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# A weak$^*$-topological dichotomy with applications in operator theory

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Abstract

Denote by $[0, \omega_1)$ the locally compact Hausdorff space consisting of all countable ordinals, equipped with the order topology, and let $C_0[0, \omega_1)$ be the Banach space of scalar-valued, continuous functions which are defined on $[0, \omega_1)$ and vanish eventually. We show that a weak$^*$-compact subset of the dual space of $C_0[0, \omega_1)$ is either uniformly Eberlein compact, or it contains a homeomorphic copy of a particular form of the ordinal interval $[0, \omega_1]$.

This dichotomy yields a unifying approach to most of the existing studies of the Banach space $C_0[0, \omega_1)$ and the Banach algebra $\mathscr{B}(C_0[0, \omega_1))$ of bounded, linear operators acting on it, and it leads to several new results, as well as to stronger versions of known ones. Specifically, we deduce that a Banach space which is a quotient of $C_0[0, \omega_1)$ can either be embedded in a Hilbert-generated Banach space, or it is isomorphic to the direct sum of $C_0[0, \omega_1)$ and a subspace of a Hilbert-generated Banach space; and we obtain several equivalent conditions describing the Loy–Willis ideal $\mathscr{M}$, which is the unique maximal ideal of $\mathscr{B}(C_0[0, \omega_1))$, including the following: an operator belongs to $\mathscr{M}$ if and only if it factors through the Banach space $(\bigoplus_{\alpha<\omega_1} C[0, \alpha])_{c_0}$. Among the consequences of these characterizations of $\mathscr{M}$ is that $\mathscr{M}$ has a bounded left approximate identity; this resolves a problem left open by Loy and Willis.

## 1. *Introduction and statement of main results*

The main motivation behind this paper is a desire to deepen our understanding of the Banach algebra $\mathscr{B}(C_0[0, \omega_1))$ of bounded, linear operators acting on the Banach space $C_0[0, \omega_1)$ of scalar-valued, continuous functions which are defined on the locally compact ordinal interval $[0, \omega_1)$ and vanish eventually. Our strategy is to begin at a topological level, where we establish a dichotomy for weak$^*$-compact subsets of the dual space of $C_0[0, \omega_1)$, and then use this dichotomy to obtain information about $C_0[0, \omega_1)$ and the operators acting on it, notably several equivalent conditions characterizing the unique maximal ideal of $\mathscr{B}(C_0[0, \omega_1))$.

The Banach space $C_0[0, \omega_1)$ is of course isometrically isomorphic to the hyperplane $\{f \in C[0, \omega_1] : f(\omega_1) = 0\}$ of the Banach space $C[0, \omega_1]$ of scalar-valued, continuous functions on the compact ordinal interval $[0, \omega_1]$, and $C_0[0, \omega_1)$ and $C[0, \omega_1]$ are isomorphic. Since our focus is on properties that are preserved under Banach-space isomorphism, we shall freely alternate between these two spaces in the following summary of the history of their study.

Semadeni [34] was the first to realize that $C[0, \omega_1]$ is an interesting Banach space, showing that it is not isomorphic to its Cartesian square. This resolved an open problem going back to Banach; another such example is given by James's quasi-reflexive Banach space, as Bessaga

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and Pełczyński [7] showed around the same time. The Banach-space structure of $C_0[0,\omega_1)$ was subsequently explored in much greater depth by Alspach and Benyamini [2], whose main conclusion is that $C_0[0,\omega_1)$ is primary, in the sense that whenever $C_0[0,\omega_1)$ is decomposed into a direct sum of two closed subspaces, one of these subspaces is necessarily isomorphic to $C_0[0,\omega_1)$.

Loy and Willis [26] initiated the study of the Banach algebra $\mathscr{B}(C[0,\omega_1])$ from an automatic-continuity point of view, proving that each derivation from $\mathscr{B}(C[0,\omega_1])$ into a Banach $\mathscr{B}(C[0,\omega_1])$-bimodule is automatically continuous. Their result was subsequently generalized by Ogden [29], who established the automatic continuity of each algebra homomorphism from $\mathscr{B}(C[0,\omega_1])$ into a Banach algebra.

Loy and Willis's starting point is the clever identification of a maximal ideal $\mathscr{M}$ of codimension 1 in $\mathscr{B}(C[0,\omega_1])$ (see equation (2.4) for the precise definition of $\mathscr{M}$), while their main technical step [26, Theorem 3.5] is the construction of a bounded right approximate identity in $\mathscr{M}$. The first- and third-named authors [21] recently showed that $\mathscr{M}$ is the unique maximal ideal of $\mathscr{B}(C[0,\omega_1])$, and named it the *Loy–Willis ideal*. We shall here give a new proof of this result, together with several new characterizations of the Loy–Willis ideal. As a consequence, we obtain that $\mathscr{M}$ has a bounded left approximate identity, thus complementing Loy and Willis's key result mentioned above.

The tools that we shall use come primarily from point-set topology and Banach space theory, and several of our results may be of independent interest to researchers in those areas, as well as to operator theorists.

Before entering into a more detailed description of this paper, let us introduce four notions that will play important roles throughout.

(i) A topological space $K$ is *Eberlein compact* if it is homeomorphic to a weakly compact subset of a Banach space; and $K$ is *uniformly Eberlein compact* if it is homeomorphic to a weakly compact subset of a Hilbert space.

(ii) A Banach space $X$ is *weakly compactly generated* if it contains a weakly compact subset whose linear span is dense in $X$; and $X$ is *Hilbert-generated* if there exists a bounded, linear operator from a Hilbert space onto a dense subspace of $X$.

These notions are relevant for our purposes primarily because the ordinal interval $[0,\omega_1]$ is one of the 'simplest' compact spaces which is not Eberlein compact. They are related as follows: uniform Eberlein compactness clearly implies Eberlein compactness, and likewise Hilbert generation implies weakly compact generation. A much deeper result, due to Amir and Lindenstrauss [3], states that a compact space $K$ is Eberlein compact if and only if the Banach space $C(K)$ is weakly compactly generated; and the other two notions enjoy a similar relationship, as detailed in Theorem 2.6. The class of Hilbert-generated Banach spaces was first studied systematically in [11].

We shall now outline how this paper is organized and state its main conclusions precisely. Section 2 contains details of our notation, key elements of previous work and some preliminary results. In Section 3, we proceed to study the weak*-compact subsets of the dual space of $C_0[0,\omega_1)$, proving in particular the following topological dichotomy.

THEOREM 1.1 (Topological dichotomy). *Exactly one of the following two alternatives holds for every weak*-compact subset $K$ of $C_0[0,\omega_1)^*$ :*

(I) *$K$ is uniformly Eberlein compact*;

(II) *$K$ contains a homeomorphic copy of $[0,\omega_1]$ of the form*

$$\{\rho+\lambda\delta_\alpha : \alpha\in D\}\cup\{\rho\}, \tag{1.1}$$

*where $\rho\in C_0[0,\omega_1)^*$, $\lambda$ is a non-zero scalar, $\delta_\alpha$ denotes the Dirac measure at $\alpha$ and $D$ is a closed and unbounded subset of $[0,\omega_1)$.*

REMARK 1.2. We are grateful to Ondřej Kalenda for showing us how the following weak version of the dichotomy, above, can be deduced from known results:

*a weak$^*$-compact subset $K$ of $C_0[0,\omega_1)^*$ is either Eberlein compact, or it contains a homeomorphic copy of $[0,\omega_1]$.*

Indeed, combining [**18**, Proposition 3.3 and Theorems 3.5 and 3.7], we see that $K$ is the continuous image of a Valdivia compact space. Hence, by [**17**], either $K$ contains a homeomorphic copy of $[0,\omega_1]$, or $K$ is Corson compact, and in the latter case $K$ is Eberlein compact by [**30**] because it is Radon–Nikodým compact. (We refer the reader to, for example, [**12**] for details of the undefined terminology used here.)

It should be stressed that this dichotomy is weaker than Theorem 1.1, not only because Eberlein compactness is weaker than its uniform counterpart, but also because the copy of $[0,\omega_1]$ found in Theorem 1.1(II) has the specific form (1.1). The latter fact will play a key role in our operator-theoretic applications of Theorem 1.1 because it enables us to understand exactly how distinct copies of $[0,\omega_1]$ obtained in this way interact. We do not know whether such results can be deduced solely from Kalenda's dichotomy, since it offers no explicit description of the copies of $[0,\omega_1]$ that can be found in the non-Eberlein case. We note, however, that Theorem 1.1 implies that inside *any* homeomorphic copy of $[0,\omega_1]$ in $C_0[0,\omega_1)^*$, there is one which has the form (1.1).

REMARK 1.3. We are grateful to the referee for pointing out that, combining Theorem 1.1 with known results, we obtain the equivalence of the following six conditions concerning a weak$^*$-compact subset $K$ of $C_0[0,\omega_1)^*$ :

(i) $K$ is not angelic (see, for example, [**12**, Definition 4.48] for the definition of this notion);
(ii) $K$ is not Corson compact (see, for example, [**12**, Definition 12.44] for the definition);
(iii) $K$ is not Eberlein compact;
(iv) $K$ is not uniformly Eberlein compact;
(v) there are $\rho \in K$, a non-zero scalar $\lambda$ and a closed and unbounded subset $D$ of $[0,\omega_1)$ such that $\rho + \lambda\delta_\alpha \in K$ for each $\alpha \in D$;
(vi) $K$ contains a homeomorphic copy of $[0,\omega_1]$.

Indeed, a proof of (i)⇒(ii) can be found, for example, in [**12**, Exercise 12.55]; (ii)⇒(iii) is a standard consequence of the work of Amir and Lindenstrauss [**3**]; (iii)⇒(iv) is trivial; (iv)⇒(v) follows immediately from Theorem 1.1; (v)⇒(vi) is easy (see Lemma 3.1 for details), and finally (vi)⇒(i) is a consequence of the fact that $\omega_1$ is not the limit of any sequence of countable ordinals.

In Section 4, we turn our attention to the structure of operators acting on $C_0[0,\omega_1)$. In the case where $T$ is a bounded, linear surjection from $C_0[0,\omega_1)$ onto an arbitrary Banach space $X$, the adjoint $T^*$ of $T$ induces a weak$^*$-homeomorphism of the unit ball of $X^*$ onto a bounded subset of $C_0[0,\omega_1)^*$ and hence the above topological dichotomy leads to the following operator-theoretic dichotomy.

THEOREM 1.4 (Operator-theoretic dichotomy). *Let $X$ be a Banach space, and suppose that there exists a bounded, linear surjection $T: C_0[0,\omega_1) \to X$. Then exactly one of the following two alternatives holds*:

(I) *$X$ embeds in a Hilbert-generated Banach space;*
(II) *the identity operator on $C_0[0,\omega_1)$ factors through $T$, and $X$ is isomorphic to the direct sum of $C_0[0,\omega_1)$ and a subspace of a Hilbert-generated Banach space.*

As another consequence of Theorem 1.1, we obtain the following result.

THEOREM 1.5. *For each bounded, linear operator $T$ on $C_0[0,\omega_1)$, there exist a unique scalar $\varphi(T)$ and a closed and unbounded subset $D$ of $[0,\omega_1)$ such that*

$$(Tf)(\alpha) = \varphi(T)f(\alpha) \quad (f \in C_0[0,\omega_1),\ \alpha \in D). \tag{1.2}$$

*Moreover, the mapping $T \mapsto \varphi(T)$ is linear and multiplicative, and it is thus a character on the Banach algebra $\mathscr{B}(C_0[0,\omega_1))$.*

We shall call this mapping $\varphi$ the *Alspach–Benyamini character* because, after having discovered the above theorem, we found out that it can also be deduced from [**2**, page 76, line 6 from below].

Theorem 1.5 is the key step towards our main operator-theoretic result, which gives a list of equivalent conditions, each describing the Loy–Willis ideal $\mathscr{M}$ of $\mathscr{B}(C_0[0,\omega_1))$.

THEOREM 1.6. *The following eight conditions are equivalent for each bounded, linear operator $T$ on $C_0[0,\omega_1)$ :*

(i) *$T$ belongs to the Loy–Willis ideal $\mathscr{M}$;*
(ii) *there is a closed and unbounded subset $D$ of $[0,\omega_1)$ such that $T^*(\delta_\alpha)(\{\alpha\}) = 0$ for each $\alpha \in D$, where $\delta_\alpha$ denotes the Dirac measure at $\alpha$;*
(iii) *$\varphi(T) = 0$, where $\varphi$ denotes the Alspach–Benyamini character introduced in Theorem 1.5;*
(iv) *$T$ factors through the Banach space $(\bigoplus_{\alpha<\omega_1} C[0,\alpha])_{c_0}$, where $C[0,\alpha]$ denotes the Banach space of scalar-valued, continuous functions on the set of ordinals not exceeding $\alpha$, equipped with the order topology;*
(v) *the range of $T$ is contained in a Hilbert-generated subspace of $C_0[0,\omega_1)$;*
(vi) *the range of $T$ is contained in a weakly compactly generated subspace of $C_0[0,\omega_1)$;*
(vii) *$T$ does not fix an isomorphic copy of $C_0[0,\omega_1)$;*
(viii) *the identity operator on $C_0[0,\omega_1)$ does not factor through $T$.*

REMARK 1.7. (1) The equivalence of conditions (i) and (viii) of Theorem 1.6 is the main result of a recent paper by the first- and third-named authors [**21**]. Our proof of Theorem 1.6 will not depend on that result, and it will thus provide an alternative proof of this equivalence.

(2) The equivalence of conditions (i) and (iv) of Theorem 1.6 disproves the conjecture stated immediately after [**21**, Equation (5.4)].

Theorem 1.6 has a number of interesting consequences, as we shall now explain. The proofs of these results will be given in the final part of Section 4. We begin with what is arguably the most important consequence of Theorem 1.6. It relies on the following notion.

DEFINITION 1.8. A net $(e_\gamma)_{\gamma\in\Gamma}$ in a Banach algebra $\mathscr{A}$ is a *bounded left approximate identity* if $\sup_{\gamma\in\Gamma}\|e_\gamma\| < \infty$ and the net $(e_\gamma a)_{\gamma\in\Gamma}$ converges to $a$ for each $a \in \mathscr{A}$. A *bounded right approximate identity* is defined analogously, and a *bounded two-sided approximate identity* is a net which is simultaneously a bounded left and right approximate identity.

A well-known theorem of Dixon [**10**, Proposition 4.1] states that a Banach algebra which has both a bounded left and a bounded right approximate identity has a bounded two-sided approximate identity. As already mentioned, Loy and Willis constructed a bounded right approximate identity in $\mathscr{M}$. Although they did not state it formally, their result immediately raises the question whether $\mathscr{M}$ has a bounded left (and hence two-sided) approximate identity. We can now provide a positive answer to this question.

Corollary 1.9. *The Loy–Willis ideal $\mathscr{M}$ contains a net $(Q_D)_{D\in\Gamma}$ of projections, each having norm at most $2$, such that, for each operator $T \in \mathscr{M}$, there is $D_0 \in \Gamma$ for which $Q_D T = T$ whenever $D \geqslant D_0$. Hence, $(Q_D)_{D\in\Gamma}$ is a bounded left approximate identity in $\mathscr{M}$.*

When discovering this result, we were surprised to find that the net $(Q_D T)_{D\in\Gamma}$ does not just converge to $T$, but it actually equals $T$ eventually. We have, however, subsequently realized that the even stronger, two-sided counterpart of this phenomenon occurs in the unique maximal ideal of the $C^*$-algebra $\mathscr{B}(H)$, where $H$ denotes the (non-separable) Hilbert space which has an orthonormal basis of cardinality $\aleph_1$; see Example 4.6 for details.

Further consequences of Theorem 1.6 include generalizations of two classical Banach-space-theoretic results. The first is Semadeni's seminal theorem [**34**], which states that $C_0[0,\omega_1)$ is not isomorphic to its square.

Corollary 1.10. *Let $m, n \in \mathbb{N}$, and suppose that $C_0[0,\omega_1)^m$ is isomorphic to either a subspace or a quotient of $C_0[0,\omega_1)^n$. Then $m \leqslant n$.*

The other is Alspach and Benyamini's main theorem [**2**, Theorem 1] as it applies to $C_0[0,\omega_1)$: this Banach space is primary.

