ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 18, Number 1, June 2014
Available online at http://acutm.math.ut.ee

Multi-norms

H. G. Dales

Abstract. We give a survey of the theory of multi-norms, based on a talk given in Tartu on 5 September 2013.

1. Introduction

A theory of multi-norms based on a normed space E was first introduced by Dales and Polyakov in [7], and there have now been several papers on this topic. The present paper is a survey of the theory, somewhat expanded from the talk in Tartu.

1.1. Some history. The work on what were to become "multi-norms" was begun in 2003, when Maxim Polyakov arrived from Moscow as a Royal Society Fellow at my then university of Leeds; Maxim had been a student of Alexander Helemskii at Moscow State University. We decided to attack together the question when various Banach left modules over group algebras were respectively projective, injective, or flat. Our work appeared in [6]. Most questions were resolved in [6], but one in particular was left open, and I first explain this.

Let G be a locally compact group. Then the group algebra $(L^1(G), \star)$ is a Banach algebra for the convolution product, and the Banach space $L^p(G)$ is a Banach left $L^1(G)$-module in a canonical way for each $p \geq 1$. The most important case is that in which $p = 2$, and so $L^p(G)$ is the Hilbert space $L^2(G)$; of course the theory of representations of $L^1(G)$ as an algebra of bounded linear operators on $L^2(G)$, which is equivalent to the theory of $L^2(G)$ as a Banach left $L^1(G)$-module, is of fundamental importance in very many areas of harmonic analysis and operator theory. The following is a consequence of the famous theory of Barry Johnson in [16].

Received February 13, 2014.
2010 *Mathematics Subject Classification.* 46B03.
Key words and phrases. Multi-norms, tensor norms, absolutely summing operators, Banach lattices, multi-bounded operators, orthogonal decompositions.
http://dx.doi.org/10.12097/ACUTM.2014.18.15

Theorem 1.1. *Suppose that G is an amenable locally compact group and that $1 < p < \infty$. Then $L^p(G)$ is an injective Banach left $L^1(G)$-module.*

Both Barry Johnson and Alexander Helemskii had asked whether the converse of this theorem holds. Our attempt in [6] concentrated on the case where G is a discrete group, and we proved the following in [6, Theorem 5.12].

Theorem 1.2. *Let G be a group, and take p with $1 < p < \infty$. Suppose that $\ell^p(G)$ is an injective Banach left $\ell^1(G)$-module. Then the group G is pseudo-amenable.*

Here we say that a group G is *pseudo-amenable* if, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, for each $n \in \mathbb{N}$ with $n \geq n_0$ and each finite subset F of G with $|F| = n$, there is a non-empty, finite subset S of G such that $|SF| < \varepsilon n\,|S|$.

This should be contrasted with the famous *Følner condition*, which characterizes when a group G is amenable: for each $\varepsilon > 0$ and each finite subset F of G, there is a non-empty, finite subset S of G such that $|Sx\Delta Sy| < \varepsilon\,|S|$ for each $x, y \in F$. Here Δ denotes the symmetric difference of two sets.

It is easy to see that every amenable group is pseudo-amenable, but we were unable to prove the converse; this still seems to be a very interesting and challenging question in the combinatorical theory of groups. A counter-example will not be easy to find because a pseudo-amenable group cannot contain $\mathbb{F}_2$, the free group on two generators, and there are not many groups that are not amenable and do not contain $\mathbb{F}_2$.

Maxim returned to Leeds in 2005 as a Marie Curie Fellow, and we decided to try again to resolve the above problem. We discovered that one could reformulate it and several others in terms of what we called “multi-norms”, and this was the seed from which the theory of multi-norms developed. Tragically, Maxim Polyakov died in Moscow in January 2006, and the work [7] that subsequently emerged was prepared in later years. Happily, the idea that this theory would resolve the above question was correct, and the following theorem is proved in [8].

We need the notions of a (p, q)-multi-norm and a multi-bounded set; these will be defined below.

Definition 1.3. Let G be a locally compact group, and take p, q such that $1 \leq p \leq q < \infty$. A continuous linear functional Λ on $L^\infty(G)$ is *left (p, q)-multi-invariant* if the set $\{s \cdot \Lambda : s \in G\}$ is multi-bounded with respect to the (p, q)-multi-norm. The group G is *left (p, q)-amenable* if there exists a left (p, q)-multi-invariant mean on $L^\infty(G)$.

The following theorem combines [8, Theorems 8.4 and 9.6].

Theorem 1.4. *Let G be a locally compact group.*

(i) *Take p, q with $1 \leq p \leq q < \infty$. Then G is amenable if and only if G is left (p, q)-amenable.*

(ii) *Take $p > 1$. Then $L^p(G)$ is injective as a Banach left $L^1(G)$-module if and only if G is amenable.*

1.2. Basic definitions. The theory of multi-norms developed a life of its own, and it seems to have applications in quite a few different arenas. The fundamental idea is that we start with a normed space $(E, \|\cdot\|)$ and define a sequence $(\|\cdot\|_n)$ of norms, where $\|\cdot\|_n$ is a norm on the Cartesian product E^n for $n \in \mathbb{N}$. Of course certain axioms must be satisfied. This scenario is reminiscent of the theory of *operator spaces*, which is now very fashionable, but it has significant differences.

Here are the basic definitions. We write $\mathbb{N}$ for the set of natural numbers, and set $\mathbb{N}_n = \{1, \ldots, n\}$ for $n \in \mathbb{N}$; the collection of permutations of the set $\mathbb{N}_n$ is denoted by $\mathfrak{S}_n$.

Definition 1.5. Let $(E, \|\cdot\|)$ be a complex normed space. A *multi-norm* on the family $\{E^n : n \in \mathbb{N}\}$ is a sequence $(\|\cdot\|_n : n \in \mathbb{N})$ such that $\|\cdot\|_n$ is a norm on E^n for each $n \in \mathbb{N}$, such that $\|x\|_1 = \|x\|$ for each $x \in E$, and such that the following Axioms (A1)–(A4) are satisfied for each $n \in \mathbb{N}$ and $\boldsymbol{x} = (x_1, \ldots, x_n) \in E^n$:

(A1) $\left\|(x_{\sigma(1)}, \ldots, x_{\sigma(n)})\right\|_n = \|\boldsymbol{x}\|_n \quad (\sigma \in \mathfrak{S}_n)$;

(A2) $\|(\alpha_1 x_1, \ldots, \alpha_n x_n)\|_n \leq (\max_{i \in \mathbb{N}_n} |\alpha_i|) \|\boldsymbol{x}\|_n \quad (\alpha_1, \ldots, \alpha_n \in \mathbb{C})$;

(A3) $\|(x_1, \ldots, x_n, 0)\|_{n+1} = \|\boldsymbol{x}\|_n$;

(A4) $\|(x_1, \ldots, x_{n-1}, x_n, x_n)\|_{n+1} = \|\boldsymbol{x}\|_n$.

In this case, $(E^n, \|\cdot\|_n) = ((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ is a *multi-normed space.*

We shall sometimes say that $(\|\cdot\|_n)$ is a multi-norm *based on E*.

In the case where $(E, \|\cdot\|)$ is a Banach space, each space $(E^n, \|\cdot\|_n)$ is a Banach space, and $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ is termed a *multi-Banach space.*

In fact, Axiom (A3) is a consequence of Axioms (A1), (A2), and (A4) [7, Proposition 2.7].

To obtain a *dual multi-norm* and a *dual multi-Banach space*, we replace Axiom (A4) with a variant, (B4):

(B4) $\|(x_1, \ldots, x_n, x_n)\|_{n+1} = \|(x_1, \ldots, x_{n-1}, 2x_n)\|_n$.

In this case, $(E^n, \|\cdot\|_n) = ((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ is a *dual multi-normed space.*

The dual space to a normed space E is denoted by E', and the closed unit ball of E is $E_{[1]}$.

Let $\|\cdot\|_n$ be a norm on E^n. Then $\|\cdot\|_n'$ is the dual norm on $(E^n, \|\cdot\|_n)'$ when this latter space is identified with $(E')^n$. The *dual* of $(E^n, \|\cdot\|_n)$ is $((E')^n, \|\cdot\|_n')$. The dual of a multi-normed space is a dual multi-Banach space, and the dual of a dual multi-normed space is a multi-Banach space. Thus the second dual of a multi-normed space based on E is a multi-normed space based on the second dual space E''. Here we have a clear difference from the theory of operator spaces; in the latter theory the dual of an operator space is an operator space.

One can ask: "What are multi-norms good for?". Here are some answers.

(1) Solving some specific questions – for example, characterizing when some modules over group algebras are injective, as above; see [8].
(2) Understanding the geometry of Banach spaces that goes beyond the shape of the unit ball.
(3) Throwing some light on absolutely summing operators.
(4) Giving a theory [7, Chapter 6] of "multi-bounded linear operators" between Banach spaces; this gives a class of bounded linear operators that subsumes various known classes, and sometimes gives new classes.
(5) Giving results about Banach lattices [7, § 6.4].
(6) Giving a theory of decompositions [7, Chapter 7] of Banach spaces, generalizing known theories.
(7) Giving a theory that "is closed in the category".

We can only glance at some of these aspects in this review.

1.3. First examples – the maximum and minimum multi-norm. Let E be a normed space, and let $(E^n, \|\cdot\|_n)$ be either a multi-normed space or a dual multi-normed space based on E. Then it is a little exercise to see that

$$\max \|x_i\| \le \|(x_1, \dots, x_n)\|_n \le \sum_{i=1}^{n} \|x_i\| \tag{1}$$

for all $x_1, \dots, x_n \in E$ and $n \in \mathbb{N}$.

