Set $A = \ell^2$ (with pointwise multiplication), take f_0 to be the sequence $(1/\sqrt{n} : n \in \mathbb{N})$, and set $B = A \oplus \mathbb{C}f_0$, as in Proposition 3.8. Then the conditions of the proposition are satisfied, and so B is a natural Banach sequence algebra on $\mathbb{N}$. Further, as a Banach space, B is reflexive, and so certainly B is an ideal in B''. However, B is not Tauberian and B does not have an approximate identity; indeed, $B_0 = A \subsetneq B$. The algebra B does not have a BPAI.

3.2. Banach function algebras on locally compact groups. Let G be a locally compact group. The *group algebra* on G is $L^1(G) = (L^1(G), \star)$ and the *measure algebra* is $M(G) = (M(G), \star)$, so that $M(G)$ is a unital Banach algebra, and $L^1(G)$ is a closed ideal in $M(G)$; the algebra $L^1(G)$ has a CAI. See [6, §3.3], for example.

Let Γ be a locally compact group with identity e_Γ and left Haar measure m_Γ. Then the *Fourier algebra* and *Fourier–Stieltjes algebra* on Γ are denoted by $A(\Gamma)$ and $B(\Gamma)$, respectively; $B(\Gamma)$ is a Banach function algebra on Γ, and $A(\Gamma)$ is a closed ideal in $B(\Gamma)$. As a Banach space, $B(\Gamma)$ is the dual of the group C^*-algebra $C^*(\Gamma)$, and so we have a weak-$*$ topology $\sigma(B(\Gamma), C^*(\Gamma))$ on $B(\Gamma)$; for details, see [15]. In the case where Γ is the dual group of a locally compact, abelian (LCA) group G, $A(T)$ and $B(\Gamma)$ are identified with the spaces of Fourier transforms of elements of $L^1(G)$ and Fourier–Stieltjes transforms of elements of $M(G)$, respectively.

The theory of Fourier and Fourier–Stieltjes algebras originates in the seminal work of Eymard [15].

More generally, take $p > 1$. Then the *Figà-Talamanca–Herz algebra*, $A_p(\Gamma)$, is described in [6, pp. 493–494]; the Fourier algebra $A(\Gamma)$ is the algebra $A_2(\Gamma)$. By [6, Theorem 4.5.31], $A_p(\Gamma)$ is a self-adjoint, natural, strongly regular Banach function algebra on Γ, and $A_p(\Gamma)$ is dense in $(C_0(\Gamma), |\cdot|_\Gamma)$. The particular maximal modular ideal

$$A_e(\Gamma) = \{f \in A(\Gamma) : f(e_\Gamma) = 0\}$$

is the *augmentation ideal* of $A(\Gamma)$.

The following proposition combines results from [38] and [3, Corollary 2.8].

PROPOSITION 3.10. *Let Γ be a locally compact group, take $p > 1$, and set $A = A_p(\Gamma)$. Then Φ_A is $\sigma(A', A'')$-closed in A' if and only if Γ is amenable.* ∎

Let A be a natural Banach function algebra. Then it is immediate (see [38, Proposition 2.8]) that Φ_A is $\sigma(A', A'')$-closed in A' whenever A has a BPAI. Hence we obtain the following result.

PROPOSITION 3.11. *Let Γ be a locally compact group, and take $p > 1$. Then the following are equivalent:*

(a) *Γ is amenable;*
(b) *$A_p(\Gamma)$ has a* BPAI*;*
(c) *$A_p(\Gamma)$ has a* BAI*;*
(d) *$A_p(\Gamma)$ has a* CAI.

Proof. The equivalence of (a), (c), and (d) is well-known; for example, see [6, Theorem 4.5.32]. Clearly (c) implies (b). Now suppose that $A = A_p(\Gamma)$ has a BPAI. Then Φ_A is $\sigma(A', A'')$-closed in A', and so Γ is amenable by Proposition 3.10. ∎

COROLLARY 3.12. *Let Γ be an amenable locally compact group, and take $p > 1$ and $x \in \Gamma$. Then $M_x(A_p(\Gamma))$ has a* BAI *of bound* 2.

Proof. We may suppose that $x = e_\Gamma$. By considering elements of the form $(\chi_V \star \chi_V)/m_\Gamma(V)$ in $A_p(\Gamma)$ for a suitable symmetric, compact neighbourhood V of e_Γ, we see that, for each neighbourhood U of e_Γ, there exists $g \in A_p(\Gamma)_{[1]}$ with $g(e_\Gamma) = 1$ and $\operatorname{supp} g \subset U$. The result follows from Proposition 2.12(i) (with $m = n = 1$), where we recall that $A_p(\Gamma)$ is strongly regular, and so $\overline{J_x(A_p(\Gamma))} = M_x(A_p(\Gamma))$. ∎

Let G be a LCA group. A subalgebra S of $(L^1(G), \star)$ with a norm $\|\cdot\|_S$ is a *Segal algebra on* G if S is dense in $(L^1(G), \|\cdot\|_1)$, if $(S, \|\cdot\|_S)$ is a Banach algebra such that $\|f\|_S \geq \|f\|_1$ $(f \in S)$, if S is invariant under all the translations S_a for $a \in G$, if $\|S_a f\|_S = \|f\|_S$ $(f \in S,\ a \in G)$, and if the map

$$a \mapsto S_a f, \quad G \to S,$$

is continuous for each $f \in S$. These Segal algebras are abstract Segal algebras with respect to the Banach function algebra $L^1(G)$, in the sense of Definition 2.1 [6, §4.5].

We now give two examples of natural Banach function algebras A such that A has a CPAI and each maximal modular ideal of A has a BPAI, but A has no BAI.

EXAMPLE 3.13. Let G be a non-discrete LCA group with dual group Γ, and take $p \geq 1$. We define

$$S_p(G) = \{f \in L^1(G) : \widehat{f} \in L^p(\Gamma)\}$$

and

$$\|f\|_{S_p} = \max\{\|f\|_1, \|\widehat{f}\|_p\} \quad (f \in S_p(G)).$$

Then $(S_p(G), \star)$ is a Segal algebra with respect to $L^1(G)$ that is identified with a natural Banach function algebra on Γ; see [6, Example 4.5.27(iii)], where the norm has a slightly different form. Then, by [6, Proposition 4.5.28], $S_p(G)^2 \subsetneq S_p(G)$, and so, by Theorem 2.2(i), $S_p(G)$ does not have a BAI. Indeed, it is clear that $S_p(G)$ does not have BRAUs. However, by [25, Theorem 3.1], $S_p(G)$ has a CPAI whenever G is also non-compact. (Note that our $S_p(G)$ is called '$A_p(G)$' in [25].)

