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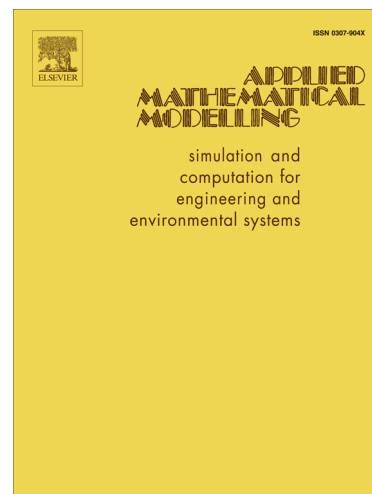
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Models of impulsive culling of mosquitoes to interrupt transmission of West Nile Virus to birds

Xiaxia Xu[1], Yanni Xiao [1*], Robert A. Cheke[2]

[1]*Department of Applied Mathematics, Xi'an Jiaotong University, Xi'an, 710049, P.R.China*
[2]*Natural Resources Institute, University of Greenwich at Medway, Central Avenue, Chatham Maritime, Chatham, Kent, ME4 4TB,UK*

Abstract

A mathematical model describing the transmission of West Nile virus (WNV) between vector mosquitoes and birds, incorporating a control strategy of culling mosquitoes and defined by impulsive differential equations is presented and its properties investigated. First, we consider a strategy of periodic impulsive culling of the mosquitoes. Theoretical results indicate that if the threshold R_0 is greater than unity the disease uniformly persists, but, if not, the disease does not necessarily become extinct. The explicit conditions determining the backward or forward bifurcation were obtained. The culling rate has a major effect on the occurrence of backward bifurcation. Analysis shows that the disease is most sensitive to mosquito-bird contacts, mosquito-culling rate and intervals between culls. The dependence of the outcomes of the culling strategy on mosquito biting rate is discussed. When the complete elimination of disease is impossible, mosquito culls are implemented once the

[1]*Corresponding author. E-mail: yxiao@mail.xjtu.edu.cn & yannixiao317@hotmail.com
Tel: +86(0)2982663938

infected birds reach a predefined but adjustable threshold value. Numerical analysis shows that the period of mosquito culling finally stabilizes at a fixed value. In addition, variations of mean prevalence of WNV in birds and the culling period are simulated.

1. Introduction

West Nile virus (WNV) is a virus of the family Flaviviridae. It mainly infects birds, and it is also known to infect humans, horses, dogs, cats, bats, chipmunks, skunks, squirrels, domestic rabbits, crocodiles and alligators [1, 2, 3, 4, 5, 6]. In the U.S.A., the disease is of serious public-health importance, with more than 30,000 cases and hundreds of deaths reported in 48 States since 1999 [7]. WNV is maintained in nature in a mosquito-bird-mosquito cycle [8, 9, 10], but the disease can be passed on to humans as a zoonotic disease when an infected mosquito changes hosts to bite people, who are dead-end hosts. The principal vectors are mosquitoes in the genera *Culex*, *Aedes*, *Anopheles* and *Ochlerotatus*, but other genera are also known to be infected in the wild. In the absence of an effective vaccine and/or treatment, anti-WNV efforts are primarily based on mosquito-reduction strategies (such as larviciding, adulticiding and elimination of breeding sites) and personal protection (based on the use of appropriate insect repellents). These measures are intensified during mosquito seasons [4, 11].

Compartmental epidemiological models have played a significant role in understanding the mechanisms of dynamical transmission of WNV. Lewis et al. studied the existence of travelling waves describing the speed of the spatial spread of the virus [12]. Lewis et al. also made a comparative study of discrete-time and continuous-time models to investigate WNV transmission [13]. Bowman et al. formulated a model system incorporating mosquito-bird-human populations for assessing control strategies against WNV [5]. In addition, ordinary differential equation models have been analyzed to investigate the threshold conditions for WNV outbreaks with back-

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ward bifurcation [2, 3, 4]. However, most of these mathematical models considering control measures on mosquitoes invariably assume that the pesticides affect mosquitoes continuously, but usually mosquito culling takes place only at certain times. It is known that impulsive differential equations can be used to describe pesticide sprays and analyse pest control strategies [14, 15, 16, 17, 18, 19]. As mosquito culling is a common method for WNV control [14, 15], we adopt it in this paper. The birds' population is considered to vary with time [3, 4, 8, 10], compared with the constant assumption in [14, 15]. Therefore our main purposes are to investigate the transmission of WNV between bird and mosquito populations with impulsive control strategies; analyze the dynamical behavior theoretically and investigate phenomena introduced by impulsive culling; determine the most rational strategy to control the transmission of WNV; and finally find out the mosquito-culling period to keep the infected mosquitoes always less than the adjustable threshold when the complete elimination of disease is impossible.

To achieve the above goals, we formulate two mathematical models, considering periodic or state-dependent pesticide sprays as control measures, to investigate the transmission of WNV between mosquitoes and birds, where the total number of birds varies with time. First, we propose impulsive differential equations, which have already been used to investigate malaria in human-mosquito populations [20], to describe the process of periodic culling of mosquitoes. Similar methods can be found in [21, 22, 23, 24, 25, 26]. Conditions for persistence of the disease and the occurrence of backward bifurcation are obtained theoretically. In order to assess the control strategy, sensitivity analysis is applied to study the contribution of each parameter on

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the disease transmission. If complete eradication of WNV is not possible, we extend our equations to a state-dependent model. The positive periodic solution with the conditions for the maximum value no more than the critical threshold is obtained numerically with a relatively high bird-mosquito contact rate. In addition, by considering resource savings and environmental protection, we can change the state-dependent impulsive control problem into a fixed-time impulsive control problem.

2. WNV control with fixed moments

We first consider the strategy of implementing periodic culling of mosquitoes at critical times, with differential equations proposed as follows. The total female mosquito population at time t, denoted by $N_m(t)$, is split into the populations of susceptible ($S_m(t)$) and infected ($I_m(t)$) mosquitoes. The susceptible mosquito number is increased via births or immigration at a constant rate Λ_m and diminished by infection – which may be acquired when uninfected mosquitoes feed from the blood of infected birds – and by death due to natural causes at a rate μ_m. The infected mosquito number is generated via the infection of susceptibles and diminished by natural deaths at a rate μ_m. $T > 0$ represents the mosquito-culling (such as spraying) interval. We assume that spraying reduces both susceptible and infected mosquitoes and $0 \leq p \leq 1$ is the proportion of those mosquitoes killed [20]. Similarly, the total bird population at time t, denoted by $N_b(t)$, is split into the populations

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of susceptible $(S_b(t))$ and infected $(I_b(t))$ birds.

$$\begin{cases} \begin{rcases} \frac{dS_m}{dt} = \Lambda_m - c\beta_{mb}\frac{I_b}{N_b}S_m - \mu_m S_m, \\[2mm] \frac{dI_m}{dt} = c\beta_{mb}\frac{I_b}{N_b}S_m - \mu_m I_m, \\[2mm] \frac{dS_b}{dt} = \Lambda_b - c\beta_{bm}\frac{S_b}{N_b}I_m - \mu_b S_b, \\[2mm] \frac{dI_b}{dt} = c\beta_{bm}\frac{S_b}{N_b}I_m - (\mu_b + d_b)I_b \end{rcases} t \neq nT, n \in \mathbb{N}, \mathbb{N} = 1, 2, ... \\[2mm] \begin{rcases} S_m(t^+) = (1-p)S_m(t), \\[2mm] I_m(t^+) = (1-p)I_m(t). \end{rcases} t = nT, n \in \mathbb{N}, \mathbb{N} = 1, 2, ... \end{cases} \quad (1)$$

where c is the average biting rate of the mosquitoes, Λ_b is the recruitment rate of birds, β_{bm} and β_{mb} are the transmission probabilities of WNV from mosquitoes to birds and from birds to mosquitoes respectively, μ_b is the natural death rate of the birds, and d_b is the WNV-induced death rate. All parameters are defined in Table 1.

3. Existence and stability of the disease-free periodic solution

First we consider the subsystem of (1) in the disease-free subspace $X_s = \{(S_m, I_m, S_b, I_b) : S_m \geq 0, I_m = 0, S_b \geq 0, I_b = 0\}$ as follows

$$\begin{cases} \begin{rcases} \frac{dS_m}{dt} = \Lambda_m - \mu_m S_m, \\[2mm] \frac{dS_b}{dt} = \Lambda_b - \mu_b S_b, \end{rcases} t \neq nT, n \in \mathbb{N}, \\[2mm] S_m(t^+) = (1-p)S_m(t), \quad t = nT, n \in \mathbb{N}. \end{cases} \quad (2)$$

Note that the bird population is free from impulse, so we know $S_b(t) \to \Lambda_b/\mu_b$, as $t \to \infty$. Without loss of generality, we let $\tilde{S}_b(t) = \Lambda_b/\mu_b$. The equations for the mosquitos in (2) are similar to those in [20], in which the decoupled impulsive differential equation for the total mosquito population was analyzed. It is not difficult to prove that the periodic solution for mosquitoes in

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(2) $(\tilde{S}_m)$, on the interval $(nT, (n+1)T]$

$$\tilde{S}_m(t) = \left(1 - \frac{pe^{-\mu_m(t-nT)}}{1-(1-p)e^{-\mu_m T}}\right)\frac{\Lambda_m}{\mu_m}, \tag{3}$$

is globally stable [20], and the following conclusion holds true.

Lemma 1. *System (2) has a unique positive T-periodic solution $(\tilde{S}_m, \tilde{S}_b)$ and for every solution $(S_m(t), S_b(t))$ of (2), $S_m(t) \to \tilde{S}_m$ and $S_b(t) \to \tilde{S}_b$ as $t \to \infty$.*

Based on the result of Lemma 1, system (1) admits the disease-free periodic solution (DFPS) $(\tilde{S}_m, 0, \tilde{S}_b, 0)$ on every impulsive interval $(nT, (n+1)T]$. To determine the stability of DFPS of system (1), we define

$$F = \begin{pmatrix} 0 & c\beta_{mb}\frac{\tilde{S}_m}{\tilde{S}_b} \\ c\beta_{bm} & 0 \end{pmatrix}, \quad V = \begin{pmatrix} \mu_m & 0 \\ 0 & \mu_b + d_b \end{pmatrix}.$$

Let $A(t)$ be a $n \times n$ matrix, $\Phi_{A(.)}(t)$ be the fundamental solution matrix of the linear ordinary differential system $x' = A(t)x$, and $r(\Phi_{A(.)}(w))$ be the spectral radius of $\Phi_{A(.)}(w)$. Let $S_m(t) = s_m(t) + \tilde{S}_m(t), I_m(t) = i_m(t), S_b(t) = s_b(t) + \tilde{S}_b(t), I_b(t) = i_b(t), x(t) = (s_m(t), s_b(t), i_m(t), i_b(t))$. Then system (1) becomes

$$\begin{cases} x'(t) &= Q(t)x(t), \quad t \neq nT, n \in \mathbb{N}, \\ x(t) &= \mathcal{P}x(t), \quad t = nT, n \in \mathbb{N}, \end{cases} \tag{3}$$

where

$$Q(t) = \begin{pmatrix} U & B \\ 0 & F-V \end{pmatrix}, \quad \mathcal{P} = \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}$$

with $U = \begin{pmatrix} -\mu_m & 0 \\ 0 & -\mu_b \end{pmatrix}, B = \begin{pmatrix} 0 & -c\beta_{mb}\frac{\tilde{S}_m}{\tilde{S}_b} \\ -c\beta_{bm} & 0 \end{pmatrix}, P = \begin{pmatrix} 1-p & 0 \\ 0 & 1 \end{pmatrix}.$

Let $\Phi_Q(t) = (\Phi_{ij})_{1\leq i,j\leq 2}$ be the fundamental matrix of $x'(t) = Q(t)x(t)$. Then

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$\Phi'_Q(t) = Q(t)\Phi_Q(t)$ with the initial value $\Phi_Q(0) = E_4$. Solving the equation gives

$$\Phi_Q(t) = \begin{pmatrix} e^{Ut} & \Phi_{12}(t) \\ 0 & \Phi_{F-V}(t) \end{pmatrix},$$

then we have

$$\mathcal{P}\Phi_Q(T) = \begin{pmatrix} Pe^{UT} & P\Phi_{12}(T) \\ 0 & P\Phi_{F-V}(T) \end{pmatrix}.$$

We can easily get that $r(Pe^{UT}) < 1$. Then the following theorem holds.

Theorem 2. *If $r(P\Phi_{F-V}(T)) < 1$ holds true, then the disease-free periodic solution $(\tilde{S}_m, 0, \tilde{S}_b, 0)$ of system (1) is locally asymptotically stable.*

Denote $R_0 = r(P\Phi_{F-V}(T))$. Then R_0 does not produce the number of birds infected by a single bird or the number of mosquitoes infected by a single mosquito. Namely it does not produce the average number of secondary infections [27]. However, it works as a threshold such that the disease persists as $R_0 > 1$.

4. Persistence of the disease

Theorem 3. *If $R_0 > 1$, the disease persists; namely, there exists $\eta > 0$ such that $\liminf\limits_{t\to\infty} I_i(t) \geq \eta > 0, i = m, b$.*

Proof. We first prove the following claim: there exists a positive constant η such that

$$\limsup_{t\to\infty} I_i(t) \geq \eta > 0, \quad i = m, b.$$

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Otherwise, there exists a $t_1 > 0$ such that $I_i(t) < \eta, i = m, b$, for all $t \geq t_1$.