It follows that there are a *maximum multi-norm* and *minimum multi-norm* based on E; they are denoted by $(\|\cdot\|_n^{\max} : n \in \mathbb{N})$ and $(\|\cdot\|_n^{\min} : n \in \mathbb{N})$, respectively, and they are defined by the property that

$$\|\boldsymbol{x}\|_n^{\min} \le \|\boldsymbol{x}\|_n \le \|\boldsymbol{x}\|_n^{\max} \quad (\boldsymbol{x} \in E^n,\ n \in \mathbb{N})$$

for every multi-norm $(\|\cdot\|_n : n \in \mathbb{N})$ based on E. The formula for $\|\cdot\|_n^{\min}$ is:

$$\|\boldsymbol{x}\|_n^{\min} = \max_{i \in \mathbb{N}_n} \|x_i\| \quad (\boldsymbol{x} = (x_1, \dots, x_n) \in E^n,\ n \in \mathbb{N}).$$

It easy to see that there is a maximum multi-norm based on E, but it is not so easy to obtain a formula for this multi-norm; one might suspect that the right-hand side of (1) would be the answer, but this gives not a multi-norm but a dual multi-norm.

In fact, the dual of $\|\cdot\|_n^{\max}$ is the weak 1-summing norm $\mu_{1,n}$ (see below) [7, Theorem 3.33], and hence

$$\|\boldsymbol{x}\|_n^{\max} = \sup\left\{\left|\sum_{j=1}^{n}\langle x_j,\, \lambda_j\rangle\right| : \mu_{1,n}(\boldsymbol{\lambda}) \leq 1\right\}, \tag{2}$$

for all $\boldsymbol{x} = (x_1, \ldots, x_n) \in E^n$ and $n \in \mathbb{N}$, where the supremum is taken over all $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in (E')^n$.

1.4. Matrix descriptions of multi-norms. Let $\mathbb{M}_{m,n}$ denote the linear space of $m \times n$ matrices over the complex numbers, $\mathbb{C}$. We may give $\mathbb{M}_{m,n}$ a norm by identifying it with the Banach space $\mathcal{B}(\ell_n^\infty, \ell_m^\infty)$ of bounded linear operators from ℓ_n^∞ to ℓ_m^∞, where $\ell_n^\infty = (\ell_n^\infty, \|\cdot\|_\infty)$ denotes the space $\mathbb{C}^n$ with

$$\|(\alpha_1, \ldots, \alpha_n)\|_\infty = \max\{|\alpha_1|, \ldots, |\alpha_n|\} \quad (\alpha_1, \ldots, \alpha_n \in \mathbb{C}).$$

Let E be a normed space. Then $\mathbb{M}_{m,n}$ acts from E^n to E^m in the obvious way. Consider a sequence $(\|\cdot\|_n)$ such that each $\|\cdot\|_n$ is a norm on E^n and such that $\|x\|_1 = \|x\|$ for each $x \in E$.

Theorem 1.6 ([7], Theorem 2.35). *This sequence of norms is a multi-norm if and only if*

$$\|a \cdot \boldsymbol{x}\|_m \leq \|a : \ell_n^\infty \to \ell_m^\infty\| \, \|\boldsymbol{x}\|_n$$

for all $m, n \in \mathbb{N}$, $a \in \mathbb{M}_{m,n}$, and $\boldsymbol{x} \in E^n$.

We could calculate $\|a\|$ for $a \in \mathbb{M}_{m,n}$ in different ways. For example, we could identify $\mathbb{M}_{m,n}$ with the Banach space $\mathcal{B}(\ell_n^p, \ell_m^q)$ for other values of p and q in $[1, \infty]$. The case where $p = q = 1$ gives a dual multi-norm. Cases of more general p and q seem to be rather messy.

1.5. Tensor-norm characterizations. We now explain how multi-norms correspond to certain tensor norms, and thus relate the theory of multi-norms to that of tensor products of Banach spaces. In fact, in retrospect, one can see that one could have started with norms on tensor products and avoided multi-norm theory, but this would seem to be a loss.

We describe briefly the connection; details are given in [8, § 3]. We write δ_i for the sequence $(\delta_{i,j} : j \in \mathbb{N})$ for $i \in \mathbb{N}$; c_0 is the Banach space of all null sequences.

Definition 1.7. Let E be a normed space. Then a norm $\|\cdot\|$ on $c_0 \otimes E$ is a *c_0-norm* if $\|\delta_1 \otimes x\| = \|x\|$ for each $x \in E$ and if the linear operator $T \otimes I_E$ is bounded on $(c_0 \otimes E, \|\cdot\|)$, with norm at most $\|T\|$, for each compact operator T on E.

In the above definition, we used compact operators T on E, but we could replace "compact" by "bounded" and obtain an equivalent definition. The c_0-norms that we obtain are "reasonable cross-norms", in the sense of [25, § 6.1].

Suppose that $\|\cdot\|$ is a c_0-norm on $c_0 \otimes E$, and set

$$\|(x_1, \dots, x_n)\|_n = \sum_{i=1}^{n} \delta_i \otimes x_i \quad (x_1, \dots, x_n \in E,\ n \in \mathbb{N})\,.$$

Then $(\|\cdot\|_n : n \in \mathbb{N})$ is a multi-norm based on E.

Let E be a Banach space. Then a norm $\|\cdot\|$ on $c_0 \otimes E$ satisfies *condition* (P) (due to Pisier) of [22, § 2, p. 12] if

$$\|(T \otimes I_E)(z)\| \leq \|T\|\,\|z\| \quad (z \in c_0 \otimes E,\, T \in \mathcal{B}(c_0))\,.$$

Such norms are exactly the c_0-norms of Definition 1.7, and so the definition of a multi-normed space corresponds to the theory in [22] of norms on $c_0 \otimes E$ satisfying property (P).

A more general and detailed version of the following theorem is given as [8, Theorem 3.4 and Corollary 3.7].

Theorem 1.8. *Let E be a normed space. Then the above construction defines a bijection from the family of c_0-norms on $c_0 \otimes E$ to the family of multi-norms based on E. The maximum and minimum multi-norm structures based on E correspond to the projective tensor norm $\|\cdot\|_\pi$ and the injective tensor norm $\|\cdot\|_\varepsilon$ on $c_0 \otimes E$, respectively.*

There is an analogous construction that gives a bijection from the family of "ℓ^1-norms on $\ell^1 \otimes E$" to the family of dual multi-norms based on E; see [8, Theorem 4.3].

2. Multi-norms and absolutely summing operators

There is a close connection between the theory of multi-norms and that of absolutely summing operators that we first describe. The latter theory has been studied by many authors; see the fine texts [11, 12, 15, 20, 25], for example.

2.1. The weak p-summing norm. We recall the definition of the weak p-summing norms on a normed space; the following standard definition was given in [7, Definition 4.1.1] and [9, § 2.3]. For further discussion, see [11, 12, 15].

Let E be a normed space, and take $p \in [1, \infty)$ and $n \in \mathbb{N}$. Following the notation of [7, 8, 15], we define $\mu_{p,n}(\boldsymbol{x})$ for $\boldsymbol{x} = (x_1, \dots, x_n) \in E^{\,n}$ by

$$\mu_{p,n}(\boldsymbol{x}) = \sup\left\{\left(\sum_{i=1}^{n} |\langle x_i,\, \lambda\rangle|^p\right)^{1/p} : \lambda \in E'_{[1]}\right\}.$$

Then $\mu_{p,n}$ is the *weak p-summing norm* (at dimension n). Now equation (2) makes sense.

Note that, for all $p \in [1, \infty)$, $n \in \mathbb{N}$, and $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$, we have

$$\mu_{p,n}(\boldsymbol{x}) = \sup\left\{\left\|\sum_{j=1}^{n} \zeta_j x_j\right\| : \zeta_1, \dots, \zeta_n \in \mathbb{C},\ \sum_{j=1}^{n} |\zeta_j|^{p'} \leq 1\right\}, \tag{3}$$

where p' is the conjugate index to p.

Let E be a normed space. Take $n \in \mathbb{N}$ and $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$, and define

$$T_{\boldsymbol{x}} : (\zeta_1, \dots, \zeta_n) \mapsto \sum_{j=1}^{n} \zeta_j x_j\,, \quad \mathbb{C}^n \to E\,.$$

It follows from (3) that

$$\mu_{p,n}(\boldsymbol{x}) = \left\|T_{\boldsymbol{x}} : \ell_n^{p'} \to E\right\|; \tag{4}$$

the map $\boldsymbol{x} \mapsto T_{\boldsymbol{x}}$, $(E^n, \mu_{p,n}) \to \mathcal{B}(\ell_n^{p'}, E)$, is an isometric linear isomorphism.

2.2. (q, p)-summing operators. Let E and F be Banach spaces, and suppose that $1 \leq p \leq q < \infty$. We recall that an operator $T \in \mathcal{B}(E, F)$ is (q, p)-*summing* if there exists a constant C such that

$$\left(\sum_{i=1}^{n} \|Tx_i\|^q\right)^{1/q} \leq C\, \mu_{p,n}(x_1, \dots, x_n) \quad (x_1, \dots, x_n \in E,\ n \in \mathbb{N})\,.$$

The smallest such constant C is denoted by $\pi_{q,p}(T)$. The set of these (q, p)-summing operators is denoted by $\Pi_{q,p}(E, F)$; it is a linear subspace of $\mathcal{B}(E, F)$, and $(\Pi_{q,p}(E, F), \pi_{q,p})$ is a Banach space; we write $(\Pi_p(E, F), \pi_p)$ for $(\Pi_{p,p}(E, F), \pi_{p,p})$.

There is a huge theory of these (q, p)-summing operators. At this point I should like to remember **Joram Lindenstrauss** (1936–2012) and **Aleksander Pełczyński** (1932–2012), two founders of the theory of summing operators, whom we have recently lost.