Thus, for example, $S_p(\mathbb{R})$ has a CPAI, but no BRAUs, for each $p \geq 1$.

Take $\gamma \in \Gamma$, say $\gamma = e_\Gamma$, and take a compact neighbourhood U of e_Γ with $m_\Gamma(U) \leq 1$. As in Corollary 3.12, there exists $g \in A(\Gamma)$ with $g(e_\Gamma) = \|g\| = 1$

and with $\operatorname{supp} g \subset U$, say $g = \widehat{f}$, where $f \in L^1(G)$ and $\|f\|_1 = 1$. Since $|g|_\Gamma \le 1$, we see that the norm of g in $L^p(\Gamma)$ is at most $m_\Gamma(U)^{1/p} \le 1$, and so $\|f\|_{S_p} \le 1$. By Proposition 2.12(ii), each maximal modular ideal in $S_p(G)$ has a BPAI of bound 2.

Example 3.14. Let G be a non-compact LCA group with dual group Γ, and take $p > 1$. We define

$$I_p(G) = L^1(G) \cap L^p(G) \quad \text{and} \quad \|f\|_{I_p(G)} = \max\{\|f\|_1, \|f\|_p\} \qquad (f \in I_p(G)).$$

Then $(I_p(G), \star)$ is a Segal algebra with respect to $L^1(G)$ that is identified with a natural Banach function algebra on Γ. By [25, Theorem 2.1], $I_p(G)$ has a CPAI. However, $I_p(G)$ does not have BRAUs and $I_p(G)$ does not factor.

Take a neighbourhood U of e_Γ. As above, there exists $g \in A(\Gamma)$ with $g(e_\Gamma) = \|g\| = 1$ and with $\operatorname{supp} g \subset U$, say $g = \widehat{f}$, where $f \in L^1(G)$ with $\|f\|_1 = 1$. Since $g \in L^1(\Gamma)$ and f is the (inverse) Fourier transform of g, we see that $f \in C_0(G)$ with $|f|_G \le 1$. Thus the norm of f in $L^p(G)$ is bounded by

$$\Big(\int_G |f(x)|^{p-1}|f(x)|\, dm_G(x)\Big)^{1/p} \le \Big(\int_G |f(x)|\, dm_G(x)\Big)^{1/p} = \|f\|_1^{1/p} = 1.$$

This shows that $f \in I_p(G)_{[1]}$. Again, this implies that each maximal modular ideal in $I_p(G)$ has a BPAI of bound 2.

Example 3.15. Let Γ be an amenable locally compact group, and take $p > 1$. We recall from Corollary 3.12 that each maximal modular ideal of $A_p(\Gamma)$ has a BAI of bound 2.

Let I be a closed ideal in $A(\Gamma)$. By combining [29, Theorem 5.3] and [19, Lemma 2.2], we see that the following are equivalent:

(a) I has a BPAI;
(b) I has the form $I(H)$ for some H in the closed coset ring of Γ;
(c) I has a BAI.

Take H to be a closed subgroup of Γ. Then it is proved by Delaporte and Derighetti [14, Theorems 10 and 11] and by Kaniuth and Lau [28, Theorem 3.4] that 2 is the best bound for a BAI in $I(H)$ whenever H is closed, normal, and non-open and when H is open and Γ/H is infinite. This applies when $H = \{e_\Gamma\}$ and Γ is infinite, and so 2 is the minimum bound of a BAI for each maximal modular ideal of $A(\Gamma)$ whenever Γ is infinite. For further results, see [20].

We next consider the best bound for a BPAI in a maximal ideal of $A(\Gamma)$ in the case where Γ is infinite, and Γ_d, the discrete version of Γ, is amenable (which implies that Γ is amenable). It is enough to consider the augmentation ideal $A_e(\Gamma)$. By [15, 2.24], $B(\Gamma)$ embeds isometrically in $B(\Gamma_d)$; since $A(\Gamma) \subset B(\Gamma)$, we may suppose that $A(\Gamma)$ is a subalgebra of $B(\Gamma_d)$.

Take (u_α) to be a BPAI in $A_e(\Gamma)$ of bound m, and consider (u_α) as a net in $B(\Gamma_d)_{[m]}$. By passing to a subnet, we may suppose that (u_α) converges in the weak-$*$ topology $\sigma(B(\Gamma_d), C^*(\Gamma_d))$, say to $u \in B(\Gamma_d)_{[m]}$. Since the evaluation functionals are in $C^*(\Gamma_d)$, convergence in this weak-$*$ topology implies pointwise convergence on Γ, and so $u(\gamma) = 1$ $(\gamma \in \Gamma \setminus \{e_\Gamma\})$ and $u(e_\Gamma) = 0$.

By Proposition 3.11, $A(\Gamma_d)$ has a CAI, say (v_β). Then (uv_β) is a net in $A_e(\Gamma_d)_{[m]}$ such that

$$\lim_\beta fuv_\beta = \lim_\beta fv_\beta = f \quad (f \in A_e(\Gamma_d)),$$

and so (uv_β) is a BAI in $A_e(\Gamma_d)$ of bound m. By the above remark, $m \geq 2$ whenever Γ_d is infinite and amenable, and so 2 is the minimum bound of a BPAI for each maximal modular ideal of $A(\Gamma)$ in this case.

In the case where Γ is a locally compact group that is not amenable, $A(\Gamma)$ does not have a BPAI; if Γ is amenable, but not amenable as a discrete group (for example, $\Gamma = SO(3)$), we do not know the best bound for a BPAI in $A(\Gamma)$.

As remarked in [20, Remark 2.5], it seems to be open whether the bound of 2 for a BAI in maximal modular ideal of $A_p(\Gamma)$ is optimal in the case where $p > 1$ and $p \neq 2$.

There is a large and significant class of Banach function algebras known as BSE algebras [37]. This class includes the Fourier algebra $A(\Gamma)$ of an amenable group Γ [29], and the two Banach algebras $S_p(G)$ and $I_p(G)$ considered in Examples 3.13 and 3.14 [25]. As will be proved in [13], the multiplier algebra of a pointwise contractive BSE algebra on a locally compact space K is isomorphic to $C^b(K)$. This shows that none of the algebras $A(\Gamma)$, $S_p(G)$, or $I_p(G)$ is pointwise contractive. In particular, as is the case here, the algebra A is certainly not pointwise contractive when the multiplier algebra of a BSE algebra is weakly sequentially complete.