By the first and third equations of system (1), we have

$$
\left\{
\begin{array}{l}
\frac{dS_m}{dt} \geq \Lambda_m - c\beta_{mb}\frac{S_m}{N_b}\eta - \mu_m S_m, \\
\frac{dS_b}{dt} \geq \Lambda_b - c\beta_{bm}\frac{S_b}{N_b}\eta - \mu_b S_b,
\end{array}
\right\} t \neq nT, n \in \mathbb{N},
$$
$$
S_m(t^+) = (1-p)S_m(t), \quad t = nT, n \in \mathbb{N}.
$$
(4)

Consider the auxiliary system

$$
\left\{
\begin{array}{l}
x_1' = \Lambda_m - c\beta_{mb}\frac{S_m}{N_b}\eta - \mu_m x_1, \\
x_2' = \Lambda_b - c\beta_{bm}\eta - \mu_b x_2,
\end{array}
\right\} t \neq nT, n \in \mathbb{N},
$$
$$
x_1(t^+) = (1-p)x_1(t), \quad t = nT, n \in \mathbb{N}.
$$
(5)

Using the same method as system (2), we obtain that system (5) admits a globally asymptotically stable positive periodic solution $\tilde{x} = (\tilde{x}_1, \tilde{x}_2)$, meanwhile $\lim_{\eta \to 0} \tilde{x} = (\tilde{S}_m, \tilde{S}_b)$. Thus there exists η_1 small enough and for any $\epsilon_1 > 0$, such that $\tilde{x}_1 \geq \tilde{S}_m - \epsilon_1$ and $\tilde{x}_2 \geq \tilde{S}_b - \epsilon_1$ for $\eta < \eta_1$. By the comparison theorem, there exists $t_2 \geq t_1$ and $\epsilon_2 > 0$, such that $S_m(t) \geq x_1(t) \geq \tilde{x}_1 - \epsilon_2 \geq \tilde{S}_m - \epsilon_1 - \epsilon_2$ and $S_b(t) \geq x_2(t) \geq \tilde{x}_2 - \epsilon_2 \geq \tilde{S}_b - \epsilon_1 - \epsilon_2$ for $t \geq t_2$.

By the second and fourth equations of system (1), we have

$$
\left\{
\begin{array}{l}
\frac{dI_m}{dt} \geq c\beta_{mb}\frac{I_b}{N_b}(\tilde{S}_m - \epsilon_1 - \epsilon_2) - \mu_m I_m, \\
\frac{dI_b}{dt} \geq c\beta_{bm}\frac{I_m}{N_b}(\tilde{S}_b - \epsilon_1 - \epsilon_2) - (\mu_b + d_b)I_b.
\end{array}
\right\} t \neq nT, n \in \mathbb{N},
$$
$$
I_m(t^+) = (1-p)I_m(t), \quad t = nT, n \in \mathbb{N}.
$$
(6)

In fact, by system (1) we have

$$
\left\{
\begin{array}{l}
\frac{dN_m}{dt} = \Lambda_m - \mu_m N_m, \\
\frac{dN_b}{dt} = \Lambda_b - \mu_b N_b - d_b I_b,
\end{array}
\right\} t \neq nT, n \in \mathbb{N},
$$
$$
N_m(t^+) = (1-p)N_m(t), \quad t = nT, n \in \mathbb{N}.
$$
(7)

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Comparing it with system (2) gives that $N_m^{(1)} \leq S_m^{(2)}(t)$ and $N_b^{(1)} \leq S_b^{(2)}(t)$, where $N_m^{(1)}$ and $N_b^{(1)}$ denote the solution of system (1), and $S_m^{(2)}(t)$ and $S_b^{(2)}(t)$ denote the solution of system (2). Since $S_m^{(2)}(t) \leq \tilde{S}_m$ and $S_b^{(2)}(t) \leq \tilde{S}_b$, we have $N_m^{(1)} \leq \tilde{S}_m$ and $N_b^{(1)} \leq \tilde{S}_b$. Then system (6) can be modified as

$$\begin{cases} \left. \begin{array}{ll} \frac{dI_m}{dt} & \geq c\beta_{mb}\frac{\tilde{S}_m}{\tilde{S}_b}I_b - \mu_m I_m, \\ \frac{dI_b}{dt} & \geq c\beta_{bm}I_m - (\mu_b + d_b)I_b. \end{array} \right\} t \neq nT, n \in \mathbb{N}, \\ I_m(t^+) = (1-p)I_m(t), \quad t = nT, n \in \mathbb{N}. \end{cases} \quad (8)$$

Consider the auxiliary system

$$\begin{cases} u'(t) & = (F-V)u(t), \quad t \neq nT, n \in \mathbb{N}, \\ u(t) & = (1-p)u(t), \quad t = nT, n \in \mathbb{N}, \end{cases} \quad (9)$$

where $u = (u_1, u_2)^\tau$. The solution of system (9) can be expressed as $u(t, nT, u(nT^+)) = \Phi_{F-V}(t-nT)u(nT^+)$. Then $u((n+1)T^+) = P\Phi_{F-V}(T)u(nT^+)$. While $R_0 > 1$, $u_1 \to \infty$ and $u_2 \to \infty$ as $t \to \infty$. Then $\lim_{t\to\infty} I_m = \infty$ and $\lim_{t\to\infty} I_b = \infty$, which contradicts with the boundedness of I_i $(i = m, b)$. Thus the claim is proved; that is, $\limsup_{t\to\infty} I_i(t) \geq \eta, i = m, b$.

From the claim, we discuss the following two possibilities.

(I) $I_i(t) > \eta$ for all large t, $i = m, b$;

(II) $I_i(t)$ oscillates about η for all large t, $i = m, b$.

If condition (I) holds, then we complete our proof. Next we will consider possibility (II). Since $\limsup_{t\to\infty} I_i(t) \geq \eta$, $i = m, b$, there exists a $t_1 \in (n_1 T, (n_1 + 1)T]$ such that $I_i(t_1) \geq \eta$, $i = m, b$. By the above discussion there exists another $t_2 \in (n_2 T, (n_2 + 1)T]$, where $n_2 - n_1 \geq 0$ is finite, such that $I_i(t_2) \geq \eta$, $i = m, b$. Then we will consider the solution of system (1) in the time interval $[t_1, t_2]$:

$$I_b' = c\beta_{bm}\frac{S_b}{N_b}I_m - (\mu_b + d_b)I_b \geq -(\mu_b + d_b)I_b.$$

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We have

$$I_b(t) \geq I_b(t_1)e^{-(\mu_b+d_b)(t-t_1)} \geq \eta e^{-(\mu_b+d_b)(t_2-t_1)} \geq \eta e^{-(\mu_b+d_b)(n_2-n_1+1)T}.$$

Moreover,

$$I'_m = c\beta_{mb}\frac{I_b}{N_b}S_m - \mu_m I_m \geq -\mu_m I_m, \quad t \neq nT,$$
$$I_m(nT^+) = (1-p)I_m(nT), \quad t = nT,$$

which gives

$$I_m(t) \geq \eta(1-p)^{n_2-n_1}e^{-\mu_m(t_2-t_1)} \geq \eta(1-p)^{n_2-n_1}e^{-\mu_m(n_2-n_1)T}.$$

Let $\eta_1 = \min\{\eta(1-p)^{n_2-n_1}e^{-\mu_m(n_2-n_1)T}, \ \eta e^{-(\mu_b+d_b)(n_2-n_1)T}\}$, then $\eta_1 > 0$ cannot be infinitely small ($n_2 - n_1 \geq 0$ is finite). We have $I_i(t) \geq \eta_1 > 0, i = m, b$.

For $t > t_2$, the same arguments can be continued. We similarly get non-infinitesimal positive η_2. Thus the sequence $\{\eta_j\}, j = 1, 2...k...$ where $\eta_k = \min\{\eta(1-p)^{n_{k+1}-n_k}e^{-\mu_m(n_{k+1}-n_k)T}, \ \eta e^{-(\mu_b+d_b)(n_{k+1}-n_k)T}\}$ is non-infinitesimal since $n_{k+1} - n_k \geq 0$ is finite. The solution of system (1) $I_i(t) \geq \eta_k > 0, i = m, b$ holds true in the time interval $[t_k, t_{k+1}], t_k \in (n_kT, (n_k+1)T], t_{k+1} \in (n_{k+1}T, (n_{k+1}+1)T]$. Let $\eta^* = \min_j \eta_j = \eta_l > 0, l \in N, \eta_l \in \{\eta_j\}, j = 1, 2...$ hence $I_i(t) \geq \eta^* > 0, i = m, b$ for all $t \geq t_1$. The proof is complete.

5. Forward and backward bifurcation of endemic periodic solutions

We now proceed to study bifurcation using the bifurcation theory of Lakmeche and Arino [28]. We let the culling rate p be the bifurcation parameter. Then define solution vector $X(t) := (S_m(t), S_b(t), I_m(t), I_b(t))$, the mapping $F(X(t)) = (F_1(X(t)), F_2(X(t)), F_3(X(t)), F_4(X(t))) : R^4 \longrightarrow R^4$ by

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the right hand side of the first four equations of system (1), and the mapping $I(p, X(t)) = (I_1(p, X(t)), I_2(p, X(t)), I_3(p, X(t)), I_4(p, X(t))) = ((1 - p)X_1(t), X_2(t), (1 - p)X_3(t), X_4(t))$. Furthermore, we define $\Phi(t, X_0), 0 < t \leq T$ to be the solution of the system consisting of the first four equations of system (1), where $X_0 = X(0)$. Then $X(T) = \Phi(T, X_0) := \Phi(X_0)$ and $X(T^+) = I(p, \Phi(X_0))$. Define the operator Ψ by

$$\Psi(p, X) := I(p, \Phi(X)),$$

where $\Psi(p, X) = (\Psi_1(p, X), \Psi_2(p, X), \Psi_3(p, X), \Psi_4(p, X))$. Denote $D_X \Psi$ the derivative of Ψ with respect to X. Then X is a periodic solution of period T for system (1) if and only if its initial value X_0 is a fixed point for $\Psi(p, X)$; namely, $\Psi(p, X_0) = X_0$. Consequently, to establish the existence of nontrivial periodic solutions of system (1), one needs to prove the existence of the nontrivial fixed point of Ψ.

Let us fix all parameters except the mosquito-culling rate p and denote by p_0 the critical culling rate, which corresponds to $r(P\Phi_{F-V}) = 1$. We are interested in the bifurcation of nontrivial periodic solutions near the disease-free periodic solution $\tilde{X} = (\tilde{S}_m, \tilde{S}_b, 0, 0)$. Assume that X_0 is the starting point for the disease-free periodic solution with the culling rate p_0. It is obvious that $\Phi_3(X_0) = \Phi_4(X_0) = 0$. To find a nontrivial periodic solution with initial value X and culling rate p, we need to solve the fixed point problem $\Psi(p, X) = X$. Denote $p = p_0 + \bar{p}$ and $X = X_0 + \bar{X}$, then the fixed point problem reads as

$$N(\bar{p}, \bar{X}) = 0, \tag{10}$$

where $N(\bar{p}, \bar{X}) = (N_1(\bar{p}, \bar{X}), N_2(\bar{p}, \bar{X}), N_3(\bar{p}, \bar{X}), N_4(\bar{p}, \bar{X})) = X_0 + \bar{X} - \Psi(p_0 +$

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$\bar{p}, X_0 + \bar{X})$. We have

$$D_X N(\bar{p}, \bar{X}) = E_4 - D_X I(p, \Phi(X)) D_X \Phi(X)). \tag{11}$$

Since
$$\frac{d}{dt}(D_X \Phi(t, X_0)) = D_X F(\Phi(t, X_0)) D_X \Phi(t, X_0), \tag{12}$$

with the initial condition $D_X \Phi(0, X_0) = E_4$ and

$\Phi(t, X_0) = (\Phi_1(t, X_0), \Phi_2(t, X_0), 0, 0)$, then (12) takes the form

$$\frac{d}{dt}(D_X \Phi(t, X_0))(t, X_0) = Q(t)(D_X \Phi(t, X_0))(t, X_0). \tag{13}$$

It can be deduced that

$$D_X N(0, O) = \begin{pmatrix} E_2 - \mathcal{P}e^{UT} & -\mathcal{P}\Phi_{12}(T) \\ 0 & E_2 - \mathcal{P}\Phi_{F-V}(T) \end{pmatrix},$$

where $O = (0, 0, 0, 0)$. A necessary condition for the bifurcation of the non-trivial periodic solution near $\tilde{X} = (\tilde{S}_m, \tilde{S}_b, 0, 0)$ is then

$$\det[D_X N(0, O)] = 0.$$

One can easily note that $\det[E_2 - \mathcal{P}e^{\mu T}] \neq 0$. Then $\det[D_X N(0, O)] = 0$ reduces to $\det[E_2 - \mathcal{P}\Phi_{F-V}(T)] = 0$. It is clear that, when $r(\mathcal{P}\Phi_{F-V}(T)) = 1$, one has $\det[E_2 - \mathcal{P}\Phi_{F-V}(T)] = 0$. Assume $r(\mathcal{P}\Phi_{F-V}(T)) = 1$ holds and we now investigate the sufficient conditions for the existence of bifurcation nontrivial T-period solutions. It is convenient for the computations to denote

$$D_X N(0, O) = \begin{pmatrix} e_0 & 0 & a_1 & b_1 \\ 0 & f_0 & c_1 & d_1 \\ 0 & 0 & a_0 & b_0 \\ 0 & 0 & c_0 & d_0 \end{pmatrix}, \quad \mathcal{A} := \begin{pmatrix} e_0 & 0 & a_1 \\ 0 & f_0 & c_1 \\ 0 & 0 & a_0 \end{pmatrix}.$$

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See Appendix A for the expression of each element in the above matrices. Then $\det[E_2 - \mathcal{P}\Phi_{F-V}(T)] = 0$ implies that there exists a constant k such that $c_0 = ka_0$ and $d_0 = kb_0$. Furthermore, we have $\dim \operatorname{Ker}(D_X N(0, O)) = 1$, and a basis in $\operatorname{Ker}(D_X N(0, O))$ is

$$Y_1 = (\frac{a_1 b_0}{a_0 e_0} - \frac{b_1}{e_0}, \frac{b_0 c_1}{a_0 f_0} - \frac{d_1}{f_0}, -\frac{b_0}{a_0}, 1);$$

we denote it as $Y_1 = (Y_{11}, Y_{12}, Y_{13}, Y_{14})$. The basis in $\operatorname{Im}(D_X N(0, O))$ are $Y_2 = (1, 0, 0, 0), Y_3 = (0, 1, 0, 0), Y_4 = (0, 0, 1, 0)$. From the decomposition

$$\mathcal{R}^4 = \operatorname{Ker}(D_X N(0, O)) \bigoplus \operatorname{Im}(D_X N(0, O)),$$

we have $\bar{X} = \alpha_1 Y_1 + \alpha_2 Y_2 + \alpha_3 Y_3 + \alpha_4 Y_4$, where $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in R$ are unique. Then equation (10) is equivalent to

$$N_i(\bar{p}, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = N_i(\bar{p}, \alpha_1 Y_1 + \alpha_2 Y_2 + \alpha_3 Y_3 + \alpha_4 Y_4) = 0, \qquad (14)$$

$i = 1, 2, 3, 4$. From (14), we have

$$\frac{D(N_1, N_2, N_3)(0, O)}{D(\alpha_2, \alpha_3, \alpha_4)} = |\mathcal{A}| \neq 0. \qquad (15)$$

Therefore, by the implicit function theorem, one may solve equation (14) as $i = 1, 2, 3$ near $(0, O)$ with respect to $\alpha_i, i = 2, 3, 4$ as functions of $\bar{p}$ and α_1, and find $\tilde{\alpha}_i = \tilde{\alpha}_i(\bar{p}, \alpha_1)$ such that $\tilde{\alpha}_i(0, 0) = 0, i = 2, 3, 4$, and

$$N_i(\bar{p}, \alpha_1) = N_i(\bar{p}, \alpha_1 Y_1 + \tilde{\alpha}_2 Y_2 + \tilde{\alpha}_3 Y_3 + \tilde{\alpha}_4 Y_4) = 0, \qquad (16)$$

$i = 1, 2, 3$. Then $N(\bar{p}, \bar{X}) = 0$ if and only if

$$N_4(\bar{p}, \alpha_1) = N_4(\bar{p}, \bar{X}(\bar{p}, \alpha_1)) = 0, \qquad (17)$$

with $\bar{X}(\bar{p}, \alpha_1) = (Y_{11}\alpha_1 + \tilde{\alpha}_2, Y_{12}\alpha_1 + \tilde{\alpha}_3, Y_{13}\alpha_1 + \tilde{\alpha}_4, \alpha_1)$.