42

The space $\Pi_{q,p}(E,F)$ is a component of an *operator ideal*: see the tremendous theory created by **Professor Albrecht Pietsch**, especially [24]. It is an honour that Professor Pietsch was in the audience for this talk in Tartu.

2.3. The (p,q)-multi-norm. Again in retrospect, it seems that the most important specific multi-norm is the "(p,q)-multi-norm".

The following definition was first given in [7, § 4.1].

Definition 2.1. Let E be a normed space, and take p, q such that $1 \leq p \leq q < \infty$. For each $n \in \mathbb{N}$ and $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$, define

$$\|\boldsymbol{x}\|_n^{(p,q)} = \sup \left\{ \left(\sum_{j=1}^n |\langle x_j, \lambda_j \rangle|^q \right)^{1/q} : \mu_{p,n}(\boldsymbol{\lambda}) \leq 1 \right\},$$

where the supremum is taken over all $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in (E')^n$.

As noted in [7, Theorem 4.1], $(\|\cdot\|_n^{(p,q)} : n \in \mathbb{N})$ is a multi-norm based on E; it is called the *(p,q)-multi-norm*.

Take $n \in \mathbb{N}$ and $\boldsymbol{x} \in E^n$. Then it is clear that $\|\boldsymbol{x}\|_n^{(p,q_1)} \leq \|\boldsymbol{x}\|_n^{(p,q_2)}$ when $1 \leq p \leq q_2 \leq q_1$ and that $\|\boldsymbol{x}\|_n^{(p_1,q)} \leq \|\boldsymbol{x}\|_n^{(p_2,q)}$ when $1 \leq p_1 \leq p_2 \leq q$.

We remark that the $(1,1)$-multi-norm $(\|\cdot\|_n^{(1,1)})$ is exactly the maximum multi-norm $(\|\cdot\|_n^{\max})$, and we note the useful fact that the (p,q)-multi-norm over E'', when restricted to E, is the (p,q)-multi-norm over E; this latter fact is a nice application of the *principle of local reflexivity*.

A key result from [9, Theorem 2.6] relates (p,q)-multi-norms to the theory of absolutely summing operators.

Theorem 2.2. *Let E be a normed space, and let $1 \leq p \leq q < \infty$. Then the (p,q)-multi-norm induces the norm on $c_0 \otimes E$ given by embedding $c_0 \otimes E$ into $\Pi_{q,p}(E', c_0)$.*

Indeed, for $n \in \mathbb{N}$ and $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$, we have

$$\|\boldsymbol{x}\|_n^{(p,q)} = \pi_{q,p}(T'_{\boldsymbol{x}} : E' \to c_0). \tag{5}$$

In the special case in which $q = p$, we can identify the norm on $c_0 \otimes E$ induced by the (p,p)-multi-norm: it is exactly the *Chevet–Saphar cross-norm*, called d_p, in [12, Chapter 12] and [25, p. 135].

We should like to identify the dual space of $(E^n, \|\cdot\|_n^{(p,q)})$ for any p, q with $1 \leq p \leq q < \infty$: an abstract description of this space is given in [7, § 4.1.4], but we lack a good concrete description of this dual space.

3. Further examples of multi-norms

3.1. The standard t-multi-norm. Let (Ω, μ) be a measure space, take $r \geq 1$, and consider the Banach space $L^r(\Omega, \mu)$, with the usual L^r-norm, which is denoted by $\|\cdot\|_{L^r}$ or just $\|\cdot\|$. In particular, we concentrate on the Banach spaces ℓ^r and $L^r[0,1]$. It is standard [1, Proposition 6.4.1] that, in the case where $L^r(\Omega)$ is an infinite-dimensional space, we can regard ℓ^r as a closed, 1-complemented subspace of $L^r(\Omega)$.

Now take r, t with $1 \leq r \leq t < \infty$. For each family $\mathbf{X} = \{X_1, \dots, X_n\}$ of pairwise-disjoint measurable subsets of Ω such that $X_1 \cup \cdots \cup X_n = \Omega$, we set

$$r_{\mathbf{X}}((f_1, \dots, f_n)) = \left(\|P_{X_1} f_1\|^t + \cdots + \|P_{X_n} f_n\|^t\right)^{1/t},$$

where $P_X : L^r(\Omega) \to L^r(X)$ is the natural projection. Finally,

$$\|(f_1, \dots, f_n)\|_n^{[t]} = \sup_{\mathbf{X}} r_{\mathbf{X}}((f_1, \dots, f_n)).$$

We see that $(\|\cdot\|_n^{[t]})$ is a multi-norm. This is the *standard t-multi-norm* (based on $L^r(\Omega)$); see [7, § 4.2]. In the special case in which $t = r$, we have the following representation theorem.

Theorem 3.1 ([8], Theorem 6.1)**.** *Let Ω be a measure space, and suppose that $1 \leq r < \infty$. Then the standard r-multi-norm based on $L^r(\Omega)$ induces the c_0-norm on $c_0 \otimes L^r(\Omega)$ which comes from identifying $c_0 \otimes L^r(\Omega)$ with a subspace of the vector-valued Banach space $L^r(\Omega, c_0)$.*

3.2. The Hilbert multi-norm. Let H be a Hilbert space. For $n \in \mathbb{N}$ and each family $\mathbf{H} = \{H_1, \dots, H_n\}$ of closed subspaces of H such that $H = H_1 \perp \cdots \perp H_n$, we set

$$r_{\mathbf{H}}((x_1, \dots, x_n)) = \left(\|P_1 x_1\|^2 + \cdots + \|P_n x_n\|^2\right)^{1/2} \quad (x_1, \dots, x_n \in H),$$

where $P_i : H \to H_i$ for $i = 1, \dots, n$ is the projection, and then set

$$\|(x_1, \dots, x_n)\|_n^H = \sup_{\mathbf{H}} r_{\mathbf{H}}((x_1, \dots, x_n)).$$

Then we obtain a multi-norm $(\|\cdot\|_n^H : n \in \mathbb{N})$ based on H. It is the *Hilbert multi-norm*.

However the Hilbert multi-norm is not really a new multi-norm because we have the following theorem [7, Theorem 4.19].

Theorem 3.2. *Let H be an infinite-dimensional Hilbert space. Then*

$$\|\boldsymbol{x}\|_n^H = \|\boldsymbol{x}\|_n^{(2,2)} \quad (\boldsymbol{x} \in H^n,\, n \in \mathbb{N}).$$

3.3. Banach lattice multi-norms. Let $(E, \|\cdot\|)$ be a complex Banach lattice. Thus E is the complexification of a real Banach lattice $E_{\mathbb{R}}$. For the theory of Banach lattices, see [2, 21, 23], for example; there is a summary of the properties of Banach lattices that seem to be relevant for multi-norm theory in [7, § 1.3].

For example, the Banach spaces $L^r(\Omega)$ (for $r \geq 1$), $L^\infty(\Omega)$, or $C(K)$ (for a compact space K), with the usual norms and the obvious lattice operations, are all (complex) Banach lattices.

Definition 3.3 ([7], Definition 4.41)**.** Let $(E, \|\cdot\|)$ be a Banach lattice. For $n \in \mathbb{N}$ and $x_1, \dots, x_n \in E$, set

$$\|(x_1, \dots, x_n)\|_n^L = \| \, |x_1| \vee \cdots \vee |x_n| \, \|$$

and

$$\|(x_1, \dots, x_n)\|_n^{DL} = \| \, |x_1| + \cdots + |x_n| \, \| \, .$$

Then $(E^n, \|\cdot\|_n^L)$ is a multi-Banach space; $(\|\cdot\|_n^L)$ is the *Banach lattice multi-norm.* Also $(E^n, \|\cdot\|_n^{DL})$ is a dual multi-Banach space; $(\|\cdot\|_n^{DL})$ is the *dual Banach lattice multi-norm.*

For each $n \in \mathbb{N}$, the dual of the norm $\|\cdot\|_n^L$ on E^n is the norm $\|\cdot\|_n^{DL}$ on $(E')^n$, and the dual of the norm $\|\cdot\|_n^{DL}$ on E^n is the norm $\|\cdot\|_n^L$ on $(E')^n$. It follows easily that the second dual of the lattice multi-norm based on E is exactly the lattice multi-norm based on E''.

For example, it follows from Theorem 3.1 that, for each $n \in \mathbb{N}$ and $f_1, \dots, f_n \in L^r(\Omega)$, we have

$$\|(f_1, \dots, f_n)\|_n^{[r]} = \| \, |f_1| \vee \cdots \vee |f_n| \, \| = \|(f_1, \dots, f_n)\|_n^L \, ,$$

and so the standard r-multi-norm coincides with the Banach lattice multi-norm on $L^r(\Omega)$.

When $r = 1$, it is well-known that $L^1(\Omega) \widehat{\otimes} E = L^1(\Omega, E)$ for any Banach space E, and so the standard 1-multi-norm on $L^1(\Omega)$ is the maximum multi-norm. Thus, for each $n \in \mathbb{N}$ and $f_1, \dots, f_n \in L^1(\Omega)$, we have

$$\|(f_1, \dots, f_n)\|_n^{\max} = \| \, |f_1| \vee \cdots \vee |f_n| \, \| = \|(f_1, \dots, f_n)\|_n^{[1]} = \|(f_1, \dots, f_n)\|_n^L \, .$$

3.4. The $[p, q]$-concave multi-norm on Banach lattices. In [3], we introduced a new class of multi-norms on general Banach lattices, and related some of these to standard t-multi-norms: these multi-norms are of interest in their own right, and also helped us to settle some questions about the equivalence of the (p, q)-multi-norms and the standard t-multi-norms on ℓ^r, as we discuss below.