Example 3.16. Let ω be a weight on an infinite LCA group G (so that $\omega : G \to [1, \infty)$ is such that $\omega(s+t) \leq \omega(s)\omega(t)$ $(s, t \in G)$ and $\omega(e_G) = 1$). Since $L^1(G)$ is not pointwise contractive, the Beurling algebra $(L^1(G, \omega), \star)$ also fails to be pointwise contractive.

The following example, which shows that a Banach function algebra can have a CPAI without having any approximate identity, is easier than the one given in Example 3.7.

Example 3.17. Take $(L^1(G), \star)$ for a non-discrete, LCA group G, and take a singular measure $\mu_0 \in M(G)_{[1]} \setminus L^1(G)$ such that $\mu_0 \star \mu_0 \in L^1(G)$; a proof that such an element μ_0 exists for each such group G is given in [24]. Since $L^1(G)$ is a closed ideal in $M(G)$, it follows that $f \star \mu_0 \in L^1(G)$ with $\|f \star \mu_0\|_1 \le \|f\|_1$ $(f \in L^1(G))$, and also $\|\mu_0 \star \mu_0\|_1 \le \|\mu_0\|$. We regard μ_0 as an element of $C_0(\Gamma)$, where Γ is the dual group to G. Set $B = L^1(G) \oplus \mathbb{C}\mu_0$. Then B satisfies the conditions given in Proposition 3.8.

In this case, $L^1(G)$ has a CAI and each maximal ideal of $L^1(G)$ has a BAI of bound 2. Thus B has a CPAI and each maximal ideal of B has a BPAI of bound 2. Since maximal modular ideals of $L^1(G)$ have BRAUs with bound 2, the same is true for B. However, B does not have an approximate identity.

4. Uniform algebras

4.1. Cole algebras. First recall that every C^*-algebra A is Arens regular and that $(A'', \Box)$ is a C^*-algebra [6, Theorem 3.2.36]; in particular, $(C_0(K)'', \Box)$ is a commutative, unital C^*-algebra, and so has the form $C(\widetilde{K})$ for a compact space $\widetilde{K}$, called the *hyper-Stonean envelope* of K. (See [7] for an extensive discussion and explicit constructions of the space $\widetilde{K}$.) Now suppose that A is a uniform algebra on a non-empty, locally compact space K. Then $(A'', \Box)$ is also Arens regular and is a closed subalgebra of $C(\widetilde{K})$.

Our results will be based on the following theorem. The proof that (c) implies (d) is surely well-known.

THEOREM 4.1. *Let A be a uniform algebra on a non-empty, compact space K, and take $x \in K$. Then the following conditions on x are equivalent:*

(a) $\varepsilon_x \in \operatorname{ex} K_A$;
(b) $x \in \Gamma_0(A)$;
(c) *M_x has a* BAI;
(d) *M_x has a* CAI.

Proof. The equivalence of (a), (b), and (c) is a special case of [6, Theorem 4.3.5], and trivially (d) implies (c).

Now suppose that (c) holds. Then M_x'' is a closed subalgebra of $C(\widetilde{K})$ and, by [6, Proposition 2.9.16(iii)], M_x'' contains an identity, say e. Clearly e is an idempotent in $C(\widetilde{K})$, and so $|e|_{\widetilde{K}} = 1$. By [6, Proposition 2.9.16(iii)] again, M_x has a CAI, giving (d). ■

Thus, by Theorem 2.2(i), M_x factors whenever x is a p-point for a uniform algebra.

Suppose that A is a uniform algebra on a non-empty, compact space K and that $x \in K$ is a peak point, say $f \in A$ peaks at x. Then $(1_K - f^n : n \in \mathbb{N})$ is a sequential BAI for M_x (with bound 2); set $f_n = 1_K - f^n$ $(n \in \mathbb{N})$. For each $n \in \mathbb{N}$, there exists $e_n \in (M_x)_{[1]}$ with $|f_n - e_n f_n|_K < 1/n$. Now take

$g \in M_x$ and $\varepsilon > 0$. Then there exists $n \in \mathbb{N}$ with $|g - f_n g|_K < \varepsilon$ and $n\varepsilon > |g|_K$, and hence $|g - e_n g|_K < 2\varepsilon$. This shows that (e_n) is a sequential CAI for M_x.

We shall show in Example 4.8(vii) that there is a uniform algebra with a BPAI, but no CPAI, so there is no 'pointwise' analogue of the implication (c)⇒(d) of the above theorem.

We now introduce the following definition.

Definition 4.2. Let A be a natural uniform algebra on a non-empty, locally compact space K. Then A is a *Cole algebra* if $\Gamma_0(A) = K$.

Of course $C_0(K)$ is a Cole algebra.

The reason for this terminology is the following. In the case where K is compact and metrizable, a natural uniform algebra on K is a Cole algebra if and only if every point of K is a peak point. It was a long-standing conjecture, called the 'peak-point conjecture', that $C(K)$ is the only Cole algebra on a compact, metrizable space K. The first counter-example is due to Cole [4], and is described in [36, §19]. An example of Basener [1], also described in [36, §19], gives a compact space K in $\mathbb{C}^2$ such that the uniform algebra $R(K)$ of all uniform limits on K of the restrictions to K of the functions which are rational on a neighbourhood of K, is a Cole algebra, but $R(K) \neq C(K)$. Further, Feinstein [16, 18] obtained examples of non-trivial Cole algebras on compact, metrizable spaces K such that they are a strong Ditkin algebra and are not regular, respectively.

Let A be a natural uniform algebra on the closed unit interval $\mathbb{I}$. Then it is a very famous question of Gel'fand whether A is necessarily equal to $C(\mathbb{I})$. It is a consequence of Rossi's local maximum modulus theorem [36, Corollary 9.14] that $\Gamma_0(A)$ is dense in $\mathbb{I}$. However, it seems to be unknown whether every such algebra is necessarily a Cole algebra, and also unknown whether every Cole algebra on $\mathbb{I}$ is necessarily trivial. It is known that strongly regular uniform algebras on $\mathbb{I}$ are trivial [39].

The following theorem is the main result of this section; it is immediate from Theorem 4.1.

Theorem 4.3. *Let A be a uniform algebra on a compact space K. Then the following are equivalent:*

(a) *A is contractive;*
(b) *A is a Cole algebra;*
(c) *M_x has a* BAI *for each $x \in K$.* ∎

4.2. Pointwise approximate identities for uniform algebras. We now consider when maximal ideals in a uniform algebras have a BPAI or a CPAI and the relationship between 'pointwise contractive' and 'contractive' for uniform algebras.