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We proceed to solving (17) next. It is obvious that $N_4(\bar{p}, \alpha_1)$ vanishes at $(0,0)$. We determine the Taylor expansion of $N_4(\bar{p}, \alpha_1)$ around $(0,0)$. First, we compute the first-order partial derivatives $\partial N_4(0,0)/\partial \alpha_1$ and $\partial N_4(0,0)/\partial \bar{p}$, and find that

$$\frac{\partial N_4(0,0)}{\partial \alpha_1} = \frac{\partial N_4(0,0)}{\partial \bar{p}} = 0.$$

(See Appendix B for details). Then it is necessary for us to compute the second-order derivatives of $N_4(\bar{p}, \alpha_1)$. Denote

$$A = \frac{\partial^2 N_4(0,0)}{\partial \bar{p}^2}, \quad B = \frac{\partial^2 N_4(0,0)}{\partial \bar{p} \partial \alpha_1}, \quad C = \frac{\partial^2 N_4(0,0)}{\partial \alpha_1^2}.$$

It can be observed from Appendix C that $A = 0$, from Appendix D that

$$\begin{aligned}
B = \quad & \frac{\Phi_1(X_0)}{e_0} \left[\frac{\partial^2 \Phi_4(X_0)}{\partial S_m \partial I_m} Y_{13} + \frac{\partial^2 \Phi_4(X_0)}{\partial S_m \partial I_b} Y_{14} \right] \\
& - \frac{k(1-p_0)\Phi_1(X_0)}{e_0} \left(\frac{\partial^2 \Phi_3(X_0)}{\partial S_m \partial I_m} Y_{13} + \frac{\partial^2 \Phi_3(X_0)}{\partial S_m \partial I_b} Y_{14} \right) + \frac{k}{1-p_0} Y_{13},
\end{aligned}$$

and from Appendix E that

$$C = \sum_{i=1}^4 \sum_{j=1}^4 \left(-\frac{\partial^2 \Phi_4(X_0)}{\partial X_i \partial X_j} + k(1-p_0)\frac{\partial^2 \Phi_3(X_0)}{\partial X_i \partial X_j} \right) Y_{1i} Y_{1j}.$$

Hence we have

$$\begin{aligned}
N_4(\bar{p}, \alpha_1) &= B\alpha_1 \bar{p} + C\frac{\alpha_1^2}{2} + o(\alpha_1, \bar{p})(\alpha_1^2 + \bar{p}^2) \\
&= \alpha_1 \left(B\bar{p} + C\frac{\alpha_1}{2} + \frac{1}{\alpha_1} o(\alpha_1, \bar{p})(\alpha_1^2 + \bar{p}^2) \right).
\end{aligned}$$

Denoting

$$\tilde{N}_4(\bar{p}, \alpha_1) = B\bar{p} + C\frac{\alpha_1}{2} + \frac{1}{\alpha_1} o(\alpha_1, \bar{p})(\alpha_1^2 + \bar{p}^2),$$

then

$$\frac{\partial \tilde{N}_4(0,0)}{\partial \alpha_1} = \frac{C}{2}.$$

Hence, for $C \neq 0$, we can use the implicit function theorem and solve the above equation near $(0,0)$ with respective to α_1 as a function of $\bar{p}$, and find

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$\alpha_1 = \alpha_1(\bar{p})$ such that $\alpha_1(0) = 0$ and $\tilde{N}_4(\bar{p}, \alpha_1(\bar{p})) = 0$. Since $\partial \tilde{N}_4(0, 0)/\partial \bar{p} = B$, we can also find $\bar{p} = \bar{p}(\alpha_1)$ such that $\tilde{N}_4(\bar{p}(\alpha_1), \alpha_1) = 0$, provided $B \neq 0$. Then, if $BC \neq 0$, we have $\alpha_1/\bar{p} \simeq -2B/C$. There is a supercritical bifurcation to a nontrivial periodic solution near the fixed point X_0 if $BC < 0$, or else a subcritical one if $BC > 0$. We know that the threshold R_0 decreases as p increases. Then a supercritical bifurcation means a backward bifurcation in the model while the subcritical bifurcation equated to a forward bifurcation in the $p - \alpha_1$ plane. Thus we have the following theorem.

Theorem 4. *As the parameter p passes through the critical value p_0, a backward bifurcation occurs if $BC < 0$, or else there will be a forward bifurcation as $BC > 0$ at $R_0 = 1$.*

6. Numerical studies

6.1. Uncertainty and Sensitivity Analysis

In this section, uncertainty and sensitivity analysis, based on the Latin Hypercube Sampling (LHS) scheme [29, 30], is applied to explore the variabilities of the outcome variables due to the uncertainty in estimating input parameters, and to examine how disease spread is sensitive to parameters [31]. We mainly investigate three outcome variables: new infections N_0 and deaths of birds D_0, and the threshold R_0. Define

$$N_0 := \frac{1}{T} \int_{nT}^{(n+1)T} c\beta_{bm} \frac{S_b(t)I_m(t)}{N_b(t)} dt,$$
$$D_0 := \frac{1}{T} \int_{nT}^{(n+1)T} d_b I_b(t) dt,$$

where n is large enough to make sure that the outcome variables stabilize at the values of the periodic solution. Although the explicit expression for R_0 cannot be obtained, we can numerically calculate it.

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Each input parameter is sampled 2000 times. We choose a normal distribution for $c, \beta_{mb}, \beta_{bm}, \mu_m, \mu_b$ and d_b with mean value and possible value range given in table 1. The remaining parameters λ_m, λ_b, p and T are assumed to be uniform in distribution due to lack of further information. The initial values are $S_m(0) = 10,000$ and $S_b(0) = 1,000$ [4], and we assume $I_m(0) = 10$ and $I_b(0) = 0$. We know that the mosquito population is replenished by 130 new adult female mosquitoes per day for every 1,000 female mosquitoes [32]. Thus the recruitment of mosquitoes should be $10,000 \times 130/1000$ at the beginning of the transmission, resulting in $\Lambda_m = 1,300$. We let $\Lambda_b = 2.1$ be the baseline value, which possibly ranges from 1.8 to 2.4 according to paper [4].

The partial rank correlation coefficients (PRCCs) between each input parameter and outcome variable, which can identify the importance of the parameter contribution to the variabilities of outcomes, together with p-values are presented in Fig.1 and Table 2. We consider absolute values of PRCC>0.4 as indicating an important correlation between an input parameter and output variables, values between 0.1 and 0.4 as moderate correlations, and values between 0 and 0.1 as not significantly different from zero [33]. First, we analyze the influence of each parameter on N_0 and find that N_0 is only sensitive to the mosquito biting rate c . The parameters with a moderate impact on N_0 are T and β_{mb}. The remaining seven parameters slightly affect N_0. Fig.1 (a) also shows that increasing Λ_b, μ_m, μ_b and p, or decreasing $c, \beta_{mb}, T, \Lambda_m, \beta_{bm}$ and d_b can lead to a decline in N_0. Second, we analyze the influence of each parameter on D_0. D_0 is most sensitive to parameter T, followed by c. Parameter Λ_m affects the outcome moderately.

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The remaining seven parameters contribute little to D_0. Finally, we consider the contribution of each parameter on the threshold R_0. It follows from Fig.1(c) and Table 2 that the three parameters with most impact on R_0 are the culling interval T, the culling rate p and the biting rate of mosquitoes c. Moreover, an increase in any of the four parameters μ_m, Λ_b, p and d_b can lead to a decline in R_0. Note that the pattern of PRCC on R_0 is a bit different from that of PRCC on the new infections N_0. This is because R_0 describes the initial transmission of the disease, whereas the new infections N_0 is calculated as time is large enough such that the disease stabilizes at the periodic level. In a summary, all of the three considered outcomes are sensitive to the parameter c. Therefore, reducing the mosquito-bird contacts by burning repellent plants in the birds' habitat or at their water sources, similar to the strategies of avoiding mosquito bites on humans [34], could effectively limit the disease transmission and weaken the WNV-introduced damage to the bird population. Other techniques such as bird-scarers or deploying falcons might also succeed in deterring birds from sensitive sites.

The LHS uncertainty technique is used to explore the effect of the uncertainty in estimating the values of the input parameters on the prediction precision of the three outcome variables. 2000 estimates are made for these three outcome variables; Fig.2 (a)-(c) show their frequency distributions. The mean value for them are $M_{N_0} = 0.9844$, $M_{D_0} = 0.0214$ and $M_{R_0} = 1.3448$ with standard deviation $S_{N_0}=0.2624$, $S_{D_0}=0.0001$ and $S_{R_0}=0.2523$, respectively. The coefficients of variation (CV), the ratio of the standard deviations to the means, were 26.66%, 0.47% and 18.76%, respectively. We consider the frequency distribution for a variable to be dispersed if CV is greater than

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10%, else to be concentrated. Fig.2 (a) indicates that the derived frequency distribution for N_0 is quite dispersed ($CV_{N_0} = 26.66\% > 10\%$), where the minimum estimate is 0.1200 and the maximum is 6.9221. Moreover, 90 percent of these estimates are less than 1.4896. From Fig.2 (b), we know that the frequency distribution for deaths of birds D_0 ranged from 0.0017 to 0.1245 and is concentrated ($CV_{D_0} = 0.47\%$). Our estimate from replicated LHS indicates that the probability of $R_0 > 1$ is 0.9014. It also follows from Fig.2 (c) that 90 percent of these estimates of R_0 are less than 1.8462, where the maximum and minimum estimates are 7.1070 and 0.9422, respectively. Moreover, the CV for D_0 ($CV_{D_0} = 0.47\%$) is the smallest among those for the three outcomes. This implies that the frequency of D_0 is the most concentrated. These analyses of D_0 enable us to generate quantitative results with which to better understand the deaths of birds caused by WNV. Estimation of the cumulative distribution function (CDF) of the three output variables based on LHS is also shown in Fig.2 (d)-(f). This uncertainty technique enables us to quantify the degree of prediction imprecision, and we can use it as a basis for comparing the expected results (three output variables) with the observed results.

6.2. Effectiveness of control strategies

To assess the effect of culling mosquitoes on controlling the spread of WNV, we numerically compute the prevalence for both birds and mosquitoes as shown in Fig.3. The mean prevalences of WNV for both mosquitoes and birds decreases as the culling rate (p) increases while the time interval for the culling of mosquitoes (T) is fixed (shown in Fig.3 (a) and (b)), or decreases as the time interval for mosquito culling T is reduced for a fixed mosquito

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culling rate p (Fig.3 (c) and (d)).

It follows from Fig.3 (a) that the prevalence of WNV in the mosquitoes declines rapidly after an impulsive culling and reaches its minimum value between two neighboring culls. Compared to the variation in the prevalence of WNV in the mosquitoes, the prevalence of WNV in the birds also decreases after the culling and reaches its own minimum value, but with a slight delay (Fig.3 (b)). This is mainly because the birds are infected by infectious mosquitoes. Thus the decreased number of infected mosquitoes leads to a decrease in infected birds delaying the birds' prevalence reaching its minimum value. Fix $p = 0.85, T = 5$ (the third case in Fig.3 (c) and (d)). Then we have $R_0 = 0.9985$ and the disease finally persists.

It follows from Fig.1(c) that R_0 is sensitive to the culling rate (p), the culling interval (T) and the biting rate of mosquitoes (c). The contour plots of Fig.4 (a)-(c) show the dependence of R_0 on parameters p and T with various mosquito biting rates. For the minimum value of the biting rate (c=0.03, shown in Fig.4(a)), either a small culling interval or a relatively large culling interval with a large culling rate can reduce the threshold to less than unity. Conversely, for middle or maximum values of the biting rate (c=0.09 and c=0.15 shown in Fig.4(b) and (c)), the threshold can be less than unity for frequent culls at a very high rate. This implies that periodically culling mosquitoes could make the threshold less than unity, but by how much and how often the intervention needs to be implemented is influenced by the biting rate of mosquitoes.

