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An optimal inventory pricing and ordering strategy subject to demand dependent on stock level and price

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Abstract: This article considers the deterministic singular optimal control problem of profit maximisation for inventory replenished at a variable rate and depleted by demand which is assumed to vary with price and stock availability. Optimal policies for the product order rate and price are derived using the maximum principle. Several initial inventory regions are identified as potential inventory states for feasible profit optimisation. Bounds on the maximum price for maximising net profit or minimising loss are obtained. Numerical simulations accompanied by phase diagrams are performed to support the theoretical findings.

Keywords: optimal control; Hamiltonian; maximum principle; transversality condition; costate variable; bang-bang control.

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1 Literature review

The main premise of this article is that the demand for a particular product is allowed to increase linearly as more stock on display becomes available. An increase in the shelf space can influence customers. An empirical evidence of this phenomenon was presented by Wolfe (1968). Balakrishnan et al. (1994) used an extension of a standard inventory-dependent demand model from the literature, and provided a convenient characterisation of products that require early replenishment. They demonstrated that the optimal cycle time is largely governed by the conventional trade-off between ordering and holding costs, whereas the reorder point relates to a promotions-oriented cost-benefit perspective. They next showed that the optimal policy yields significantly higher profits than cost-based inventory policies, underscoring the importance of profit-driven inventory management. Optimal ordering strategies adopting the economic order quantity (*EOQ*) were addressed by Baker and Urban (1988), Urban (1992, 1995) and Gerchak and Wang (1994). Xu and Li (2008) concerned with the joint management problem of production control and dynamic pricing to balance the finished goods inventory and market demand in a make-to-stock manufacturing system using a long-run average profit criterion. In the system, the production rate was random, with a controllable mean rate, and the demand was Markov, with a changeable mean rate which depended on the sale price. The management issue was how to dynamically adjust the production rate and the selling price to maximise the long-run average profit. They discovered that the optimal policy of dynamic pricing and production control over an infinite horizon was a matter of thresholds. An effective algorithm was also suggested. Sana et al. (2009) introduced a model dealing with an *EOQ* or economic production quantity (*EPQ*) model where the effect of 'advertising', 'sales price' and 'stock-display' was investigated. It was developed both for deteriorating and ameliorating items in the light of capacity constraint for storage facility and the limitation of the budget for advertising. The associated average profit function was maximised by calculus method and was also illustrated by some numerical data for the test problem. Roy et al. (2010) considered a stochastic economic production lot size (*EPLS*) model with price sensitive demand when the price is random. They investigated a production model with finite replenishment and maximised the profit over a finite time horizon allowing shortages. Sahoo at al. (2010) developed a deterministic model with constant deterioration and demand rate a function of the selling price. The model was solved allowing for inventory shortages and time dependent holding costs. Chang et al. (2010) dealt with the problem of determining the optimal selling price and order quantity simultaneously for the *EOQ* model for items with deterioration. They also assumed that the demand rate was dependent on both the selling price and the limited stock level on display and derived algorithms to maximise profit. Saha et al. (2010) considered day-to-day time-based competition, with the unit selling price of a high-tech product declining significantly over its short life cycle. In this paper, the authors introduced dynamic pricing to traditional *EPQ* models for time and price sensitive products with the objective of maximising total profit and proved that the total profit is a concave function of selling price within fixed planning horizon. A solution procedure was presented to determine optimal prices, optimal number of production cycles, optimal lot size and optimal profit simultaneously. Shah and Soni (2011) analysed a continuous review inventory system with an objective to determine an (r, Q) policy which minimises the cost function. In their study they considered the demand rate, holding cost and shortage cost are sensitive to imprecise selling price. The proposed model assumed that shortages are allowed and completely backlogged and the lead time is fuzzy in nature. In a stochastic setting, Wei and Zhao (2011) dealt with the optimal pricing decision problem of a fuzzy closed-loop supply chain with retail competition where the fuzziness is associated with the customer demands, the remanufacturing cost and the collecting cost. By using game theory and fuzzy theory, the optimal decision on wholesale price, retail prices and remanufacturing rate were explored respectively under the centralised and the decentralised decision scenarios, and the expressions for them were also established. Shi et al. (2011a) developed a mathematical model to maximise the overall profit of a manufacturing system by simultaneously determining the selling price, the production quantities for brand-new products and remanufactured products, and the acquisition price of used products. Through a numerical example, the impacts of the uncertainties of both demand and return on the production plan, selling price, and the acquisition price of used products were analysed. Shi et al. (2011b) also studied a manufacturing system with uncertain demand with the objective to maximise the manufacturer's expected profit by jointly determining the production quantities of brand-new products, the quantities of remanufactured products and the acquisition prices of the used products, subject to a capacity constraint. Salvierri et al. (2014) presented a stochastic version of the economic lot sizing problem with pricing. The control variables of the stochastic problem are the production quantities and cycle lengths for each product. The recourse variables are the sales prices and the external purchase quantities in each production cycle. A solution method based on simulation, decomposition, and column generation was proposed and tested using a number of designed experiments. The method was found to produce very close to optimal solutions quickly.