Let $(E, \|\cdot\|)$ be a (complex) Banach lattice. We write $|x|$ for the modulus of an element $x \in E$. Take $n \in \mathbb{N}$ and an n-tuple $(x_1, \dots, x_n)$ in E^n. Recall that, for each $p \geq 1$, we can define the element $\left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p} \in E$ by the Krivine calculus; see [7, § 1.3.1] and [21, II.1.d].

Definition 3.4. Let $(E, \|\cdot\|)$ be a Banach lattice, take $p, q \geq 1$, and take $n \in \mathbb{N}$. For each $\boldsymbol{x} \in E^n$, define

$$\|\boldsymbol{x}\|_n^{[p,q]} = \sup \left\{ \left(\sum_{j=1}^{n} |\langle x_j,\, \lambda_j \rangle|^q \right)^{1/q} : \left\| \left(\sum_{j=1}^{n} |\lambda_j|^p \right)^{1/p} \right\| \leq 1 \right\},$$

where the supremum is taken over $\lambda_1, \ldots, \lambda_n \in E'$.

As noted in [3, Theorem 3.4], $(\|\cdot\|_n^{[p,q]} : n \in \mathbb{N})$ is a multi-norm based on E whenever $1 \leq p \leq q < \infty$; it is called the (p,q)-*concave multi-norm* based on E. It is easy to see that $\|\boldsymbol{x}\|_n^{[p,q]} \leq \|\boldsymbol{x}\|_n^{(p,q)}$ $(\boldsymbol{x} \in E^n,\ n \in \mathbb{N})$.

These multi-norms are closely related to the (q,p)-*concave* operators on E that are described on [12, p. 330].

We have the following key relationship between a standard t-multi-norm and certain concave multi-norms.

Theorem 3.5 ([3], Theorem 3.9)**.** *Suppose that* $1 \leq r \leq t < \infty$, *and define* v *by the formula* $1/v = 1/r - 1/t$. *Then the standard* t*-multi-norm is equal to the* $[1, v']$*-concave multi-norm on* ℓ^r.

3.5. A representation theorem. In all mathematical theories we would like to find a "universal representation theorem" that gives a rather concrete representation for all objects in an axiomatically-defined class. Thus, for example, the great Gel'fand–Naimark theorem shows how to represent each axiomatically-defined C^*-algebra as a norm-closed, $*$-closed subalgebra of the C^*-algebra $\mathcal{B}(H)$ for some Hilbert space H [5, Theorem 3.2.29]. Again, let E be a normed space. An *operator space* based on E is a sequence $(\|\cdot\|_n)$ of "matricial norms", where $\|\cdot\|_n$ is defined on $\mathbb{M}_n(E)$, the space of $n \times n$-matrices with terms in E, such that the sequence satisfies certain "Ruan's axioms"; see [4, 13, 14], for example. The concrete representation of an abstract operator space is given by *Ruan's theorem*, which represents each such system as a closed subspace of $\mathcal{B}(H)$ for some Hilbert space H, the matricial norms being recovered in a canonical way. These representation theorems are enormously useful.

There is a "universal representation theorem" for multi-Banach spaces and for dual multi-Banach spaces; for example, it represents an arbitrary multi-norm as closed subspace of a Banach lattice in such a way that the multi-norm is a restriction to the subspace of a Banach lattice multi-norm. However it must be confessed that, so far, this representation theorem has not proved to be very useful.

43

Clause (i) of the following theorem is basically a theorem of Pisier, as given in a thesis of a student, Marcolino Nhani [22, Théorème 2.1]; it shows that our multi-normed spaces are the "sous-espaces de treillis" of [22, Définition 3.1]. There is a simplified proof in [10]; clause (ii) is a new dual version from [10].

Theorem 3.6. (i) *Let $(E^n, \|\cdot\|_n)$ be a multi-Banach space. Then there is a Banach lattice X, a closed subspace Y of X, and an isometric isomorphism $J : E \to Y$ such that*

$$\|(Jx_1, \dots, Jx_n)\|_n^L = \|(x_1, \dots, x_n)\|_n \quad (x_1, \dots, x_n \in E,\, n \in \mathbb{N})\,.$$

(ii) *Let $(E^n, \|\cdot\|_n)$ be a dual multi-Banach space. Then there is a Banach lattice X, a closed subspace Y of X, and an isometric isomorphism $J : E \to X/Y$ such that*

$$\|(Jx_1, \dots, Jx_n)\|_n^{DL} = \|(x_1, \dots, x_n)\|_n \quad (x_1, \dots, x_n \in E,\, n \in \mathbb{N})\,.$$

3.6. An associated sequence. Let E be a normed space, and let $(\|\cdot\|_n)$ be a multi-norm based on E. We can define a *rate of growth* sequence $(\varphi_n(E))$ by setting

$$\varphi_n(E) = \sup\{\|(x_1, \dots, x_n)\|_n : x_1, \dots, x_n \in E_{[1]}\} \quad (n \in \mathbb{N})\,.$$

In particular $(\varphi_n^{\max}(E))$ is the sequence associated with the maximum multi-norm based on E.

Trivially, $1 \le \varphi_n(E) \le n$ for all $n \in \mathbb{N}$ and

$$\varphi_{m+n}(E) \le \varphi_m(E) + \varphi_n(E)$$

for all $m, n \in \mathbb{N}$.

The sequence $(\varphi_n(E))$ depends on the multi-norm that is based on E (even though the notation does not show this), but the sequence $(\varphi_n^{\max}(E))$ is intrinsic to the Banach space E; it can be shown quite easily that $\varphi_n^{\max}(E)$ is equal to

$$\sup\left\{\sum_{j=1}^{n} \|\lambda_j\| : \sum_{j=1}^{n} |\langle x,\, \lambda_j\rangle| \le 1 \ \ (x \in E_{[1]})\right\}$$

for each $n \in \mathbb{N}$, where $\lambda_1, \dots, \lambda_n \in E'$.

The following is a natural example.

Theorem 3.7. (i) *For each $p \in [1, 2]$, we have $\varphi_n^{\max}(\ell^p) = n^{1/p}$ $(n \in \mathbb{N})$.*

(ii) *For each $p \in [2, \infty]$, there is a constant C_p such that*

$$\sqrt{n} \le \varphi_n^{\max}(\ell^p) \le C_p\sqrt{n} \quad (n \in \mathbb{N})\,.$$

Further, $C_2 = 1$ and $C_\infty = \sqrt{2}$.

For this and further, related examples, see [7, § 3.6].

In fact, the bound $\sqrt{n}$ of the above example is a general truth; the following theorem follows from Dvoretzky's famous theorem on almost spherical sections.

Theorem 3.8 ([7], Theorem 3.58)**.** *Let E be an infinite-dimensional normed space. Then*

$$\sqrt{n} \leq \varphi_n^{\max}(E) \leq n \quad (n \in \mathbb{N}) .$$

The rate of growth constants $\varphi_n^{(p,q)}(E)$ associated with a (p,q)-multi-norm are equal to known constants.

Theorem 3.9. *Let E be a Banach space, and suppose that $1 \leq p \leq q < \infty$ and $n \in \mathbb{N}$. Then $\varphi_n^{(p,q)}(E) = \pi_{q,p}^{(n)}(E')$.*

4. Multi-bounded operators

Let E and F be Banach spaces. Then of course the "natural morphisms" form the Banach space $\mathcal{B}(E,F)$ of all bounded, equivalently, continuous, operators from E to F. Now suppose that $(E^n, \|\cdot\|_n)$ and $(F^n, \|\cdot\|_n)$ are multi-Banach spaces based on E and F, respectively. What are the "natural morphisms"? Do they form a natural multi-Banach space?

4.1. Definitions. Of course, the "natural morphisms" will be the "multi-bounded operators"; we first define these.

Definition 4.1 ([7], §§ 5.2, 6.1)**.** Let $(E^n, \|\cdot\|_n)$ be a multi-normed space. A subset B of E is *multi-bounded* if

$$c_B := \sup_{n\in\mathbb{N}} \sup \{\|(x_1, \dots, x_n)\|_n : x_1, \dots, x_n \in B\} < \infty .$$

A sequence (x_n) in E is *multi-null* if, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\sup\{\|(x_{n+1}, \dots, x_{n+k})\| : k \in \mathbb{N}\} < \varepsilon \quad (n \geq n_0) .$$

The concept of "multi-null sequence" captures some well-known properties in our examples. For example, let E be a Banach lattice, and consider the Banach lattice multi-norm based on E. Then a multi-null sequence (x_n) is "order-null", in the sense that there is a decreasing sequence (u_n) in E^+ such that $|x_n| \leq u_n$ $(n \in \mathbb{N})$ and $\inf_{n\in\mathbb{N}} u_n = 0$ in E^+ [7, Theorem 5.14]; for many Banach lattices, including the examples $L^r(\Omega)$, the converse is true [7, Theorem 5.15]. For Banach lattices E which are "monotonically bounded", in the sense that every increasing net in $E^+_{[1]}$ is bounded above, a subset B is multi-bounded (with respect to the Banach lattice multi-norm) if and only if it is order-bounded, in the sense that there exists $y \in E^+$ such that $|x| \leq y \ (x \in B)$.

Definition 4.2 ([7], §§ 6.1.3, 6.1.4)**.** Let $(E^n, \|\cdot\|_n)$ and $(F^n, \|\cdot\|_n)$ be two multi-normed spaces. An operator $T \in \mathcal{B}(E,F)$ is *multi-bounded* if $T(B)$ is multi-bounded in F whenever B is multi-bounded in E, and T *multi-continuous* if (Tx_n) is a multi-null sequence in F whenever (x_n) is a multi-null sequence in E.