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7. WNV control with a critical threshold

The WNV transmission model with a fixed periodic control strategy was investigated and the results obtained above imply that the infected birds and mosquitoes can be completely eradicated under suitable conditions. In reality, complete eradication of WNV by culling mosquitoes is generally not possible. A good control programme should reduce the infected birds to levels acceptable to the public. Thus we will use a state-dependent impulsive model in this section to show how this can be achieved. The culling interventions are applied when the infected birds reach a critical threshold (CT). Then p proportion of both susceptible and infected mosquitoes are removed when the culling such as spraying is implemented. We thus have the following model:

$$
\begin{cases}
\left.
\begin{aligned}
\frac{dS_m}{dt} &= \Lambda_m - c\beta_{mb}\frac{I_b}{N_b}S_m - \mu_m S_m, \\
\frac{dI_m}{dt} &= c\beta_{mb}\frac{I_b}{N_b}S_m - \mu_m I_m, \\
\frac{dS_b}{dt} &= \Lambda_b - c\beta_{bm}\frac{S_b}{N_b}I_m - \mu_b S_b, \\
\frac{dI_b}{dt} &= c\beta_{bm}\frac{S_b}{N_b}I_m - (\mu_b + d_b)I_b
\end{aligned}
\right\} I_b < CT, \\[2pt]
\left.
\begin{aligned}
S_m(t^+) &= (1-p)S_m(t), \\
I_m(t^+) &= (1-p)I_m(t).
\end{aligned}
\right\} I_b = CT.
\end{cases}
\tag{18}
$$

The definitions of variables and parameters are the same as those in system (1). We let the critical threshold (CT) be 340 and fix other parameters as in Table 1. It appears from Fig.5 that there exists a periodic solution with bird population size no larger than the critical threshold CT. Interestingly, the impulsive period quickly stabilizes at a fixed value, denoted by T_c (here $T_c \approx 4.13$). That is, the number of infected birds will not exceed the critical threshold if we carry out the culling measure at each time point T_c. Of course, if we implement the impulsive culling in a time interval which is less

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than T_c, then the maximum level of the infected birds will be reduced and thus the number of infected mosquitos, and *vice versa*.

Note that, as we pointed out in the last section, the biting rate of mosquitoes is a key parameter that greatly influences the outcomes. If parameter c is chosen small, the disease may either die out or persist at a low level such that the number of infected birds cannot exceed the CT and then it is, of course, unnecessary to initiate the intervention. If parameter c is chosen relatively large, the number of infected birds frequently reaches the threshold CT and the cull strategy is then correspondingly implemented. In particular, as c increases, the intervention may be implemented more frequently given CT and the culling rate.

In the following, we will investigate the variations in the period of the process T_C and the mean prevalence of diseased birds MP_b with respect to the culling proportion p based on the critical threshold being $CT = 340$, shown in Fig.6 (a), or with respect to the critical threshold CT based on the culling proportion being $p = 0.75$, shown in Fig.6 (b). Here we define the mean disease prevalence for birds MP_b in a period as

$$\mathrm{MP}_b := \frac{1}{T_c} \int_{nT_c}^{(n+1)T_c} \frac{I_b(t)}{S_b(t) + I_b(t)} dt.$$

Fig.6 (a) shows that there will be a distinct rise in the period T_c, and the mean prevalence of WNV in birds decreases as the culling proportion p increases. This is because the mean values for both the number of infected mosquitoes and the prevalence of infected mosquitoes decrease as the culling proportion increases, and then the incidence of birds decreases accordingly, leading to a longer time for the infected birds population to rebound back to the threshold, which results in longer period cycles. Moreover the mean

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WNV prevalences for the birds population also decreases according to the decreased incidence of birds. We know from Fig.6 (b) that both the culling period T_c and the mean WNV prevalence for birds increase as the threshold CT increases. Without doubt, a long period between cullings will be convenient for a control programme to implement the strategy. However, it is really not beneficial for disease control with a high prevalence. Thus choosing a suitable threshold is quite essential to curb the disease spreading.

8. Conclusion and Discussion

WNV transmission has been investigated through different types of mathematical models. Impulsive differential equations, on the one hand, can fully reflect the actual control situation, and, on the other hand, they can guide the operator to implement the impulsive control strategy accurately and conveniently. Hu et al. [14] and Gourley et al. [15] proposed impulsive equations investigating the transmission of WNV where the mosquitoes are subject to culling and the bird population is assumed to be constant. We extend this modelling approach to include both a non-constant bird population and impulsive culling regimes. In particular, we consider the strategy of culling mosquitoes at fixed time or when the infected birds reach a critical threshold.

We note that many papers have been published on impulsive interventions of disease or pest control [17, 18, 19, 35, 36, 37], in some of which the explicit expressions of thresholds for persistence of systems were obtained. However, most of the cited systems can be simplified such that the equations of infected individuals related to the calculation are actually one-dimensional. In our study, we consider a model of more complex cross-infection between two

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populations, and determine the threshold that can completely govern the dynamics of the system. Namely, the disease-free periodic solution is locally asymptotically stable if $R_0 < 1$ while it is uniformly persistent if $R_0 > 1$. Moreover, system (1) may exhibit a backward bifurcation of nontrivial periodic solution branching from the disease-free periodic solution at $R_0 = 1$, depending on the sign of the expression BC. It follows from Fig.7 that both a disease-free periodic solution and an endemic periodic solution are feasible and bi-stable. The culling rate greatly affects the occurrence of a backward bifurcation and Fig.8 shows the variation in R_0 with culling rate p. In the case that the parameters are chosen such that the non-impulsive system proposed in [3] undergoes a backward bifurcation, the impulsive system (1) initially exhibits backward bifurcation and the bifurcated periodic solution appears as the culling rate increase from zero to the critical value 0.1877, then the bifurcated periodic solution disappears as p exceeds the critical value. Thus the bifurcated periodic solution exists when p lies in the interval $(0, 0.1877)$, as shown in Fig.8 (a). On the other hand, in the case that the parameters are chosen such that the non-impulsive system proposed in [3] persists, it is obvious that the threshold R_0 of impulsive system (1) is initially greater than 1 as p is sufficiently small. But the backward bifurcation occurs as p exceeds 0.2088 and the bifurcated periodic solution is feasible till p reaches 0.7871. Thus bifurcated periodic solution is present when p is in the interval of $(0.2088, 0.7871)$, as shown in Fig.8 (b). This extends the corresponding conclusion for the system without impulsive intervention [3].

In order to completely eradicate WNV, our results suggest that the strategy of frequent culling with a relative high rate should be implemented when

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the mosquito-bird contact rate c is relatively high. When the complete erad-
ication is not possible, our main purpose becomes keeping the infected birds
below a threshold for ecological damage. A state-dependent model is then
formulated. Numerical simulation shows that there exists a positive peri-
odic solution with the maximum value of the infected birds no larger than
the previously chosen level. The period of this periodic solution T_c can be
numerically obtained, then we can successfully control the infected bird pop-
ulation at a level no larger than the critical threshold if we implement the
culling programme at each T_c time point. Consequently, the state-dependent
impulsive control problem can be changed into a fixed-time impulsive control
problem, which is easily analyzed. Therefore we only need to know the ini-
tial data on bird and mosquito numbers instead of repeatedly observing and
calculating the number of birds in reality. Then we can control the disease
with a culling strategy applied at fixed moments, significantly reducing costs
[23].

Sensitivity analysis shows that the disease transmission is sensitive to
the mosquito biting rate c. Thus reducing mosquito-bird contacts is sug-
gested to be an effective method to reduce the disease transmission. This
method of reducing contacts is also advocated in [32, 38, 39], in which the
authors studied the spread of malaria between humans and mosquitoes. Con-
sider non-impulsive mosquito culling which has been investigated in [5]. The
threshold

$$R_{0c} = \sqrt{\frac{c^2 \beta_{mb} \beta_{bm}}{(\mu_b + d_b)\mu_m} \frac{\Lambda_m/\mu_m}{\Lambda_b/\mu_b}}$$

gives the geometric mean of the number of birds infected by the vector and the
number of vectors infected by a bird [27]. It is clear from the mathematical

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formula that R_{0c} is an increasing function of the parameters Λ_m, c, β_{mb}, β_{bm} and μ_b, and a decreasing function of the parameters Λ_b, μ_m, d_b. The result is consistent with the sensitivity results of R_0 for model (1). In fact, it is not difficult to observe that R_{0c} is positively correlated with the ratio of mosquito to bird size at the disease-free level $(\Lambda_m/\mu_m)/(\Lambda_b/\mu_b)$, which implies that a reduction in mosquito density (a reduction in Λ_m or a rise in μ_m) or a rise in bird density (a rise in Λ_b or a reduction in μ_b) can help to control the epidemic. This is consistent with our sensitivity results for the new infections or the threshold R_0. Lewis also concludes from his continuous model in [12] that a reduction in bird density can exacerbate the epidemic.

It is important to emphasize that the factors of seasonal variation in mosquito population size, the diversity of bird species, the dispersion of both birds and mosquitoes, and vertical transmission of the virus in the mosquito population, affect the dynamics of both mosquitoes and birds and hence disease spread between mosquitoes and birds. We leave these topics for future work.

Acknowledgements

Appendix A: The expression for each element of $D_X N(0, O)$

It is clear from equation (5) that

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial\Phi_3}{\partial X_1} &= -\mu_m\frac{\partial\Phi_3}{\partial X_1} + c\beta_{mb}\frac{\tilde{S}_m}{\tilde{S}_b}\frac{\partial\Phi_4}{\partial X_1}, & \frac{\partial\Phi_3}{\partial X_1}(0, X_0) &= 0, \\
\frac{d}{dt}\frac{\partial\Phi_4}{\partial X_1} &= c\beta_{bm}\frac{\partial\Phi_3}{\partial X_1} - (\mu_b + d_b)\frac{\partial\Phi_4}{\partial X_1}, & \frac{\partial\Phi_4}{\partial X_1}(0, X_0) &= 0, \\
\frac{d}{dt}\frac{\partial\Phi_3}{\partial X_2} &= -\mu_m\frac{\partial\Phi_3}{\partial X_2} + c\beta_{mb}\frac{\tilde{S}_m}{\tilde{S}_b}\frac{\partial\Phi_4}{\partial X_2}, & \frac{\partial\Phi_3}{\partial X_2}(0, X_0) &= 0, \\
\frac{d}{dt}\frac{\partial\Phi_4}{\partial X_2} &= c\beta_{bm}\frac{\partial\Phi_3}{\partial X_2} - (\mu_b + d_b)\frac{\partial\Phi_4}{\partial X_2}, & \frac{\partial\Phi_4}{\partial X_2}(0, X_0) &= 0.
\end{aligned}
$$

Thus one obtains

$$
\frac{\partial\Phi_i}{\partial X_j}(t, X_0) \equiv 0, i = 3, 4, j = 1, 2
$$

for $0 \le t < T$. Then, we have

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial\Phi_1}{\partial X_1} &= -\mu_m\frac{\partial\Phi_1}{\partial X_1}, & \frac{\partial\Phi_1}{\partial X_1}(0, X_0) &= 1, \\
\frac{d}{dt}\frac{\partial\Phi_1}{\partial X_2} &= -\mu_m\frac{\partial\Phi_1}{\partial X_2}, & \frac{\partial\Phi_1}{\partial X_2}(0, X_0) &= 0, \\
\frac{d}{dt}\frac{\partial\Phi_2}{\partial X_1} &= -\mu_b\frac{\partial\Phi_2}{\partial X_1}, & \frac{\partial\Phi_2}{\partial X_1}(0, X_0) &= 1, \\
\frac{d}{dt}\frac{\partial\Phi_2}{\partial X_2} &= -\mu_b\frac{\partial\Phi_2}{\partial X_2}, & \frac{\partial\Phi_2}{\partial X_2}(0, X_0) &= 0.
\end{aligned}
$$

It is obvious that

$$
\frac{\partial\Phi_1}{\partial X_1}(t, X_0) = e^{-\mu_m t}, \quad \frac{\partial\Phi_1}{\partial X_2}(t, X_0) = 0,
$$

$$
\frac{\partial\Phi_2}{\partial X_1}(t, X_0) = 0, \quad \frac{\partial\Phi_2}{\partial X_2}(t, X_0) = e^{-\mu_b t}.
$$

Consequently, one obtains

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial\Phi_3}{\partial X_3} &= -\mu_m\frac{\partial\Phi_3}{\partial X_3} + c\beta_{mb}\frac{\tilde{S}_m}{\tilde{S}_b}\frac{\partial\Phi_4}{\partial X_3}, & \frac{\partial\Phi_3}{\partial X_3}(0, X_0) &= 1, \\
\frac{d}{dt}\frac{\partial\Phi_4}{\partial X_3} &= c\beta_{bm}\frac{\partial\Phi_3}{\partial X_3} - (\mu_b + d_b)\frac{\partial\Phi_4}{\partial X_3}, & \frac{\partial\Phi_4}{\partial X_3}(0, X_0) &= 0, \\
\frac{d}{dt}\frac{\partial\Phi_3}{\partial X_4} &= -\mu_m\frac{\partial\Phi_3}{\partial X_4} + c\beta_{mb}\frac{\tilde{S}_m}{\tilde{S}_b}\frac{\partial\Phi_4}{\partial X_4}, & \frac{\partial\Phi_3}{\partial X_4}(0, X_0) &= 0, \\
\frac{d}{dt}\frac{\partial\Phi_4}{\partial X_4} &= c\beta_{bm}\frac{\partial\Phi_3}{\partial X_4} - (\mu_b + d_b)\frac{\partial\Phi_4}{\partial X_4}, & \frac{\partial\Phi_4}{\partial X_4}(0, X_0) &= 1, \\
\frac{d}{dt}\frac{\partial\Phi_1}{\partial X_3} &= -\mu_m\frac{\partial\Phi_1}{\partial X_3} - c\beta_{mb}\frac{\tilde{S}_m}{\tilde{S}_b}\frac{\partial\Phi_4}{\partial X_3}, & \frac{\partial\Phi_1}{\partial X_3}(0, X_0) &= 0, \\
\frac{d}{dt}\frac{\partial\Phi_2}{\partial X_3} &= -c\beta_{bm}\frac{\partial\Phi_3}{\partial X_3} - \mu_b\frac{\partial\Phi_2}{\partial X_3}, & \frac{\partial\Phi_2}{\partial X_3}(0, X_0) &= 0, \\
\frac{d}{dt}\frac{\partial\Phi_1}{\partial X_4} &= -\mu_m\frac{\partial\Phi_1}{\partial X_4} - c\beta_{mb}\frac{\tilde{S}_m}{\tilde{S}_b}\frac{\partial\Phi_4}{\partial X_4}, & \frac{\partial\Phi_1}{\partial X_4}(0, X_0) &= 0, \\
\frac{d}{dt}\frac{\partial\Phi_2}{\partial X_4} &= -c\beta_{bm}\frac{\partial\Phi_3}{\partial X_4} - \mu_b\frac{\partial\Phi_2}{\partial X_4}, & \frac{\partial\Phi_2}{\partial X_4}(0, X_0) &= 0,
\end{aligned}
$$