In this paper the demand rate is a composite function of both inventory and price. It is linear with respect to the inventory and a convex function of the product price. Datta and Paul (2001) analysed a multi-cycle replenishment inventory system with the demand rate being determined by price and stock level and treated the mark-up rate and number of orders as the decision variables. They assumed that demand rate was given by $d(x, k) = f(k)x^{\beta}$, where $0 < \beta < 1$and k is the mark-up rate and two different expressions for $f(k)$; $f(k) = ae^{-bk}$, a form adopted by Gallego and Rysin (1994) and $f(k) = ak^{-b}$, a form used by Arcelus and Srinivasan (1987). Demand with linear price dependence with profit maximisation using a quadratic performance was also investigated by Jørgensen and Kort (2002). The convexity introduced in the demand growth model in this work allows for a more gradual slowdown in the demand rate as price increases compared to the linear case in Jørgensen and Kort (2002. The objective in this article is to maximise the net profit from selling the product incorporating linear holding and ordering costs over a finite horizon with restrictions on both order rate and price. A similar objective in an infinite horizon setting was used by Khmelnitsky and Gerchak (2002), who did not however factor in demand-price interdependence but instead treated the demand rate as function of both inventory and time. The contribution of this work is the synthesis of optimal replenishment and pricing policies towards achieving the objective of maximising net profit from selling a single product whose stock display and pricing affects demand for it. Upper bounds are imposed on both the benchmark maximum price and order rate. Three regimes are identified in which the replenishment rate is either at its maximum or lowest and product pricing is adjusted continuously towards realisation of the stated objective.

2 The optimal control model

The demand for an item is a separable function of the inventory in stock at time t, $x(t)$, and the price, p. Specifically, the demand rate, $d(x, p)$, is given by

$$d(x,p) = ax(b-p)^2 \tag{1}$$

where a is a suitably chosen demand growth parameter and b is the maximum product price possible, $p \leq b$. The demand rate will increase as x increases, $\frac{\partial d}{\partial x} > 0,$ and will decline as p increases, $\frac{\partial d}{\partial p} < 0,$ with demand ceasing to grow altogether when either $x = 0$ or $p = b$, $d(0, p) = d(x, b) = 0$. As price increases, the demand rate will decline with the available stock,

$$\frac{\partial^2 d}{\partial p \partial x} < 0.$$

The inventory evolves according to the differential equation

$$\dot{x} = u - d(x,p) = u - ax(b-p)^2 \tag{2}$$

where u is the order (replenishment) rate, limited to a maximum rate, U, $0 \leq u \leq U$, $0 \leq p \leq b$ and $x(0) = x_0$.

The objective is to select an order and pricing policy so as to maximise the net profit over a finite horizon T:

$$\sup_{u,p} \left(\int_0^T \left(pd(x,p) - hx - cu \right) dt \right) \tag{3}$$

where c is the unit order cost and h is the unit holding cost, $c < b$.