PROPOSITION 4.4. *Let A be a uniform algebra on a non-empty, locally compact space. Suppose that A has a sequential* BPAI. *Then A has a sequential* CAI.

Proof. There is a sequence (f_n) in A such that $f_n(x) \to 1$ as $n \to \infty$ for each $x \in K$ and with $\sup |f_n|_K \leq m$, say. Take $f \in A$. Then $ff_n - f \to 0$ pointwise on K as $n \to \infty$, and so, by the dominated convergence theorem,

$$\lim_{n\to\infty} \int_K (ff_n - f)(x)\, d\mu(x) = 0$$

for each positive measure μ on K. Thus $\lim_{n\to\infty}\langle ff_n - f, \lambda\rangle = 0$ for each $\lambda \in A'$. By [6, Proposition 2.9.14(iii)], A has a BAI of bound m. Now A is a maximal ideal in the uniform algebra $A^\sharp$ on K_∞, and A has a BAI with respect to this uniform norm, and so, by Theorem 4.1, A has a CAI, which we may take to be sequential. ■

We next recall the definition of a 'Gleason part' for a uniform algebra. The first lemma is [36, Lemma 16.1]; here ρ denotes the hyperbolic metric on $\overline{\mathbb{D}}$.

LEMMA 4.5. *Let A be a natural uniform algebra on a compact space K, and take $x, y \in K$. Then the following are equivalent:*

(a) $\|\varepsilon_x - \varepsilon_y\| < 2$;
(b) *there exists $c \in (0,1)$ such that $|f(x)| < c|f|_K$ $(f \in M_y)$;*
(c) *there is a constant $M > 0$ such that $\rho(f(x), f(y)) \leq M$ for each $f \in A_{[1]}$.* ■

Now define $x \sim y$ for $x, y \in K$ if x and y satisfy the conditions of the lemma. It follows that $\sim$ is an equivalence relation on K; the equivalence classes with respect to this relation are the *Gleason parts* for A. We equip K with the topology induced by the *Gleason metric* δ, where

$$\delta(x,y) = \|\varepsilon_x - \varepsilon_y\| \quad (x, y \in K).$$

The Gleason parts form a partition of K, and each part is σ-compact with respect to the Gleason metric. Clearly $\{x\}$ is a one-point Gleason part whenever x is a p-point.

For a discussion of Gleason parts, see [21, Chapter VI].

The following result seems to have been unnoticed so far.

THEOREM 4.6. *Let A be a natural uniform algebra on a compact space K, and take $x \in K$. Then the following are equivalent:*

(a) *$\{x\}$ is a one-point Gleason part;*
(b) *M_x has a* CPAI;
(c) *for each $y \in K \setminus \{x\}$, there is a sequence (f_n) in M_x such that $|f_n|_K \leq 1$ $(n \in \mathbb{N})$ and $f_n(y) \to 1$ as $n \to \infty$.*

Proof. The equivalence of (a) and (c) is immediate from Lemma 4.5, and clearly (b) implies (c).

Now suppose that (c) holds, and take a finite set $F = \{y_1, \dots, y_m\}$ in $K \setminus \{x\}$. Take B to be the closed subalgebra of A consisting of the functions in A that are constant on F, and set $M = M_{y_1} \cap \dots \cap M_{y_m}$, so that M is a maximal ideal of B and B is a natural uniform algebra on the compact set L formed by identifying the points $y_1, \dots, y_m$, say this point is $y_F \in L$.

For each $j \in \mathbb{N}_m$, there is a sequence $(f_{j,n} : n \in \mathbb{N})$ in $(M_{y_j})_{[1]}$ with $\lim_{n\to\infty} f_{j,n}(x) = 1$. Set

$$f_n = f_{1,n} \cdots f_{m,n} \quad (n \in \mathbb{N}).$$

Then $f \in M$ and $\lim_{n\to\infty} f_n(x) = 1$, and so $x \nsim y_F$ in L. Hence there exists (h_n) in $(M_x)_{[1]}$ with $\lim_{n\to\infty} h_n(y_F) = 1$. We see that

$$\lim_{n\to\infty} h_n(y_j) = 1 \quad (j \in \mathbb{N}_m)$$

as sequence in A, and so M_x has a CPAI, giving (b). ■

Let A be a natural uniform algebra on a compact space K, and take $x \in K$. Suppose that M_x has a BPAI. Then x is isolated in K with respect to the Gleason metric.

The following immediate consequence of the above theorem is the second main result of this section.

THEOREM 4.7. *Let A be a natural uniform algebra on a compact space K. Then A is pointwise contractive if and only if each Gleason part in K is a singleton.* ■

We now present various uniform algebras; they show that all possibilities not excluded by previous theorems do occur.

EXAMPLE 4.8. (i) Let $A = A(\overline{\mathbb{D}})$ be the disc algebra, as in Example 2.4, and, for $z \in \overline{\mathbb{D}}$, set $M_z = \{f \in A : f(z) = 0\}$. Take $z \in \mathbb{D}$ and $f \in M_z$. By the Schwarz–Pick theorem,

$$|f(w)| \le \frac{|w-z|}{|1-\overline{w}z|}|f|_{\overline{\mathbb{D}}} \quad (w \in \mathbb{D}).$$

Thus there exists $\delta > 0$ such that $|f(w)| < 1/2$ whenever $|w - z| < \delta$, and so M_z does not have a BPAI.

Take $z \in \overline{\mathbb{D}}$. In this example, M_z has a BPAI if and only if M_z has a CPAI if and only if z is a peak point.

(ii) Let A be a uniform algebra on a compact set K, and take $x \in K$. It is also possible to have $x \in \Gamma(A)$, but such that M_x does not have a BPAI. Indeed, let $K = \overline{\mathbb{D}} \times \mathbb{I}$ and take A to be the 'tomato can algebra' [36, Example 7.8], so that A is the uniform algebra of all continuous functions f

on K such that the function $z \mapsto f(z,1)$, $\overline{\mathbb{D}} \to \mathbb{C}$, belongs to $A(\overline{\mathbb{D}})$. Then

$$\Gamma_0(A) = \{(z,t) \in K : 0 \leq t < 1\} \cup \{(z,1) \in K : z \in \mathbb{T}\}$$

and $\Gamma(A) = K$. The set $K \setminus \Gamma_0(A) = \{(z,1) : z \in \mathbb{D}\}$ is a Gleason part, and again we see that M_x has a BPAI if and only if M_x has a CPAI if and only if x is a peak point, where $x \in K$; if $x \in K \setminus \Gamma_0(A)$, then $M_x^2 = \overline{M_x^2} \subsetneq M_x$, and so M_x does not have an approximate identity.