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We solve the above equations and denote

$$
\begin{aligned}
e_0 &= 1 - (1 - p_0)\frac{\partial \Phi_1}{\partial X_1}(T, X_0) = 1 - (1 - p_0)e^{-\mu_m T}, \\
f_0 &= 1 - \frac{\partial \Phi_2}{\partial X_2}(T, X_0) = 1 - e^{-\mu_b T}. \\
a_1 &= -(1 - p_0)\frac{\partial \Phi_1}{\partial X_3}(T, X_0), \quad b_1 = -(1 - p_0)\frac{\partial \Phi_1}{\partial X_4}(T, X_0), \\
c_1 &= -\frac{\partial \Phi_2}{\partial X_3}(T, X_0), \quad d_1 = -\frac{\partial \Phi_2}{\partial X_4}(T, X_0), \\
a_0 &= 1 - (1 - p_0)\frac{\partial \Phi_3}{\partial X_3}(T, X_0), \quad b_0 = -(1 - p_0)\frac{\partial \Phi_3}{\partial X_4}(T, X_0), \\
c_0 &= -\frac{\partial \Phi_4}{\partial X_3}(T, X_0), \quad d_0 = 1 - \frac{\partial \Phi_4}{\partial X_4}(T, X_0).
\end{aligned}
$$

Appendix B: The first-order partial derivatives of $N_4(\bar{p}, \alpha_1)$

We can easily get

$$
\begin{aligned}
\frac{\partial N_4(0,0)}{\partial \alpha_1} &= \frac{\partial N_4(0,0)}{\partial S_m}\left(Y_{11} + \frac{\partial \tilde{\alpha}_2(0,0)}{\partial \alpha_1}\right) + \frac{\partial N_4(0,0)}{\partial S_b}\left(Y_{12} + \frac{\partial \tilde{\alpha}_3(0,0)}{\partial \alpha_1}\right) \\
&\quad + \frac{\partial N_1(0,0)}{\partial I_m}\left(Y_{13} + \frac{\partial \tilde{\alpha}_4(0,0)}{\partial \alpha_1}\right) + \frac{\partial N_4(0,0)}{\partial I_b}Y_{14}, \\
\frac{\partial N_4(0,0)}{\partial \bar{p}} &= -\frac{\partial \Phi_4(X_0)}{\partial S_m}\frac{\partial \tilde{\alpha}_2(0,0)}{\partial \bar{p}} - \frac{\partial \Phi_4(X_0)}{\partial S_b}\frac{\partial \tilde{\alpha}_3(0,0)}{\partial \bar{p}} - \frac{\partial \Phi_4(X_0)}{\partial I_m}\frac{\partial \tilde{\alpha}_4(0,0)}{\partial \bar{p}}.
\end{aligned}
\tag{19}
$$

From the equation of (16) as $i = 1$, we have

$$
0 = \frac{\partial N_1}{\alpha_1} = \frac{\partial N_1}{\partial S_m}Y_{11} + \frac{\partial N_1}{\partial S_b}Y_{12} + \frac{\partial N_1}{\partial I_m}Y_{13} + \frac{\partial N_1}{\partial I_b}Y_{14} + \frac{\partial N_1}{\partial S_m}\frac{\partial \tilde{\alpha}_2}{\partial \alpha_1} + \frac{\partial N_1}{\partial S_b}\frac{\partial \tilde{\alpha}_3}{\partial \alpha_1} + \frac{\partial N_1}{\partial I_m}\frac{\partial \tilde{\alpha}_4}{\partial \alpha_1}.
\tag{20}
$$

Since Y_1 is a basis in $\mathrm{Ker}(D_X N(0, O))$, namely

$$
\frac{\partial N_i(0,0)}{\partial S_m}Y_{11} + \frac{\partial N_i(0,0)}{\partial S_b}Y_{12} + \frac{\partial N_i(0,0)}{\partial I_m}Y_{13} + \frac{\partial N_i(0,0)}{\partial I_b}Y_{14} = 0, \quad i = 1, 2, 3, 4.
\tag{21}
$$

Thus we can deduce from (20) and (21) that

$$
e_0\frac{\partial \tilde{\alpha}_2(0,0)}{\partial \alpha_1} + 0 \cdot \frac{\partial \tilde{\alpha}_3(0,0)}{\partial \alpha_1} + a_1\frac{\partial \tilde{\alpha}_4(0,0)}{\partial \alpha_1} = 0.
\tag{22}
$$

Similarly, from the equation of (16) as $i = 2, 3$, we can obtain that

$$
\begin{aligned}
0 \cdot \frac{\partial \tilde{\alpha}_2(0,0)}{\partial \alpha_1} + f_0\frac{\partial \tilde{\alpha}_3(0,0)}{\partial \alpha_1} + c_1\frac{\partial \tilde{\alpha}_4(0,0)}{\partial \alpha_1} = 0, \\
0 \cdot \frac{\partial \tilde{\alpha}_2(0,0)}{\partial \alpha_1} + 0 \cdot \frac{\partial \tilde{\alpha}_3(0,0)}{\partial \alpha_1} + a_0\frac{\partial \tilde{\alpha}_4(0,0)}{\partial \alpha_1} = 0.
\end{aligned}
\tag{23}
$$

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It is obvious from (22) and (23) that

$$\frac{\partial \tilde{\alpha}_2(0,0)}{\partial \alpha_1} = \frac{\partial \tilde{\alpha}_3(0,0)}{\partial \alpha_1} = \frac{\partial \tilde{\alpha}_4(0,0)}{\partial \alpha_1} = 0. \tag{24}$$

Considering equation (16) as $i = 1$, we have

$$N_1(\bar{p}, \alpha_1) = X_{01} + Y_{11}\alpha_1 + \tilde{\alpha}_2 - (1 - p_0 - \bar{p})\Phi_1(p_0 + \bar{p}, X_0 + \bar{X}(\bar{p}, \alpha_1)), \tag{25}$$

with $X_0 = (X_{01}, X_{02}, X_{03}, X_{04})$ and $\bar{X} = (\bar{X}_1, \bar{X}_2, \bar{X}_3, \bar{X}_4)$. Thus one obtains

$$\begin{aligned} 0 = \frac{\partial N_1(0,0)}{\partial \bar{p}} &= \frac{\partial \tilde{\alpha}_2(0,0)}{\partial \bar{p}} + \Phi_1(p_0, X_0) - (1 - p_0) \sum_{i=1}^{3} \frac{\partial \Phi_1(p_0, X_0)}{\partial X_i} \frac{\partial \tilde{\alpha}_{i+1}(0,0)}{\partial \bar{p}} \\ &= \Phi_1(p_0, X_0) + \left(e_0 \frac{\partial \tilde{\alpha}_2(0,0)}{\partial \bar{p}} + 0 \cdot \frac{\partial \tilde{\alpha}_3(0,0)}{\partial \bar{p}} + a_1 \frac{\partial \tilde{\alpha}_4(0,0)}{\partial \bar{p}} \right). \end{aligned} \tag{26}$$

One can similarly obtain from equation (16) as $i = 2, 3$ that

$$\begin{aligned} \frac{\partial N_2(0,0)}{\partial \bar{p}} &= 0 \cdot \frac{\partial \tilde{\alpha}_2(0,0)}{\partial \bar{p}} + f_0 \frac{\partial \tilde{\alpha}_3(0,0)}{\partial \bar{p}} + c_1 \frac{\partial \tilde{\alpha}_4(0,0)}{\partial \bar{p}} = 0, \\ \frac{\partial N_3(0,0)}{\partial \bar{p}} &= \Phi_3(p_0, X_0) + 0 \cdot \frac{\partial \tilde{\alpha}_2(0,0)}{\partial \bar{p}} + 0 \cdot \frac{\partial \tilde{\alpha}_3(0,0)}{\partial \bar{p}} + a_0 \frac{\partial \tilde{\alpha}_4(0,0)}{\partial \bar{p}} = 0 \end{aligned} \tag{27}$$

Moreover, $\Phi_3(p_0, X_0) = 0$ holds since X_0 is the starting point of disease-free periodic solution $\tilde{X}$. It can be deduced from (26) and (27) that

$$\frac{\partial \tilde{\alpha}_2(0,0)}{\partial \bar{p}} = \frac{-\Phi_1(X_0)}{e_0}, \quad \frac{\partial \tilde{\alpha}_3(0,0)}{\partial \bar{p}} = \frac{\partial \tilde{\alpha}_4(0,0)}{\partial \bar{p}} = 0. \tag{28}$$

Since

$$\frac{\partial N_4(0,0)}{\partial S_m} = \frac{\partial N_4(0,0)}{\partial S_b} = 0,$$

we can thus observe from (19), (21), (24) and (28) that

$$\frac{\partial N_4(0,0)}{\partial \alpha_1} = \frac{\partial N_4(0,0)}{\partial \bar{p}} = 0.$$

Appendix C: The second-order partial derivatives of $N_4(\bar{p}, \alpha_1)$ with respect

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to p

From equation (12) we have

$$\frac{d}{dt}\frac{\partial \Phi_3(t,X_0)}{\partial X_1} = \frac{\partial F_3(\tilde{X}(t))}{\partial X_1}\frac{\partial \Phi_1(t,X_0)}{\partial X_1} + \frac{\partial F_3(\tilde{X}(t))}{\partial X_2}\frac{\partial \Phi_2(t,X_0)}{\partial X_1} + \frac{\partial F_3(\tilde{X}(t))}{\partial X_3}\frac{\partial \Phi_3(t,X_0)}{\partial X_1}$$
$$+ \frac{\partial F_3(\tilde{X}(t))}{\partial X_4}\frac{\partial \Phi_4(t,X_0)}{\partial X_1}.$$

Then

$$\frac{d}{dt}\frac{\partial^2 \Phi_3(t,X_0)}{\partial X_1^2} = \frac{\partial F_3^2(\tilde{X}(t))}{\partial X_1^2}\frac{\partial \Phi_1(t,X_0)}{\partial X_1} + \frac{\partial F_3(\tilde{X}(t))}{\partial X_1}\frac{\partial \Phi_1^2(t,X_0)}{\partial X_1^2} + \frac{\partial F_3^2(\tilde{X}(t))}{\partial X_1 \partial X_2}\frac{\partial \Phi_2(t,X_0)}{\partial X_1}$$
$$+ \frac{\partial F_3(\tilde{X}(t))}{\partial X_2}\frac{\partial \Phi_2^2(t,X_0)}{\partial X_1^2} + \frac{\partial F_3^2(\tilde{X}(t))}{\partial X_1 \partial X_3}\frac{\partial \Phi_3(t,X_0)}{\partial X_1} + \frac{\partial F_3(\tilde{X}(t))}{\partial X_3}\frac{\partial \Phi_3^2(t,X_0)}{\partial X_1^2}$$
$$+ \frac{\partial F_3^2(\tilde{X}(t))}{\partial X_1 \partial X_4}\frac{\partial \Phi_4(t,X_0)}{\partial X_1} + \frac{\partial F_3(\tilde{X}(t))}{\partial X_4}\frac{\partial \Phi_4^2(t,X_0)}{\partial X_1^2}.$$

It is obvious that

$$\frac{\partial F_3^2(\tilde{X}(t))}{\partial X_1^2} = \frac{\partial F_3(\tilde{X}(t))}{\partial X_1} = \frac{\partial F_3^2(\tilde{X}(t))}{\partial X_1 \partial X_2} = \frac{\partial F_3^2(\tilde{X}(t))}{\partial X_1 \partial X_3} = \frac{\partial \Phi_4(t,X_0)}{\partial X_1} = 0.$$

Thus

$$\frac{d}{dt}\frac{\partial^2 \Phi_3(t,X_0)}{\partial X_1^2} = \frac{\partial F_3(\tilde{X}(t))}{\partial X_3}\frac{\partial \Phi_3^2(t,X_0)}{\partial X_1^2} + \frac{\partial F_3(\tilde{X}(t))}{\partial X_4}\frac{\partial \Phi_4^2(t,X_0)}{\partial X_1^2}. \tag{29}$$

We can similarly obtain

$$\frac{d}{dt}\frac{\partial^2 \Phi_4(t,X_0)}{\partial X_1^2} = \frac{\partial F_4(\tilde{X}(t))}{\partial X_3}\frac{\partial \Phi_3^2(t,X_0)}{\partial X_1^2} + \frac{\partial F_4(\tilde{X}(t))}{\partial X_4}\frac{\partial \Phi_4^2(t,X_0)}{\partial X_1^2}. \tag{30}$$

With the initial conditions

$$\frac{\partial^2 \Phi_3(0, X_0)}{\partial X_1^2} = \frac{\partial^2 \Phi_4(0, X_0)}{\partial X_1^2} = 0,$$

it can be deduced from (29) and (30) that

$$\frac{\partial^2 \Phi_3(t, X_0)}{\partial X_1^2} = \frac{\partial^2 \Phi_4(t, X_0)}{\partial X_1^2} = 0. \tag{31}$$

The same method can be adopted to get that

$$\frac{\partial^2 \Phi_3(t, X_0)}{\partial X_2^2} = \frac{\partial^2 \Phi_4(t, X_0)}{\partial X_2^2} = 0, \tag{32}$$