The Hamiltonian is defined as

$$\mathcal{H}(x,u,p,\lambda) = (p-\lambda)ax(b-p)^2 - hx + (\lambda - c)uf_i = f_{\min} + \beta\left(f_{\max} - f_{\min}\right)p_i^{(0)} \tag{4}$$

where λ is the costate variable measuring the shadow price of the inventory variable, x (Chiang 1992). The Pontryagin maximum principle conditions are

$$\frac{\partial \mathcal{H}}{\partial u} = 0 \text{ and } \frac{\partial \mathcal{H}}{\partial p} = 0 \text{ optimality conditions} \tag{5}$$

$$\dot{\lambda} = -\frac{\partial \mathcal{H}}{\partial x} \text{ trajectory of } \lambda(t) \tag{6}$$

$$\lambda(T) = 0 \text{ transversality condition} \tag{7}$$

The maximum principle conditions (5) are necessary but not sufficient for maximising the Hamiltonian. Arrow's sufficiency theorem must be applied to the Hamiltonian to ensure sufficiency by proving the concavity of the Hamiltonian (Kamien and Schwartz 1971). Arrow's theorem states that the maximised Hamiltonian, $\mathcal{H}^*$, must exist for all x and be concave in x for all t. In the autonomous case like this one, the Hamiltonian is constant on the optimal trajectory.

The optimality condition, $\frac{\partial \mathcal{H}}{\partial u} = 0,$ cannot be strictly applied because the Hamiltonian is linear in *u*. Since $\frac{\partial \mathcal{H}}{\partial u} = \lambda - c,$ this is a case of bang-bang control insofar as the optimal ordering policy is concerned. We have then for the optimal ordering policy, u^*:

$$u^* = \begin{cases} 0 \ \text{ if } \lambda < c \\ \text{indeterminate if } \lambda = c \\ U \ \text{ if } \lambda > c \end{cases} \tag{8}$$

The second optimality condition, $\frac{\partial \mathcal{H}}{\partial p} = 0,$ yields the optimal pricing policy

$$p^* = \frac{b + 2\lambda}{3} \tag{9}$$

From (6) we have

$$\dot{\lambda} = h - a(p - \lambda)(b - p)^2 \tag{10}$$

The price, *p*, must be, by definition, non-negative. This restriction leads to the identification of four distinct regions *A*, *B*, *C*, and *D*. Each region is dealt with separately.

2.1 Region A

$$\lambda \le -\frac{b}{2}, \ p^* = 0, \ u^* = 0$$
$$\dot{x} = -ab^2 x, \ \dot{\lambda} = ab^2 \lambda + h$$

Solving the two differential equations is a straightforward matter:

$$x(t) = x_0 e^{-ab^2 t} \tag{11}$$

$$\lambda(t) = \left(k_1 + \frac{h}{ab^2}\right) e^{ab^2 t} - \frac{h}{ab^2} \tag{12}$$

where k_1 is the integration constant, $k_1 = \lambda(0) > -\frac{h}{ab^2}.$

The Hamiltonian on the optimal path is

$$\mathcal{H}^*(x, \lambda) = -\left(h + \lambda ab^2\right)x = -x_0\left(h + k_1 ab^2\right)$$

In this region the optimal policy is to allow inventory to deplete without replenishment, by pricing it low. For the costate variable to keep increasing, $\dot{\lambda} = h + ab^2\lambda > 0,$ or

$\lambda(t) \geq -\frac{h}{ab^2}$. Since $\lambda(t) \leq -\frac{b}{2}$, the following key restriction on the maximum price, b, must be met within region A:

$$b \leq \left(\frac{2h}{a}\right)^{\frac{1}{3}} \tag{13}$$

2.2 Region B

$$-\frac{b}{2} < \lambda \leq c,\ p^* = \frac{b+2\lambda}{3},\ u^* = 0$$

$$\dot{x} = -\frac{4ax}{9}(b-\lambda)^2,\ \dot{\lambda} = h - \frac{4a}{27}(b-\lambda)^3$$

The solution to the costate differential equation is given by the implicit expression

$$\ln\left(\frac{\gamma - b}{\lambda(t)+\gamma-b}\right) + \int_t^T \frac{2\gamma + b - \lambda(t)}{(\lambda(t)-b)^2 - (\lambda(t)-b)\gamma + \gamma^2} = (4ah^2)^{\frac{1}{3}}(T-t) \tag{14}$$

where

$$\gamma = \frac{3}{2}\left(\frac{2h}{a}\right)^{\frac{1}{3}} \tag{15}$$

and k_2 is the integration constant. Since in this region $\lambda(t)$ assumes values between $-\frac{b}{2}$ and c the transversality condition, $\lambda(T) = 0$, may be utilised to determine k_2. By virtue of the transversality condition region B is the terminal region.