Corollary 1.11. *For each bounded, linear projection $P$ on $C_0[0,\omega_1)$, either the kernel of $P$ is isomorphic to $C_0[0,\omega_1)$ and the range of $P$ is isomorphic to a subspace of the Banach space $(\bigoplus_{\alpha<\omega_1} C[0,\alpha])_{c_0}$, or vice versa.*

To state another Banach-space-theoretic consequence of Theorem 1.6, we require the following notion. A Banach space $X$ is *complementably homogeneous* if, whenever $W$ is a closed subspace of $X$ such that $W$ is isomorphic to $X$, there exists a closed, complemented subspace $Y$ of $X$ such that $Y$ is isomorphic to $X$ and $Y$ is contained in $W$.

Corollary 1.12. *The Banach space $C_0[0,\omega_1)$ is complementably homogeneous.*

This conclusion may also be deduced from [**2**, Lemma 1.2 and Proposition 2].

Combining Theorem 1.6 with the techniques developed by Willis [**36**], we obtain a very short proof of Ogden's main theorem [**29**, Theorem 6.18] as it applies to the ordinal $\omega_1$.

Corollary 1.13 (Ogden). *Each algebra homomorphism from $\mathscr{B}(C_0[0,\omega_1))$ into a Banach algebra is automatically continuous.*

Our final result relies on a suitable modification of work of the third-named author [**25**] and involves the following two purely algebraic notions. The *commutator* of a pair of elements $a$

and $b$ of an algebra $\mathscr{A}$ is given by $[a,b] = ab - ba$. A *trace* on $\mathscr{A}$ is a scalar-valued, linear mapping $\tau$ defined on $\mathscr{A}$ such that $\tau(ab) = \tau(ba)$ for each pair $a, b \in \mathscr{A}$.

Corollary 1.14. *Every operator belonging to the Loy–Willis ideal is the sum of at most three commutators. Hence, a scalar-valued, linear mapping $\tau$ defined on $\mathscr{B}(C_0[0,\omega_1))$ is a trace (if and) only if $\tau$ is a scalar multiple of the Alspach–Benyamini character. In particular, each trace on $\mathscr{B}(C_0[0,\omega_1))$ is automatically continuous.*

Remark 1.15. Building on Corollary 1.14, one can prove that the $K_0$-group of the Banach algebra $\mathscr{B}(C_0[0,\omega_1))$ is isomorphic to $\mathbb{Z}$ by arguments similar to those given in [**23**, Section 4]. A full proof of this result will be given in [**20**].

## 2. *Preliminaries*

### 2.1. *General conventions*

Our notation and terminology are fairly standard. We shall now outline our most important conventions. Let $X$ be a Banach space, always supposed to be over the scalar field $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We write $B_X$ for the closed unit ball of $X$, and denote by $\langle\,\cdot\,,\,\cdot\,\rangle$ the duality bracket between $X$ and its dual space $X^*$; we identify $X$ with its canonical image in the bidual space $X^{**}$.

By an *operator*, we understand a bounded, linear mapping between Banach spaces. We write $\mathscr{B}(X)$ for the Banach algebra of all operators on $X$, and $\mathscr{B}(X,Y)$ for the Banach space of all operators from $X$ to some other Banach space $Y$. For an operator $T \in \mathscr{B}(X,Y)$, we denote by $T^* \in \mathscr{B}(Y^*, X^*)$ its adjoint, while $I_X$ is the identity operator on $X$. We say that an operator $T \in \mathscr{B}(X,Y)$ is *bounded below* by a constant $c > 0$ if $\|Tx\| \geqslant c\|x\|$ for each $x \in X$.

Given Banach spaces $W$, $X$, $Y$ and $Z$ and operators $S \in \mathscr{B}(W,X)$ and $T \in \mathscr{B}(Y,Z)$, we say that $S$ *factors through* $T$ if $S = UTR$ for some operators $R \in \mathscr{B}(W,Y)$ and $U \in \mathscr{B}(Z,X)$. The following elementary characterization of the operators that the identity operator factors through is well known and easy to verify directly, or it can be deduced from [**24**, Lemma 3.6].

Lemma 2.1. *Let $X, Y$ and $Z$ be Banach spaces, and let $T : Y \to Z$ be an operator. Then the identity operator on $X$ factors through $T$ if and only if $Y$ contains a closed subspace $W$ such that*:

(i) *$W$ is isomorphic to $X$*;
(ii) *the restriction of $T$ to $W$ is bounded below*;
(iii) *the image of $W$ under $T$ is complemented in $Z$.*

For a Hausdorff space $K$, we write $C(K)$ for the vector space of scalar-valued, continuous functions on $K$. In the case where $K$ is locally compact, $C_0(K)$ denotes the subspace consisting of those functions $f \in C(K)$ which vanish eventually, in the sense that the set $\{x \in K : |f(x)| \geqslant \varepsilon\}$ is compact for each $\varepsilon > 0$. Then $C_0(K)$ is a Banach space with respect to the supremum norm. Alternatively, one may define $C_0(K)$ as $C_0(K) = \{f \in C(\tilde{K}) : f(\infty) = 0\}$, where $\tilde{K} = K \cup \{\infty\}$ is the one-point compactification of $K$. By the Riesz Representation Theorem, we may identify the dual space of $C_0(K)$ with the Banach space of scalar-valued, regular Borel measures on $K$, and we shall therefore freely use measure-theoretic terminology and notation when dealing with functionals on $C_0(K)$. Given $x \in K$, we denote by $\delta_x$ the Dirac measure at $x$.

Lower-case Greek letters such as $\alpha$, $\beta$, $\gamma$, $\xi$, $\eta$ and $\zeta$ denote ordinals. The first infinite ordinal is $\omega$, while the first uncountable ordinal is $\omega_1$. By convention, we consider 0 a limit ordinal. Given a pair of ordinals $\alpha \leqslant \beta$, we write $[\alpha, \beta]$ and $[\alpha, \beta)$ for the sets of ordinals $\gamma$ such that $\alpha \leqslant \gamma \leqslant \beta$ and $\alpha \leqslant \gamma < \beta$, respectively.

For a non-zero ordinal $\alpha$, we equip the ordinal interval $[0, \alpha)$ with the order topology, which turns it into a locally compact Hausdorff space that is compact if and only if $\alpha$ is a successor ordinal. (According to the standard construction of the ordinals, the interval $[0, \alpha)$ is equal to the ordinal $\alpha$; we use the symbol $[0, \alpha)$ to emphasize its structure as a topological space.) Since $[0, \alpha)$ is scattered, a classical result of Rudin [**33**] states that each regular Borel measure on $[0, \alpha)$ is purely atomic, so that the Riesz Representation Theorem implies that the dual space of $C_0[0, \alpha)$ is isometrically isomorphic to the Banach space $\ell_1([0, \alpha))$ of scalar-valued, absolutely summable functions defined on $[0, \alpha)$ via the mapping

$$g \longmapsto \sum_{\beta<\alpha} g(\beta)\delta_\beta, \quad \ell_1([0, \alpha)) \to C_0[0, \alpha)^*. \tag{2.1}$$

This implies in particular that we can associate with each operator $T$ on $C_0[0, \alpha)$ a scalar-valued $[0, \alpha) \times [0, \alpha)$-matrix $(T_{\beta,\gamma})_{\beta,\gamma<\alpha}$ which has absolutely summable rows. The $\beta$th row of this matrix is simply the Rudin representation of the functional $T^*\delta_\beta$; that is, $(T_{\beta,\gamma})_{\gamma<\alpha}$ is the uniquely determined element of $\ell_1([0, \alpha))$ such that

$$T^*\delta_\beta = \sum_{\gamma<\alpha} T_{\beta,\gamma}\delta_\gamma. \tag{2.2}$$

This matrix representation plays an essential role in the original definition of the Loy–Willis ideal, which is our next topic.

## 2.2. *The Loy–Willis ideal*

Suppose that $\alpha = \omega_1 + 1$ in the notation of the previous paragraph, and note that $C_0[0, \omega_1 + 1) = C[0, \omega_1]$. Using the fact that every scalar-valued, continuous function on $[0, \omega_1]$ is eventually constant, Loy and Willis [**26**, Proposition 3.1] showed that, for each operator $T$ on $C[0, \omega_1]$, the function corresponding to the final column of its matrix,

$$k^T_{\omega_1}\colon\ \beta \longmapsto T_{\beta,\omega_1}, \quad [0, \omega_1] \to \mathbb{K}, \tag{2.3}$$

is continuous on $[0, \omega_1)$, and the limit $\lim_{\beta\to\omega_1} T_{\beta,\omega_1}$ always exists. (This is clearly the best possible conclusion because the function corresponding to final column of the matrix associated with the identity operator is equal to the indicator function $\mathbf{1}_{\{\omega_1\}}$, which is discontinuous at $\omega_1$.) Hence, as Loy and Willis observed, the set

$$\mathscr{M} = \{T \in \mathscr{B}(C[0, \omega_1]) : k^T_{\omega_1} \text{ is continuous}\} \tag{2.4}$$

is a subspace of $\mathscr{B}(C[0, \omega_1])$ of codimension 1. Since the composition of operators on $C[0, \omega_1]$ corresponds to matrix multiplication, in the sense that

$$(ST)_{\alpha,\gamma} = \sum_{\beta\leqslant\omega_1} S_{\alpha,\beta}T_{\beta,\gamma} \quad (S, T \in \mathscr{B}(C[0, \omega_1]),\ \alpha, \gamma \in [0, \omega_1]),$$

$\mathscr{M}$ is a left ideal, named the *Loy–Willis ideal* in [**21**]. Having codimension 1, $\mathscr{M}$ is automatically a maximal and two-sided ideal of $\mathscr{B}(C[0, \omega_1])$.

Consequently, the Banach algebra $\mathscr{B}(C_0[0, \omega_1))$ also contains a maximal ideal of codimension 1 because it is isomorphic to $\mathscr{B}(C[0, \omega_1])$. Loy and Willis's definition (2.4) does not carry over to $\mathscr{B}(C_0[0, \omega_1))$ because the matrix of an operator on $C_0[0, \omega_1)$ has no final column. Instead we may define the Loy–Willis ideal of $\mathscr{B}(C_0[0, \omega_1))$ as follows. (A more explicit definition will emerge after Proposition 2.5.) Choose an isomorphism $U$ of $C[0,\omega_1]$ onto $C_0[0,\omega_1)$, and declare that an operator $T$ on $C_0[0,\omega_1)$ belongs to the Loy–Willis ideal of $\mathscr{B}(C_0[0,\omega_1))$ if and only if the operator $U^{-1}TU$ on $C[0,\omega_1]$ belongs the original Loy–Willis ideal (2.4). Since the latter is a two-sided ideal, this definition is independent of the choice of the isomorphism $U$. We shall denote by $\mathscr{M}$ the Loy–Willis ideal of $\mathscr{B}(C_0[0,\omega_1))$ defined in this way; this should not cause any confusion with the original Loy–Willis ideal given by (2.4).

## 2.3. *The ideal of Semadeni operators*

The purpose of this section is to introduce a new closed operator ideal (in the sense of Pietsch), which generalizes Loy and Willis's definition (2.4) to operators between arbitrary Banach spaces.

For a Banach space $X$, Semadeni [**34**, p. 82] considered the subspace

$$X_s = \{F \in X^{**} : \langle f_n, F\rangle \to 0 \text{ as } n \to \infty \text{ for each weak}^*\text{-null sequence } (f_n)_{n\in\mathbb{N}} \text{ in } X^*\}$$

consisting of the weak$^*$-sequentially continuous functionals on $X^*$. This is clearly a closed subspace of $X^{**}$, and it contains the canonical copy of $X$.

LEMMA 2.2. *Let $X$ and $Y$ be Banach spaces. Then $T^{**}[X_s] \subseteq Y_s$ for each operator $T : X \to Y$.*

*Proof.* Given $F \in X_s$ and a weak$^*$-null sequence $(f_n)_{n\in\mathbb{N}}$ in $Y^*$, the weak$^*$-continuity of $T^*$ implies that the sequence $(T^*f_n)_{n\in\mathbb{N}}$ is weak$^*$-null in $X^*$. Therefore, the sequence $(\langle T^*f_n, F\rangle)_{n\in\mathbb{N}} = (\langle f_n, T^{**}F\rangle)_{n\in\mathbb{N}}$ tends to 0, which shows that $T^{**}F \in Y_s$. □

DEFINITION 2.3. Let $X$ and $Y$ be Banach spaces. An operator $T : X \to Y$ is a *Semadeni operator* if $T^{**}$ maps $X_s$ into the canonical copy of $Y$ in $Y^{**}$. We write $\mathscr{S}\!em(X,Y)$ for the set of Semadeni operators from $X$ to $Y$.

PROPOSITION 2.4. *The class $\mathscr{S}\!em$ is a closed operator ideal which contains the ideal of weakly compact operators.*

*Proof.* Let $W$, $X$, $Y$ and $Z$ be Banach spaces. We see immediately that $\mathscr{S}\!em(X,Y)$ is a closed subspace of $\mathscr{B}(X,Y)$ because $Y$ is a closed subspace of $Y^{**}$. Moreover, given $R \in \mathscr{B}(W,X)$, $S \in \mathscr{S}\!em(X,Y)$ and $T \in \mathscr{B}(Y,Z)$, we may apply Lemma 2.2 to obtain

$$(TSR)^{**}[W_s] \subseteq T^{**}[S^{**}[X_s]] \subseteq T^{**}[Y] = T[Y] \subseteq Z,$$

which shows that $TSR \in \mathscr{S}\!em(W,Z)$. Finally, Gantmacher's characterization of the weakly compact operators as those operators $T : X \to Y$ for which $T^{**}[X^{**}] \subseteq Y$ (see, for example, [**28**, Theorem 3.5.8]) implies that every weakly compact operator is a Semadeni operator. □

PROPOSITION 2.5. *The Loy–Willis ideal $\mathscr{M}$ defined by (2.4) is equal to $\mathscr{S}\!em(C[0,\omega_1])$.*

*Proof.* By (2.1), we may identify $C[0,\omega_1]^{**}$ with the Banach space $\ell_\infty([0,\omega_1])$ of scalar-valued, bounded functions defined on $[0,\omega_1]$. Under this identification, we have

$$C[0,\omega_1]_s = C[0,\omega_1] \oplus \mathbb{K}\mathbf{1}_{\{\omega_1\}}. \tag{2.5}$$

This may be deduced from [**34**, Theorem 1], but for completeness we shall give a short argument. Each function $F \in C[0,\omega_1]_s$ is sequentially continuous, and hence continuous at each countable ordinal. Consequently, the net $(F(\alpha))_{\alpha<\omega_1}$ is Cauchy and thus convergent, and the function $F + (\lim_{\alpha\to\omega_1} F(\alpha) - F(\omega_1))\mathbf{1}_{\{\omega_1\}}$ is continuous on $[0,\omega_1]$, which shows that $F$ belongs to the right-hand side of (2.5).

Conversely, it suffices to verify that $\mathbf{1}_{\{\omega_1\}} \in C[0,\omega_1]_s$. To this end, suppose that $(\mu_n)_{n\in\mathbb{N}}$ is a weak$^*$-null sequence in $C[0,\omega_1]^*$. By (2.1), each $\mu_n$ has countable support, so that we can choose a countable ordinal $\alpha$ such that $\mu_n(\{\beta\}) = 0$ for each $n \in \mathbb{N}$ and each $\beta \in [\alpha+1,\omega_1)$. Then we have $\langle \mu_n, \mathbf{1}_{\{\omega_1\}}\rangle = \langle \mathbf{1}_{[\alpha+1,\,\omega_1]}, \mu_n\rangle \to 0$ as $n \to \infty$ by the continuity of $\mathbf{1}_{[\alpha+1,\,\omega_1]}$, and the conclusion follows.

For each $T \in \mathscr{B}(C[0,\omega_1])$, $T^{**}$ leaves the canonical copy of $C[0,\omega_1]$ invariant, so that (2.5) implies that $T \in \mathscr{S}\!em(C[0,\omega_1])$ if and only if $T^{**}\mathbf{1}_{\{\omega_1\}} \in C[0,\omega_1]$, which in turn is equivalent to $T \in \mathscr{M}$ because $k^T_{\omega_1} = T^{**}\mathbf{1}_{\{\omega_1\}}$ by (2.2), where $k^T_{\omega_1}$ is the function given by (2.3). $\square$

Since the Banach spaces $C_0[0,\omega_1)$ and $C[0,\omega_1]$ are isomorphic, Propositions 2.4 and 2.5 imply that $\mathscr{S}\!em(C_0[0,\omega_1))$ is equal to the Loy–Willis ideal $\mathscr{M}$ of $\mathscr{B}(C_0[0,\omega_1))$, thus providing a more explicit definition of this ideal than the one given above.