(iii) Let K be a compact plane set, and consider the natural uniform algebra $R(K)$ on K. Take $x \in K$. Then $\{x\}$ is a one-point Gleason part if and only x is a peak point [36, Corollary 26.14], and so M_x has a BAI if and only if M_x has a CPAI if and only if x is a peak point for $R(K)$. By [36, Corollary 26.15], $R(K) = C(K)$ if and only if each point of K is a one-point Gleason part, and so $R(K) = C(K)$ if and only if $R(K)$ is pointwise contractive. It follows from [36, Corollary 26.12] that x is not isolated in the Gleason metric whenever x is not a peak point, and so M_x has a BPAI if and only if it has a CPAI. (By [36, Corollary 26.13] each Gleason part for $R(K)$ that is not a singleton has positive plane area.)

(iv) There are natural, separable uniform algebras A on a compact space K that have one-point parts $\{x\}$ for some $x \in K \setminus \Gamma(A)$. For such points x, the maximal ideal M_x has a CPAI, but no BAI. For example, the uniform algebra $\mathfrak{A}_\alpha$ of [36, Theorem 18.1] has this property.

(v) Let H^∞ be the uniform algebra of all bounded analytic functions on $\mathbb{D}$, so that H^∞ is non-separable. The (large) character space of H^∞ is denoted by Φ; it is studied in [22, Chapter 10]. Since H^∞ is a logmodular algebra on its Šilov boundary, every point of $\Gamma = \Gamma(H^\infty)$ has a unique representing measure on Γ, and this unique representing measure must be the point mass. Consequently, each point of Γ is a p-point, and hence a one-point Gleason part. In fact, each Gleason part for H^∞ is either a one-point part or an analytic disc and there are one-point Gleason parts that are not in $\Gamma(H^\infty)$. Suppose that $\{x\}$ is a one-point part with $x \in \Phi$. Then M_x factors [22, Theorem 2.4].

(vi) The following example is given in [8, Theorem 2.3]. There is a natural, separable uniform algebra on a compact, metric space K such that each point of K is a one-point Gleason part, so that A is pointwise contractive, but $\Gamma(A) \subsetneq K$, so that A is not a Cole algebra, and hence not contractive. Thus some maximal ideals of A are pointwise contractive uniform algebras without a BAI. The existence of such an example also follows from Theorem 2.5 in [4].

(vii) In [17, Theorem 2.1], Feinstein constructed a separable, regular, natural uniform algebra A on a compact space K such that there is a two-point Gleason part, say $\{x_1, x_2\}$, and such that all other points of K are one-point Gleason parts.

Set $M = M_{x_1}$, and take a finite set F in $K \setminus \{x_1\}$, say F is a subset of a set of the form $\{x_2, \dots, x_n\}$, where $x_2, \dots, x_n$ are distinct. Take $f_2 \in M_{x_2}$ with $f_2(x_1) = 1$, and set $m = |f_2|_K$. Fix $\varepsilon \in (0, 1)$, and take $\delta > 0$ such that $mn\delta < \varepsilon$. For $j = 3, \dots, n$, take $f_j \in M_{x_j}$ with $|f_j(x_1)| > 1-\delta$ and $|f_j|_K = 1$, and set $f = f_2 f_3 \cdots f_n \in A$, so that $|f|_K \le m$ and $|f(x_1) - 1| < mn\delta < \varepsilon$. Finally, set $g = f(x_1)1_K - f$, so that $g \in M$ with $|g(x_j)| > 1 - \varepsilon$ and $|g|_K \le m+1+\varepsilon$. It follows that M has a BPAI with bound $m+1$. Similarly, M_{x_2} has a BPAI, and each other point of K has a CPAI.

Thus, in this example, each maximal ideal has a BPAI (with a uniform bound), but the algebra is not pointwise contractive.

(viii) In Examples 5.13 and 5.16 of [35], there are natural uniform algebras A on compact spaces K and points $x \in K \setminus \Gamma(A)$ such that $\{x\}$ is a one-point Gleason part and M_x^2 is not dense in M_x; in particular, M_x does not factor. In these cases, the uniform algebra M_x has a CPAI, but no approximate identity. We are grateful to Alexander Izzo for pointing out this example.

We are grateful to Joel Feinstein for pointing out the following example. We obtain a strong form of a CPAI in a maximal ideal M_x such that, for each non-empty, finite set F disjoint from x, there is a function $f \in (M_x)_{[1]}$ that attains the value 1 at each point in F.

THEOREM 4.9. *There is a natural, pointwise contractive uniform algebra A on a non-empty, compact, metrizable space K such that a maximal ideal M of A does not have a* BAI, *and $\Gamma_0(A) \subsetneq K$.*

Proof. Let A be the uniform algebra on a compact, metrizable space K that is constructed in [16, Theorem 5.1]: the algebra A is natural and has the property that there exists a point $x \in K$ such that $\Gamma_0(A) = K \setminus \{x\}$, and so $A \neq C(K)$.

Let F be a non-empty, finite subset of $K \setminus \{x\}$. Then F is a peak set, and so there exists a function $f \in A$ such that $f(y) = 1$ $(y \in F)$ and $|f(y)| < 1$ $(y \in K \setminus F)$. Set $a = f(x)$, so that $a \in \mathbb{D}$. By composing f with the function $\psi_a(z) = \zeta(z - a)/(1 - \overline{a}z)$ $(z \in \overline{\mathbb{D}})$ for suitable $\zeta \in \mathbb{T}$, we may suppose that $f \in M_x$. Thus M_x has a CAI. Each point $y \in K \setminus \{x\}$ is a peak point for A, and so M_y has a CAI. Thus A is pointwise contractive.

Since $x \notin \Gamma_0(A)$, it follows from Theorem 4.1 that M_x does not have a BAI. Thus A is not contractive, and A is not a Cole algebra. ∎

We do not know whether there is a natural uniform algebra A on a compact space K such that every point of K is a one-point part, but M does not have an approximate identity for some maximal ideal M in A.

5. Banach function algebras on closed intervals. We now present two natural, unital Banach function algebras on intervals of $\mathbb{R}$. The first example gives a pointwise contractive Ditkin algebra such that one maximal ideal does not have BRAUs, and hence has no BAI. This maximal ideal is an abstract Segal algebra with respect to $C_0((0,1])$. The example is easier than that of Jones and Lahr, and a small variation is also stronger in that it is pointwise contractive, but has no approximate identity. The example is developed from a suggestion of Charles Read.