The mathematical relationship between $x(t)$ and $\lambda(t)$ can be established by solving

$$\frac{dx}{d\lambda} = \frac{-\frac{4ax}{9}(b-\lambda)^2}{h - \frac{4a}{27}(b-\lambda)^3} = \frac{-\frac{4ax}{9}}{\frac{h}{(b-\lambda)^2} - \frac{4a}{27}(b-\lambda)}$$

whence $x\dot{\lambda} = K$, where K is a constant determinable by boundary conditions. The Hamiltonian is given by

$$\mathcal{H}(x,\lambda) = \left(\frac{4a}{27}(b-\lambda)^3 - h\right)x.$$

Within B there exist two distinct sub-regions; B_1 where $-\frac{b}{2} < \lambda(t) < 0$, and region B_2 where $0 < \lambda(t) \leq c$, with the transversality condition delineating the common boundary between the two areas. The inventory dynamics within each sub-region is discussed below:

- B_1: the costate must keep increasing from negative values towards 0. For this to occur $\dot{\lambda} = h - \frac{4a}{27}(b-\lambda)^3 > 0$, or $b - \lambda < \gamma$. Since $\lambda(t) < 0$ throughout, the following restriction on the maximum price, b, must be imposed in B_1:

$$b < \gamma \tag{16}$$

The pricing range in this region is $\left(0, \frac{b}{3}\right]$.

- B_2: the costate must keep decreasing from positive values towards 0. For this to occur $\dot{\lambda} = h - \frac{4a}{27}(b-\lambda)^3 < 0$, or $b - \lambda > \gamma$. Since $\lambda(t) \le c$ throughout, the following restriction on the maximum price, b, must be imposed in B_2:

$$b \ge c + \gamma \tag{17}$$

The pricing range in this region is $\left(\frac{b}{3}, \frac{b+2c}{3}\right]$.

2.3 Region C

$$c < \lambda \le b, \;\; p^* = \frac{b+2\lambda}{3}, \;\; u^* = U$$

$$\dot{x} = U - \frac{4ax}{9}(b-\lambda)^2, \;\; \dot{\lambda} = h - \frac{4a}{27}(b-\lambda)^3$$

This region is adjacent to B_2 and for B_2 to be reached $\lambda(t)$ must decrease, so the condition (17) applies here as well. The (x, λ) system however, possesses the unique equilibrium:

$$\begin{aligned} \bar{x} &= \frac{9U}{4a\gamma^2} \\ \bar{\lambda} &= b - \gamma \end{aligned} \tag{18}$$

The Jacobian matrix possesses two real eigenvalues, equal in magnitude but of opposite sign, $\mu_1 = \frac{4a}{9}\gamma^2, \mu_2 = -\frac{4a}{9}\gamma^2$. The equilibrium is a saddle point, and so it is unstable. The perturbations, $(\tilde{x}, \tilde{\lambda})$, from the equilibrium $(\bar{x}, \bar{\lambda})$ obey

$$\begin{aligned} \dot{\tilde{x}} &= \mu_2 \tilde{x} + \frac{2U}{\gamma}\tilde{\lambda} \\ \dot{\tilde{\lambda}} &= \mu_1 \tilde{\lambda} \end{aligned}$$