### 2.4. *Uniform Eberlein compactness*

Recall from the second page of the Introduction that a topological space is uniformly Eberlein compact if it is homeomorphic to a weakly compact subset of a Hilbert space. There are many equivalent ways to describe this notion. We have collected those that are most relevant for our purposes in the following theorem, which combines work of Benyamini, Rudin and Wage [**5**] and Benyamini and Starbird [**6**].

THEOREM 2.6 (Benyamini–Rudin–Wage and Benyamini–Starbird). *The following four conditions are equivalent for a compact Hausdorff space $K$:*

(i) *$K$ is uniformly Eberlein compact*;
(ii) *the Banach space $C(K)$ is Hilbert-generated*;
(iii) *the unit ball of $C(K)^*$ is uniformly Eberlein compact in the weak$^*$-topology*;
(iv) *there exists a family $\mathscr{F} = \bigcup_{n\in\mathbb{N}} \mathscr{F}_n$ of open $F_\sigma$-subsets of $K$ such that*:
  (1) *whenever $x, y \in K$ are distinct, some $G \in \mathscr{F}$ separates $x$ and $y$, in the sense that either ($x \in G$ and $y \notin G$) or ($y \in G$ and $x \notin G$)*;
  (2) $\sup_{x\in K} |\{G \in \mathscr{F}_n : x \in G\}|$ *is finite for each $n \in \mathbb{N}$.*

Another important theorem that we shall require is the following internal characterization of the Banach spaces which embed in a Hilbert-generated Banach space. It is closely related to the equivalence of conditions (ii) and (iii) of Theorem 2.6. We refer the reader, for example, to [**15**, Theorem 6.30] for a proof.

THEOREM 2.7. *A Banach space $X$ embeds in a Hilbert-generated Banach space if and only if the unit ball of $X^*$ is uniformly Eberlein compact in the weak$^*$-topology.*

In contrast, we have the following well-known result for $C_0[0,\omega_1)$.

THEOREM 2.8. *The Banach space $C_0[0,\omega_1)$ does not embed in any weakly compactly generated Banach space.*

*Proof* (*Sketch*). Every weakly compactly generated Banach space is weakly Lindelöf (see, for example, [**12**, Theorem 12.35]), and this property is inherited by closed subspaces. However, $C_0[0,\omega_1)$ is not weakly Lindelöf (see, for example, [**12**, Theorem 12.40]). $\square$

Combining this result with Amir and Lindenstrauss's theorem that a compact space $K$ is Eberlein compact if and only if $C(K)$ is weakly compactly generated, we obtain the following conclusion, which can also be proved directly by more elementary means (see, for example, [**12**, Exercises 12.58–59]).

Corollary 2.9. *The ordinal interval* $[0,\omega_1]$ *is not Eberlein compact.*

2.5. *$c_0$-direct sums and the Banach space $E_{\omega_1}$*

By the *$c_0$-direct sum* of a family $(X_j)_{j\in J}$ of Banach spaces, we understand the Banach space

$$\Big(\bigoplus_{j\in J} X_j\Big)_{c_0} = \Big\{ f\colon J\to \bigcup_{j\in J} X_j : f(j)\in X_j\ (j\in J) \text{ and the set } \{j\in J : \|f(j)\|\geqslant \varepsilon\} \text{ is finite for each } \varepsilon>0 \Big\}.$$

In the case where $X_j = X$ for each $j\in J$, we write $c_0(J,X)$ instead of $(\bigoplus_{j\in J} X_j)_{c_0}$. We note that, for any index set $M$, the formula

$$(Uf)(j)(m) = f(m)(j) \quad (m\in M,\ j\in J) \tag{2.6}$$

defines an isometric isomorphism $U$ of $c_0(M,(\bigoplus_{j\in J} X_j)_{c_0})$ onto $(\bigoplus_{j\in J} c_0(M,X_j))_{c_0}$, as is easy to verify. The notion of a $c_0$-direct sum is relevant for our purposes mainly due to the central role that the Banach space

$$E_{\omega_1} = \Big(\bigoplus_{\alpha<\omega_1} C[0,\alpha]\Big)_{c_0} \tag{2.7}$$

plays. We shall now record some of its basic properties for later reference.

Lemma 2.10. *Let $(X_j)_{j\in J}$ be a family of Hilbert-generated Banach spaces. Then the Banach space $(\bigoplus_{j\in J} X_j)_{c_0}$ is Hilbert-generated.*

*Proof.* For each $j\in J$, choose a Hilbert space $H_j$ and an operator $T_j\colon H_j\to X_j$ of norm 1 such that the range of $T_j$ is dense in $X_j$. The formula $(x_j)_{j\in J}\mapsto (T_jx_j)_{j\in J}$ then defines an operator of norm 1 from the Hilbert space $(\bigoplus_{j\in J} H_j)_{\ell_2}$ onto a dense subspace of $(\bigoplus_{j\in J} X_j)_{c_0}$. $\square$

Corollary 2.11. *The Banach space $E_{\omega_1}$ is Hilbert-generated.*

*Proof.* This follows immediately from Lemma 2.10 because, for each countable ordinal $\alpha$, the Banach space $C[0,\alpha]$ is separable, and it is thus Hilbert-generated. $\square$

Lemma 2.12. *The Banach space $E_{\omega_1}$ is isomorphic to the $c_0$-direct sum of countably many copies of itself.*

*Proof.* It is known that $C[0,\alpha]$ is isomorphic to $c_0(\mathbb{N},C[0,\alpha])$ for each $\alpha\in[\omega,\omega_1)$, with the Banach–Mazur distance bounded uniformly in $\alpha$ (see, for example, [**32**, Theorem 2.24]). Hence, using the isomorphism (2.6), we have

$$c_0(\mathbb{N},E_{\omega_1})\cong\Bigl(\bigoplus_{\alpha<\omega_1}c_0(\mathbb{N},C[0,\alpha])\Bigr)_{c_0}\cong\Bigl(\bigoplus_{\alpha<\omega_1}C[0,\alpha]\Bigr)_{c_0}=E_{\omega_1},$$

as desired. □

Remark 2.13. Let $L$ be the disjoint union of a family $(L_j)_{j\in J}$ of locally compact Hausdorff spaces. Then $L$ is a locally compact Hausdorff space, and $C_0(L)$ is isometrically isomorphic to $(\bigoplus_{j\in J}C_0(L_j))_{c_0}$ (see, for example, [**9**, Exercise 9, p. 191]). Hence, the Banach space $E_{\omega_1}$ given by (2.7) is isometrically isomorphic to $C_0(L_0)$, where $L_0$ denotes the disjoint union of the compact ordinal intervals $[0,\alpha]$ for $\alpha<\omega_1$.

## 2.6. *Club subsets*

A closed and unbounded subset of $[0,\omega_1)$ is called a *club subset*. Hence, a subset $D$ of $[0,\omega_1)$ is a club subset if and only if $D$ is uncountable and $D\cup\{\omega_1\}$ is closed in $[0,\omega_1]$. The collection

$$\mathscr{D}=\{D\subseteq[0,\omega_1):D\text{ contains a club subset of }[0,\omega_1)\}$$

is a filter on the set $[0,\omega_1)$, and $\mathscr{D}$ is countably complete, in the sense that $\bigcap\mathscr{C}$ belongs to $\mathscr{D}$ for each countable subset $\mathscr{C}$ of $\mathscr{D}$.

The following lemma is a variant of [**2**, Lemma 1.1(c)–(d)], tailored to suit our applications. Its proof is fairly straightforward, so we omit the details.

Lemma 2.14. *Let $D$ be a club subset of $[0,\omega_1)$.*

(i) *The order isomorphism $\psi_D:[0,\omega_1)\to D$ is a homeomorphism, and hence the composition operator $U_D:g\mapsto g\circ\psi_D$ is an isometric isomorphism of $C_0(D)$ onto $C_0[0,\omega_1)$.*

(ii) *The mapping*

$$\pi_D:\ \alpha\longmapsto\min(D\cap[\alpha,\omega_1)),\quad[0,\omega_1)\to D,\tag{2.8}$$

*is an increasing retraction, and hence the composition operator $S_D:g\mapsto g\circ\pi_D$ is a linear isometry of $C_0(D)$ into $C_0[0,\omega_1)$.*

(iii) *Let $\iota_D:D\to[0,\omega_1)$ denote the inclusion mapping. Then the composition operator $R_D:f\mapsto f\circ\iota_D$ is a linear surjection of norm 1 of $C_0[0,\omega_1)$ onto $C_0(D)$, and $R_DS_D=I_{C_0(D)}$.*

(iv) *The operator $P_D=S_DR_D\in\mathscr{B}(C_0[0,\omega_1))$ is a projection of norm 1 such that*

$$\ker P_D=\{f\in C_0[0,\omega_1):f(\alpha)=0\ (\alpha\in D)\},\tag{2.9}$$

*and the range of $P_D$,*

$$\mathscr{R}_D=\{f|_D\circ\pi_D:f\in C_0[0,\omega_1)\},\tag{2.10}$$

*is isometrically isomorphic to $C_0[0,\omega_1)$.*

(v) *Suppose that $D\neq[0,\omega_1)$. Then $[0,\omega_1)\setminus D=\bigcup_{\alpha<\gamma}[\xi_\alpha,\eta_\alpha)$ for some ordinal $\gamma\in[1,\omega_1]$ and some sequences $(\xi_\alpha)_{\alpha<\gamma}$ and $(\eta_\alpha)_{\alpha<\gamma}$, where $\xi_\alpha$ is either 0 or a countable successor ordinal, $\eta_\alpha\in D$ and $\xi_\alpha<\eta_\alpha<\xi_{\alpha+1}$ for each $\alpha$, and $\ker P_D$ is isometrically isomorphic to $(\bigoplus_{\alpha<\gamma}C_0[\xi_\alpha,\eta_\alpha))_{c_0}$.*

Corollary 2.15. *For each club subset $D$ of $[0,\omega_1)$, $\ker P_D$ is Hilbert-generated and isometrically isomorphic to a complemented subspace of $E_{\omega_1}$.*

*Proof.* The result is trivial if $D=[0,\omega_1)$ because $\ker P_D=\{0\}$ in this case. Otherwise Lemma 2.14(v) applies, and the conclusion follows from Lemma 2.10 and (2.7). $\square$

Corollary 2.16. *There exists a club subset $D$ of $[0,\omega_1)$ such that $\ker P_D$ is isometrically isomorphic to $E_{\omega_1}$.*

*Proof.* We can inductively define a strictly increasing transfinite sequence $(\xi_\alpha)_{\alpha<\omega_1}$ of countable ordinals by $\xi_0=0$ and $\xi_\alpha=\sup_{\beta<\alpha}(\xi_\beta+\beta)+2$ for each $\alpha\in[1,\omega_1)$. Then the ordinal interval $[\xi_\alpha,\xi_\alpha+\alpha]$ is open for each $\alpha<\omega_1$, so that $D=[0,\omega_1)\setminus\bigcup_{\alpha<\omega_1}[\xi_\alpha,\xi_\alpha+\alpha]$ is a proper club subset of $[0,\omega_1)$. In the notation of Lemma 2.14(v), we have $\gamma=\omega_1$ and $\eta_\alpha=\xi_\alpha+\alpha+1$ for each $\alpha<\omega_1$, and hence the conclusion follows. $\square$

Lemma 2.17. *Let $D_1$ and $D_2$ be club subsets of $[0,\omega_1)$. Then*

$$\ker P_{D_1}\cap\ker P_{D_2}=\ker P_{D_1\cup D_2}\quad\textit{and}\quad \mathscr{R}_{D_1}\cap\mathscr{R}_{D_2}=\mathscr{R}_{D_1\cap D_2}.$$

*Proof.* The first identity is an immediate consequence of (2.9).

To verify the second, suppose first that $f\in\mathscr{R}_{D_1}\cap\mathscr{R}_{D_2}$. Given $\alpha\in[0,\omega_1)$, an easy transfinite induction shows that $f(\beta)=f(\alpha)$ for each $\beta\in[\alpha,\pi_{D_1\cap D_2}(\alpha)]$, so that in particular we have $f(\alpha)=f(\pi_{D_1\cap D_2}(\alpha))$, and hence $f\in\mathscr{R}_{D_1\cap D_2}$.

Conversely, for each $\alpha\in[0,\omega_1)$, we see that $D_1\cap[\pi_{D_1}(\alpha),\omega_1)=D_1\cap[\alpha,\omega_1)$. Consequently, $D_1\cap D_2\cap[\pi_{D_1}(\alpha),\omega_1)=D_1\cap D_2\cap[\alpha,\omega_1)$, so that $\pi_{D_1\cap D_2}(\pi_{D_1}(\alpha))=\pi_{D_1\cap D_2}(\alpha)$, and therefore, for each $f\in C_0[0,\omega_1)$, we have

$$(P_{D_1}P_{D_1\cap D_2}f)(\alpha)=f(\pi_{D_1\cap D_2}(\pi_{D_1}(\alpha)))=f(\pi_{D_1\cap D_2}(\alpha))=(P_{D_1\cap D_2}f)(\alpha).$$

This proves that $\mathscr{R}_{D_1\cap D_2}\subseteq\mathscr{R}_{D_1}$. A similar argument shows that $\mathscr{R}_{D_1\cap D_2}\subseteq\mathscr{R}_{D_2}$. $\square$

## 3. *The proof of Theorem* 1.1

Lemma 3.1. *Let $D$ be a club subset of $[0,\omega_1)$, and let $\rho\in C_0[0,\omega_1)^*$ and $\lambda\in\mathbb{K}\setminus\{0\}$. Then the mapping $\sigma_{\rho,\lambda}\colon D\cup\{\omega_1\}\to C_0[0,\omega_1)^*$ given by $\sigma_{\rho,\lambda}(\alpha)=\rho+\lambda\delta_\alpha$ for $\alpha\in D$ and $\sigma_{\rho,\lambda}(\omega_1)=\rho$ is injective and continuous with respect to the weak$^*$-topology on its codomain. Hence, its range, which is equal to $\{\rho+\lambda\delta_\alpha:\alpha\in D\}\cup\{\rho\}$, is homeomorphic to $[0,\omega_1]$.*

*Proof.* It is a standard and easily verifiable fact that the mapping $\tau\colon D\cup\{\omega_1\}\to C_0[0,\omega_1)^*$ given by $\tau(\alpha)=\delta_\alpha$ for $\alpha\in D$ and $\tau(\omega_1)=0$ is a continuous injection with respect to the weak$^*$-topology on its codomain, and hence the same is true for $\sigma_{\rho,\lambda}$. The final clause follows because the compact space $D\cup\{\omega_1\}$ is homeomorphic to $[0,\omega_1]$ by (an easy modification of) Lemma 2.14(i), and the weak$^*$-topology on $C_0[0,\omega_1)^*$ is Hausdorff. $\square$

Definition 3.2. A subset $S$ of $[0,\omega_1)$ is *stationary* if $S\cap D\neq\emptyset$ for each club subset $D$ of $[0,\omega_1)$.

Stationary sets have many interesting topological and combinatorial properties, as detailed in [**16**, **22**], for instance. We shall require only the following result, which is due to Fodor [**13**].

THEOREM 3.3 (Pressing Down Lemma). *Let $S$ be a stationary subset of $[0, \omega_1)$, and let $f : S \to [0, \omega_1)$ be a function which satisfies $f(\alpha) < \alpha$ for each $\alpha \in S$. Then $S$ contains a subset $S'$ which is stationary and for which $f|_{S'}$ is constant.*

We can now explain how the proof of Theorem 1.1 is structured: it consists of three parts, set out in the following lemma. Theorem 1.1 follows immediately from it, using Lemma 3.1.

LEMMA 3.4. *Let $K$ be a weak$^*$-compact subset of $C_0[0, \omega_1)^*$.*

(i) *Exactly one of the following two alternatives holds*:
(I) *there is a club subset $D$ of $[0, \omega_1)$ such that*

$$\mu([\alpha, \omega_1)) = 0 \quad (\mu \in K,\, \alpha \in D); \tag{3.1}$$

(II) *the set*

$$\{\alpha \in [0, \omega_1) : \mu([\alpha, \omega_1)) \neq 0 \text{ for some } \mu \in K\} \tag{3.2}$$

*is stationary.*

(ii) *Condition* (I) *is satisfied if and only if $K$ is uniformly Eberlein compact.*
(iii) *Condition* (II) *is satisfied if and only if there exist $\rho \in K$, $\lambda \in \mathbb{K} \setminus \{0\}$ and a club subset $D$ of $[0, \omega_1)$ such that $\rho + \lambda\delta_\alpha \in K$ for each $\alpha \in D$.*

In the proof, we shall require the following elementary observations. The first of these is an easy consequence of the isomorphism (2.1), and so we omit its proof.