As one would expect, a linear map from E to F is multi-continuous if and only if it is multi-bounded [7, Theorem 6.14].

The set of multi-bounded operators from E to F is a linear subspace $\mathcal{M}(E,F)$ of $\mathcal{B}(E,F)$.

Definition 4.3 ([7], § 6.1.3)**.** Let $(E^n, \|\cdot\|_n)$ and $(F^n, \|\cdot\|_n)$ be two multi-normed spaces, and let $T \in \mathcal{M}(E,F)$. Then

$$\|T\|_{mb} = \sup\{c_{T(B)} : B \subset E,\ c_B \leq 1\}\,.$$

We must check that $\|T\|_{mb} < \infty$ for each $T \in \mathcal{M}(E,F)$. Explicitly, we have

$$\|T\|_{mb} = \sup_n \ \sup\left\{\frac{\|(Tx_1,\dots,Tx_n)\|_n}{\|(x_1,\dots,x_n)\|_n} : (x_1,\dots,x_n) \neq 0\right\} < \infty\,, \tag{6}$$

and so T is multi-bounded if and only if $\|T\|_{mb} = \sup_{n\in\mathbb{N}} \left\|T^{(n)}\right\| < \infty$, where $T^{(n)}$ is the n^{th}-amplification of T. In this case, we have

$$\|(Tx_1,\dots,Tx_n)\|_n \leq \|T\|_{mb} \|(x_1,\dots,x_n)\|_n \quad (x_1,\dots,x_n \in E,\ n \in \mathbb{N})\,.$$

Let $\mathcal{N}(E,F)$ denote the Banach space of nuclear operators from E to F, taken with the nuclear norm ν.

Theorem 4.4 ([7], Theorem 6.15)**.** *Let $(E^n, \|\cdot\|_n)$ and $(F^n, \|\cdot\|_n)$ be two multi-normed spaces, and suppose that F is a Banach space. Then*

$$(\mathcal{M}(E,F), \|\cdot\|_{mb})$$

is a Banach space. Further, $\mathcal{N}(E,F) \subset \mathcal{M}(E,F)$, and $\|T\|_{mb} \leq \nu(T)$ for each $T \in \mathcal{N}(E,F)$.

We can have $\mathcal{M}(E,F) = \mathcal{B}(E,F)$ and $\mathcal{M}(F,E) = \mathcal{N}(F,E)$ for two multi-normed spaces $(E^n, \|\cdot\|_n)$ and $(F^n, \|\cdot\|_n)$, and so there is no "multi-Banach isomorphism theorem".

We find it to be interesting to determine the space $\mathcal{M}(E,F)$ for various examples of multi-normed spaces $(E^n, \|\cdot\|_n)$ and $(F^n, \|\cdot\|_n)$. For example, suppose that E and F are Banach lattices, and we consider the lattice multi-norms based on E and F. An operator $T \in \mathcal{B}(E,F)$ is *order-bounded* if $T(B)$ is order-bounded in F whenever B is an order-bounded set in F, and T is *regular* if it is a linear combination of positive operators from E to F; see [7, § 1.3.4]. The spaces of regular and order-bounded operators are denoted by $\mathcal{B}_r(E,F)$ and $\mathcal{B}_b(E,F)$, respectively; clearly $\mathcal{B}_r(E,F) \subset \mathcal{B}_b(E,F)$, and often (but not always) these two spaces are the same.

In the case where $F = E$, the spaces $\mathcal{B}_r(E)$ and $\mathcal{B}_b(E)$ are Banach subalgebras of $\mathcal{B}(E)$; I find these examples to be interesting and surprisingly little studied; they are not mentioned in [5].

Here is a typical theorem.

Theorem 4.5 ([7], § 6.4)**.** (i) *Let E and F be Banach lattices, and consider the lattice multi-norms based on E and F. Then $\mathcal{B}_b(E,F) \subset \mathcal{M}(E,F)$; in the case where F is monotonically bounded, $\mathcal{M}(E,F) = \mathcal{B}_b(E,F)$.*

(ii) *Let $E = \ell^r$ and $F = \ell^s$, where $r, s \geq 1$, and consider the Banach lattice or, equivalently, the standard r- and s-multi-norms on E and F, respectively. Then $\mathcal{M}(E,F) = \mathcal{B}_r(E,F)$.*

More generally, one could consider the spaces $E = \ell^r$ and $F = \ell^s$ with standard t- and u-multi-norms, respectively, where $1 \leq r \leq t < \infty$ and $1 \leq s \leq u < \infty$. Is the space $\mathcal{M}(E,F)$ of interest in this case?

4.2. The multi-bounded multi-norm. Let $(E^n, \|\cdot\|_n)$ and $(F^n, \|\cdot\|_n)$ be two multi-normed spaces. As we suggested, we would wish there to be a natural multi-norm based on $\mathcal{M}(E,F)$. In fact, it is given as follows.

Definition 4.6 ([7], Definition 6.18)**.** Let $(E^n, \|\cdot\|_n)$ and $(F^n, \|\cdot\|_n)$ be two multi-normed spaces, and take $n \in \mathbb{N}$ and $T_1, \dots, T_n \in \mathcal{M}(E,F)$. Then

$$\|(T_1, \dots, T_n)\|_n^{mb} = \sup\{c_{T_1(B)\cup\cdots\cup T_n(B)} : B \subset E,\ c_B \leq 1\}\,.$$

Here is a somewhat more explicit formula for $\|(T_1, \dots, T_n)\|_n^{mb}$:

$$\|(T_1, \dots, T_n)\|_n^{mb} = \sup \|(T_i x_j : i \in \mathbb{N}_n,\, j \in \mathbb{N}_k)\|_{nk}\ ,$$

where the supremum is taken over $x_1, \dots, x_k \in E$ with $\|(x_1, \dots, x_k)\|_k \leq 1$.

The theorem that follows easily is the following.

Theorem 4.7 ([7], Theorem 6.20)**.** *Let $(E^n, \|\cdot\|_n)$ and $(F^n, \|\cdot\|_n)$ be two multi-normed spaces. Then $\|\cdot\|_n^{mb}$ is a norm on the linear space $\mathcal{M}(E,F)^n$, and*

$$((\mathcal{M}(E,F)^n, \|\cdot\|_n^{mb}) : n \in \mathbb{N})$$

is a multi-normed space with $\|T\|_1^{mb} = \|T\|_{mb}$; it is a multi-Banach space in the case where F is a Banach space.

There is a natural concept of a "multi-Banach algebra"; the obvious example is the multi-normed space $(\mathcal{M}(E,F)^n, \|\cdot\|_n^{mb})$, with the "product" defined by

$$(S_1,\ldots,S_n)\cdot(T_1,\ldots,T_n)=(S_1T_1,\ldots,S_nT_n)$$

for $S_1,\ldots,S_n,\, T_1,\ldots,T_n \in \mathcal{M}(E,F))$ There are other natural examples, for example related to group algebras. It is hoped that this idea will be investigated in the future.

5. Equivalences of multi-norms

The natural notion of the equivalence of two multi-norms was given in [7, § 2.2.4]. We would like to know when two (p,q)-multi-norms are mutually equivalent, especially on the Banach spaces of the form $L^r(\Omega)$; also, it seems that a (p,q)-multi-norm is rarely equivalent to a standard t-multi-norm, and we conjectured in [9] that this is never the case. The investigation of these questions seems to involve some quite delicate calculations in the classical theory of absolutely summing operators.

Some preliminary results on equivalences were given in [7]. The question was taken up more seriously in [9], and rather a large number of results was obtained; some of the remaining questions are resolved in [3]. However one or two aggravating points still remain open.

5.1. Definitions. We first give the basic definitions.

Definition 5.1. Let $(E,\|\cdot\|)$ be a normed space. Then $\mathcal{E}_E$ is the family of all multi-norms based on E. Suppose that $(\|\cdot\|_n^1 : n\in\mathbb{N})$ and $(\|\cdot\|_n^2 : n\in\mathbb{N})$ belong to $\mathcal{E}_E$. Then

$$(\|\cdot\|_n^1)\le(\|\cdot\|_n^2)\quad\text{if}\quad \|\boldsymbol{x}\|_n^1\le\|\boldsymbol{x}\|_n^2\quad(\boldsymbol{x}\in E^n,\, n\in\mathbb{N})\,,$$

and $(\|\cdot\|_n^2 : n\in\mathbb{N})$ *dominates* $(\|\cdot\|_n^1 : n\in\mathbb{N})$, written $(\|\cdot\|_n^1)\preccurlyeq(\|\cdot\|_n^2)$, if there is a constant $C>0$ such that

$$\|\boldsymbol{x}\|_n^1\le C\,\|\boldsymbol{x}\|_n^2\quad(\boldsymbol{x}\in E^n,\, n\in\mathbb{N})\,; \tag{7}$$

the two multi-norms are *equivalent*, written

$$(\|\cdot\|_n^1 : n\in\mathbb{N})\cong(\|\cdot\|_n^2 : n\in\mathbb{N})\quad\text{or}\quad(\|\cdot\|_n^1)\cong(\|\cdot\|_n^2)\,,$$

if each dominates the other.

Clearly equivalent multi-norms have equivalent rates of growth (via the sequences $(\varphi_n(E))$), but the converse does not hold.

5.2. The Hilbert space multi-norm. We first give a result about the Hilbert space multi-norm based on a Hilbert space H. We have noted that it was proved in [7] that the Hilbert space multi-norm is equal to the $(2,2)$-multi-norm; at that stage we guessed that it would be equivalent to the maximum multi-norm because we could not think of any larger multi-norm. This turned out to be correct; the proof in [8, §4.1] involves the "little Grothendieck theorem" and obtains the best constant of equivalence.