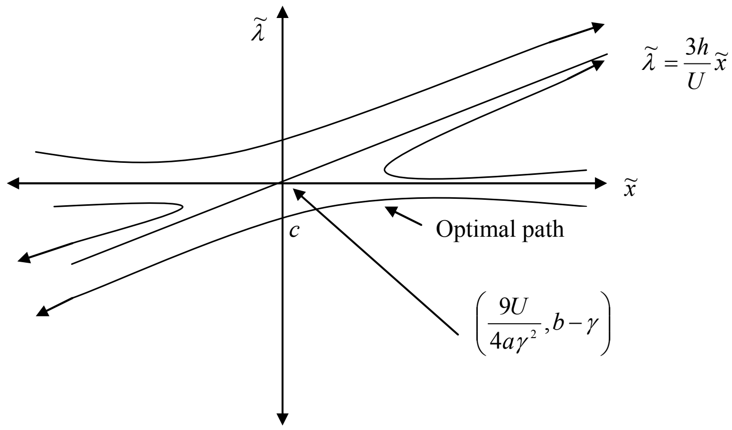
with solutions

$$\tilde{x}(t) = \left(\tilde{x}_0 - \frac{U}{3h}\tilde{\lambda}_0\right)e^{\mu_2 t} + \frac{U}{3h}\tilde{\lambda}(t)$$
$$\tilde{\lambda}(t) = \tilde{\lambda}_0 e^{\mu_1 t}.$$

The phase plane equation representing the slope of the phase path is

$$\frac{d\tilde{\lambda}}{d\tilde{x}} = \frac{\mu_1\tilde{\lambda}}{\mu_2\tilde{x} + \dfrac{2U}{\gamma}\tilde{\lambda}}.$$

Figure 1 Phase paths around the saddle equilibrium



The slopes of the asymptotic isoclines are 0 and $\frac{3h}{U}$. The stable manifold is the $\tilde{\lambda} = 0$ axis and the unstable manifold is the straight line $\tilde{\lambda}(t) = \frac{3h}{U}\tilde{x}(t)$. As $t \to -\infty\, \tilde{x}(t) \to \pm\infty$ and $\tilde{\lambda}(t) \to 0$. As $t \to +\infty$, $\tilde{x}(t) \to \frac{U}{3h}\tilde{\lambda}(t)$ and $\tilde{\lambda}(t) \to \infty$. Figure 1 displays the four possible phase paths. Only the path below the $\tilde{x}$-axis that crosses the $\tilde{\lambda}$-axis at c and the $\tilde{x}$-axis at $\frac{9U}{4a\gamma^2}$ is the optimal trajectory away from the C region towards the B_2 region. The initial inventory values, x_0, that qualify for such transition are those with values greater than $\frac{9U}{4a\gamma^2}$ whereas the shadow price must not exceed $b - \gamma$. As illustrated in Figure 1 there exists a whole range of solutions $x(t)$, $\lambda(t)$ for the given system but only one trajectory that satisfies the condition $\lambda(\tau) = c, x(\tau) = \frac{9U}{4a\gamma^2}$ for some time, τ, $\tau \leq T$. This solution implicitly depends on the initial state inventory x_0 as well as the finite time horizon T. Thus the costate value is in fact $\lambda(t) = \lambda(x_0, t, T)$. The pricing range in this region is $\left(\frac{b+2c}{3}, b-\frac{2\gamma}{3}\right)$.

In this region the optimal Hamiltonian constant is given by

$$\mathcal{H}(x,U,\lambda) = \left(\frac{4a}{27}(b-\lambda)^3 - h\right)x + U(\lambda - c).$$

2.4 Region D

$$\lambda > b,\ p^* = b,\ u^* = U, \dot{x} = U, \dot{\lambda} = h$$

Region D must be ruled out as part of the optimal policy because the inventory and the shadow price grow linearly in time, both starting from positive values, and consequently the transversality condition (7) cannot be met. Besides the demand has grown to a halt and the inventory grows linearly which is altogether unrealistic. As such, none of the other three regions, A, B or C, are accessible from D.