The second example exhibits a contractive Banach function algebra that is not equivalent to a uniform algebra.

EXAMPLE 5.1. Consider the set A of functions $f \in C(\mathbb{I})$ such that

$$I(f) := \int_0^1 \frac{|f(t) - f(0)|}{t}\, dt < \infty.$$

Clearly A is a self-adjoint, linear subspace of $C(\mathbb{I})$ containing the polynomials, and so A is uniformly dense in $C(\mathbb{I})$. Indeed, A is 'large', in that it contains all the Banach function algebras $(\mathrm{Lip}_\alpha(\mathbb{I}), \|\cdot\|_\alpha)$ (for $0 < \alpha \leq 1$). Also, A contains each $f \in C(\mathbb{I})$ with $\operatorname{supp} f \subset (0,1]$.

For $f \in A$, define

$$\|f\| = |f|_{\mathbb{I}} + I(f).$$

Clearly $(A, \|\cdot\|)$ is a normed space; we recall a standard fact that it is complete. Indeed, take (f_n) to be a Cauchy sequence in $(A, \|\cdot\|)$. Then there exists $f \in C(\mathbb{I})$ such that $|f_n - f|_{\mathbb{I}} \to 0$ as $n \to \infty$. Take $\varepsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that

$$|f_m - f_n|_I + I(f_m - f_n) < \varepsilon \quad (m, n \geq n_0).$$

By Fatou's lemma, $I(f_m - f) \leq \liminf_{n\to\infty} I(f_m - f_n) \leq \varepsilon$ for each $m \geq n_0$. We see that $I(f) \leq I(f_{n_0}) + \varepsilon$, and so $f \in A$; further,

$$\|f_m - f\| \leq 2\varepsilon \quad (m \geq n_0),$$

and hence (f_n) converges to f in $(A, \|\cdot\|)$.

Our first *claim* is that the set A is a subalgebra of $C(\mathbb{I})$ and, further, that $\|fg\| \leq \|f\|\,\|g\|$ $(f, g \in A)$. Indeed, for $f, g \in A$, we have

$$|(fg)(t) - (fg)(0)| \leq |f|_{\mathbb{I}}|g(t) - g(0)| + |g|_{\mathbb{I}}|f(t) - f(0)| \quad (t \in \mathbb{I}),$$

and so $I(fg) \leq |f|_{\mathbb{I}} I(g) + |g|_{\mathbb{I}} I(f)$, which implies the claim. Also, $\|1_{\mathbb{I}}\| = 1$, and so $(A, \|\cdot\|)$ is a Banach function algebra on $\mathbb{I}$. Set $M = M_0(A)$. Then M is also an ideal in $C_0((0,1])$ and

$$\|fg\| \leq \|f\|\,|g|_{\mathbb{I}} \quad (f \in M,\ g \in C_0((0,1])).$$

Our second *claim* is that A is natural on $\mathbb{I}$. By [6, Proposition 4.1.5(ii)], it suffices to show that each $f \in A$ that is such that $f(t) \neq 0$ $(t \in \mathbb{I})$ is invertible in A. We may suppose, without loss of generality, that $f(0) = 1$, say $f = 1 + g$, where $g \in M$, and that $1/f = 1 + h$, where $h \in C(\mathbb{I})$ and $h(0) = 0$. Choose $\delta > 0$ such that $|g(t)| < 1/2$ $(t \in [0, \delta])$. Since $|h(t)| \leq 2|g(t)|$ $(t \in [0, \delta])$, we have $\int_0^\delta (|h(t)|/t)\, dt < \infty$; clearly,

$$\int_\delta^1 \frac{|h(t)|}{t}\, dt \leq |h|_{\mathbb{I}} \log(1/\delta) < \infty,$$

and so $h \in M$ and $1/f \in A$, giving the claim.

Our third *claim* is that the norm of A is not equivalent to the uniform norm. To see this, take $n \in \mathbb{N}$, and define f_n to be linear on $[0, 1/n]$ and equal to 1 on $[1/n, 1]$. Then $f_n \in A$ and

$$\|f_n\| = 1 + \int_0^{1/n} n\, dt + \int_{1/n}^1 \frac{1}{t}\, dt = 2 + \log n,$$

whereas $|f_n|_{\mathbb{I}} = 1$. This gives the claim. Alternatively, take

$$h_0(t) = \frac{1}{\log(1/t)} \qquad (t \in (0, 1]), \tag{5.1}$$

with $h_0(0) = 0$. Then $h_0 \in C(\mathbb{I})$, but $h_0 \notin A$. Indeed, for each $n \in \mathbb{N}$ and $f \in M$ with

$$|1 - f(x)| < 1/2 \qquad (1/n \leq x \leq 1),$$

we have $\|f\| \geq (\log n)/2$, and so M does not have BRAUs.

Take $t_0 \in \mathbb{I}$, and take (g_n) to be as in Definition 2.5, so that (g_n) is a sequence in $J_{t_0}(A)$, and take $f \in M_{t_0}$. We *claim* that $\|f - fg_n\| \to 0$ as $n \to \infty$. This is immediate for $t_0 > 0$. In the case where $t_0 = 0$, fix $\varepsilon > 0$ and take $\delta > 0$ such that $|f(t)| < \varepsilon$ $(0 \leq t \leq \delta)$ and

$$\int_0^\delta \frac{|f(t)|}{t}\, dt < \varepsilon.$$

Then, for $n > 1/\delta$, we see that $\|f - fg_n\| < 2\varepsilon$. The claim follows. Thus the natural Banach function algebra A is a Ditkin algebra, and hence strongly regular. The sequence (g_n) (for $t_0 = 0$) is an approximate identity for M and $C_0((0, 1])$, and so M is an abstract Segal algebra with respect to $C_0((0, 1])$. For $t_0 > 0$, we see that $\|g_n\| = 1 + O(1/n)$ as $n \to \infty$, and so the sequence $(g_n/\|g_n\| : n \in \mathbb{N})$ is a CAI for M_{t_0}. This implies that $M_{t_0} = M_{t_0}^{[2]} = \overline{J}_{t_0}$.

Our fourth *claim* is that the algebra of polynomials (restricted to $\mathbb{I}$) is dense in A. Indeed, to see this, it suffices to show that, given $f \in J_0$ and $\varepsilon > 0$, there is a polynomial p such that $\|f - p\| < \varepsilon$. For this, we first define $g(t) = f(t)/t$ $(t \in \mathbb{I})$ (with $g(0) = 0$), so that $g \in C(\mathbb{I})$. There is a polynomial q such that $|g-q|_{\mathbb{I}} < \varepsilon/2$. Set $p(t) = tq(t)$ $(t \in \mathbb{I})$, so that p is a polynomial; clearly $\|f-p\| < \varepsilon$, as required.