Theorem 5.2. *Let H be an infinite-dimensional, complex Hilbert space. Then*

$$\|\boldsymbol{x}\|_n^H = \|\boldsymbol{x}\|_n^{(2,2)} \leq \|\boldsymbol{x}\|_n^{\max} \leq \frac{2}{\sqrt{\pi}}\, \|\boldsymbol{x}\|_n^{(2,2)} \quad (\boldsymbol{x} \in H^n,\, n \in \mathbb{N})\,;$$

the constant $2/\sqrt{\pi}$ is best-possible in this inequality.

One can fix $n \in \mathbb{N}$ and try to find the best constant c_n such that

$$\|\boldsymbol{x}\|_n^{\max} \leq c_n\, \|\boldsymbol{x}\|_n^{(2,2)} \quad (\boldsymbol{x} \in H^n)\,.$$

It is not too hard to see that $c_1 = c_2 = 1$, but the calculation of c_3 seems to be very challenging; with a lot of effort, it was discovered in [8, § 4.1] that $c_3 = 1$ and that $c_4 > 1$; this seems to give some new information about the geometry of Hilbert spaces.

5.3. Equivalence of two (p,q)-multi-norms. Let E be a normed space, and take p, q with $1 \leq p \leq q < \infty$. The question of the equivalence of two (p,q)-multi-norms on E can be reduced to a question about (q,p)-summing operators from E.

Theorem 5.3 ([8], Corollary 2.9)**.** *Let E be a Banach space, and suppose that $1 \leq p_1 \leq q_1 < \infty$ and $1 \leq p_2 \leq q_2 < \infty$. Then the following are equivalent:*

(a) $(\|\cdot\|_n^{(p_1,q_1)} : n \in \mathbb{N}) \cong (\|\cdot\|_n^{(p_2,q_2)} : n \in \mathbb{N})$ *on E;*

(b) $\Pi_{q_1,p_1}(E', c_0) = \Pi_{q_2,p_2}(E', c_0)\,.$

One might expect that one could now just consult the classical literature to determine when clause (b), above, holds. However it seems that earlier theory gives only some partial results.

Look at the "triangle" $\mathcal{T} = \{(p,q) : 1 \leq p \leq q < \infty\}$. For $c \in [0,1)$, look at the curve $\mathcal{C}_c$:

$$\mathcal{C}_c = \left\{ (p,q) \in \mathcal{T} : \frac{1}{p} - \frac{1}{q} = c \right\}.$$

Take $r \in (1, \infty)$. Then the curve $\mathcal{C}_{1/r}$ meets the line $p = 1$ at the point $(1, r')$, and the union of these curves is $\mathcal{T}$.

Two points $P_1 = (p_1, q_1)$ and $P_2 = (p_2, q_2)$ in $\mathcal{T}$ are *equivalent for a normed space* E if the two corresponding multi-norms $(\|\cdot\|_n^{(p_1,q_1)})$ and $(\|\cdot\|_n^{(p_2,q_2)})$ based on E are equivalent.

Our first main question is: When are two points in $\mathcal{T}$ equivalent for ℓ^r (where $r \geq 1$)?

The following is a fairly easy consequence of the theory of absolutely summing operators, as given in [12, Theorem 10.4], for example.

Theorem 5.4. *Let E be a normed space, and suppose that (p_1, q_1) and (p_2, q_2) belong to $\mathcal{T}$. Then $(\|\cdot\|_n^{(p_2,q_2)}) \leq (\|\cdot\|_n^{(p_1,q_1)})$ whenever both $q_1 \leq q_2$ and $1/p_1 - 1/q_1 \leq 1/p_2 - 1/q_2$.*

The following calculation gives us a start. It will show non-equivalence between some (p, q)-multi-norms.

We calculate $\|(\delta_1, \dots, \delta_n)\|_n^{(p,q)}$ acting on ℓ^r (for $r \geq 1$ and $(p, q) \in \mathcal{T}$). The answer is:

$$\left\{ \begin{array}{ll} n^{1/r+1/q-1/p} & \text{when } p < r \text{ and } 1/p - 1/q \leq 1/r, \\ \\ 1 & \text{when } 1/p - 1/q > 1/r, \\ \\ n^{1/q} & \text{when } p \geq r\,. \end{array} \right\}$$

There are similar calculations involving $\|(f_1, \dots, f_n)\|_n^{(p,q)}$, where

$$f_i = \frac{1}{n^{1/r}}(\zeta^{-i}, \zeta^{-2i}, \dots, \zeta^{-ni}, 0, 0, \dots)$$

and $\zeta = \exp(2\pi\mathrm{i}/n)$.

The techniques used in the proofs of equivalence include the following. First, the *generalized Hölder's inequality* gives the next lemma; here E is any Banach space.

Lemma 5.5. *Take p, q_1, q_2 such that $1 \leq p \leq q_1 < q_2$. Then, for each $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$, the number $\|\boldsymbol{x}\|_n^{(p,q_2)}$ is equal to*

$$\sup \left\{ \|(\zeta_1 x_1, \dots, \zeta_n x_n)\|_n^{(p,q_1)} : \sum_{j=1}^{n} |\zeta_j|^u \leq 1 \right\},$$

where u satisfies $1/u = 1/q_1 - 1/q_2$.

Second, the famous *Khintchine's inequality* gives the following involving the *Rademacher functions* r_n. See [1, Theorem 6.2.3], for example.

Theorem 5.6. *For each $u > 0$, there exist constants A_u and B_u such that*

$$A_u\left(\sum_{j=1}^n |\alpha_j|^2\right)^{1/2} \le \left(\int_0^1 \left|\sum_{j=1}^n \alpha_j r_j(t)\right|^u \mathrm{d}t\right)^{1/u} \le B_u\left(\sum_{j=1}^n |\alpha_j|^2\right)^{1/2}$$

for all $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ and all $n \in \mathbb{N}$.

Third, we need of course to use a result connected with Grothendieck's constant K_G; we use the following factorization theorem of *Grothendieck* from [12, Lemma 2.23], for example.

Theorem 5.7. *Let $F = L^s(\Omega)$, where Ω is a measure space and $s \ge 1$, and let $n \in \mathbb{N}$. Take $u > s$ and $u = 2$ in the cases where $s > 2$ and $s \in [1, 2]$, respectively. Then there is a constant $K_u > 0$ such that, for each $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in F^n$ with $\mu_{1,n}(\boldsymbol{\lambda}) = 1$, there exist $\zeta_1, \ldots, \zeta_n \in \mathbb{C}$ and $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_n) \in F^n$ such that:*

(i) $\lambda_j = \zeta_j \nu_j \ (j \in \mathbb{N}_n)$;

(ii) $\sum_{j=1}^n |\zeta_j|^u \le 1$;

(iii) $\mu_{u',n}(\boldsymbol{\nu}) \le K_u$.

In the case where $s \in [1, 2]$, we can take $K_u = K_G$.

Finally, we need the identification of the spaces $\Pi_{q,p}(H)$ for a Hilbert space H in terms of various Schatten classes, essentially as given in [12]. For details, see [9, Theorem 3.15].

These preliminaries allow us to obtain a full solution of the equivalence question when $r = 1$ and when $r \ge 2$. Here is the solution when $r \ge 2$; see also Figure 1.

Theorem 5.8. *Let Ω be a measure space such that $E := L^r(\Omega)$ is an infinite-dimensional space, where $r \ge 2$. Then the triangle $\mathcal{T}$ is decomposed into the following (mutually disjoint) equivalence classes:*

(1) *the region $\mathcal{T}_{\min} := A_r = \{(p,q) \in \mathcal{T} \colon 1/p - 1/q \ge 1/2\}$;*

(2) *the curves $\mathcal{T}_c := \{(p,q) \in \mathcal{C}_c \colon 1 \le p \le 2\}$, for $c \in (0, 1/2)$;*

(3) *the line segment $\mathcal{T}_{\max} := \{(p,p) \colon 1 \le p \le 2\}$;*

(4) *the singletons $\mathcal{T}_{(p,q)} := \{(p,q)\}$ for $(p,q) \in \mathcal{T}$ with $p > 2$.*

Moreover:

(5) *there is a constant $K > 0$ such that*

$$\|\cdot\|_n^{\min} \le \|\cdot\|_n^{(p,q)} \le \|\cdot\|_n^{(1,2)} \le K\,\|\cdot\|_n^{\min} \quad (n \in \mathbb{N}),$$

and so the (p,q)-multi-norm is equivalent to the minimum multi-norm for E, for each $(p,q) \in \mathcal{T}_{\min}$;

45

(6) *for each* $c \in (0, 1/2)$ *and each* $(p, q) \in \mathcal{T}_c$*, we have*

$$\|\cdot\|_n^{(2,2/(1-2c))} \leq \|\cdot\|_n^{(p,q)} \leq \|\cdot\|_n^{(1,1/(1-c))} \leq K_G \|\cdot\|_n^{(2,2/(1-2c))} \quad (n \in \mathbb{N});$$

(7) *for each* $(p, p) \in \mathcal{T}_{\max}$*, the* (p, p)*-multi-norm is equivalent to the maximum multi-norm for* E*, and the* $(1, 1)$*-multi-norm is equal to the maximum multi-norm.*

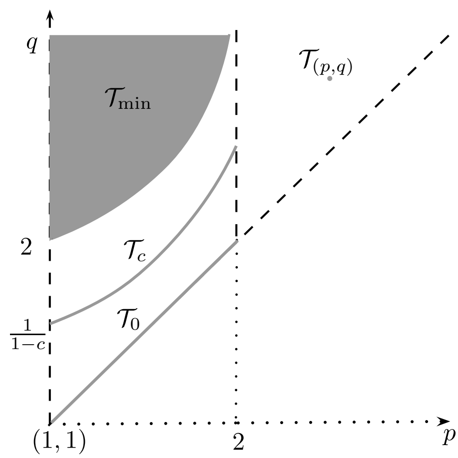

FIGURE 1. The various mutually disjoint equivalence classes of (p, q)-multi-norms on $L^r(\Omega)$ for $r \geq 2$.