3 Theoretical implementation of the control policies

If the shadow price is negative it is not profitable to replenish the stock ($u^* = 0$) and the price is set low ($p^* = 0$) in order to get rid of the stock. When the shadow price assumes values in the range $\left[-\frac{b}{2}, b-\gamma\right]$ variable pricing kicks in, with maximum replenishment only when $c < \lambda < b$. Region B is the terminal region being the only one for which the transversality condition (7) is met. The transitions $A \to C \to B$ and $C \to A \to B$ are both precluded by the continuity property of the costate variable. There are therefore two possible regional transitions, $A \to B_1$ and $C \to B_2$. The time of transition from region A, where the restriction on the maximum price, $b \le \frac{2\gamma}{3}$, holds, to B_1, $t_{A\to B_1}$, is determined from (7) with $\lambda(t_{A\to B_1}) = -\frac{b}{2}$. The time of transition from region C, where the restriction on the maximum price, $b > \gamma + c$, holds to B_2, $t_{C\to B_2}$, is determined again from (7) with $\lambda(t_{C\to B_2}) = c$. No transition will occur when $-\frac{b}{2} < \lambda < c$ as in this case the inventory is already in region B. Numerical solutions to the differential solutions have to be carried out backwards as the optimal control is cast as a two-point boundary problem. The optimal solution to the objective functional, $V^*(x_0, 0) = \sup_{u,p}\left(\int_0^T \left(pd(x,p) - hx - cu\right)dt\right)$, can be worked out easily in any region by solving the trivial differential equation, $-\frac{\partial V^*}{\partial t} = \mathcal{H}^*$ for constant $\mathcal{H}^*$, in each region.

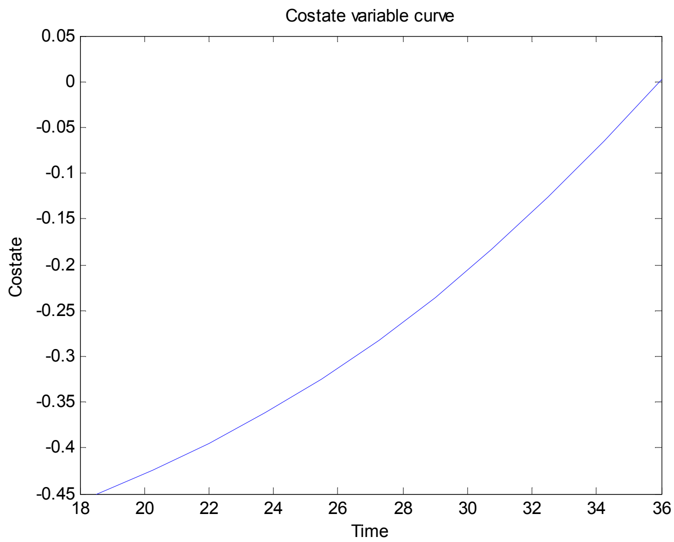
4 A numerical implementation of the control policies

We fix the main parameters, $a = 0.1$, $h = 0.05$, $c = 0.6$, $U = 10$. Then from (15) we get $\gamma = \frac{3}{2}\left(\frac{2h}{a}\right)^{\frac{1}{3}} = 1.5.$ We provide optimal solutions for two different regional transitions: $A \to B_1$ and $C \to B_2$.

4.1 $A \to B_1$

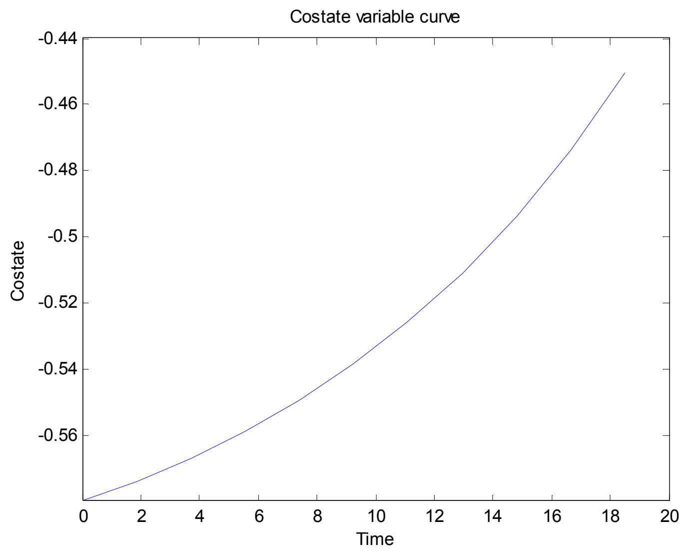
Let $b = 0.9 < \left(\frac{2h}{a}\right)^{\frac{1}{3}} = 1,$ so that condition (13) is upheld. Let the length of the finite planning horizon be $T = 36$. The condition (16), $b < \gamma$, within region B_1 is supplanted by (13). We work out the numerical solution to the costate differential equation, $\dot{\lambda} = 0.05 - \frac{0.4}{27}(0.9 - \lambda)^3,$ working backwards from $\lambda(36) = 0$ to $\lambda(t_{A \to B_1}) = -\frac{b}{2} = -0.45,$ where $t_{A \to B_1} = 18.5.$ The costate curve is depicted in Figure 2.