LEMMA 3.5. *Let $\mu \in C_0[0, \omega_1)^*$, and let $\alpha \in [\omega, \omega_1)$ be a limit ordinal. Then, for each $\varepsilon > 0$, there exists an ordinal $\alpha_0 < \alpha$ such that $|\mu([\beta, \alpha))| < \varepsilon$ whenever $\beta \in [\alpha_0, \alpha)$.*

LEMMA 3.6. (i) *Let $\{S_n : n \in \mathbb{N}\}$ be a countable family of subsets of $[0, \omega_1)$ such that $\bigcup_{n\in\mathbb{N}} S_n$ is a stationary subset of $[0, \omega_1)$. Then $S_n$ is stationary for some $n \in \mathbb{N}$.*

(ii) *Let $S$ be a stationary subset of $[0, \omega_1)$, and let $D$ be a club subset of $[0, \omega_1)$. Then $S \cap D$ is stationary.*

*Proof.* (i) Suppose contrapositively that, for each $n \in \mathbb{N}$, the set $S_n$ is not stationary, and take a club subset $D_n$ of $[0, \omega_1)$ such that $S_n \cap D_n = \emptyset$. Then $D = \bigcap_{n\in\mathbb{N}} D_n$ is a club subset of $[0, \omega_1)$ such that $(\bigcup_{n\in\mathbb{N}} S_n) \cap D = \emptyset$, which shows that $\bigcup_{n\in\mathbb{N}} S_n$ is not stationary.
(ii) We have $(S \cap D) \cap D' = S \cap (D \cap D') \neq \emptyset$ for each club subset $D'$ of $[0, \omega_1)$ because $D \cap D'$ is a club subset, and hence $S \cap D$ is stationary. □

LEMMA 3.7. *Let $K$ be a scattered, locally compact space. Then the unit ball of $C_0(K)^*$ is weak$^*$-sequentially compact.*
*In particular, the unit ball of $C_0[0, \omega_1)^*$ is weak$^*$-sequentially compact.*

*Proof.* This is essentially known, see, for example, [**12**, Exercise 12.12], so we shall only outline a short argument, based on standard results; alternatively, one can give a more elementary, but longer, proof using (2.1). Since $K$ is scattered, $C_0(K)$ is an Asplund space by [**12**, Theorem 12.29]. Consequently, the unit ball of $C_0(K)^*$ in its weak*-topology is Radon–Nikodym compact, and it is thus weak*-sequentially compact. □

*Proof of Lemma* 3.4. Let $S$ denote the set given by (3.2). Since weak*-compact sets are bounded, we may suppose that $K$ is contained in the unit ball of $C_0[0,\omega_1)^*$.

Part (i) is clear because (II) is simply the negation of (I).

The proofs of (ii) and (iii) are organized as follows. First, we establish the forward implication of (ii), and then we observe that it has the backward implication of (iii) as an immediate consequence. Next, we prove the forward implication of (iii), and finally we deduce the backward implication of (ii) from it.

(ii), $\Rightarrow$. Suppose that $D$ is a club subset of $[0,\omega_1)$ such that (3.1) holds. Replacing $D$ with its intersection with the club subset of limit ordinals in $[\omega,\omega_1)$, we may additionally suppose that $D$ consists entirely of infinite limit ordinals. In the case of real scalars, let $\Delta$ be the collection of open intervals $(q_1,q_2)\subseteq\mathbb{R}$, where $q_1<q_2$ are rational and $0\notin(q_1,q_2)$. Otherwise $\mathbb{K}=\mathbb{C}$, in which case we define $\Delta$ as the collection of open rectangles $(q_1,q_2)\times(r_1,r_2)\subseteq\mathbb{C}$, where $q_1<q_2$ and $r_1<r_2$ are rational and $0=(0,0)\notin(q_1,q_2)\times(r_1,r_2)$. In both cases, $\Delta$ is countable, so that we can find a bijection $\delta\colon\mathbb{N}\to\Delta$. Moreover, we shall fix a bijection $\sigma\colon\mathbb{N}\to\mathbb{N}\times\mathbb{N}$.

For technical reasons, it is convenient to introduce a new limit ordinal which is the predecessor of 0, and which we shall therefore suggestively denote by $-1$. Set $D'=D\cup\{-1\}$. For each $\alpha\in D'$, we define $\alpha^+=\pi_D(\alpha+1)=\min(D\cap[\alpha+1,\omega_1))\in D$, using the notation of Lemma 2.14(ii). Let $\Gamma_\alpha$ denote the set of ordered pairs $(\xi,\eta)$ of ordinals such that $\alpha\leqslant\xi<\eta<\alpha^+$, and choose a bijection $\gamma_\alpha\colon\mathbb{N}\to\Gamma_\alpha$. We can then define a bijection by

$$\tau_\alpha=(\gamma_\alpha\times\delta)\circ\sigma\colon\mathbb{N}\to\Gamma_\alpha\times\Delta\quad(\alpha\in D').$$

Hence, for each $n\in\mathbb{N}$ and $\alpha\in D'$, we can express $\tau_\alpha(n)$ as $\tau_\alpha(n)=(\xi,\eta,R)$, where $(\xi,\eta)\in\Gamma_\alpha$ and $R\in\Delta$, and $R$ depends only on $n$, not on $\alpha$. With $(\xi,\eta,R)$ thus determined by $\alpha$ and $n$, we define

$$G_\alpha^n=\{\mu\in K:\mu([\xi+1,\eta])\in R\}.$$

This is a relatively weak*-open $F_\sigma$-subset of $K$ because $R$ is an open $F_\sigma$-subset of $\mathbb{K}$ and the indicator function $\mathbf{1}_{[\xi+1,\eta]}$ is continuous. Let $\mathscr{F}_n=\{G_\alpha^n:\alpha\in D'\}$. We shall now complete the proof of the forward implication of (ii) by verifying that the family $\mathscr{F}=\bigcup_{n\in\mathbb{N}}\mathscr{F}_n$ satisfies conditions (1)–(2) of Theorem 2.6(iv).

(1). Suppose that $\mu,\nu\in K$ are distinct. Since $\mu$ and $\nu$ are purely atomic, we can choose $\alpha\in[0,\omega_1)$ such that $\mu(\{\alpha\})\neq\nu(\{\alpha\})$. There are two separate cases to consider.

Suppose first that $\alpha$ belongs to $D$. Then, as $\alpha^+$ also belongs to $D$, (3.1) implies that

$$\mu([\alpha+1,\alpha^+))=\mu([\alpha,\omega_1))-\mu(\{\alpha\})-\mu([\alpha^+,\omega_1))=-\mu(\{\alpha\}),$$

and similarly $\nu([\alpha+1,\alpha^+))=-\nu(\{\alpha\})$. Hence, using Lemma 3.5, we can find $\eta\in[\alpha+1,\alpha^+)$ such that $\mu([\alpha+1,\eta])\neq\nu([\alpha+1,\eta])$. By interchanging $\mu$ and $\nu$ if necessary, we may suppose that $\mu([\alpha+1,\eta])\neq0$. Then there is $R\in\Delta$ such that $\mu([\alpha+1,\eta])\in R$ and $\nu([\alpha+1,\eta])\notin R$. Since $(\alpha,\eta)\in\Gamma_\alpha$, we may define $n=\tau_\alpha^{-1}(\alpha,\eta,R)\in\mathbb{N}$. It now follows that $\mu\in G_\alpha^n$ and $\nu\notin G_\alpha^n$, as desired.

Secondly, in the case where $\alpha\notin D$, we can take $\beta\in D'$ such that $\beta<\alpha<\beta^+$. (This is where the introduction of the new ordinal $-1$ is useful.) We claim that there is an ordinal $\xi\in[\beta,\alpha)$ such that $\mu([\xi+1,\alpha])\neq\nu([\xi+1,\alpha])$. Indeed, if $\alpha$ is the successor of some ordinal $\xi\in[-1,\omega_1)$, then $[\xi+1,\alpha]=\{\alpha\}$, and so the claim follows from the choice of $\alpha$. Otherwise $\alpha$ is an infinite limit ordinal, in which case we can find $\xi\in[\beta,\alpha)$ such that

$$\max\{|\mu([\xi+1,\alpha))|,|\nu([\xi+1,\alpha))|\}<\frac{|\mu(\{\alpha\})-\nu(\{\alpha\})|}{2},$$

by Lemma 3.5. This implies that

$$|\mu([\xi+1,\alpha]) - \nu([\xi+1,\alpha])| \geqslant |\mu(\{\alpha\}) - \nu(\{\alpha\})| - |\mu([\xi+1,\alpha))| - |\nu([\xi+1,\alpha))| > 0,$$

which establishes the claim. As before, we may suppose that $\mu([\xi+1,\alpha]) \neq 0$ by interchanging $\mu$ and $\nu$ if necessary, so that we can choose $R \in \Delta$ such that $\mu([\xi+1,\alpha]) \in R$ and $\nu([\xi+1,\alpha]) \notin R$. Since $(\xi,\alpha) \in \Gamma_\beta$, we may define $n = \tau_\beta^{-1}(\xi,\alpha,R) \in \mathbb{N}$, and then $\mu \in G_\beta^n$ and $\nu \notin G_\beta^n$, as desired.

(2). Assume towards a contradiction that $\sup_{\mu\in K} |\{G \in \mathscr{F}_n : \mu \in G\}|$ is infinite for some $n \in \mathbb{N}$, and write $(n_1,n_2) = \sigma(n) \in \mathbb{N}^2$. We shall focus on the case of complex scalars as it is slightly more complicated than the real case. Let $R = \delta(n_2) = (q_1,q_2) \times (r_1,r_2) \in \Delta$. Since $(0,0) \notin R$, either $0 \notin (q_1,q_2)$ or $0 \notin (r_1,r_2)$. Suppose that we are in the first case, and choose $m \in \mathbb{N}$ such that $m \cdot \min\{|q_1|,|q_2|\} > 1$. By the assumption, we can find $\mu \in K$ and ordinals $\alpha_1 < \alpha_2 < \cdots < \alpha_m$ in $D'$ such that $\mu \in \bigcap_{j=1}^m G_{\alpha_j}^n$. Letting $(\xi_j,\eta_j) = \gamma_{\alpha_j}(n_1) \in \Gamma_{\alpha_j}$ for each $j \in \{1,\dots,m\}$, we have $\tau_{\alpha_j}(n) = (\xi_j,\eta_j,R)$, so that $\mu([\xi_j+1,\eta_j]) \in R$ because $\mu \in G_{\alpha_j}^n$. The fact that

$$\alpha_1 \leqslant \xi_1 < \eta_1 < \alpha_1^+ \leqslant \alpha_2 \leqslant \xi_2 < \eta_2 < \alpha_2^+ \leqslant \cdots \leqslant \alpha_m \leqslant \xi_m < \eta_m < \alpha_m^+$$

implies that the intervals $[\xi_1+1,\eta_1], [\xi_2+1,\eta_2], \dots, [\xi_m+1,\eta_m]$ are disjoint, and hence

$$\begin{aligned} 1 \geqslant \|\mu\| &\geqslant \left|\mu\left(\bigcup_{j=1}^m [\xi_j+1,\eta_j]\right)\right| = \left|\sum_{j=1}^m \mu([\xi_j+1,\eta_j])\right| \geqslant \left|\operatorname{Re}\sum_{j=1}^m \mu([\xi_j+1,\eta_j])\right| \\ &\geqslant m \cdot \min\{|q_1|,|q_2|\} > 1, \end{aligned}$$

which is clearly absurd. The case where $0 \notin (r_1,r_2)$ is very similar: we simply replace $\min\{|q_1|,|q_2|\}$ and the real part with $\min\{|r_1|,|r_2|\}$ and the imaginary part, respectively.

The case where $\mathbb{K} = \mathbb{R}$ is also similar, but easier, because there is no need to pass to the real part in the above calculation.

(iii), $\Leftarrow$, is an easy consequence of the previous implications. Suppose contrapositively that condition (II) is not satisfied. Then, by (i), condition (I) holds, so that $K$ is uniformly Eberlein compact by what we have just proved. Every weak*-closed subset of $K$ is therefore also uniformly Eberlein compact, and hence Corollary 2.9 implies that no subset of $K$ is homeomorphic to $[0,\omega_1]$. The desired conclusion now follows from Lemma 3.1.

(iii), $\Rightarrow$. Suppose that the set $S$ given by (3.2) is stationary. Since

$$S = \bigcup_{n\in\mathbb{N}} \left\{\alpha \in [0,\omega_1) : |\mu([\alpha,\omega_1))| > \frac{1}{n} \text{ for some } \mu \in K\right\},$$

Lemma 3.6(i) implies that the set

$$S_0 = \{\alpha \in [0,\omega_1) : |\mu([\alpha,\omega_1))| > \varepsilon_0 \text{ for some } \mu \in K\}$$

is stationary for some $\varepsilon_0 > 0$. Replacing $S_0$ with its intersection with the club subset of limit ordinals in $[\omega,\omega_1)$, we may in addition suppose that $S_0$ consists entirely of infinite limit ordinals by Lemma 3.6(ii). For each $\alpha \in S_0$, take $\mu_\alpha \in K$ such that $|\mu_\alpha([\alpha,\omega_1))| > \varepsilon_0$. Lemma 3.5 implies that $|\mu_\alpha|([f(\alpha),\alpha)) < \varepsilon_0/3$ for some ordinal $f(\alpha) \in [0,\alpha)$, where $|\mu_\alpha|$ denotes the total variation of $\mu_\alpha$, that is, the positive measure on $[0,\omega_1)$ given by $|\mu_\alpha|(B) = \sum_{\beta\in B} |\mu_\alpha(\{\beta\})|$ for each $B \subseteq [0,\omega_1)$. By Theorem 3.3, $S_0$ contains a subset $S'$ which is stationary and for which $f|_{S'}$ is constant, say $f(\alpha) = \zeta_0$ for each $\alpha \in S'$.

Define $\mathbb{L} = \mathbb{Q}$ if $\mathbb{K} = \mathbb{R}$ and $\mathbb{L} = \{q + ri : q,r \in \mathbb{Q}\}$ if $\mathbb{K} = \mathbb{C}$, so that $\mathbb{L}$ is a countable, dense subfield of $\mathbb{K}$. For each $\alpha \in S'$ and $k \in \mathbb{N}$, choose a non-empty, finite subset $F_{\alpha,k}$ of $[0,\omega_1)$ and scalars $q^\beta_{\alpha,k} \in \mathbb{L}$ for $\beta \in F_{\alpha,k}$ such that

$$\mu_{\alpha,k} = \sum_{\beta\in F_{\alpha,k}} q^\beta_{\alpha,k}\delta_\beta \in C_0[0,\omega_1)^* \tag{3.3}$$

has norm at most 1 and satisfies

$$\|\mu_{\alpha,k} - \mu_\alpha\| < \min\left\{\frac{\varepsilon_0}{3}, \frac{1}{k}\right\}. \tag{3.4}$$

Suppose that $(\alpha_k)_{k\in\mathbb{N}}$ is a sequence in $S'$ such that $(\mu_{\alpha_k,k})_{k\in\mathbb{N}}$ weak*-converges to a limit $\nu \in C_0[0,\omega_1)^*$. Then we claim that

$$\nu = \mathrm{w}^*\text{-}\lim_k \mu_{\alpha_k}, \tag{3.5}$$

a conclusion which we shall require towards the end of the proof. Indeed, for each $\varepsilon > 0$ and $g \in C_0[0,\omega_1)$, we can choose $k_0 \in \mathbb{N}$ such that $k_0 > 2\|g\|/\varepsilon$ and $|\langle g, \mu_{\alpha_k,k} - \nu\rangle| < \varepsilon/2$ whenever $k \geqslant k_0$, and hence

$$|\langle g, \mu_{\alpha_k} - \nu\rangle| \leqslant |\langle g, \mu_{\alpha_k} - \mu_{\alpha_k,k}\rangle| + |\langle g, \mu_{\alpha_k,k} - \nu\rangle| < \frac{\|g\|}{k} + \frac{\varepsilon}{2} < \varepsilon \quad (k \geqslant k_0).$$

For each $k \in \mathbb{N}$, $\{F_{\alpha,k} : \alpha \in S'\}$ is an uncountable collection of finite sets, so by the $\Delta$-system Lemma (see [**35**], or [**16**, Theorem 9.18] for an exposition), we can find a subset $\Delta_k$ of $[0,\omega_1)$ and an uncountable subset $A_k$ of $S'$ such that

$$F_{\alpha,k} \cap F_{\beta,k} = \Delta_k \quad (\alpha, \beta \in A_k,\ \alpha \neq \beta). \tag{3.6}$$

Set $\zeta_1 = \sup(\{\zeta_0\} \cup \bigcup_{k\in\mathbb{N}} \Delta_k) + 1 \in [1,\omega_1)$, and fix $k \in \mathbb{N}$ for a while. We shall now repeatedly pass to suitably chosen uncountable subsets of $A_k$ in order to arrange that the elements have certain useful additional properties.