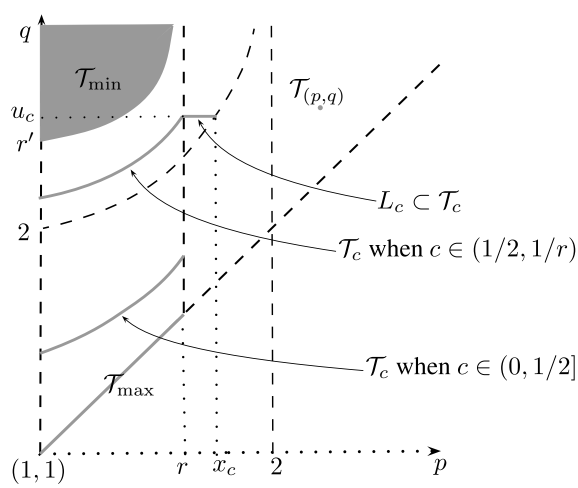

FIGURE 2. The various mutually inequivalent sets of (p, q)-multi-norms on $L^r(\Omega)$ for $1 < r < 2$.

The corresponding picture in the more difficult case in which $1 < r < 2$ is the following. Here our knowledge is incomplete. Consider the following sets and Figure 2:

$\mathcal{T}_{\min} = \{(p,q) \in \mathcal{T} \colon 1/p - 1/q \geq 1/r\}$;

$\mathcal{T}_c := \{(p,q) \in \mathcal{C}_c \colon 1 \leq p \leq r\} \cup \{(p,u_c) \colon r \leq p \leq x_c\}$ where $1/r - 1/u_c = c$ and $1/x_c - 1/u_c = 1/2$ for some $c \in (1/2, 1/r)$;

the curves $\mathcal{T}_c := \{(p,q) \in \mathcal{C}_c \colon 1 \leq p \leq r\}$ for some $c \in (0, 1/2]$;

the line segment $\mathcal{T}_{\max} := \{(p,p) \colon 1 \leq p < r\}$;

the singletons $\mathcal{T}_{(p,q)} := \{(p,q)\}$ for $(p,q) \in \mathcal{T}$ with either $p = q = r$ or both $p > r$ and $1/p - 1/q < 1/2$.

No two points in distinct sets are equivalent, and it may well be that these are exactly the equivalence classes. However, we cannot say whether the points (r, u_c) and (x_c, u_c) are equivalent when $1/2 < c < 1/r$. Now consider the points on the curve $\mathcal{C}_c$ with $1 \leq p \leq r$; the left-hand point of this curve is $(1, 1/(1-c))$, and each such point with $1 \leq p < r$ is equivalent to $(1, 1/(1-c))$. This leaves open the question whether the point (r, u_c) is equivalent to $(1, 1/(1-c))$. An old example of Kwapień [19] shows that this is not the case for $c = 0$, and it is proved in [3] that it is true for $c \in (1/2, 1/r)$, but we do not know what happens when $c \in (0, 1/2]$.

5.4. Equivalence of a (p,q)-multi-norm and the standard t-multi-norm. Fix the space ℓ^r, where $r \geq 1$, and fix $t \geq r$, so the standard t-multi-norm on ℓ^r is defined. We wish to determine the two sets

$$B_{r,t} := \left\{(p,q) \in \mathcal{T} : (\|\cdot\|_n^{[t]}) \preccurlyeq (\|\cdot\|_n^{(p,q)})\right\}$$

and

$$D_{r,t} := \left\{(p,q) \in \mathcal{T} : (\|\cdot\|_n^{(p,q)}) \preccurlyeq (\|\cdot\|_n^{[t]})\right\}.$$

Clearly there is a (p,q)-multi-norm which is equivalent to the standard t-multi-norm on ℓ^r if and only if these regions have a non-empty intersection, and so we wish to determine these two sets.

First, it is easy to see that

$$B_{r,t} = \{(p,q) \in \mathcal{T} : 1/p - 1/q \leq 1/r - 1/t,\ q \leq t\}.$$

However it is harder to determine the set $D_{r,t}$. Again this is rather easy in the case where $r \geq 2$: we have

$$D_{r,t} = \{(p,q) \in \mathcal{T} : 1/p - 1/q \geq 1/2\},$$

and so the the two sets $B_{r,t}$ and $D_{r,t}$ are indeed disjoint.

Now suppose that $1 < r < 2$, and define v by $1/v = 1/r - 1/t$. Again, provided that $v > 2$, the (p,q)-multi-norms on ℓ^r are never equivalent to the standard t-multi-norm on ℓ^r.

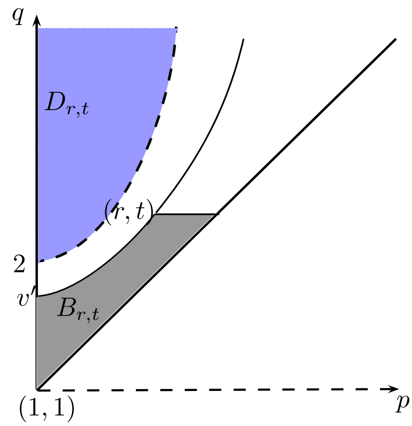

FIGURE 3. The set $B_{r,t}$ and (the possible range for) the set $D_{r,t}$ when $1 < r < 2$, $t \geq r$, and $1/r - 1/t \leq 1/2$. When $r \geq 2$, the set $D_{r,t}$ contains the dotted line.

But now we finally come to the case where $1 < r < 2$ and $v \leq 2$. Here we can use the notion of $[p,q]$-concave multi-norms and some deep theorems of Maurey given in [12] to see that

$$\left\{(p,q) \in \mathcal{T} : \frac{1}{p} - \frac{1}{q} \geq \frac{1}{r} - \frac{1}{t}\right\} \subset D_{r,t} \subset \left\{(p,q) \in C_{r,t} : \frac{1}{p} - \frac{1}{q} \geq \frac{1}{2}\right\}.$$

The situation is shown in the next diagram:

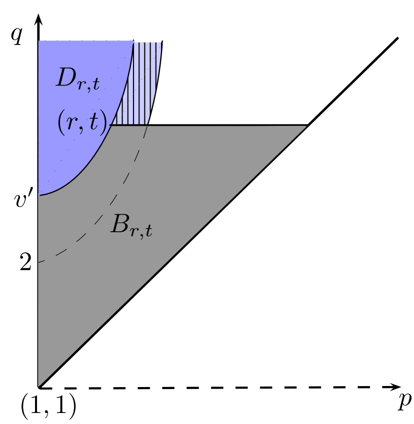

FIGURE 4. The set $B_{r,t}$ and (the possible range for) the set $D_{r,t}$ when $1 < r < 2$, $t \geq r$, and $1/r - 1/t > 1/2$

Thus we see that, in the case where $1 < r < 2$ and $1/r - 1/t > 1/2$, the points (p,q) such that $1/p - 1/q = 1/r - 1/t$ and $1 \leq p \leq r$ are indeed such that the (p,q)-multi-norm is equivalent to the standard t-multi-norm on ℓ^r, thus refuting a conjecture in [9].

6. Decompositions and multi-duals

We conclude with some remarks about decompositions of Banach spaces, taken from [7, Chapter 7]. This theory was developed as a step towards finding a good notion of a "multi-dual space". Throughout E is a Banach space.

6.1. Decompositions. Let $E = E_1 \oplus \cdots \oplus E_k$ be a direct sum decomposition of E. Then the decomposition is *hermitian* if

$$\|\zeta_1 x_1 + \cdots + \zeta_k x_k\| \leq \|x_1 + \cdots + x_k\|$$

whenever $|\zeta_1| \dots, |\zeta_k| \leq 1$ and $x_1 \in E_1, \dots, x_k \in E_k$.

The reason for this terminology is that the decomposition is hermitian if and only if the projections $P_j : E \to E_j$ are each hermitian operators (in the sense of numerical range theory).

For example, take $p \in [1, \infty]$ with $p \neq 2$, and let $\ell^p = E_1 \oplus \cdots \oplus E_k$ be an hermitian decomposition. Then there exist subsets $S_1, \dots, S_k$ of $\mathbb{N}$ such that $E_j = \ell^p(S_j)$ $(j \in \mathbb{N}_k)$.

We now consider decompositions related to multi-norms.

Definition 6.1. Let $(E^n, \|\cdot\|_n)$ be a multi-normed space, let $k \in \mathbb{N}$, and let $E = E_1 \oplus \cdots \oplus E_k$ be a direct sum decomposition of E. The decomposition is *small* (with respect to the multi-norm) if

$$\|P_1 x_1 + \cdots + P_k x_k\| \leq \|(x_1, \dots, x_k)\|_k \quad (x_1, \dots, x_k \in E)\,,$$

and the decomposition is *orthogonal* (with respect to the multi-norm) if

$$\|(y_1, \dots, y_j)\|_j = \|(x_1, \dots, x_k)\|_k$$

whenever $x_1 \in E_1, \dots, x_k \in E_k$ and there is a partition $\{S_j : j \in \mathbb{N}_k\}$ of $\mathbb{N}_n$ such that $y_j = \sum\{x_i : i \in S_j\}$ for each $j \in \mathbb{N}_k$.