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and

$$\frac{\partial^2 \Phi_3(t, X_0)}{\partial X_1 \partial X_2} = \frac{\partial^2 \Phi_3(t, X_0)}{\partial X_2 \partial X_1} = \frac{\partial^2 \Phi_4(t, X_0)}{\partial X_1 \partial X_2} = \frac{\partial^2 \Phi_4(t, X_0)}{\partial X_2 \partial X_1} = 0. \qquad (33)$$

Based on the second equation of (24), one obtains that

$$
\begin{aligned}
0 = \ & \frac{\partial^2 N_3(0,0)}{\partial \bar{p}^2} = \frac{\partial}{\partial \bar{p}} \frac{\partial N_3(0,0)}{\partial \bar{p}} \\
= \ & \frac{\partial^2 \tilde{\alpha}_4(0,0)}{\partial \bar{p}^2} + \sum_{i=1}^{3} \frac{\partial \Phi_3(p_0, X_0)}{\partial X_i} \frac{\partial \tilde{\alpha}_{i+1}(0,0)}{\partial \bar{p}} + \frac{\partial \Phi_3(p_0, X_0)}{\partial \bar{p}} \\
& -(1 - p_0) \cdot \frac{\partial}{\partial \bar{p}} \left(\frac{\partial \Phi_3(p_0, X_0)}{\partial X_1} \frac{\partial \tilde{\alpha}_2(0,0)}{\partial \bar{p}} + \frac{\partial \Phi_3(p_0, X_0)}{\partial X_2} \frac{\partial \tilde{\alpha}_3(0,0)}{\partial \bar{p}} + \frac{\partial \Phi_3(p_0, X_0)}{\partial X_3} \frac{\partial \tilde{\alpha}_4(0,0)}{\partial \bar{p}} \right) \\
= \ & \frac{\partial^2 \tilde{\alpha}_4(0,0)}{\partial \bar{p}^2} - (1 - p_0) \left(\frac{\partial \Phi_3(p_0, X_0)}{\partial X_1} \frac{\partial^2 \tilde{\alpha}_2(0,0)}{\partial \bar{p}^2} + \frac{\partial \Phi_3(p_0, X_0)}{\partial X_2} \frac{\partial^2 \tilde{\alpha}_3(0,0)}{\partial \bar{p}^2} \right. \\
& + \frac{\partial \Phi_3(p_0, X_0)}{\partial X_3} \frac{\partial^2 \tilde{\alpha}_4(0,0)}{\partial \bar{p}^2} \\
& + \frac{\partial \tilde{\alpha}_2(0,0)}{\partial \bar{p}} \left[\frac{\partial^2 \Phi_3(p_0, X_0)}{\partial X_1^2} \frac{\partial \tilde{\alpha}_2(0,0)}{\partial \bar{p}} + \frac{\partial^2 \Phi_3(p_0, X_0)}{\partial X_1 \partial X_2} \frac{\partial \tilde{\alpha}_3(0,0)}{\partial \bar{p}} + \frac{\partial^2 \Phi_3(p_0, X_0)}{\partial X_1 \partial X_3} \frac{\partial \tilde{\alpha}_4(0,0)}{\partial \bar{p}} \right] \\
& + \frac{\partial \tilde{\alpha}_3(0,0)}{\partial \bar{p}} \left[\frac{\partial^3 \Phi_3(p_0, X_0)}{\partial X_1 \partial X_2} \frac{\partial \tilde{\alpha}_2(0,0)}{\partial \bar{p}} + \frac{\partial^2 \Phi_3(p_0, X_0)}{\partial X_2^2} \frac{\partial \tilde{\alpha}_3(0,0)}{\partial \bar{p}} + \frac{\partial^2 \Phi_3(p_0, X_0)}{\partial X_2 \partial X_3} \frac{\partial \tilde{\alpha}_4(0,0)}{\partial \bar{p}} \right] \\
& + \left. \frac{\partial \tilde{\alpha}_4(0,0)}{\partial \bar{p}} \left[\frac{\partial^3 \Phi_3(p_0, X_0)}{\partial X_1 \partial X_3} \frac{\partial \tilde{\alpha}_2(0,0)}{\partial \bar{p}} + \frac{\partial^2 \Phi_3(p_0, X_0)}{\partial X_2 \partial X_3} \frac{\partial \tilde{\alpha}_3(0,0)}{\partial \bar{p}} + \frac{\partial^2 \Phi_3(p_0, X_0)}{\partial X_3^2} \frac{\partial \tilde{\alpha}_4(0,0)}{\partial \bar{p}} \right] \right) \\
& + \frac{\partial \Phi_3(p_0, X_0)}{\partial X_1} \frac{\partial \tilde{\alpha}_2(0,0)}{\partial \bar{p}} + \frac{\partial \Phi_3(p_0, X_0)}{\partial X_2} \frac{\partial \tilde{\alpha}_3(0,0)}{\partial \bar{p}} + \frac{\partial \Phi_3(p_0, X_0)}{\partial X_3} \frac{\partial \tilde{\alpha}_4(0,0)}{\partial \bar{p}}.
\end{aligned}
$$
$$(34)$$

Submitting (28) and (31) into (34), it can be deduced that

$$\frac{\partial^2 \tilde{\alpha}_4(0,0)}{\partial \bar{p}^2} = 0. \qquad (35)$$

We can easily get that

$$
\begin{aligned}
\frac{\partial^2 N_4(0,0)}{\partial \bar{p}^2} = \ & \frac{\partial}{\partial \bar{p}} \left(-\frac{\partial \Phi_4(X_0)}{\partial S_m} \frac{\partial \tilde{\alpha}_2(0,0)}{\partial \bar{p}} - \frac{\partial \Phi_4(X_0)}{\partial S_b} \frac{\partial \tilde{\alpha}_3(0,0)}{\partial \bar{p}} - \frac{\partial \Phi_4(X_0)}{\partial I_m} \frac{\partial \tilde{\alpha}_4(0,0)}{\partial \bar{p}} \right) \\
= \ & -\frac{\partial \tilde{\alpha}_2(0,0)}{\partial \bar{p}} \frac{\partial}{\partial \bar{p}} \left(\frac{\partial \Phi_4(X_0)}{\partial S_m} \right) - \frac{\partial \Phi_4(X_0)}{\partial S_m} \frac{\partial^2 \tilde{\alpha}_2(0,0)}{\partial \bar{p}^2} - \frac{\partial \tilde{\alpha}_3(0,0)}{\partial \bar{p}} \frac{\partial}{\partial \bar{p}} \left(\frac{\partial \Phi_4(X_0)}{\partial S_b} \right) \\
& -\frac{\partial \Phi_4(X_0)}{\partial S_b} \frac{\partial^2 \tilde{\alpha}_3(0,0)}{\partial \bar{p}^2} - \frac{\partial \tilde{\alpha}_4(0,0)}{\partial \bar{p}} \frac{\partial}{\partial \bar{p}} \left(\frac{\partial \Phi_4(X_0)}{\partial I_m} \right) - \frac{\partial \Phi_4(X_0)}{\partial I_m} \frac{\partial^2 \tilde{\alpha}_4(0,0)}{\partial \bar{p}^2} \\
= \ & -\frac{\partial \tilde{\alpha}_2(0,0)}{\partial \bar{p}} \left(\frac{\partial^2 \Phi_4(X_0)}{\partial^2 S_m} \frac{\partial \tilde{\alpha}_2(0,0)}{\partial \bar{p}} + \frac{\partial^2 \Phi_4(X_0)}{\partial S_m \partial S_b} \frac{\partial \tilde{\alpha}_3(0,0)}{\partial \bar{p}} + \frac{\partial^2 \Phi_4(X_0)}{\partial S_m \partial I_m} \frac{\partial \tilde{\alpha}_4(0,0)}{\partial \bar{p}} \right) \\
& -\frac{\partial \Phi_4(X_0)}{\partial I_m} \frac{\partial^2 \tilde{\alpha}_4(0,0)}{\partial \bar{p}^2}.
\end{aligned}
$$

Submitting (28), (31) and (35) into the above equation, it is obvious that

$$A = \frac{\partial^2 N_4(0,0)}{\partial \bar{p}^2} = 0.$$

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Appendix D: The second-order partial derivatives of $N_4(\bar{p}, \alpha_1)$ with respect to p and α_1

We will first calculate the value of $\partial^2 \tilde{\alpha}_4(0,0)/\partial\bar{p}\partial\alpha_1$.

$$
\begin{aligned}
0 = \ & \frac{\partial^2 N_3(0,0)}{\partial\bar{p}\partial\alpha_1} = \frac{\partial}{\partial\alpha_1}\frac{\partial N_3(0,0)}{\partial\bar{p}} \\
= \ & \frac{\partial^2 \tilde{\alpha}_4(0,0)}{\partial\bar{p}\partial\alpha_1} + \frac{\partial\Phi_3(p_0,X_0)}{\partial\alpha_1} \\
& -(1-p_0)\cdot\frac{\partial}{\partial\alpha_1}\left(\frac{\partial\Phi_3(p_0,X_0)}{\partial X_1}\frac{\partial\tilde{\alpha}_2(0,0)}{\partial\bar{p}} + \frac{\partial\Phi_3(p_0,X_0)}{\partial X_2}\frac{\partial\tilde{\alpha}_3(0,0)}{\partial\bar{p}} + \frac{\partial\Phi_3(p_0,X_0)}{\partial X_3}\frac{\partial\tilde{\alpha}_4(0,0)}{\partial\bar{p}}\right) \\
= \ & \frac{\partial^2 \tilde{\alpha}_4(0,0)}{\partial\bar{p}\partial\alpha_1} + \frac{\partial\Phi_3(p_0,X_0)}{\partial X_1}\left(Y_{11} + \frac{\partial\tilde{\alpha}_2(0,0)}{\partial\alpha_1}\right) \\
& + \frac{\partial\Phi_3(p_0,X_0)}{\partial X_2}\left(Y_{12} + \frac{\partial\tilde{\alpha}_3(0,0)}{\partial\alpha_1}\right) + \frac{\partial\Phi_3(p_0,X_0)}{\partial X_3}\left(Y_{13} + \frac{\partial\tilde{\alpha}_4(0,0)}{\partial\alpha_1}\right) + \frac{\partial\Phi_3(p_0,X_0)}{\partial X_4}Y_{14} \\
& (1-p_0)\cdot\left(\frac{\partial\Phi_3(p_0,X_0)}{\partial X_1}\frac{\partial^2\tilde{\alpha}_2(0,0)}{\partial\bar{p}\partial\alpha_1} + \frac{\partial\tilde{\alpha}_2(0,0)}{\partial\bar{p}}\left[\frac{\partial^2\Phi_3(p_0,X_0)}{\partial X_1^2}(Y_{11} + \frac{\partial\tilde{\alpha}_2(0,0)}{\partial\alpha_1})\right.\right. \\
& + \frac{\partial^2\Phi_3(p_0,X_0)}{\partial X_2\partial X_1}(Y_{12} + \frac{\partial^2\tilde{\alpha}_3(0,0)}{\partial\alpha_1}) + \frac{\partial^2\Phi_3(p_0,X_0)}{\partial X_3\partial X_1}(Y_{13} + \frac{\partial\tilde{\alpha}_4(0,0)}{\partial\alpha_1}) + \frac{\partial^2\Phi_3(p_0,X_0)}{\partial X_4\partial X_1}Y_{14}\bigg] \\
& + \frac{\partial\Phi_3(p_0,X_0)}{\partial X_2}\frac{\partial^2\tilde{\alpha}_3(0,0)}{\partial\alpha_1\partial\bar{p}} + \frac{\partial\tilde{\alpha}_3(0,0)}{\partial\bar{p}}\frac{\partial}{\partial\alpha_1}(\frac{\partial\Phi_3(p_0,X_0)}{\partial X_2}) \\
& \left.+ \frac{\partial\Phi_3(p_0,X_0)}{\partial X_3}\frac{\partial^2\tilde{\alpha}_4(0,0)}{\partial\alpha_1\partial\bar{p}} + \frac{\partial\tilde{\alpha}_4(0,0)}{\partial\bar{p}}\frac{\partial}{\partial\alpha_1}(\frac{\partial\Phi_3(p_0,X_0)}{\partial X_3})\right).
\end{aligned}
\tag{36}
$$

Once again, by submitting (24), (28), (31) and (33) into (36), we can thus deduce that

$$
\begin{aligned}
\frac{\partial\tilde{\alpha}_4^2(0,0)}{\partial\bar{p}\partial\alpha_1} = \ & \frac{-(1-p_0)\Phi_1(X_0)}{a_0 e_0}\left(\frac{\partial\Phi_3^2(X_0)}{\partial S_m\partial I_m}Y_{13} + \frac{\partial\Phi_3^2(X_0)}{\partial S_m\partial I_b}Y_{14}\right) \\
& -\frac{1}{e_0}\big(\frac{\partial\Phi_3(X_0)}{\partial I_m}Y_{13} + \frac{\partial\Phi_3(X_0)}{\partial I_b}Y_{14}\big).
\end{aligned}
\tag{37}
$$