The fact that $A_k = \bigcup_{n\in\mathbb{N}}\{\alpha \in A_k : |F_{\alpha,k}| = n\}$ implies that, for some $n_k \in \mathbb{N}$, there is an uncountable subset $A'_k$ of $A_k$ such that $|F_{\alpha,k}| = n_k$ for each $\alpha \in A'_k$. Let $\theta_{\alpha,k} : \{1,\dots,n_k\} \to F_{\alpha,k}$ be the unique order isomorphism for each $\alpha \in A'_k$, and recall from (3.3) that $q^\beta_{\alpha,k} \in \mathbb{L}$ for $\beta \in F_{\alpha,k}$ are the coefficients of $\mu_{\alpha,k}$. Since $\mathbb{L}$ is countable and

$$A'_k = \bigcup_{q_1,\dots,q_{n_k}\in\mathbb{L}} \{\alpha \in A'_k : q^{\theta_{\alpha,k}(j)}_{\alpha,k} = q_j \text{ for each } j \in \{1,\dots,n_k\}\},$$

we can find $q_{1,k},\dots,q_{n_k,k} \in \mathbb{L}$ and an uncountable subset $A''_k$ of $A'_k$ such that

$$q^{\theta_{\alpha,k}(j)}_{\alpha,k} = q_{j,k} \quad (j \in \{1,\dots,n_k\},\ \alpha \in A''_k). \tag{3.7}$$

Set $A'''_k = A''_k \cap [\zeta_1,\omega_1)$, which is an uncountable subset of $A''_k$, and assume towards a contradiction that $\Delta_k = F_{\alpha,k}$ for some $\alpha \in A'''_k$. Then, by (3.3), $\mu_{\alpha,k}$ is supported on $\Delta_k$, which is contained in $[0,\zeta_1)$, so that $\mu_{\alpha,k}([\alpha,\omega_1)) = 0$. Hence, we have

$$\|\mu_\alpha - \mu_{\alpha,k}\| \geqslant |(\mu_\alpha - \mu_{\alpha,k})([\alpha,\omega_1))| = |\mu_\alpha([\alpha,\omega_1))| > \varepsilon_0.$$

This, however, contradicts (3.4), and therefore we conclude that

$$\Delta_k \subsetneq F_{\alpha,k} \quad (\alpha \in A'''_k). \tag{3.8}$$

For each $\beta \in [0,\omega_1)$, the set

$$B^\beta_k = \{\alpha \in A'''_k : \min(F_{\alpha,k} \setminus \Delta_k) \leqslant \beta < \alpha\} = \bigcup_{\gamma\in[0,\beta]\setminus\Delta_k} \{\alpha \in A'''_k \cap [\beta+1,\omega_1) : \gamma \in F_{\alpha,k}\}$$

is countable because each of the sets on the right-hand side contains at most one element by (3.6). Hence, $A'''_k \cap [\beta+1,\omega_1) \setminus B^\beta_k$ is uncountable, and it is thus non-empty; that is, for each $\beta \in [0,\omega_1)$, we can find $\alpha \in A'''_k \cap [\beta+1,\omega_1)$ such that $\min(F_{\alpha,k} \setminus \Delta_k) > \beta$. A straightforward induction based on this observation yields a strictly increasing transfinite sequence $(\alpha_\xi)_{\xi<\omega_1}$ in $A_k'''$ such that $\sup(\{\zeta_1\}\cup\bigcup_{\eta<\xi}(F_{\alpha_\eta,k}\setminus\Delta_k)) < \min(F_{\alpha_\xi,k}\setminus\Delta_k)$ for each $\xi\in[0,\omega_1)$, and consequently $A_k'''' = \{\alpha_\xi : \xi\in[0,\omega_1)\}$ is an uncountable subset of $A_k'''$ such that

$$\sup\Biggl(\{\zeta_1\}\cup\bigcup_{\beta\in A_k''''\cap[0,\alpha)}(F_{\beta,k}\setminus\Delta_k)\Biggr) < \min(F_{\alpha,k}\setminus\Delta_k)\quad(\alpha\in A_k''''). \tag{3.9}$$

Set $m_k = |\Delta_k|$, so that $m_k < n_k$ by (3.8), and define $\lambda_k = \sum_{j=m_k+1}^{n_k} q_{j,k}\in\mathbb{L}$. By (3.3), (3.7) and (3.9), we have $\lambda_k = \mu_{\alpha,k}([\zeta_1,\omega_1))$ for each $\alpha\in A_k''''$, and hence

$$\begin{aligned}1 \geqslant \|\mu_{\alpha,k}\| \geqslant |\lambda_k| &\geqslant |\mu_\alpha([\zeta_1,\omega_1))| - \|\mu_{\alpha,k}-\mu_\alpha\|\\ &\geqslant |\mu_\alpha([\alpha,\omega_1))| - |\mu_\alpha|([\zeta_1,\alpha)) - \|\mu_{\alpha,k}-\mu_\alpha\| > \varepsilon_0 - \frac{\varepsilon_0}{3} - \frac{\varepsilon_0}{3} = \frac{\varepsilon_0}{3}.\end{aligned}$$

Therefore, after passing to a subsequence, we may suppose that $(\lambda_k)_{k\in\mathbb{N}}$ converges to a limit $\lambda\in\mathbb{K}$, where $1\geqslant|\lambda|\geqslant\varepsilon_0/3>0$. (We must of course also pass to the same subsequence of all the other objects (including $F_{\alpha,k}$, $\mu_{\alpha,k}$, $\Delta_k$, $A_k$, $A_k'$, …) that we have chosen dependent on $k$. This does not affect any of the identities, above; indeed, (3.4) is the only one which depends explicitly on $k$, and it clearly remains true after we pass to a subsequence.)

Choose ordinals $\beta_{1,k}<\cdots<\beta_{m_k,k}$ such that $\Delta_k = \{\beta_{1,k},\ldots,\beta_{m_k,k}\}$, and define

$$\rho_k = \sum_{j=1}^{m_k} q_{j,k}\delta_{\beta_{j,k}} \in C_0[0,\omega_1)^*. \tag{3.10}$$

Since $\|\rho_k\|\leqslant\sum_{j=1}^{n_k}|q_{j,k}| = \|\mu_{\alpha,k}\|\leqslant 1$ for each $\alpha\in A_k''''$, Lemma 3.7 implies that, by passing to subsequences once more, we may suppose that $(\rho_k)_{k\in\mathbb{N}}$ weak*-converges to a limit $\rho\in C_0[0,\omega_1)^*$.

Our next aim is to show that, for each sequence $(\alpha_k)_{k\in\mathbb{N}}$ which belongs to the set

$$\mathfrak{D} = \Biggl\{(\alpha_k)_{k\in\mathbb{N}} : \alpha_k\in A_k'''',\ \alpha_k<\alpha_{k+1}\ \text{and}\ \max(F_{\alpha_k,k}\setminus\Delta_k) < \min(F_{\alpha_{k+1},k+1}\setminus\Delta_{k+1})\\ \text{for each } k\in\mathbb{N},\ \text{and}\ \sup_{k\in\mathbb{N}}\alpha_k = \sup\bigcup_{k\in\mathbb{N}}(F_{\alpha_k,k}\setminus\Delta_k)\Biggr\}, \tag{3.11}$$

the sequence $(\mu_{\alpha_k})_{k\in\mathbb{N}}$ weak*-converges to $\rho+\lambda\delta_\alpha$, where $\alpha = \sup_{k\in\mathbb{N}}\alpha_k\in[0,\omega_1)$. By (3.5), it suffices to show that $(\mu_{\alpha_k,k})_{k\in\mathbb{N}}$ weak*-converges to $\rho+\lambda\delta_\alpha$. To verify this, let $\varepsilon>0$ and $g\in C_0[0,\omega_1)$ be given. We may suppose that $\|g\|\leqslant1$. Choose $k_1\in\mathbb{N}$ such that $|\lambda-\lambda_k|<\varepsilon/3$ and $|\langle g,\rho-\rho_k\rangle|<\varepsilon/3$ whenever $k\geqslant k_1$. Since $g$ is continuous at $\alpha$, which is a limit ordinal, we can find $\beta_0\in[0,\alpha)$ such that $|g(\beta)-g(\alpha)|<\varepsilon/3$ for each $\beta\in[\beta_0,\alpha]$. By the definition of $\mathfrak{D}$, we can take $k_2\in\mathbb{N}$ such that $F_{\alpha_k,k}\setminus\Delta_k\subseteq[\beta_0,\alpha]$ whenever $k\geqslant k_2$, and thus $|g(\beta)-g(\alpha)|<\varepsilon/3$ for each $\beta\in\bigcup_{k\geqslant k_2}(F_{\alpha_k,k}\setminus\Delta_k)$. Now we have

$$|\langle g,\mu_{\alpha_k,k}-\rho-\lambda\delta_\alpha\rangle| \leqslant |\langle g,\mu_{\alpha_k,k}-\rho_k-\lambda_k\delta_\alpha\rangle| + |\langle g,\rho-\rho_k\rangle| + |\langle g,(\lambda-\lambda_k)\delta_\alpha\rangle|,$$

where the second and third terms are both less than $\varepsilon/3$ provided that $k\geqslant k_1$. To estimate the first term, we observe that $\theta_{\alpha_k,k}(j) = \beta_{j,k}$ for each $j\in\{1,\ldots,m_k\}$. Consequently (3.3), (3.7) and (3.10) imply that $\mu_{\alpha_k,k}-\rho_k = \sum_{j=m_k+1}^{n_k} q_{j,k}\delta_{\theta_{\alpha_k,k}(j)}$, where $\theta_{\alpha_k,k}(j)\in F_{\alpha_k,k}\setminus\Delta_k$ for each

$j \in \{m_k+1,\dots,n_k\}$, and therefore we have

$$|\langle g, \mu_{\alpha_k,k} - \rho_k - \lambda_k\delta_\alpha\rangle| = \left|\sum_{j=m_k+1}^{n_k} q_{j,k} g(\theta_{\alpha_k,k}(j)) - \sum_{j=m_k+1}^{n_k} q_{j,k} g(\alpha)\right|$$
$$\leqslant \sum_{j=m_k+1}^{n_k} |q_{j,k}|\cdot|g(\theta_{\alpha_k,k}(j)) - g(\alpha)| < \frac{\varepsilon}{3}$$

provided that $k \geqslant k_2$. Hence, we conclude that $(\mu_{\alpha_k})_{k\in\mathbb{N}}$ weak*-converges to $\rho + \lambda\delta_\alpha$.

This implies in particular that $\rho+\lambda\delta_\alpha \in K$ for each $\alpha$ belonging to the set

$$D_0 = \left\{\sup_{k\in\mathbb{N}} \alpha_k : (\alpha_k)_{k\in\mathbb{N}} \in \mathfrak{D}\right\}. \tag{3.12}$$

To prove that $D_0$ is unbounded, let $\beta \in [0,\omega_1)$ be given. For each $k \in \mathbb{N}$, (3.9) implies that the transfinite sequence $(\min(F_{\alpha,k}\setminus\Delta_k))_{\alpha\in A_k''''}$ is strictly increasing, and it is thus unbounded. We can therefore inductively construct a sequence $(\alpha_k)_{k\in\mathbb{N}}$ in $[\beta,\omega_1)$ such that $\alpha_k \in A_k''''$ and

$$\max(\{\alpha_k\}\cup(F_{\alpha_k,k}\setminus\Delta_k)) < \min(\{\alpha_{k+1}\}\cup(F_{\alpha_{k+1},k+1}\setminus\Delta_{k+1})) \quad (k\in\mathbb{N}).$$

We claim that this sequence $(\alpha_k)_{k\in\mathbb{N}}$ belongs to $\mathfrak{D}$. Of the conditions in (3.11), only the final one is not immediately obvious; it, however, follows from the intertwining relation

$$\alpha_k < \min(F_{\alpha_{k+1},k+1}\setminus\Delta_{k+1}) \leqslant \max(F_{\alpha_{k+1},k+1}\setminus\Delta_{k+1}) < \alpha_{k+2} \quad (k\in\mathbb{N}).$$

Consequently, we have $\sup D_0 \geqslant \sup_{k\in\mathbb{N}}\alpha_k \geqslant \beta$, as desired.

Hence, the closure $D$ of $D_0$ in $[0,\omega_1)$ is a club subset. The unboundedness of $D_0$ implies that the net $(\delta_\alpha)_{\alpha\in D_0}$ is weak*-null, so that $\rho = \text{w}^*\text{-}\lim_\alpha(\rho+\lambda\delta_\alpha)\in K$. Since each $\alpha\in D$ is the limit of a sequence $(\gamma_n)_{n\in\mathbb{N}}$ in $D_0$, we have $\rho+\lambda\delta_\alpha = \text{w}^*\text{-}\lim_n(\rho+\lambda\delta_{\gamma_n})\in K$. This completes the proof of the forward implication of (iii).

(ii), $\Leftarrow$, now follows easily by contraposition, just as (iii), $\Leftarrow$, did. Indeed, suppose that condition (I) is not satisfied. Then, by (i), condition (II) is satisfied, so that Lemma 3.1 and the forward implication of (iii) imply that $K$ contains a subset which is homeomorphic to $[0,\omega_1]$. Hence, $K$ is not (uniformly) Eberlein compact by Corollary 2.9. □

Remark 3.8. We completed the proof of (Lemma 3.4(iii)). $\Rightarrow$). by passing to the closure $D$ of the set $D_0$ given by (3.12). In fact, $D_0$ is already closed, so that $D_0 = D$. To verify this, we observe that each $\beta$ in the closure of $D_0$ is countable, and it is thus the limit of a sequence $(\beta^j)_{j\in\mathbb{N}}$ in $D_0$. We may suppose that $(\beta^j)_{j\in\mathbb{N}}$ is strictly increasing. For each $j\in\mathbb{N}$, take $(\alpha_k^j)_{k\in\mathbb{N}}\in\mathfrak{D}$ such that $\beta^j = \sup_{k\in\mathbb{N}}\alpha_k^j$. The sequence $(\min(\{\alpha_k^j\}\cup(F_{\alpha_k^j,k}\setminus\Delta_k)))_{k\in\mathbb{N}}$ then increases strictly to the limit $\beta^j$, so we may inductively choose a strictly increasing sequence $(k_j)_{j\in\mathbb{N}}$ of integers such that $k_1 = 1$ and

$$\beta^j < \min(\{\alpha_{k_{j+1}}^{j+1}\}\cup(F_{\alpha_{k_{j+1}}^{j+1},k_{j+1}}\setminus\Delta_{k_{j+1}})) \quad (j\in\mathbb{N}). \tag{3.13}$$

We now claim that the sequence $(\gamma_\ell)_{\ell\in\mathbb{N}}$ given by

$$(\alpha_1^1,\alpha_2^1,\dots,\alpha_{k_2-1}^1,\alpha_{k_2}^2,\alpha_{k_2+1}^2,\dots,\alpha_{k_3-1}^2,\alpha_{k_3}^3,\dots,\alpha_{k_j-1}^{j-1},\alpha_{k_j}^j,\alpha_{k_j+1}^j,\dots,\alpha_{k_{j+1}-1}^j,\alpha_{k_{j+1}}^{j+1},\dots)$$

belongs to $\mathfrak{D}$. Indeed, for $\ell\in\mathbb{N}$, let $j\in\mathbb{N}$ be the unique number such that $k_j\leqslant\ell<k_{j+1}$. We then have $\gamma_\ell = \alpha_\ell^j \in A_\ell''''$. If $\ell \neq k_{j+1}-1$, then $\gamma_{\ell+1} = \alpha_{\ell+1}^j$, in which case the inequalities $\gamma_\ell < \gamma_{\ell+1}$ and $\max(F_{\gamma_\ell,\ell}\setminus\Delta_\ell) < \min(F_{\gamma_{\ell+1},\ell+1}\setminus\Delta_{\ell+1})$ are both immediate from (3.11). Otherwise $\ell = k_{j+1}-1$, and by (3.13), we find

$$\gamma_\ell = \alpha_\ell^j < \sup_{k\in\mathbb{N}}\alpha_k^j = \beta^j < \alpha_{k_{j+1}}^{j+1} = \gamma_{\ell+1}$$

and

$$\max(F_{\gamma_\ell,\ell} \setminus \Delta_\ell) < \beta^j < \min(F_{\alpha^{j+1}_{k_{j+1}},k_{j+1}} \setminus \Delta_{k_{j+1}}) = \min(F_{\gamma_{\ell+1},\ell+1} \setminus \Delta_{\ell+1}).$$

These intertwining relations imply that $\sup_{\ell\in\mathbb{N}} \gamma_\ell = \beta = \sup \bigcup_{\ell\in\mathbb{N}}(F_{\gamma_\ell,\ell} \setminus \Delta_\ell)$, which shows that $(\gamma_\ell)_{\ell\in\mathbb{N}} \in \mathfrak{D}$, and hence $\beta \in D_0$, as required.