Let $E = E_1 \oplus \cdots \oplus E_k$ be an orthogonal decomposition of E. Then certainly

$$\|(x_1, \dots, x_k)\|_k = \|x_1 + \cdots + x_k\| \quad (x_1 \in E_1, \dots, x_k \in E_k)\,.$$

It is a little exercise to see that a small decomposition is always an orthogonal decomposition and that an orthogonal decomposition is an hermitian decomposition of the Banach space E. It is an aggravating fact that I do not know an example of an orthogonal decomposition that is not a small decomposition; if the two concepts were always the same the theory would be cleaner.

Here are some examples.

Theorem 6.2. *Let K be a compact space, and let $C(K) = E_1 \oplus \cdots \oplus E_k$ be a direct sum decomposition of $C(K)$. Then the following are equivalent:*

(a) $E_j = C(K_j)$ $(j \in \mathbb{N}_k)$ *for some partition* $\{K_1, \ldots, K_k\}$ *of K into clopen subspaces;*

(b) *the decomposition is small with respect to the lattice multi-norm;*

(c) *the decomposition is orthogonal with respect to the lattice multi-norm;*

(d) *the decomposition is hermitian.*

Theorem 6.3. *Let H be a Hilbert space, and let $H = H_1 \oplus \cdots \oplus H_k$ be a direct sum decomposition of H. Then the following are equivalent:*

(a) *the decomposition is orthogonal in the classical sense;*

(b) *the decomposition is small with respect to the Hilbert multi-norm;*

(c) *the decomposition is orthogonal with respect to the Hilbert multi-norm;*

(d) *the decomposition is hermitian.*

Now suppose that E is a Banach lattice and that we are considering the lattice multi-norm based on E. It is easy to see that every band decomposition of E is a small decomposition, and hence an orthogonal decomposition. I wondered if the converse of this statement held: if so, the concepts of "band decomposition" and "orthogonal decomposition with respect to the lattice multi-norm" would coincide for Banach lattices. This required the following theorem; it was a surprise to me that the result was not standard in the Banach-lattice literature, but this seemed not to be the case. Happily, I asked the late Nigel Kalton if the result were true, and he provided a very elegant new proof; the result follows from the following theorem.

Theorem 6.4. *Let $E = E_1 \oplus \cdots \oplus E_k$ be a direct sum decomposition of a Banach lattice E. Suppose that*

$$\|x_1 + \cdots + x_k\| = \|\,|x_1| \vee \cdots \vee |x_k|\,\| \quad (x_j \in E_j,\, j \in \mathbb{N}_k)\,.$$

Then the decomposition is a band decomposition.

This provides the following result.

Theorem 6.5. *Let $E = E_1 \oplus \cdots \oplus E_k$ be a direct sum decomposition of a Banach lattice E. Then the following are equivalent:*

(a) *the decomposition is orthogonal with respect to the lattice multi-norm;*

(b) *the decomposition is small with respect to the lattice multi-norm;*

(c) *the decomposition is a band decomposition.*

In fact, Kalton [17, Theorems 5.4 and 5.5] proved the following stronger and considerably deeper result.

Theorem 6.6. *Let $E = F \oplus G$ be a direct sum decomposition of a Banach lattice E.*

(i) *Suppose that the decomposition is hermitian. Then*

$$\|x + y\| = \left\|(|x|^2 + |y|^2)^{1/2}\right\| \quad (x \in F,\, y \in G)\,.$$

(ii) *Suppose that, for some $p \in [1, \infty)$ with $p \neq 2$, we have*

$$\|x + y\| = \left\|(|x|^p + |y|^p)^{1/p}\right\| \quad (x \in F,\, y \in G)\,.$$

Then the decomposition is a band decomposition.

6.2. Multi-dual spaces. We now consider how to form the "multi-dual" of a multi-normed space.

Let $(E^n, \|\cdot\|_n)$ be a multi-normed space. It is tempting to regard $\mathcal{M}(E, \mathbb{C})$ as the "multi-dual" of the space E. However recall that $\mathcal{M}(E, \mathbb{C}) = E'$ when we regard $\mathbb{C}$ as having its unique multi-norm structure, and that, as a multi-normed space, $\mathcal{M}(E, \mathbb{C})$ has just the minimum multi-norm. Thus the approach of using this multi-normed space as a "multi-dual" is not satisfactory.

A second temptation is to look at the family $((E')^n, \|\cdot\|_n')$ for a multi-normed space $(E^n, \|\cdot\|_n)$, where $\|\cdot\|_n'$ is the dual of the norm $\|\cdot\|_n$. But this is an even worse failure: $(\|\cdot\|_n' : n \in \mathbb{N})$ is a dual multi-norm, not a multi-norm, on $\{(E')^n : n \in \mathbb{N}\}$.

In [7], we give a different approach, using the notion of orthogonal decompositions. It is somewhat complicated to describe even the definition of the "multi-dual of a multi-normed space" (maybe the theory can be simplified?), and we conclude by merely giving one theorem.

Theorem 6.7. *Take $p \geq 1$ with conjugate index q. Then the multi-dual of $((\ell^p)^n, \|\cdot\|_n^{[p]})$ is $((\ell^q)^n, \|\cdot\|_n^{[q]})$.*

Acknowledgements

I was very pleased to be invited to report on this work at the conference *Kangro-100: Methods of Analysis and Algebra, Tartu, Estonia*, and I am very grateful for the generous and friendly hospitality and superb organisation of the hosts, especially Professor Eve Oja and Dr. Kristel Mikkor: many thanks!

Also thanks to Hung Le Pham for preparing the diagrams.

References

[1] F. Albiac and N. J. Kalton, *Topics in Banach Space Theory*, Graduate Texts in Mathematics **233**, Springer, New York, 2006.

[2] C. D. Aliprantis and O. Burkinshaw, *Positive Operators*, Pure and Applied Mathematics **119**, Academic Press, Orlando, 1985; reprinted, Springer, Dordrecht, 2006.

[3] O. Blasco, H. G. Dales, and H. Le Pham, *Equivalences involving (p, q)-multi-norms*, in preparation.
[4] D. Blecher and C. Le Merdy, *Operator Algebras and Their Modules - an Operator Space Approach*, London Mathematical Society Monographs **30**, Clarendon Press, Oxford, 2004.
[5] H. G. Dales, *Banach Algebras and Automatic Continuity*, London Mathematical Society Monographs **24**, Clarendon Press, Oxford, 2000.
[6] H. G. Dales and M. E. Polyakov, *Homological properties of modules over group algebras*, Proc. London Math. Soc. (3) **89** (2004), 390–426.
[7] H. G. Dales and M. E. Polyakov, *Multi-normed spaces*, Dissertationes Math. (Rozprawy Mat.) **488** (2012), 1–165.
[8] H. G. Dales, M. Daws, H. L. Pham, and P. Ramsden, *Multi-norms and the injectivity of $L^p(G)$*, J. London Math. Soc. (2) **86** (2012), 779–809.
[9] H. G. Dales, M. Daws, H. L. Pham, and P. Ramsden, *Equivalence of multi-norms*, Dissertationes Math. (Rozprawy Mat.) **498** (2014), 1–53.
[10] H. G. Dales, N. J. Laustsen, and V. Troitsky, *Multi-norms and Banach lattices*, in preparation.
[11] A. Defant and K. Floret, *Tensor Norms and Operator Ideals*, North-Holland Mathematics Studies **176**, North-Holland, Amsterdam, 1993.
[12] J. Diestel, H. Jarchow, and A. Tonge, *Absolutely Summing Operators*, Cambridge Studies in Advanced Mathematics **43**, Cambridge University Press, 1995.
[13] E. G. Effros and Z.-J. Ruan, *Operator Spaces*, London Mathematical Society Monographs **23**, The Clarendon Press, Oxford University Press, New York, 2000.
[14] A. Ya. Helemskii, *Quantum Functional Analysis. Non-coordinate Approach*, University Lecture Series **56**, American Mathematical Society, Providence, Rhode Island, 2010.
[15] G. J. O. Jameson, *Summing and Nuclear Norms in Banach Space Theory*, London Mathematical Society Student Texts **8**, Cambridge University Press, 1987.
[16] B. E. Johnson, *Cohomology in Banach Algebras*, Memoirs of the American Mathematical Society **127**, American Mathematical Society, Providence, 1972.
[17] N. J. Kalton, *Hermitian operators on complex Banach lattices and a problem of Garth Dales*, J. London Math. Soc. (2) **86** (2012), 641–656.
[18] S. Kwapień, *Some remarks on (p, q)-absolutely summing operators in ℓ_p-spaces*, Studia Math. **29** (1968), 327–337.
[19] S. Kwapień, *On a theorem of L. Schwartz and its applications to absolutely summing operators*, Studia Math. **38** (1970), 193–201.
[20] J. Lindenstrauss and A. Pełczyński, *Absolutely summing operators in $\mathcal{L}_p$-spaces and their applications*, Studia Math. **29** (1968), 275–326.
[21] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces. I and II*, Classics in Mathematics, Springer-Verlag, Berlin, 1996.
[22] J. L. Marcolini Nhani, *La structure des sous-espaces de treillis*, Dissertationes Math. (Rozprawy Mat.) **397** (2001), 1–50.
[23] P. Meyer-Nieberg, *Banach Lattices*, Springer-Verlag, Berlin, 1991.
[24] A. Pietsch, *Operator Ideals*, North Holland Mathematical Library **20**, North-Holland, Amsterdam – New York, 1980.
[25] R. Ryan, *Introduction to Tensor Products of Banach Spaces*, Springer Monographs in Mathematics, Springer-Verlag, London, 2002.

Department of Mathematics and Statistics, University of Lancaster, Lancaster LA1 4YF, UK
E-mail address: `g.dales@lancaster.ac.uk`