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It can be calculated that

$$
\begin{aligned}
\frac{\partial^2 N_4(0,0)}{\partial \bar{p} \partial \alpha_1} =\ & \frac{\partial}{\partial \alpha_1} \left(-\frac{\partial \Phi_4(X_0)}{\partial S_m} \frac{\partial \tilde{\alpha}_2(0,0)}{\partial \bar{p}} - \frac{\partial \Phi_4(X_0)}{\partial S_b} \frac{\partial \tilde{\alpha}_3(0,0)}{\partial \bar{p}} - \frac{\partial \Phi_4(X_0)}{\partial I_m} \frac{\partial \tilde{\alpha}_4(0,0)}{\partial \bar{p}} \right) \\
=\ & -\frac{\partial \Phi_4(X_0)}{\partial S_m} \frac{\partial^2 \tilde{\alpha}_2(0,0)}{\partial \bar{p} \partial \alpha_1} - \frac{\partial \tilde{\alpha}_2(0,0)}{\partial \bar{p}} \frac{\partial}{\partial \alpha_1} \left(\frac{\partial \Phi_4(X_0)}{\partial S_m} \right) \\
& -\frac{\partial \Phi_4(X_0)}{\partial S_b} \frac{\partial^2 \tilde{\alpha}_3(0,0)}{\partial \bar{p} \partial \alpha_1} - \frac{\partial \tilde{\alpha}_3(0,0)}{\partial \bar{p}} \frac{\partial}{\partial \alpha_1} \left(\frac{\partial \Phi_4(X_0)}{\partial S_b} \right) \\
& -\frac{\partial \Phi_4(X_0)}{\partial I_m} \frac{\partial^2 \tilde{\alpha}_4(0,0)}{\partial \bar{p} \partial \alpha_1} - \frac{\partial \tilde{\alpha}_4(0,0)}{\partial \bar{p}} \frac{\partial}{\partial \alpha_1} \left(\frac{\partial \Phi_4(X_0)}{\partial I_m} \right) \\
=\ & -\frac{\partial \tilde{\alpha}_2(0,0)}{\partial \bar{p}} \frac{\partial}{\partial \alpha_1} \left(\frac{\partial \Phi_4(X_0)}{\partial S_m} \right) - \frac{\partial \Phi_4(X_0)}{\partial I_m} \frac{\partial^2 \tilde{\alpha}_4(0,0)}{\partial \bar{p} \partial \alpha_1} \ . \\
=\ & -\frac{\partial \tilde{\alpha}_2(0,0)}{\partial \bar{p}} \left[\frac{\partial^2 \Phi_4(X_0)}{\partial S_m^2} \left(Y_{11} + \frac{\partial \tilde{\alpha}_2(0,0)}{\partial \alpha_1} \right) + \frac{\partial^2 \Phi_4(X_0)}{\partial S_m \partial S_b} \left(Y_{12} + \frac{\partial \tilde{\alpha}_3(0,0)}{\partial \alpha_1} \right) \right. \\
& \left. \frac{\partial^2 \Phi_4(X_0)}{\partial S_m \partial I_m} \left(Y_{13} + \frac{\partial \tilde{\alpha}_4(0,0)}{\partial \alpha_1} \right) + \frac{\partial^2 \Phi_4(X_0)}{\partial S_m \partial I_b} Y_{14} \right] - \frac{\partial \Phi_4(X_0)}{\partial I_m} \frac{\partial^2 \tilde{\alpha}_4(0,0)}{\partial \bar{p} \partial \alpha_1} \\
=\ & -\frac{\partial \tilde{\alpha}_2(0,0)}{\partial \bar{p}} \left[\frac{\partial^2 \Phi_4(X_0)}{\partial S_m \partial I_m} Y_{13} + \frac{\partial^2 \Phi_4(X_0)}{\partial S_m \partial I_b} Y_{14} \right] - \frac{\partial \Phi_4(X_0)}{\partial I_m} \frac{\partial^2 \tilde{\alpha}_4(0,0)}{\partial \bar{p} \partial \alpha_1} \\
=\ & \frac{\Phi_1(X_0)}{e_0} \left[\frac{\partial^2 \Phi_4(X_0)}{\partial S_m \partial I_m} Y_{13} + \frac{\partial^2 \Phi_4(X_0)}{\partial S_m \partial I_b} Y_{14} \right] + k a_0 \frac{\partial^2 \tilde{\alpha}_4(0,0)}{\partial \bar{p} \partial \alpha_1}
\end{aligned}
$$

Submitting (37) into the above equation, one obtains

$$
\begin{aligned}
\frac{\partial^2 N_4(0,0)}{\partial \bar{p} \partial \alpha_1} =\ & \frac{\Phi_1(X_0)}{e_0} \left[\frac{\partial^2 \Phi_4(X_0)}{\partial S_m \partial I_m} Y_{13} + \frac{\partial^2 \Phi_4(X_0)}{\partial S_m \partial I_b} Y_{14} \right] \\
& -\frac{k(1-p_0)\Phi_1(X_0)}{e_0} \left(\frac{\partial^2 \Phi_3(X_0)}{\partial S_m \partial I_m} Y_{13} + \frac{\partial^2 \Phi_3(X_0)}{\partial S_m \partial I_b} Y_{14} \right) + \frac{k}{1-p_0} Y_{13}.
\end{aligned}
$$

Appendix E: The second-order partial derivatives of $N_4(\bar{p}, \alpha_1)$ with respect to α_1

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By tedious calculation we have

$$
\begin{aligned}
\frac{\partial^2 N_4(0,0)}{\partial \alpha_1^2} =\ & \frac{\partial}{\partial \alpha_1}\left(\frac{\partial N_4(0,0)}{\partial S_m}(Y_{11} + \frac{\partial \tilde{\alpha}_2(0,0)}{\partial \alpha_1}) + \frac{\partial N_4(0,0)}{\partial S_b}(Y_{12} + \frac{\partial \tilde{\alpha}_3(0,0)}{\partial \alpha_1})\right. \\
& \left. + \frac{\partial N_1(0,0)}{\partial I_m}(Y_{13} + \frac{\partial \tilde{\alpha}_4(0,0)}{\partial \alpha_1}) + \frac{\partial N_4(0,0)}{\partial I_b}Y_{14}\right) \\
=\ & (Y_{11} + \frac{\partial \tilde{\alpha}_2(0,0)}{\partial \alpha_1}) \cdot \frac{\partial}{\partial \alpha_1}(\frac{\partial N_4(0,0)}{\partial S_m}) + \frac{\partial N_4(0,0)}{\partial S_m}\frac{\partial^2 \tilde{\alpha}_2(0,0)}{\partial \alpha_1^2} \\
& + (Y_{12} + \frac{\partial \tilde{\alpha}_3(0,0)}{\partial \alpha_1}) \cdot \frac{\partial}{\partial \alpha_1}(\frac{\partial N_4(0,0)}{\partial S_b}) + \frac{\partial N_4(0,0)}{\partial S_b}\frac{\partial^2 \tilde{\alpha}_3(0,0)}{\partial \alpha_1^2} \\
& + (Y_{13} + \frac{\partial \tilde{\alpha}_4(0,0)}{\partial \alpha_1}) \cdot \frac{\partial}{\partial \alpha_1}(\frac{\partial N_4(0,0)}{\partial I_m}) + \frac{\partial N_4(0,0)}{\partial I_m}\frac{\partial^2 \tilde{\alpha}_4(0,0)}{\partial \alpha_1^2} \\
& + Y_{14} \cdot \frac{\partial}{\partial \alpha_1}(\frac{\partial N_4(0,0)}{\partial I_b}) \\
=\ & \frac{\partial}{\partial \alpha_1}\left(Y_{11}\frac{\partial N_4(0,0)}{\partial S_m} + Y_{12}\frac{\partial N_4(0,0)}{\partial S_b} + Y_{13}\frac{\partial N_4(0,0)}{\partial I_m}\right. \\
& \left. + Y_{14}\frac{\partial N_4(0,0)}{\partial I_b}\right) + \frac{\partial N_4(0,0)}{\partial I_m}\frac{\partial^2 \tilde{\alpha}_4(0,0)}{\partial \alpha_1^2} \\
=\ & Y_{11}\left(\frac{\partial^2 N_4(0,0)}{\partial S_m^2}(Y_{11} + \frac{\partial \tilde{\alpha}_2(0,0)}{\partial \alpha_1}) + \frac{\partial^2 N_4(0,0)}{\partial S_m \partial S_b}(Y_{12} + \frac{\partial \tilde{\alpha}_3(0,0)}{\partial \alpha_1})\right. \\
& \left. + \frac{\partial^2 N_4(0,0)}{\partial S_m \partial I_m}(Y_{13} + \frac{\partial \tilde{\alpha}_4(0,0)}{\partial \alpha_1}) + \frac{\partial^2 N_4(0,0)}{\partial S_m \partial I_b}Y_{14}\right) \\
& + Y_{12}\left(\frac{\partial^2 N_4(0,0)}{\partial S_m \partial S_b}(Y_{11} + \frac{\partial \tilde{\alpha}_2(0,0)}{\partial \alpha_1}) + \frac{\partial^2 N_4(0,0)}{\partial S_b^2}(Y_{12} + \frac{\partial \tilde{\alpha}_3(0,0)}{\partial \alpha_1})\right. \\
& \left. + \frac{\partial^2 N_4(0,0)}{\partial S_b \partial I_m}(Y_{13} + \frac{\partial \tilde{\alpha}_4(0,0)}{\partial \alpha_1}) + \frac{\partial^2 N_4(0,0)}{\partial S_b \partial I_b}Y_{14}\right) \\
& + Y_{13}\left(\frac{\partial^2 N_4(0,0)}{\partial S_m \partial I_m}(Y_{11} + \frac{\partial \tilde{\alpha}_2(0,0)}{\partial \alpha_1}) + \frac{\partial^2 N_4(0,0)}{\partial S_b \partial I_m}(Y_{12} + \frac{\partial \tilde{\alpha}_3(0,0)}{\partial \alpha_1})\right. \\
& \left. + \frac{\partial^2 N_4(0,0)}{\partial I_m^2}(Y_{13} + \frac{\partial \tilde{\alpha}_4(0,0)}{\partial \alpha_1}) + \frac{\partial^2 N_4(0,0)}{\partial I_m \partial I_b}Y_{14}\right) \\
& + Y_{14}\left(\frac{\partial^2 N_4(0,0)}{\partial S_m \partial I_b}(Y_{11} + \frac{\partial \tilde{\alpha}_2(0,0)}{\partial \alpha_1}) + \frac{\partial^2 N_4(0,0)}{\partial S_b \partial I_b}(Y_{12} + \frac{\partial \tilde{\alpha}_3(0,0)}{\partial \alpha_1})\right. \\
& \left. + \frac{\partial^2 N_4(0,0)}{\partial I_m \partial I_b}(Y_{13} + \frac{\partial \tilde{\alpha}_4(0,0)}{\partial \alpha_1}) + \frac{\partial^2 N_4(0,0)}{\partial I_b^2}Y_{14}\right) + \frac{\partial N_4(0,0)}{\partial I_m}\frac{\partial^2 \tilde{\alpha}_4(0,0)}{\partial \alpha_1^2} \\
=\ & \sum_{i=1}^{4}\sum_{j=1}^{4}\frac{\partial^2 N_4(0,0)}{\partial X_i \partial X_j} + c_0\frac{\partial^2 \tilde{\alpha}_4(0,0)}{\partial \alpha_1^2}
\end{aligned}
$$

$$(38)$$

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Considering the equation (16) as $i = 1$, we have

$$
\begin{aligned}
0 =\ & \frac{\partial^2 N_1(0,0)}{\partial \alpha_1^2} = \frac{\partial}{\partial \alpha_1}\left(\frac{\partial N_1(0,0)}{\partial \alpha_1}\right) \\
=\ & \left(Y_{11} + \frac{\partial \tilde{\alpha}_2(0,0)}{\partial \alpha_1}\right)\left(\frac{\partial^2 N_1(0,0)}{\partial X_1^2}\left(Y_{11} + \frac{\partial \tilde{\alpha}_2(0,0)}{\partial \alpha_1}\right) + \frac{\partial^2 N_1(0,0)}{\partial X_1 \partial X_2}\left(Y_{12} + \frac{\partial \tilde{\alpha}_3(0,0)}{\partial \alpha_1}\right)\right. \\
& \left. + \frac{\partial^2 N_1(0,0)}{\partial X_1 \partial X_3}\left(Y_{13} + \frac{\partial \tilde{\alpha}_4(0,0)}{\partial \alpha_1}\right) + \frac{\partial^2 N_1(0,0)}{\partial X_1 \partial X_4}Y_{14}\right) \\
& + \left(Y_{12} + \frac{\partial \tilde{\alpha}_3(0,0)}{\partial \alpha_1}\right)\left(\frac{\partial^2 N_1(0,0)}{\partial X_1 \partial X_2}\left(Y_{11} + \frac{\partial \tilde{\alpha}_2(0,0)}{\partial \alpha_1}\right) + \frac{\partial^2 N_1(0,0)}{\partial X_2^2}\left(Y_{12} + \frac{\partial \tilde{\alpha}_3(0,0)}{\partial \alpha_1}\right)\right. \\
& \left. + \frac{\partial^2 N_1(0,0)}{\partial X_2 \partial X_3}\left(Y_{13} + \frac{\partial \tilde{\alpha}_4(0,0)}{\partial \alpha_1}\right) + \frac{\partial^2 N_1(0,0)}{\partial X_2 \partial X_4}Y_{14}\right) \\
& + \left(Y_{13} + \frac{\partial \tilde{\alpha}_4(0,0)}{\partial \alpha_1}\right)\left(\frac{\partial^2 N_1(0,0)}{\partial X_1 \partial X_3}\left(Y_{11} + \frac{\partial \tilde{\alpha}_2(0,0)}{\partial \alpha_1}\right) + \frac{\partial^2 N_1(0,0)}{\partial X_2 \partial X_3}\left(Y_{12} + \frac{\partial \tilde{\alpha}_3(0,0)}{\partial \alpha_1}\right)\right. \\
& \left. + \frac{\partial^2 N_1(0,0)}{\partial X_3^2}\left(Y_{13} + \frac{\partial \tilde{\alpha}_4(0,0)}{\partial \alpha_1}\right) + \frac{\partial^2 N_1(0,0)}{\partial X_3 \partial X_4}Y_{14}\right) \\
& + Y_{14}\left(\frac{\partial^2 N_1(0,0)}{\partial X_1 \partial X_4}\left(Y_{11} + \frac{\partial \tilde{\alpha}_2(0,0)}{\partial \alpha_1}\right) + \frac{\partial^2 N_1(0,0)}{\partial X_2 \partial X_4}\left(Y_{12} + \frac{\partial \tilde{\alpha}_3(0,0)}{\partial \alpha_1}\right)\right. \\
& \left. + \frac{\partial^2 N_1(0,0)}{\partial X_3 \partial X_4}\left(Y_{13} + \frac{\partial \tilde{\alpha}_4(0,0)}{\partial \alpha_1}\right) + \frac{\partial^2 N_1(0,0)}{\partial X_4^2}Y_{14}\right) \\
& + \frac{\partial N_1(0,0)}{\partial X_1}\frac{\partial^2 \tilde{\alpha}_2(0,0)}{\partial \alpha_1^2} + \frac{\partial N_1(0,0)}{\partial X_2}\frac{\partial^2 \tilde{\alpha}_3(0,0)}{\partial \alpha_1^2} + \frac{\partial N_1(0,0)}{\partial X_3}\frac{\partial^2 \tilde{\alpha}_4(0,0)}{\partial \alpha_1^2}.
\end{aligned}
$$