The idea that a result like Theorem 1.1 might be true was inspired by a note from Richard Smith (private communication). The following corollary confirms a conjecture that he proposed therein.

Corollary 3.9. *Let $K$ be a weak$^*$-compact subset of $C_0[0,\omega_1)^*$ such that there exists a continuous surjection from $K$ onto $[0,\omega_1]$. Then $K$ contains a homeomorphic copy of $[0,\omega_1]$.*

*Proof.* By a classical result of Benyamini, Rudin and Wage [**5**], the continuous image of an Eberlein compact space is Eberlein compact. Since $[0,\omega_1]$ is not Eberlein compact, $K$ cannot be Eberlein compact, and we are therefore in case (II) of Theorem 1.1. □

Example 3.10. The purpose of this example is to show that the dichotomy stated in Lemma 3.4(i) is no longer true if condition (II) is replaced with the condition

(II′) the set $S = \{\alpha \in [0,\omega_1) : \mu([\alpha+1,\omega_1)) \neq 0 \text{ for some } \mu \in K\}$ is stationary.

Indeed, let $\Lambda$ be the set of all countable limit ordinals. Then $K = \{\delta_\alpha - \delta_{\alpha+1} : \alpha \in \Lambda\} \cup \{0\}$ is a bounded and weak$^*$-closed subset of $C_0[0,\omega_1)^*$, and it is thus weak$^*$-compact. Moreover, $K$ satisfies condition (I) because (3.1) holds for the club subset $D = \Lambda$ (and $K$ is therefore uniformly Eberlein compact by Lemma 3.4(ii)), but $K$ also satisfies (II′) because $\Lambda \subseteq S$, and each club subset $D'$ of $[0,\omega_1)$ intersects $\Lambda$, so that $S \cap D' \neq \emptyset$; that is, $S$ is stationary. Hence, conditions (I) and (II′) are not mutually exclusive.

Finally in this section we shall show that Theorem 1.1 is optimal in the following precise sense: the dichotomy would no longer be true if, in condition (I), we replace the collection of all uniformly Eberlein compact subsets of $C_0[0,\omega_1)^*$ with a strictly smaller family.

Proposition 3.11. *Every uniformly Eberlein compact space which contains a dense subset of cardinality at most $\aleph_1$ is homeomorphic to a weak$^*$-compact subset of $C_0[0,\omega_1)^*$.*

*Proof.* As in Example 3.10, denote by $\Lambda$ the set of all countable limit ordinals. Then every uniformly Eberlein compact space which contains a dense subset of cardinality at most $\aleph_1$ embeds in the closed unit ball $B_{\ell_2(\Lambda)}$ of the Hilbert space

$$\ell_2(\Lambda) = \left\{ f : \Lambda \to \mathbb{K} : \sum_{\alpha\in\Lambda} |f(\alpha)|^2 < \infty \right\},$$

endowed with the weak topology. Hence, it will suffice to prove that the mapping given by

$$\theta : \; f \longmapsto \sum_{\alpha\in\Lambda} f(\alpha)|f(\alpha)|(\delta_\alpha - \delta_{\alpha+1}), \quad B_{\ell_2(\Lambda)} \to C_0[0,\omega_1)^*, \tag{3.14}$$

is a weak–weak$^*$-continuous injection. The injectivity is easy to verify. Suppose that the net $(f_j)_{j\in J}$ in $B_{\ell_2(\Lambda)}$ converges weakly to $f$, and let $\varepsilon > 0$ and $g \in C_0[0,\omega_1)$ be given. Since the indicator functions $\mathbf{1}_{[0,\alpha]}$ for $\alpha \in [0,\omega_1)$ span a norm-dense subspace of $C_0[0,\omega_1)$, it suffices to consider the case where $g = \mathbf{1}_{[0,\,\alpha]}$ for some $\alpha \in [0,\omega_1)$. Now

$$\langle \mathbf{1}_{[0,\,\alpha]}, \theta(h)\rangle = \begin{cases} h(\alpha)|h(\alpha)| & \text{if } \alpha \in \Lambda \\ 0 & \text{otherwise} \end{cases} \quad (h \in B_{\ell_2(\Lambda)}),$$

so that we may suppose that $\alpha \in \Lambda$. Choosing $j_0 \in J$ such that $|f(\alpha) - f_j(\alpha)| < \varepsilon/2$ whenever $j \geqslant j_0$, we obtain

$$\begin{aligned} |\langle \mathbf{1}_{[0,\,\alpha]}, \theta(f) - \theta(f_j)\rangle| &= |f(\alpha)|f(\alpha)| - f_j(\alpha)|f_j(\alpha)|| \\ &\leqslant 2\max\{|f(\alpha)|, |f_j(\alpha)|\}|f(\alpha) - f_j(\alpha)| < \varepsilon \quad (j \geqslant j_0), \end{aligned}$$

which proves that $(\theta(f_j))_{j\in J}$ weak*-converges to $\theta(f)$. □

Remark 3.12. One may wonder whether a more operator-theoretic approach is possible in the proof of Proposition 3.11. The natural extension to $\ell_2(\Lambda)$ of the mapping $\theta$ given by (3.14) is clearly not linear. This is no coincidence because in fact *no* weak–weak*-continuous, linear mapping $T : \ell_2(\Lambda) \to C_0[0,\omega_1)^*$ is injective. To verify this, we first observe that $T$ is weakly compact because its domain is reflexive, and hence $T$ is compact because its codomain has the Schur property. Moreover, using the reflexivity of $\ell_2(\Lambda)$ once more, we see that the weak–weak*-continuity of $T$ implies that $T = S^*$ for some operator $S : C_0[0,\omega_1) \to \ell_2(\Lambda)$. Schauder's theorem shows that $S$ is compact, so that its range is separable. In particular, the range of $S$ is not dense in $\ell_2(\Lambda)$, and therefore $T = S^*$ is not injective.

## 4. *Operator theory on* $C_0[0,\omega_1)$

The following lemma represents the core of our proof of Theorem 1.4.

Lemma 4.1. *Let $X$ be a Banach space, and suppose that there exists a surjective operator $T : C_0[0,\omega_1) \to X$. Then exactly one of the following two alternatives holds:*

(I) *$X$ embeds in a Hilbert-generated Banach space;*
(II) *there exists a club subset $D$ of $[0,\omega_1)$ such that the restriction of $T$ to the subspace $\mathscr{R}_D$ given by (2.10) is bounded below, and $T[\mathscr{R}_D]$ is complemented in $X$.*

*Proof.* Let $B_{X^*}$ be the closed unit ball of $X^*$. The weak*-continuity of $T^*$ implies that the subset $K = T^*[B_{X^*}]$ of $C_0[0,\omega_1)^*$ is weak*-compact, so by Theorem 1.1, we have

(I′) $K$ is uniformly Eberlein compact;
(II′) there exist $\rho \in K$, $\lambda \in \mathbb{K} \setminus \{0\}$, and a club subset $D$ of $[0,\omega_1)$ such that $\rho + \lambda\delta_\alpha \in K$ for each $\alpha \in D$.

Since $T^*$ is injective by the assumption, its restriction to $B_{X^*}$ is a weak* homeomorphism onto $K$. Hence, in case (I′), $B_{X^*}$ is uniformly Eberlein compact in its weak*-topology, and so $X$ embeds in a Hilbert-generated Banach space by Theorem 2.7.

Otherwise we are in case (II′), so that there are functionals $g, g_\alpha \in B_{X^*}$ such that $T^*g = \rho$ and $T^*g_\alpha = \rho + \lambda\delta_\alpha$ for each $\alpha \in D$. Given $x \in X$, we define a mapping $Sx : [0,\omega_1) \to \mathbb{K}$ by

$$(Sx)(\alpha) = \langle x, g_{\pi_D(\alpha)} - g\rangle \quad (\alpha \in [0,\omega_1)), \tag{4.1}$$

where $\pi_D : [0,\omega_1) \to D$ is the retraction defined by (2.8). Choosing $f \in C_0[0,\omega_1)$ such that $Tf = x$, we have

$$(Sx)(\alpha) = \langle Tf, g_{\pi_D(\alpha)} - g\rangle = \langle f, T^* g_{\pi_D(\alpha)} - T^* g\rangle = \lambda f(\pi_D(\alpha)) = \lambda(P_D f)(\alpha) \qquad (4.2)$$

for each $\alpha \in [0,\omega_1)$, so that $Sx = \lambda P_D f$, where $P_D$ is the projection defined in Lemma 2.14(iv). Since $\lambda P_D f \in C_0[0,\omega_1)$, we see that (4.1) defines a mapping $S : X \to C_0[0,\omega_1)$, which is linear by the linearity of the functionals $g_{\pi_D(\alpha)}$ and $g$. Moreover, $S$ is bounded because the Open Mapping Theorem implies that there exists a constant $C > 0$, dependent only on the surjective operator $T$, such that, for each $x \in X$, there exists $f \in C_0[0,\omega_1)$ with $\|f\| \leqslant C\|x\|$ and $Tf = x$. Then we have $Sx = \lambda P_D f$ by (4.2), and hence $\|Sx\| = |\lambda|\,\|P_D f\| \leqslant |\lambda|\,\|f\| \leqslant |\lambda| C\|x\|$, as desired. Another application of (4.2) shows that $S(T(P_D f)) = \lambda P_D(P_D f) = \lambda P_D f$ for each $f \in C_0[0,\omega_1)$, so that $STP_D = \lambda P_D$. Consequently, $\|Tf\| \geqslant |\lambda|\,\|S\|^{-1}\|f\|$ for each $f \in \mathscr{R}_D$, and the operator $\lambda^{-1}TP_D S \in \mathscr{B}(X)$ is a projection with range $T[\mathscr{R}_D]$; thus (II) is satisfied.

Finally, the conditions (I) and (II) are mutually exclusive because (II) implies that $T$ induces an isomorphic embedding of $\mathscr{R}_D \cong C_0[0,\omega_1)$ in $X$, and hence (I) cannot be satisfied by Theorem 2.8. □

*Proof of Theorem* 1.4. Suppose that (I) is not satisfied. Then, by Lemma 4.1, we can find a club subset $D$ of $[0,\omega_1)$ and a projection $Q \in \mathscr{B}(X)$ such that $T|_{\mathscr{R}_D}$ is bounded below and $Q[X] = T[\mathscr{R}_D]$. Hence, the identity operator on $C_0[0,\omega_1)$ factors through $T$ by Lemma 2.1 because $\mathscr{R}_D$ is isomorphic to $C_0[0,\omega_1)$ by Lemma 2.14(iv).

We shall complete the proof that (II) is satisfied by showing that $\ker Q$ embeds in a Hilbert-generated Banach space. Assume the contrary, and apply Lemma 4.1 to the surjective operator $U : f \mapsto (I_X - Q)Tf$, $C_0[0,\omega_1) \to \ker Q$, to obtain a club subset $D'$ of $[0,\omega_1)$ such that $U|_{\mathscr{R}_{D'}}$ is bounded below by $c > 0$, say. Then we have

$$c\|f\| \leqslant \|Uf\| = \|(I_X - Q)Tf\| = 0 \quad (f \in \mathscr{R}_D \cap \mathscr{R}_{D'}),$$

so that $\mathscr{R}_D \cap \mathscr{R}_{D'} = \{0\}$. This, however, contradicts Lemma 2.17.

Theorem 2.8 shows that conditions (I) and (II) are mutually exclusive. □

Remark 4.2. Not all quotients of $C_0[0,\omega_1)$ are subspaces of $C_0[0,\omega_1)$. This follows from a result of Alspach [**1**], which says that $C[0,\omega^\omega]$ has a quotient $X$ which does not embed in $C[0,\alpha]$ for any countable ordinal $\alpha$. Since $C_0[0,\omega_1)$ contains a complemented copy of $C[0,\omega^\omega]$, $X$ is also a quotient of $C_0[0,\omega_1)$. However, $X$ does not embed in $C_0[0,\omega_1)$ because $X$ is separable (being a quotient of a separable space), and each separable subspace of $C_0[0,\omega_1)$ embeds in $C[0,\alpha]$ for some countable ordinal $\alpha$ (see, for example, [**21**, Lemma 4.2]).

Lemma 4.3. *Let $\theta$ be a continuous mapping from $[0,\omega_1]$ into a Hausdorff space $K$. Then exactly one of the following two alternatives holds*:

(I) $[0,\omega_1]$ *contains a closed, uncountable subset $F_1$ such that $\theta|_{F_1}$ is constant*;
(II) $[0,\omega_1]$ *contains a closed, uncountable subset $F_2$ such that $\theta|_{F_2}$ is injective, and $\theta|_{F_2}$ is thus a homeomorphism onto $\theta[F_2]$.*

*Proof.* Suppose that $K$ contains a point $x$ whose pre-image under $\theta$ is uncountable. Then (I) is satisfied for $F_1 = \theta^{-1}[\{x\}]$.

Otherwise the pre-image under $\theta$ of each point of $K$ is at most countable. In this case, we shall inductively construct a strictly increasing transfinite sequence $(\alpha_\xi)_{\xi<\omega_1}$ in $[1,\omega_1)$ such that

$$\{\beta \in [0,\omega_1) : \theta(\beta) = \theta(\omega_1) \text{ or } \theta(\beta) = \theta(\alpha_\eta) \text{ for some } \eta < \xi\} \subseteq [0,\alpha_\xi) \quad (\xi \in [0,\omega_1)) \tag{4.3}$$

and

$$\sup_{\eta<\xi} \alpha_\eta \in \{\alpha_\eta : \eta \leqslant \xi\} \quad (\xi \in [1,\omega_1)). \tag{4.4}$$

To start the induction, let $\alpha_0 = (\sup \theta^{-1}[\{\omega_1\}]) + 1 \in [1,\omega_1)$.

Now assume inductively that, for some $\xi \in [1,\omega_1)$, a strictly increasing sequence $(\alpha_\eta)_{\eta<\xi}$ of countable ordinals has been chosen in accordance with (4.3) and (4.4), and define

$$\alpha_\xi = \sup\{\beta + 1 : \theta(\beta) = \theta(\alpha_\eta) \text{ for some } \eta < \xi\} \in \left[\sup_{\eta<\xi}(\alpha_\eta + 1), \omega_1\right).$$

Then (4.3) is certainly satisfied for $\xi$. To verify (4.4), let $\gamma = \sup_{\eta<\xi} \alpha_\eta$. The conclusion is clear if this supremum is attained. Otherwise $\xi$ is a limit ordinal, and we claim that $\gamma = \alpha_\xi$. Since $\gamma \leqslant \alpha_\xi$, it suffices to show that $\beta + 1 \leqslant \gamma$ whenever $\beta \in [0,\omega_1)$ satisfies $\theta(\beta) = \theta(\alpha_\zeta)$ for some $\zeta < \xi$. Now $\zeta + 1 < \xi$ because $\xi$ is a limit ordinal, so that (4.3) holds when $\xi$ is replaced with $\zeta + 1$ by the induction hypothesis; consequently, $\beta + 1 \leqslant \alpha_{\zeta+1} < \gamma$, and the induction continues.