The ideal M does not have a BAI. Indeed, our fifth *claim* is the slightly stronger fact that M^2 has infinite codimension in M, and so the space of point derivations at 0 is infinite dimensional. To see this, first take $f \in M^2$, say $f = \sum_{j=1}^{k} g_j h_j$, where $g_1, \dots, g_k, h_1, \dots, h_k \in M$, and set

$$u = \sum_{j=1}^{k} (|g_j| + |h_j|).$$

Then $u \in M$ and $|f(t)| \le u(t)^2$ $(t \in \mathbb{I})$. We apply this with f_α defined by

$$f_\alpha(t) = \frac{1}{(\log(1/t))^\alpha} \quad (t \in (0,1]),$$

with $f_\alpha(0) = 0$, for $\alpha > 0$. Then $f_\alpha \in M$ whenever $\alpha > 1$. Suppose that $f_\alpha \in M^2$. Then there exists $u_\alpha \in M$ such that $|f_\alpha(t)| \le u_\alpha(t)^2$ $(t \in \mathbb{I})$, and so $u_\alpha(t) \ge 1/(\log(1/t))^{\alpha/2}$ $(t \in (0,1])$. This implies that $\alpha > 2$, and so $f_\alpha \in M \setminus M^2$ for $\alpha \in (1,2]$. It follows easily that the set

$$\{f_\alpha + M^2 : \alpha \in (1,2]\}$$

is linearly independent in M/M^2, giving the claim.

We have seen that the maximal ideal M does not have a BAI. However, our sixth *claim* is that it has a CPAI. Indeed, take F to be a finite subset of $(0,1]$, say $F = \{t_1, \dots, t_k\}$. For each $n \in \mathbb{N}$ and $i \in \mathbb{N}_k$, take $f_{n,i}$ to be the restriction to $\mathbb{I}$ of the function such that $f_{n,i}(x_i) = 1$, such that $f_{n,i} = 0$ outside the interval $[t_i - 1/n, t_i + 1/n]$, and such that $f_{n,i}$ is linear on $[t_i - 1/n, t_i]$ and $[t_i, t_i + 1/n]$, and set $f_{F,n} = \sum_{i=1}^{k} f_{n,i}$. Then clearly $f_{F,n}(t) = 1$ $(t \in F)$ and $\|f_{F,n}\| = 1 + O(1/n)$ as $n \to \infty$. The claim follows. (We note that our CPAI in M is a net, not a sequence.) It follows that A is pointwise contractive.

Define h_0 as in equation (5.1) above, so that $h_0 \in C(\mathbb{I}) \setminus M$ and $h_0^2 \in M$, and then set $B = M \oplus \mathbb{C}h_0$. We have $\|fh_0\| \le \|f\||h_0|_{\mathbb{I}}$ $(f \in M)$, and so, by multiplying h_0 by a suitable positive constant, we may suppose that h_0 satisfies the conditions of Proposition 3.8. Since M is pointwise contractive, so is B. However, B does not have any approximate identity.

Example 5.2. We now present a Banach function algebra on the circle $\mathbb{T}$, but, for notational convenience, we identify $C(\mathbb{T})$ with the subalgebra of $C([-1,1])$ consisting of functions $f \in C([-1,1])$ with

$$\exp(-\pi i f(-1)) = \exp(\pi i f(1)).$$

Addition and subtraction in $[-1,1]$ are taken modulo $[-1,1]$.

We fix a constant α with $1 < \alpha < 2$.

For $t \in [-1,1]$, the *shift* of $f \in C(\mathbb{T})$ by t is defined by

$$(S_t f)(s) = f(s-t) \quad (s \in [-1,1]);$$

the *oscillation* of f is

$$\omega_f(t) = \sup\{|f(s) - (S_t f)(s)| : s \in [-1,1]\} \quad (t \in [-1,1]),$$

and

$$\Omega_f(t) = \|f - S_t f\|_1 = \int_{-1}^{1} |f(s) - f(s-t)|\, ds \quad (t \in [-1,1]).$$

Then we define

$$I(f) = \int_{-1}^{1} \frac{\Omega_f(t)}{|t|^\alpha}\, dt.$$

We note that the function $t \mapsto \Omega_f(t)$, $[-1,1] \to \mathbb{R}^+$, is continuous, and so $I(f)$ is well defined (in $[0,\infty]$). Also $\Omega_f = \Omega_{1-f}$, and so $I(f) = I(1-f)$.

We now define A to be the space of functions $f \in C(\mathbb{T})$ with $I(f) < \infty$, and set

$$\|f\| = |f|_{\mathbb{T}} + I(f) \quad (f \in A).$$

Clearly A is a self-adjoint, linear subspace of $C(\mathbb{T})$, and $(A, \|\cdot\|)$ is a normed space. Further, $(A, \|\cdot\|)$ is complete, and so is a Banach space. For let (f_n) be a Cauchy sequence in $(A, \|\cdot\|)$. Then there exists $f \in C(\mathbb{T})$ with $f_n \to f$ uniformly on $\mathbb{T}$ as $n \to \infty$. We have $\Omega_{f_n}(t) \to \Omega_f(t)$ as $n \to \infty$ for each $t \in [-1,1]$, and so it again follows from Fatou's lemma that $f_n \to f$ in $(A, \|\cdot\|)$ as $n \to \infty$.

We see that $S_t f \in A$ with $\|S_t f\| = \|f\|$ for each $f \in A$ and $t \in [-1,1]$, and so the algebra A is homogeneous on the circle.

Suppose that $f \in C(\mathbb{T})$ with $\omega_f(t) = O(t^\gamma)$ for some $\gamma > \alpha - 1$. Then $f \in A$. In particular, A contains the trigonometric polynomials, and so A is uniformly dense in $C(\mathbb{T})$.

Our first *claim* is that $(A, \|\cdot\|)$ is a Banach function algebra on $\mathbb{T}$. Indeed, take $f, g \in A$ and $t \in [-1,1]$. For each $s \in [-1,1]$, we have

$$|(fg - S_t(fg))(s)| \le |f|_{\mathbb{T}} |(g - S_t g)(s)| + |g|_{\mathbb{T}} |(f - S_t f)(s)|,$$

and so $\Omega_{fg}(t) \le |f|_{\mathbb{T}} \Omega_g(t) + |g|_{\mathbb{T}} \Omega_f(t)$. It follows that

$$I(fg) \le |f|_{\mathbb{T}} I(g) + |g|_{\mathbb{T}} I(f) \quad \text{and hence} \quad \|fg\| \le \|f\| \|g\|.$$

Further, $\|1_{\mathbb{T}}\| = 1$. The claim follows.