Figure 2 $\lambda(t)$ in region B_1 between $t_{A \to B_1} = 18.5$ and $T = 36$ (see online version for colours)



In region A the costate variable is explicitly given by $\lambda(t) = (\lambda(0) + 0.6173)e^{0.081t} - 0.6173$. At $t = 18.5$, $\lambda(18.5) = -0.45$, hence $\lambda(0) \approx -0.58 < -0.45$, so condition (13) is met. The analytic solution for the costate variable in region A is $\lambda(t) = 0.0373e^{0.081t} - 0.6173$, shown in Figure 3.

Figure 3 $\lambda(t)$ in region A between $t = 0$ and $t_{A\to B_1} = 18.5$ (see online version for colours)

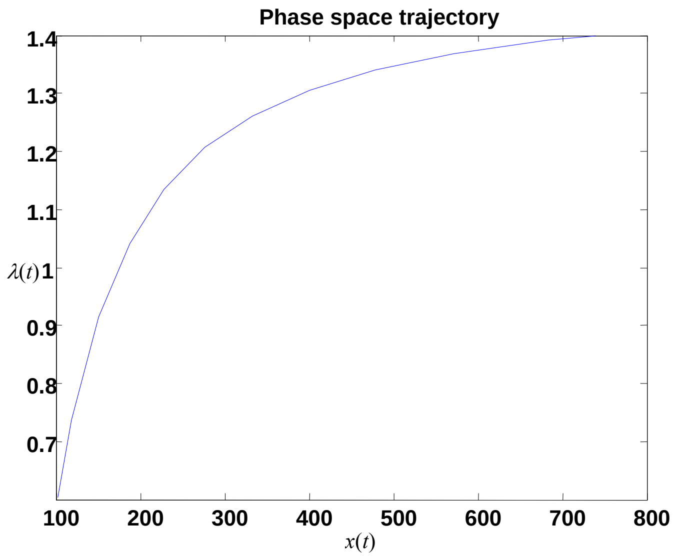


The pricing policy is to set the price at 0 whilst in region A and adjust it upwards gradually from 0 to 0.3 whilst in region B_1. For an initial inventory, x_0, in A, the optimal Hamiltonian in this region is $\mathcal{H}^*(x_0) = -x_0(h + k_1ab^2) = -0.003x_0$, and the optimal value is $V^*(x_0) = -0.003x_0t_{A\to B_1} = -0.056x_0$, signifying a financial loss by getting rid of inventory at zero price. For an initial inventory, $x_0 = 740$, this loss amounts to approximately $V^*(740) = -0.056 * 740 \approx -41.44$. In region B_1 the inventory continues to decline but at variable selling price. The inventory level at the transition point is around $x(18.5) \approx 165$ units. In region B_1 the decline slows down settling to an approximate final value of $x(36) \approx 57$. The optimal Hamiltonian in region B_1 is $\mathcal{H}^* \approx -2.235$ amounting to a further net loss of $V^*(165) = -2.235 * (36 - 18.5) \approx -39$. The net aggregate loss for the transition $A \to B_1$ is $V^*(740) + V^*(165) \approx -80.44$, which represents the minimum possible (optimal) net loss given the parameters used.