Let $F_2 = \{\alpha_\xi : \xi < \omega_1\} \cup \{\omega_1\}$; this set is uncountable because $(\alpha_\xi)_{\xi<\omega_1}$ is strictly increasing, and it is closed by (4.4). Moreover, $\theta|_{F_2}$ is injective by (4.3), so that (II) is satisfied.

Finally, to see that conditions (I) and (II) are mutually exclusive, assume towards a contradiction that $[0,\omega_1]$ contains closed, uncountable subsets $F_1$ and $F_2$ such that $\theta|_{F_1}$ is constant and $\theta|_{F_2}$ is injective. Then $F_1 \cap F_2$ is uncountable and contains $\omega_1$, so that $\theta(\alpha) = \theta(\omega_1)$ for each $\alpha \in F_1 \cap F_2$ by the choice of $F_1$. This, however, contradicts the injectivity of $\theta|_{F_2}$. □

*Proof of Theorem* 1.5. Let $T \in \mathscr{B}(C_0[0,\omega_1))$. The following 'function-free' reformulation of (1.2) will enable us to simplify certain calculations somewhat:

$$T^*\delta_\alpha = \varphi(T)\delta_\alpha \quad (\alpha \in D). \tag{4.5}$$

To prove that there exist $\varphi(T) \in \mathbb{K}$ and a club subset $D$ of $[0,\omega_1)$ such that this identity is satisfied, we consider the composite mapping $\theta$ given by

$$\begin{tikzcd} {[0,\omega_1]} \arrow[rr, dashed, "\theta"] \arrow[dr, "\sigma_{0,1}"'] & & C_0[0,\omega_1)^* \\ & C_0[0,\omega_1)^*, \arrow[ur, "T^*"'] & \end{tikzcd}$$

where both copies of $C_0[0,\omega_1)^*$ are equipped with the weak$^*$-topology, and $\sigma_{0,1}$ is the injection defined in Lemma 3.1, that is, $\sigma_{0,1}(\alpha) = \delta_\alpha$ for $\alpha \in [0,\omega_1)$ and $\sigma_{0,1}(\omega_1) = 0$. Since $\sigma_{0,1}$ and $T^*$ are continuous, so is $\theta$. We may therefore apply Lemma 4.3 to conclude that $[0,\omega_1]$ contains a closed, uncountable subset $F$ such that either $\theta|_F$ is constant, or $\theta|_F$ is injective. Note that $\omega_1 \in F$, and $F \setminus \{\omega_1\}$ is a club subset of $[0,\omega_1)$.

If $\theta|_F$ is constant, then we have

$$T^*\delta_\alpha = (T^* \circ \sigma_{0,1})(\alpha) = \theta(\alpha) = \theta(\omega_1) = (T^* \circ \sigma_{0,1})(\omega_1) = T^*0 = 0 \quad (\alpha \in F \setminus \{\omega_1\}),$$

so that (4.5) is satisfied for $D = F \setminus \{\omega_1\}$ and $\varphi(T) = 0$.

Otherwise $\theta|_F$ is injective, in which case $\widetilde{\theta} : \alpha \mapsto \theta(\alpha),\ F \to \theta[F]$, is a homeomorphism. Since $F$ is homeomorphic to $[0,\omega_1]$, which is not Eberlein compact, Theorem 1.1 implies that there exist $\rho \in \theta[F]$, $\varphi(T) \in \mathbb{K} \setminus \{0\}$ and a club subset $D'$ of $[0,\omega_1)$ such that $\rho + \varphi(T)\delta_\alpha \in \theta[F]$

for each $\alpha \in D'$; let $\eta_\alpha = \tilde{\theta}^{-1}(\rho + \varphi(T)\delta_\alpha)$ for $\alpha \in D'$. Then $(\eta_\alpha)_{\alpha\in D'}$ is an uncountable net of distinct elements of $F$, and it converges to $\tilde{\theta}^{-1}(\rho)$. The only possible limit of such a net is $\omega_1$, so that $\rho = \theta(\omega_1) = 0$, and we have

$$T^*\delta_{\eta_\alpha} = \theta(\eta_\alpha) = \varphi(T)\delta_\alpha \quad (\alpha \in D'). \tag{4.6}$$

We shall now show that $D = \{\alpha \in [0,\omega_1) : T^*\delta_\alpha = \varphi(T)\delta_\alpha\}$ is a club subset of $[0,\omega_1)$; this will complete the existence part of the proof because (4.5) is evidently satisfied for this choice of $D$.

Suppose that the net $(\alpha_j)_{j\in J}$ in $D$ converges to $\alpha \in [0,\omega_1)$. Then the net $(\delta_{\alpha_j})_{j\in J}$ weak*-converges to $\delta_\alpha$, and therefore, by the weak*-continuity of $T^*$, we have

$$T^*\delta_\alpha = \mathrm{w}^*\text{-}\lim_j T^*\delta_{\alpha_j} = \mathrm{w}^*\text{-}\lim_j \varphi(T)\delta_{\alpha_j} = \varphi(T)\delta_\alpha.$$

Hence, $\alpha \in D$, so that $D$ is closed.

To prove that $D$ is unbounded, let $\gamma \in [0,\omega_1)$ be given. For each $\beta \in [0,\omega_1)$, the sets $D' \cap [0,\beta]$ and $\{\alpha \in D' : \eta_\alpha \leqslant \beta\}$ are both countable, so that the complement in $D'$ of their union, which is equal to $\{\alpha \in D' \cap [\beta+1,\omega_1) : \eta_\alpha > \beta\}$, is uncountable, and it is thus non-empty. Using this, we can inductively construct a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $D' \cap [\gamma+1,\omega_1)$ such that

$$\max\{\alpha_n, \eta_{\alpha_n}\} < \min\{\alpha_{n+1}, \eta_{\alpha_{n+1}}\} \quad (n \in \mathbb{N}).$$

Let $\alpha = \sup_{n\in\mathbb{N}} \alpha_n$. Then $\alpha \in D' \cap [\gamma+1,\omega_1)$, and both of the sequences $(\alpha_n)_{n\in\mathbb{N}}$ and $(\eta_{\alpha_n})_{n\in\mathbb{N}}$ converge to $\alpha$. Hence, $(\delta_{\alpha_n})_{n\in\mathbb{N}}$ and $(\delta_{\eta_{\alpha_n}})_{n\in\mathbb{N}}$ both weak*-converge to $\delta_\alpha$, so that

$$T^*\delta_\alpha = \mathrm{w}^*\text{-}\lim_n T^*\delta_{\eta_{\alpha_n}} = \mathrm{w}^*\text{-}\lim_n \varphi(T)\delta_{\alpha_n} = \varphi(T)\delta_\alpha,$$

by the weak*-continuity of $T^*$ and (4.6). This proves that $\alpha \in D$, and so $D$ is unbounded.

We shall next prove that the scalar $\varphi(T)$ is uniquely determined by the operator $T$. Suppose that $\varphi_1(T)$ and $\varphi_2(T)$ are scalars such that

$$(Tf)(\alpha) = \varphi_j(T)f(\alpha) \quad (\alpha \in D_j,\ f \in C_0[0,\omega_1),\ j = 1,2),$$

for some club subsets $D_1$ and $D_2$ of $[0,\omega_1)$. Then $D_1 \cap D_2$ is a club subset, and it is thus non-empty. Choosing $\alpha \in D_1 \cap D_2$, we obtain $(T\mathbf{1}_{[0,\alpha]})(\alpha) = \varphi_j(T)\mathbf{1}_{[0,\alpha]}(\alpha) = \varphi_j(T)$ for $j = 1,2$, so that $\varphi_1(T) = \varphi_2(T)$, as required.

Consequently, we can define a mapping $\varphi\colon T \mapsto \varphi(T), \mathscr{B}(C_0[0,\omega_1)) \to \mathbb{K}$, which is non-zero because $\varphi(I_{C_0[0,\omega_1)}) = 1$. To see that $\varphi$ is an algebra homomorphism, let $\lambda \in \mathbb{K}$ and $T_1, T_2 \in \mathscr{B}(C_0[0,\omega_1))$ be given, and choose club subsets $D_1$ and $D_2$ of $[0,\omega_1)$ such that

$$T_j^*\delta_\alpha = \varphi(T_j)\delta_\alpha \quad (\alpha \in D_j,\ j = 1,2).$$

Then, for each $\alpha$ belonging to the club subset $D_1 \cap D_2$ of $[0,\omega_1)$, we have

$$(\lambda T_1 + T_2)^*\delta_\alpha = \lambda T_1^*\delta_\alpha + T_2^*\delta_\alpha = (\lambda\varphi(T_1) + \varphi(T_2))\delta_\alpha$$

and

$$(T_1T_2)^*\delta_\alpha = T_2^*(T_1^*\delta_\alpha) = T_2^*(\varphi(T_1)\delta_\alpha) = \varphi(T_1)\varphi(T_2)\delta_\alpha,$$

so that $\varphi(\lambda T_1 + T_2) = \lambda\varphi(T_1) + \varphi(T_2)$ and $\varphi(T_1T_2) = \varphi(T_1)\varphi(T_2)$ by the definition of $\varphi$. $\square$

*Proof of Theorem* 1.6. We begin by showing that conditions (ii) and (iii) are equivalent.

Suppose that $D$ is a club subset of $[0,\omega_1)$ such that $T^*(\delta_\alpha)(\{\alpha\}) = 0$ for each $\alpha \in D$. By Theorem 1.5, we can find a club subset $D'$ of $[0,\omega_1)$ such that $T^*\delta_\alpha = \varphi(T)\delta_\alpha$ for each $\alpha \in D'$. Then, choosing $\alpha \in D \cap D'$ (which is possible because $D \cap D'$ is a club subset, and it is thus non-empty), we obtain $0 = T^*\delta_\alpha(\{\alpha\}) = \varphi(T)\delta_\alpha(\{\alpha\}) = \varphi(T)$, which establishes (iii).

Conversely, suppose that $\varphi(T)=0$, so that there exists a club subset $D$ of $[0,\omega_1)$ such that $T^*\delta_\alpha = 0$ for each $\alpha\in D$. Then clearly $T^*(\delta_\alpha)(\{\alpha\})=0$ for each $\alpha\in D$, so that (ii) is satisfied.

We shall next prove that conditions (iii)–(viii) are equivalent.

(iii)⇒(iv). Suppose that $\varphi(T)=0$, and choose a club subset $D$ of $[0,\omega_1)$ such that $(Tf)(\alpha)=0$ for each $f\in C_0[0,\omega_1)$ and each $\alpha\in D$. Then (2.9) shows that the range of $T$ is contained in $\ker P_D$, so that $T$ factors through $E_{\omega_1}$ by Corollary 2.15.

(iv)⇒(v). Suppose that $T=SR$, where $R\in\mathscr{B}(C_0[0,\omega_1),E_{\omega_1})$ and $S\in\mathscr{B}(E_{\omega_1},C_0[0,\omega_1))$. Then the range of $T$ is contained in the range of $S$, and the closure of the range of $S$ is Hilbert-generated because the domain of $S$ is Hilbert-generated by Corollary 2.11.

The implication (v)⇒(vi) is clear because every Hilbert-generated Banach space is weakly compactly generated.

The implications (vi)⇒(vii)⇒(viii)⇒(iii) are all proved by contraposition.

(vi)⇒(vii). Suppose that, for some $U\in\mathscr{B}(C_0[0,\omega_1))$, the composite operator $TU$ is an isomorphism onto its range. Then Theorem 2.8 shows that the range of $TU$, and hence the range of $T$, cannot be contained in any weakly compactly generated Banach space.

(vii)⇒(viii). Suppose that $STR = I_{C_0[0,\omega_1)}$ for some operators $R,S\in\mathscr{B}(C_0[0,\omega_1))$. Then the operator $TR$ is bounded below, and it is thus an isomorphism onto its range, so that $T$ fixes an isomorphic copy of $C_0[0,\omega_1)$.

(viii)⇒(iii). Suppose that $\varphi(T)\neq 0$. By rescaling, we may suppose that $\varphi(T)=1$, so that we can choose a club subset $D$ of $[0,\omega_1)$ such that $(Tf)(\alpha)=f(\alpha)$ for each $f\in C_0[0,\omega_1)$ and each $\alpha\in D$. Then, in the notation of Lemma 2.14, we have a commutative diagram, which implies that (viii) is not satisfied:

$$\begin{tikzcd}
C_0[0,\omega_1) \arrow[rrr, "I_{C_0[0,\omega_1)}"] \arrow[dr, "U_D^{-1}"] \arrow[dd, dashed] & & & C_0[0,\omega_1) \\
& C_0(D) \arrow[r, "I_{C_0(D)}"] \arrow[dl, "S_D"] & C_0(D) \arrow[ur, "U_D"] & \\
C_0[0,\omega_1) \arrow[rrr, "T"] & & & C_0[0,\omega_1) \arrow[ul, "R_D"] \arrow[uu, dashed]
\end{tikzcd}$$

Indeed, the commutativity of the upper trapezium is clear, while for the lower one, we find

$$(R_DTS_Dg)(\alpha) = T(g\circ\pi_D)(\alpha) = (g\circ\pi_D)(\alpha) = g(\alpha)\quad (g\in C_0(D),\ \alpha\in D).$$

Finally, to see that conditions (i) and (iii) are equivalent, we note that, on the one hand, the Loy–Willis ideal $\mathscr{M}$ is a maximal ideal of $\mathscr{B}(C_0[0,\omega_1))$ by its definition. On the other, the implication (viii)⇒(iii), which has just been established, shows that the identity operator belongs to the ideal generated by any operator which is not in $\ker\varphi$, so that $\ker\varphi$ is the *unique* maximal ideal of $\mathscr{B}(C_0[0,\omega_1))$. Hence, $\ker\varphi = \mathscr{M}$. □

As we have already noted in the final paragraph of the proof, above, Theorem 1.6 has the following important consequence, which generalizes [**21**, Theorem 1.1].

Corollary 4.4. *The ideal* $\mathscr{M} = \mathscr{S}\!\mathit{em}(C_0[0,\omega_1)) = \ker\varphi$ *is the unique maximal ideal of* $\mathscr{B}(C_0[0,\omega_1))$.

Remark 4.5. The first-named author and Kochanek [**19**] have recently introduced the notion of a *weakly compactly generated operator* as an operator whose range is contained in a weakly compactly generated subspace of its codomain, and they have shown that the collection $\mathscr{WCG}$ of all weakly compactly generated operators forms a closed operator ideal in the sense of Pietsch.

Analogously, let us say that an operator is *Hilbert-generated* if its range is contained in a Hilbert-generated subspace of its codomain. An argument along the same lines as the proof of [**19**, Theorem 2.1] establishes that the collection $\mathscr{HG}$ of all Hilbert-generated operators defines a closed operator ideal in the sense of Pietsch, and the equivalence of conditions (i), (v) and (vi) of Theorem 1.6 shows that

$$\mathscr{M} = \mathscr{HG}(C_0[0,\omega_1)) = \mathscr{WCG}(C_0[0,\omega_1)).$$

*Proof of Corollary* 1.9. Let $\Gamma$ denote the set of all club subsets of $[0,\omega_1)$, ordered by reverse inclusion. This order is filtering upward because $D \cap D' \in \Gamma$ is a majorant for any pair $D, D' \in \Gamma$, and hence Theorem 1.6, Lemma 2.14(iv) and Corollary 2.15 imply that

$$Q_D = I_{C_0[0,\,\omega_1)} - P_D \in \mathscr{M} \quad (D \in \Gamma)$$

defines a net of projections, each having norm at most 2. As in the proof of Theorem 1.6, (iii)⇒(iv), we see that, for each $T \in \mathscr{M}$, there is a club subset $D_0$ of $[0,\omega_1)$ such that $P_{D_0}T = 0$. Equation (2.9) then shows that $P_D T = 0$ for each $D \subseteq D_0$; that is, $Q_D T = T$ whenever $D \geqslant D_0$. □

Example 4.6. Consider the Hilbert space $H = \{f : [0,\omega_1) \to \mathbb{K} : \sum_{\alpha<\omega_1} |f(\alpha)|^2 < \infty\}$. The work of Gramsch [**14**] and Luft [**27**] shows that the set $\mathscr{X}(H)$ of operators on $H$ having separable range is the unique maximal ideal of $\mathscr{B}(H)$. (In fact, Gramsch and Luft proved that the entire lattice of closed ideals of $\mathscr{B}(H)$ is given by

$$\{0\} \subset \mathscr{K}(H) \subset \mathscr{X}(H) \subset \mathscr{B}(H),$$

but we do not require the full strength of their result.) Since $\mathscr{B}(H)$ is a $C^*$-algebra, each of its non-zero closed ideals has a bounded two-sided approximate identity consisting of positive operators of norm 1. The purpose of this example is to show that, in the case of $\mathscr{X}(H)$, we have a bounded two-sided approximate identity $(P_L)_{L\in\Gamma}$ consisting of self-adjoint projections such that $P_L T = T = TP_L$ eventually for each $T \in \mathscr{X}(H)$. We note in passing that algebras which contain a net with this property have been studied in a purely algebraic context by Ara and Perera [**4**, Definition 1.4] and Pedersen and Perera [**31**, Section 4].