Our second *claim* is that A is natural on $[-1,1]$. Indeed, take a function $f \in A$ such that $|f(t)| \ge \delta > 0$ $(t \in [-1,1])$. Then we see easily that $I(1/f) \le \delta^2 I(f) < \infty$, and so $1/f \in A$. Thus A is natural.

Our third *claim* is that $A \ne C(\mathbb{T})$. For this, we define

$$e_n(s) = \exp(i\pi n s) \quad (s \in [-1,1])$$

for $n \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$, we have

$$\Omega_{e_n}(t) = \int_{-1}^{1} |e^{i\pi ns} - e^{i\pi n(s-t)}|\,ds = 2|1 - e^{i\pi nt}| \quad (t \in [-1,1]),$$

and so

$$I(e_n) = 2\int_{-1}^{1} \frac{|1 - e^{i\pi nt}|}{|t|^\alpha}\,dt \geq 2(\pi n)^{\alpha-1}\int_{\pi/2}^{\pi} \frac{|1 - e^{iu}|}{|u|^\alpha}\,du = Cn^{\alpha-1}$$

for some constant $C > 0$. Thus $\|e_n\| \geq Cn^{\alpha-1} \to \infty$ as $n \to \infty$, whereas $|e_n|_{\mathbb{T}} = 1$ for each $n \in \mathbb{N}$, and so the claim follows.

Our fourth *claim* is that the Banach function algebra A is contractive. Since A is homogeneous on $\mathbb{T}$, it suffices to show that the maximal ideal $M := \{f \in A : f(0) = 0\}$ has a CAI. For this, define

$$\Delta_n(s) = \max\{1 - n|s|, 0\} \quad (s \in [-1,1],\ n \in \mathbb{N}).$$

Take $n \in \mathbb{N}$. Suppose first that $|t| \leq 1/n$. Then $|\Delta_n(s) - \Delta_n(s-t)| \leq n|t|$ for $|s| \leq 2/n$ and $\Delta_n(s) = \Delta_n(s-t) = 0$ for $|s| \geq 2/n$, and so

$$\Omega_{\Delta_n}(t) \leq 2\int_0^{2/n} n|t|\,ds = 4|t|.$$

Second, suppose that $|t| \geq 1/n$. Then

$$\Omega_{\Delta_n}(t) \leq 2\int_{-1}^{1} \Delta_n(s)\,ds = \frac{2}{n}.$$

Hence

$$I(\Delta_n) \leq 8\int_0^{1/n} t^{1-\alpha}\,dt + \frac{2}{n}\int_{1/n}^{1} t^{-\alpha}\,dt = O\left(\frac{1}{n^{2-\alpha}}\right) \to 0 \quad \text{as } n \to \infty$$

because $\alpha < 2$, and so $\|1 - \Delta_n\| = \|\Delta_n\| = 1 + o(1)$ as $n \to \infty$.

In fact, we can suppose that $I(\Delta_n) \leq 1$ and $\|\Delta_n\| \leq 2$ for all $n \in \mathbb{N}$.

Finally, we show that $(1 - \Delta_n : n \in \mathbb{N})$ is an approximate identity for the maximal ideal M. Certainly $1 - \Delta_n \in M$ $(n \in \mathbb{N})$. Now write

$$I_\delta(f) = \int_{-\delta}^{\delta} \frac{\Omega_f(t)}{|t|^\alpha}\,dt, \qquad J_\delta(f) = 2\int_{\delta}^{1} \frac{\Omega_f(t)}{|t|^\alpha}\,dt \quad (f \in A)$$

for each $\delta > 0$, so that $I(f) = I_\delta(f) + J_\delta(f)$ $(f \in A)$. Fix $f \in M$ and $\varepsilon > 0$, and then choose $\delta > 0$ such that $I_\delta(f) < \varepsilon$ and $|f|_{[-\delta,\delta]} < \varepsilon$. Then

$$I_\delta(f\Delta_n) \leq |\Delta_n|_{\mathbb{T}} I_\delta(f) + |f|_{[-\delta,\delta]} I(\Delta_n) < 2\varepsilon.$$

Next choose $n_0 \in \mathbb{N}$ such that $n_0\delta > 1$ and

$$\int_{-1/n_0}^{1/n_0} |f(t)|\,dt < \varepsilon\delta^\alpha.$$

Take $n \geq n_0$. Then $|f\Delta_n|_{[-1,1]} \leq |f|_{[-\delta,\delta]} < \varepsilon$ and

$$\Omega_{f\Delta_n}(t) \leq 2 \int_{-1/n}^{1/n} |f(s)|\,ds < 2\varepsilon\delta^\alpha \qquad (t \in [-1,1]),$$

and so

$$J_\delta(f\Delta_n) < 4\varepsilon\delta^\alpha \int_\delta^1 \frac{dt}{t^\alpha} < 4\varepsilon.$$

Hence $\|f\Delta_n\| \leq \varepsilon + 2\varepsilon + 4\varepsilon = 7\varepsilon$ $(n \geq n_0)$. Thus $f \cdot (1 - \Delta_n) \to f$ in $(A, \|\cdot\|)$ as $n \to \infty$, so that $(1 - \Delta_n : n \in \mathbb{N})$ is indeed an approximate identity for M.

We conclude that $((1 - \Delta_n)/\|1 - \Delta_n\| : n \in \mathbb{N})$ is a CAI in M, and so A is contractive.

The above example is somewhat related to the 'remarkable homogeneous Banach algebra' of Pisier [32], as discussed in [31]. In that example, in the formulation of [31, Theorem 2.1], $\Omega_f(t)$ is replaced by $\Psi_f(t)$, where Ψ_f is the increasing rearrangement of the function $t \mapsto \|f - S_t f\|_2$, and the function $1/t^\alpha$ on $(0, 1]$ is replaced by $1/t(\log(1/t))^{1/2}$. However, this example is not contractive.

Acknowledgements. The authors thank David Blecher, Joel Feinstein, Alexander Izzo, and Charles Read for enjoyable and informative conversations. The first author is grateful to the second author and Koç University for generous hospitality on two occasions.

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Received October 29, 2014
Revised version April 22, 2015 (8117)