4.2 $C \to B_2$

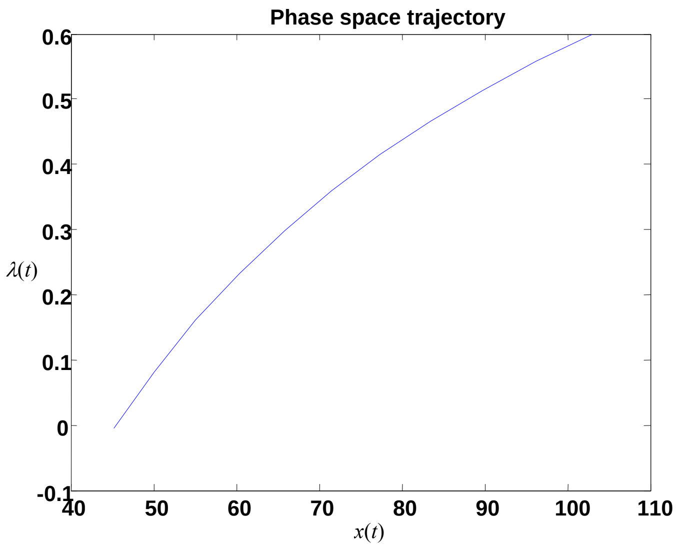
Let $b = 3 > c + \gamma = 0.6 + 1.5 = 2.1$, so condition (17) is met and let the initial inventory be again $x_0 = 740 > \bar{x} = \dfrac{9U}{4a\gamma^2} = \dfrac{90}{0.9} = 100.$ If the region B_2 is to be accessible from C, $\lambda(0)$ must be less than $\bar{\lambda} = b - \gamma = 1.5,$ otherwise the costate variable cannot reach $\lambda = c = 0.6$ at the time of transition $t_{C\to B_2}$, as Figure 1 roughly illustrates. Let $\lambda(0) = 1.4$, and the inventory begins to decline to keep up with demand whilst being replenished at the maximum rate, $U = 10$. The time of transition to region B_2 as well as the inventory level at the same time are found numerically, $t_{C\to B_2} \approx 17.6$, $x(t_{C\to B_2}) \approx 103$. Figure 4 depicts the phase path (x, λ) in region C.

Figure 4 Phase space path (x, λ) in region C (see online version for colours)



The pricing policy range is (1.4, 2).The optimal Hamiltonian constant in region C is $\mathcal{H}^* \approx 15.9$, yielding a net profit of $V^*(740) = 15.9 * 17.6 \approx 279.84$. At time $t_{C \to B_2} \approx 17.6$ region B_2 is entered, replenishment ceases and the inventory continues to decline reaching $x(20.2) \approx 45$ at $T = 20.2$ where $\lambda(20.2) = 0$. The pricing range in B_2 is (1, 1.4], and the optimal Hamiltonian constant is $\mathcal{H}^* \approx 15.9$, yielding a net profit of $V^*(103) = 15.9 * (20.2 - 17.6) \approx 41.34$. The net aggregate profit for the transition $C \to B_2$ is $V^*(740) + V^*(103) \approx 279.84 + 41.34 \approx 321.18$. The final Figure 5 depicts the phase path (x, λ) in region B_2.

Figure 5 Phase space path (x, λ) in region B_2 (see online version for colours)



5 Discussion

In this work we have formulated an optimal inventory control problem and outlined its full solution when the demand is simultaneously dependent on both stock and price and the order rate as well as the maximum price are bounded. The key findings of this paper were the three regions singled out as being feasible starting points for net profit maximisation along with pricing restrictions within each one of them. The subdivision of the state-space was based on the value of the costate variable representing the imputed value of one unit of stock. In each region a specific bound on the maximum price was imposed in order to achieve the formulated objective. The bounds assigned involved all three parameters of the problem; a, the rate of increase in demand with increasing stock, h, the unit holding cost, and c, the unit order cost. Such restrictions enable the retailer to set the benchmark maximum price accordingly, given the current inventory level and the imputed value of one unit of stock, so that net profit maximisation is realised.

Our results were obtained numerically for the most part. The linear dependence of demand on inventory facilitated the analysis somewhat. A more appropriate form could be concave, for instance ax^{β}, $0 < \beta < 1$, allowing for variable returns to the inventory level. In this case calculation of the key transition states between regions would be solely reliant on numerical analysis although the optimal laws would be qualitatively unchanged. Yet another form could be one that induces saturation in the demand rate when the inventory level reaches some a priori designated high value. It is certainly a worthwhile research program to improve on and extend the work presented here.

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