Let $\Gamma$ denote the set of all closed, separable subspaces of $H$, ordered by inclusion. This order is filtering upward because $\overline{L+M} \in \Gamma$ majorizes the pair $L, M \in \Gamma$. For $L \in \Gamma$, let $P_L \in \mathscr{X}(H)$ be the orthogonal projection which has range $L$. Suppose that $T \in \mathscr{X}(H)$, and denote by $T^\star$ the Hilbert-space adjoint of $T$. We have $T^\star \in \mathscr{X}(H)$ because each closed ideal of a $C^*$-algebra is self-adjoint, and therefore $M = \overline{T[H] + T^\star[H]}$ belongs to $\Gamma$. Now, for each $L \in \Gamma$ such that $L \supseteq M$, we see that $P_L T = T$ and $P_L T^\star = T^\star$, from which the desired conclusion follows by taking the adjoint of the latter equation.

*Proof of Corollary* 1.10. Assume towards a contradiction that, for some natural numbers $m > n$, there exists either an operator $R : C_0[0,\omega_1)^m \to C_0[0,\omega_1)^n$ which is bounded below, or an operator $T : C_0[0,\omega_1)^n \to C_0[0,\omega_1)^m$ which is surjective. We shall focus on the first case; the other is very similar. The proof is best explained if we represent the operator $R : C_0[0,\omega_1)^m \to C_0[0,\omega_1)^n$ by the operator-valued $(n\times m)$-matrix $(R_{j,k})_{j,k=1}^{n,m}$ given by $R_{j,k} = Q_j^{(n)} R J_k^{(m)} \in \mathscr{B}(C_0[0,\omega_1))$, where $Q_j^{(n)} : C_0[0,\omega_1)^n \to C_0[0,\omega_1)$ and $J_k^{(m)} : C_0[0,\omega_1) \to C_0[0,\omega_1)^m$ denote the $j$th coordinate projection and $k$th coordinate embedding, respectively.

Using elementary column operations, we can reduce the scalar-valued matrix $S = (\varphi(R_{j,k}))_{j,k=1}^{n,m}$ to column-echelon form; that is, we can find an invertible, scalar-valued

$(m \times m)$-matrix $U$ such that $SU$ has column-echelon form. Since $m > n$, the final column of $SU$ must be zero. Consequently, each operator in the final column of the matrix $RU$ belongs to $\mathscr{M}$, which is equal to the operator ideal $\mathscr{HG}(C_0[0, \omega_1))$ introduced in Remark 4.5; hence we have $RUJ_m^{(m)} \in \mathscr{HG}(C_0[0, \omega_1), C_0[0, \omega_1)^n)$. This, however, contradicts Theorem 2.8 because the operators $J_m^{(m)}$, $U$ and $R$ are all bounded below, and therefore the range of $RUJ_n^{(n)}$ is isomorphic to its domain $C_0[0, \omega_1)$. □

*Proof of Corollary* 1.11. Since $P$ is idempotent, we have $\varphi(P) \in \{0, 1\}$. We shall consider the case where $\varphi(P) = 0$; the case where $\varphi(P) = 1$ is similar, just with $P$ and $I_{C_0[0, \omega_1)} - P$ interchanged. Let $X = \ker P$ and $Y = P[C_0[0, \omega_1)]$. As in the proof of Theorem 1.6, (iii)⇒(iv), we see that $Y$ is contained in $\ker P_D$ for some club subset $D$ of $[0, \omega_1)$. By Corollary 2.15, $\ker P_D$ is isomorphic to a complemented subspace of $E_{\omega_1}$, so that the same is true for $Y$, say $E_{\omega_1} \cong Y \oplus Z$ for some Banach space $Z$. Using Lemma 2.12, we obtain

$$E_{\omega_1} \cong c_0(\mathbb{N}, E_{\omega_1}) \cong c_0(\mathbb{N}, Y \oplus Z) \cong c_0(\mathbb{N}, Y \oplus Z) \oplus Y \cong E_{\omega_1} \oplus Y,$$

where we recall that $c_0(\mathbb{N}, W)$ denotes the $c_0$-direct sum of countably many copies of the Banach space $W$. Consequently, $C_0[0, \omega_1) \cong C_0[0, \omega_1) \oplus Y$ because $C_0[0, \omega_1)$ contains a complemented subspace isomorphic to $E_{\omega_1}$ by Corollary 2.16. Theorem 1.4 implies that $X$ contains a complemented subspace which is isomorphic to $C_0[0, \omega_1)$, so that $X \cong W \oplus C_0[0, \omega_1)$ for some Banach space $W$, and hence we have

$$X \cong W \oplus C_0[0, \omega_1) \cong W \oplus C_0[0, \omega_1) \oplus Y \cong X \oplus Y = C_0[0, \omega_1),$$

as required. □

*Proof of Corollary* 1.12. Let $U$ be an isomorphism of $C_0[0, \omega_1)$ onto a closed subspace $W$ of $C_0[0, \omega_1)$, and consider the operator $T = JU \in \mathscr{B}(C_0[0, \omega_1))$, where $J : W \to C_0[0, \omega_1)$ denotes the natural inclusion. Then $T$ fixes an isomorphic copy of $C_0[0, \omega_1)$. Hence, by Theorem 1.6, we can find operators $R$ and $S$ on $C_0[0, \omega_1)$ such that $STR = I_{C_0[0, \omega_1)}$. This implies that $TR$ is an isomorphism of $C_0[0, \omega_1)$ onto its range $Y$, which is contained in $W$, and $TRS$ is a projection of $C_0[0, \omega_1)$ onto $Y$. □

*Proof of Corollary* 1.13. This proof follows closely that of [**36**, Proposition 8], where any unexplained terminology can also be found.

We begin by showing that Willis's ideal of compressible operators on $C_0[0, \omega_1)$, as defined in [**36**, p. 252], is equal to the Loy–Willis ideal $\mathscr{M}$. Indeed, [**36**, Proposition 2] and Corollary 1.10 show that the identity operator on $C_0[0, \omega_1)$ is not compressible, so that the ideal of compressible operators is proper, and it is thus contained in $\mathscr{M}$ by Corollary 4.4. Conversely, each operator $T \in \mathscr{M}$ is compressible by [**36**, Proposition 1] because $T$ factors through a complemented subspace $X$ of $C_0[0, \omega_1)$ such that $X \cong E_{\omega_1}$, and Lemma 2.12 implies that $X$ is isomorphic to the $c_0$-direct sum of countably many copies of itself.

Next, we observe that null sequences in $\mathscr{M}$ factor, in the sense that, for each norm-null sequence $(T_n)_{n \in \mathbb{N}}$ in $\mathscr{M}$, there are $R \in \mathscr{M}$ and a norm-null sequence $(S_n)_{n \in \mathbb{N}}$ in $\mathscr{M}$ such that $T_n = RS_n$ for each $n \in \mathbb{N}$. This is a standard consequence of Cohen's Factorization Theorem (see, for example, [**8**, Corollary I.11.2]), which applies because $\mathscr{M}$ has a bounded left approximate identity. We do not, however, need Cohen's Factorization Theorem to draw this conclusion. Indeed, for each $n \in \mathbb{N}$, Theorems 1.5 and 1.6 imply that there is a club subset $D_n$ of $[0, \omega_1)$ such that $T_n^* \delta_\alpha = 0$ for each $\alpha \in D_n$. Then $D = \bigcap_{n \in \mathbb{N}} D_n$ is a club subset of $[0, \omega_1)$ such that $T_n^* \delta_\alpha = 0$ for each $\alpha \in D$ and $n \in \mathbb{N}$. This implies that $P_D T_n = 0$ for each $n \in \mathbb{N}$ by (2.9), so that the operator $R = I_{C_0[0, \omega_1)} - P_D$ satisfies $RT_n = T_n$ for each $n \in \mathbb{N}$; and $R$ belongs to $\mathscr{M}$ by Corollary 2.15 and Theorem 1.6.

Now consider an algebra homomorphism $\theta$ from $\mathscr{B}(C_0[0,\omega_1))$ into some Banach algebra $\mathscr{C}$. Then [**36**, Proposition 7] implies that the continuity ideal of $\theta|_{\mathscr{M}}$ contains $\mathscr{M}$, so that the mapping $S \mapsto \theta(RS)$, $\mathscr{M} \to \mathscr{C}$, is continuous for each fixed $R \in \mathscr{M}$. Since null sequences factor, as shown above, this proves the continuity of $\theta|_{\mathscr{M}}$, and thus of $\theta$, because $\mathscr{M}$ has finite codimension in $\mathscr{B}(C_0[0,\omega_1))$. □

*Proof of Corollary* 1.14. Each operator in $\mathscr{M}$ factors through the Banach space $E_{\omega_1}$ by Theorem 1.6. Lemma 2.12 states that $E_{\omega_1}$ is isomorphic to the $c_0$-direct sum of countably many copies of itself, so that [**25**, Proposition 3.7] implies that each operator on $E_{\omega_1}$ is the sum of at most two commutators, and therefore each operator which factors through $E_{\omega_1}$ is the sum of at most three commutators by [**25**, Lemma 4.5].

Suppose that $\tau$ is a trace on $\mathscr{B}(C_0[0,\omega_1))$. Then $\tau$ vanishes on each commutator, so that $\mathscr{M} \subseteq \ker\tau$ by the result established in the first paragraph of the proof. Hence, we have $\tau(T + \lambda I_{C_0[0,\omega_1)}) = \lambda\tau(I_{C_0[0,\omega_1)}) = \varphi(T + \lambda I_{C_0[0,\omega_1)})\tau(I_{C_0[0,\omega_1)})$ for each $T \in \mathscr{M}$ and $\lambda \in \mathbb{K}$ because $\varphi(T) = 0$ by Theorem 1.6. The converse implication is clear. □

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## *References*


**1.** D. Alspach, 'A quotient of $C(\omega^\omega)$ which is not isomorphic to a subspace of $C(\alpha)$, $\alpha < \omega_1$', *Israel J. Math.* 35 (1980) 49–60.

**2.** D. Alspach and Y. Benyamini, 'Primariness of spaces of continuous functions on ordinals', *Israel J. Math.* 27 (1977) 64–92.

**3.** D. Amir and J. Lindenstrauss, 'The structure of weakly compact sets in Banach spaces', *Ann. of Math.* 88 (1968) 35–46.

**4.** P. Ara and F. Perera, 'Multipliers of von Neumann regular rings', *Comm. Algebra* 28 (2000) 3359–3385.

**5.** Y. Benyamini, M. E. Rudin and M. Wage, 'Continuous images of weakly compact subsets of Banach spaces', *Pacific J. Math.* 70 (1977) 309–324.

**6.** Y. Benyamini and T. Starbird, 'Embedding weakly compact sets into Hilbert space', *Israel J. Math.* 23 (1976) 137–141.

**7.** C. Bessaga and A. Pełczyński, 'Banach spaces non-isomorphic to their Cartesian squares. I', *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys.* 8 (1960) 77–80.

**8.** F. F. Bonsall and J. Duncan, *Complete normed algebras*, Ergebnisse der Mathematik und ihrer Grenzgebiete 80 (Springer, New York, 1973).

**9.** J. B. Conway, *A course in functional analysis*, 2nd edn, Graduate Texts in Mathematics 96 (Springer, New York, 1990).

**10.** P. G. Dixon, 'Approximate identities in normed algebras', *Proc. London Math. Soc.* 26 (1973) 485–496.

**11.** M. Fabian, G. Godefroy, P. Hájek and V. Zizler, 'Hilbert-generated Banach spaces', *J. Funct. Anal.* 200 (2003) 301–323.

**12.** M. Fabian, P. Habala, P. Hájek, V. Montesinos Santalucía, J. Pelant and V. Zizler, *Functional analysis and infinite-dimensional geometry*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC 8 (Springer, New York, 2001).

**13.** G. Fodor, 'Eine Bemerkung zur Theorie der regressiven Funktionen', *Acta Sci. Math. Szeged* 17 (1956) 139–142.

**14.** B. Gramsch, 'Eine Idealstruktur Banachscher Operatoralgebren', *J. reine angew. Math.* 225 (1967) 97–115.

**15.** P. Hájek, V. Montesinos Santalucía, J. Vanderwerff and V. Zizler, *Biorthogonal systems in Banach spaces*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC 26 (Springer, New York, 2008).

**16.** T. Jech, *Set theory*, The third millennium edition, revised and expanded, Springer Monographs in Mathematics (Springer, Berlin, 2003).

**17.** O. F. K. Kalenda, 'Embedding of the ordinal segment $[0,\omega_1]$ into continuous images of Valdivia compacta', *Comment. Math. Univ. Carolin.* 40 (1999) 777–783.

**18.** O. F. K. Kalenda, 'On the class of continuous images of Valdivia compacta', *Extracta Math.* 18 (2003) 65–80.

**19.** T. Kania and T. Kochanek, 'The ideal of weakly compactly generated operators acting on a Banach space', *J. Operator Theory* 71 (2014) 101–124.
**20.** T. Kania, P. Koszmider and N. J. Laustsen, 'Banach spaces whose algebra of bounded operators has the integers as their $K_0$-group', Preprint, 2014, arXiv:1303.2606.
**21.** T. Kania and N. J. Laustsen, 'Uniqueness of the maximal ideal of the Banach algebra of bounded operators on $C([0,\omega_1])$', *J. Funct. Anal.* 262 (2012) 4831–4850.
**22.** K. Kunen, *Set theory. An introduction to independence proofs*, Studies in Logic and the Foundations of Mathematics 102 (North-Holland, Amsterdam, 1980).
**23.** N. J. Laustsen, '$K$-Theory for the Banach algebra of operators on James's quasi-reflexive Banach spaces', *K-Theory* 23 (2000) 115–127.
**24.** N. J. Laustsen, 'Maximal ideals in the algebra of operators on certain Banach spaces', *Proc. Edinb. Math. Soc.* 45 (2002) 523–546.
**25.** N. J. Laustsen, 'Commutators of operators on Banach spaces', *J. Operator Theory* 48 (2002) 503–514.
**26.** R. J. Loy and G. A. Willis, 'Continuity of derivations on $\mathscr{B}(E)$ for certain Banach spaces $E$', *J. London Math. Soc.* 40 (1989) 327–346.
**27.** E. Luft, 'The two-sided closed ideals of the algebra of bounded linear operators of a Hilbert space', *Czechoslovak Math. J.* 18 (1968) 595–605.
**28.** R. E. Megginson, *An introduction to Banach space theory*, Graduate Texts in Mathematics 183 (Springer, New York, 1998).
**29.** C. Ogden, 'Homomorphisms from $B(C(\omega_\eta))$', *J. London Math. Soc.* 54 (1996) 346–358.
**30.** J. Orihuela, W. Schachermayer and M. Valdivia, 'Every Radon–Nikodým Corson compact space is Eberlein compact', *Studia Math.* 98 (1991) 157–174.
**31.** G. K. Pedersen and F. Perera, 'Inverse limits of rings and multiplier rings', *Math. Proc. Cambridge Philos. Soc.* 139 (2005) 207–228.
**32.** H. P. Rosenthal, 'The Banach spaces $C(K)$', *Handbook of the geometry of Banach spaces*, vol. 2 (North-Holland, Amsterdam, 2003) 1547–1602.
**33.** W. Rudin, 'Continuous functions on compact spaces without perfect subsets', *Proc. Amer. Math. Soc.* 8 (1957) 39–42.
**34.** Z. Semadeni, 'Banach spaces non-isomorphic to their Cartesian squares. II', *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys.* 8 (1960) 81–84.
**35.** N. A. Shanin, 'A theorem from the general theory of sets', *C. R. (Doklady) Acad. Sci. URSS (N.S.)* 53 (1946) 399–400.
**36.** G. Willis, 'Compressible operators and the continuity of homomorphisms from algebras of operators', *Studia Math.* 115 (1995) 251–259.

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