Submitting (24) into the above equation, it can be deduced that

$$
\begin{aligned}
\frac{\partial N_1(0,0)}{\partial X_1}\frac{\partial^2 \tilde{\alpha}_2(0,0)}{\partial \alpha_1^2} + \frac{\partial N_1(0,0)}{\partial X_3}\frac{\partial^2 \tilde{\alpha}_4(0,0)}{\partial \alpha_1^2} =\ & -\sum_{i=1}^4 \sum_{j=1}^4 \frac{\partial^2 N_1(0,0)}{\partial X_i \partial X_j}Y_{1i}Y_{1j}. \\
=\ & (1 - p_0)\sum_{i=1}^4 \sum_{j=1}^4 \frac{\partial^2 \Phi_1(p_0,X_0)}{\partial X_i \partial X_j}Y_{1i}Y_{1j}.
\end{aligned}
\tag{39}
$$

We can similarly get from equation (16) as $i = 2, 3$ that

$$
\begin{aligned}
\frac{\partial N_2(0,0)}{\partial X_2}\frac{\partial^2 \tilde{\alpha}_3(0,0)}{\partial \alpha_1^2} + \frac{\partial N_2(0,0)}{\partial X_3}\frac{\partial^2 \tilde{\alpha}_4(0,0)}{\partial \alpha_1^2} =\ & -\sum_{i=1}^4 \sum_{j=1}^4 \frac{\partial^2 N_2(0,0)}{\partial X_i \partial X_j}Y_{1i}Y_{1j}. \\
=\ & \sum_{i=1}^4 \sum_{j=1}^4 \frac{\partial^2 \Phi_2(p_0,X_0)}{\partial X_i \partial X_j}Y_{1i}Y_{1j}. \\
\frac{\partial N_3(0,0)}{\partial X_3}\frac{\partial^2 \tilde{\alpha}_4(0,0)}{\partial \alpha_1^2} =\ & -\sum_{i=1}^4 \sum_{j=1}^4 \frac{\partial^2 N_3(0,0)}{\partial X_i \partial X_j}Y_{1i}Y_{1j}. \\
=\ & (1 - p_0)\sum_{i=1}^4 \sum_{j=1}^4 \frac{\partial^2 \Phi_3(p_0,X_0)}{\partial X_i \partial X_j}Y_{1i}Y_{1j}.
\end{aligned}
\tag{40}
$$

By solving equations (39) and (40), we can get the values of $\partial^2 \tilde{\alpha}_i(0,0)/\partial \alpha_1^2, i = 2, 3, 4$, and submit it as $i = 4$ into (38), one obtains

$$
\begin{aligned}
\frac{\partial^2 N_4(0,0)}{\partial \alpha_1^2} =\ & \sum_{i=1}^4 \sum_{j=1}^4 \left(\frac{\partial^2 N_4(0,0)}{\partial X_i \partial X_j} - k\frac{\partial^2 N_3(0,0)}{\partial X_i \partial X_j}\right)Y_{1i}Y_{1j} \\
=\ & \sum_{i=1}^4 \sum_{j=1}^4 \left(-\frac{\partial^2 \Phi_4(X_0)}{\partial X_i \partial X_j} + k(1 - p_0)\frac{\partial^2 \Phi_3(X_0)}{\partial X_i \partial X_j}\right)Y_{1i}Y_{1j}.
\end{aligned}
$$

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Table 1. Input parameter sample values for simulation

Para	Description	Value (range)	Source
Λ_m	recruitment rate of mosquitoes	1300 (1200,1400)	see text
Λ_b	recruitment rate of birds	2.1 (1.8,2.4)	[4]
c	biting rate of mosquitoes	0.09 (0.03,0.15)	[11]
β_{mb}	transmission probability from bird to mosquito	0.16 (0.08,0.24)	[3],[11]
β_{bm}	transmission probability from mosquito to bird	0.88 (0.80,0.96)	[5],[11]
μ_m	natural death rate of mosquitoes	0.029 (0.016,0.042)	[10]
μ_b	natural death rate of birds	0.001 (0.0005,0.0015)	[5]
d_b	WNV-introduced death rate of birds	0.005 (0.0045,0.0055)	[3]
p	culling rate of mosquitoes	0.75 (0.5,1)	assumed
T	mosquito culling interval	10 (5,15)	assumed

Note: More than 99.73% of the data will fall within 2.58 standard deviations (2.58σ) of the mean for a

normal distribution. We choose the standard deviation (σ) such that 2.58σ equals half of the range of

the value for each parameter (take Λ_m for example, $2.58\sigma=100$).

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Table 2. Partial rank correlation coefficients illustrating the dependence of the three variables (new infections N_0, Death D_0 and R_0) on each parameter. The values of each parameter used in the simulations are listed as in Table 1. Here we denote p as zero if p is smaller than 0.0001.

Para	N_0		D_0		R_0	
	PRCC	p-values	PRCC	p-values	PRCC	p-values
Λ_m	0.0047	0.8346	0.2687	0.2071	0.0501	0.0251
Λ_b	-0.0336	0.1329	-0.0238	0.2881	-0.0876	0.0001
c	0.9278	0	0.6346	0	0.4679	0
β_{mb}	0.1249	0	0.0633	0.0046	0.1836	0
β_{bm}	0.0799	0.0003	0.0682	0.0023	0.0508	0.0230
μ_m	-0.0701	0.0017	-0.0401	0.0728	-0.0401	0.0732
μ_b	-0.0052	0.8163	-0.0217	0.3318	0.1291	0
d_b	0.0265	0.2363	0.0750	0.0008	-0.0015	0.9475
p	-0.0105	0.6382	-0.0164	0.4642	-0.4920	0
T	0.2687	0	0.7166	0	0.6484	0
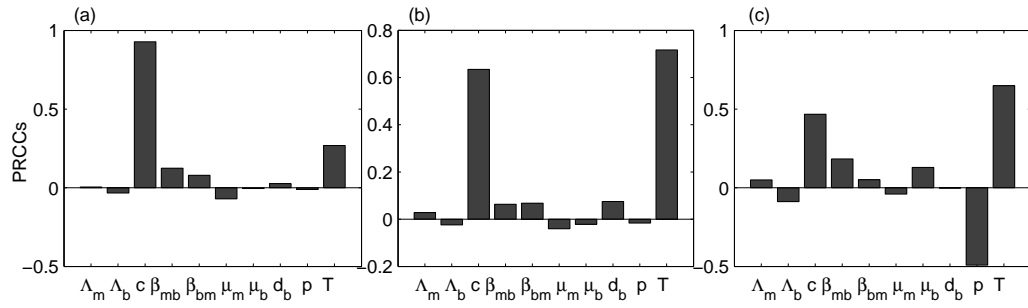
Figure 1: Sensitivity results based on Latin hypercube sampling. PRCCs illustrating the dependence of the three outcome variables ((a) new infections N_0, (b) death D_0 and (c) R_0) on the ten input parameters, respectively.
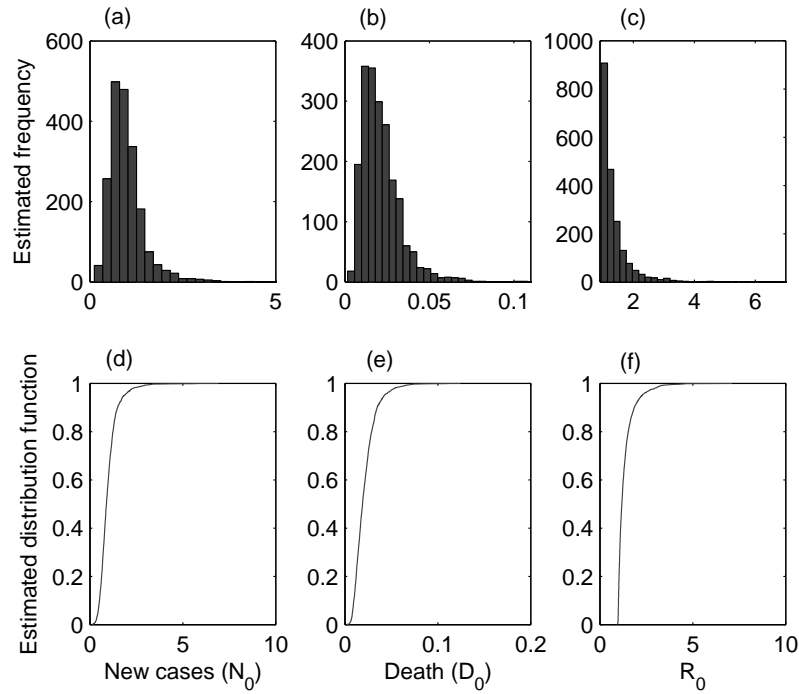
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Figure 2: Uncertainty results based on Latin hypercube sampling. The first row consists of the frequency plots for (a) new infections in birds, (b) bird deaths and (c) the threshold R_0. The second row shows estimates of CDFs for output variables ((d) new infections N_0, (e) bird deaths D_0 and (f) R_0).

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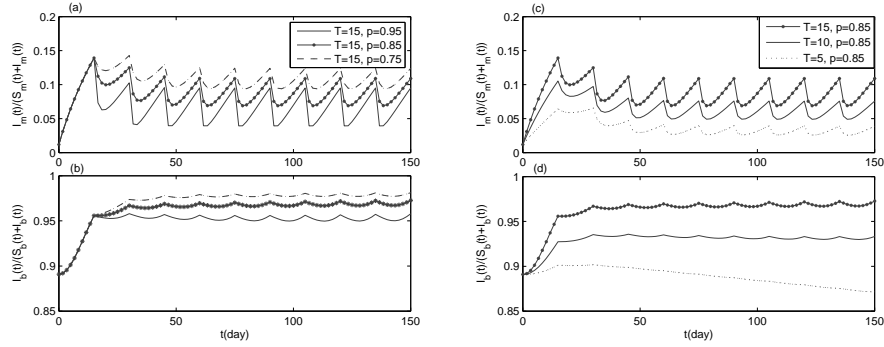
Figure 3: WNV prevalence for both mosquitoes (a),(c) and birds (b),(d). For (a) and (b) the impulse-time interval is fixed and the mosquito culling rate varies. For (c) and (d), the mosquito culling rate is fixed while the impulse time interval varies.
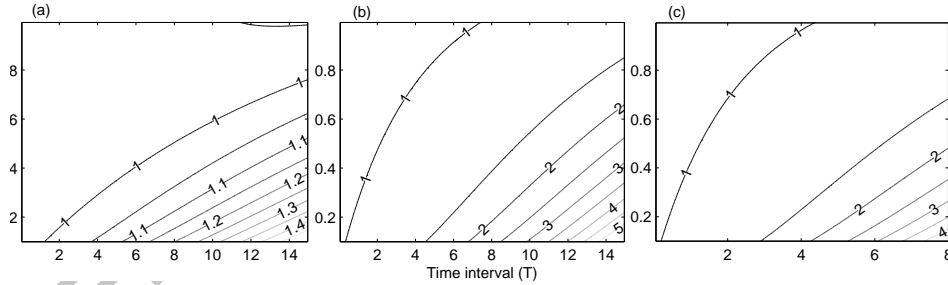
Figure 4: Contour plots of R_0. Plot contours of R_0 versus the culling time interval T and the culling rate p with mosquito-biting rate c equal to (a) 0.03, (b) 0.09 and (c) 0.15. Other parameters are chosen as in Table 1.
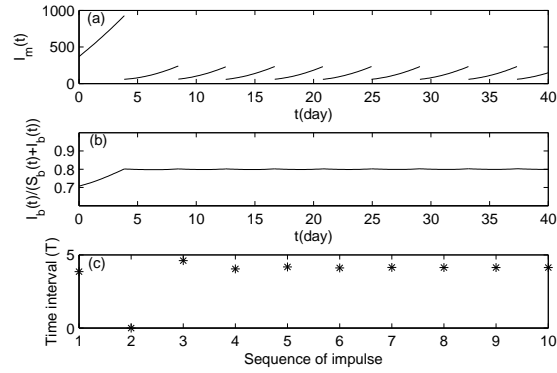
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Figure 5: Plots of (a) infected mosquitoes, (b) mean WNV prevalence for birds, and (c) the pulse period for the threshold $CT = 340$. Parameters are chosen as in Table 1.
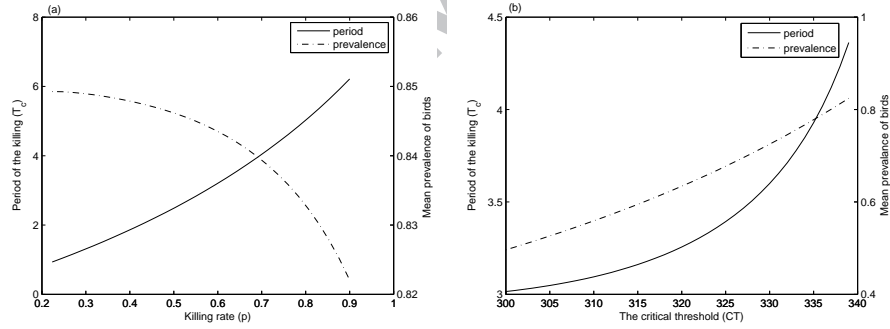

Figure 6: Plots of stable impulsive period (T_c) and mean WNV prevalence in birds for the state-dependent model. (a) Variations in impulsive period T_c and mean WNV prevalence in birds with the culling proportion p for fixed CT=500, (b) Variations in impulsive period T_c and mean WNV prevalence in birds with the threshold CT for fixed $p = 0.75$.
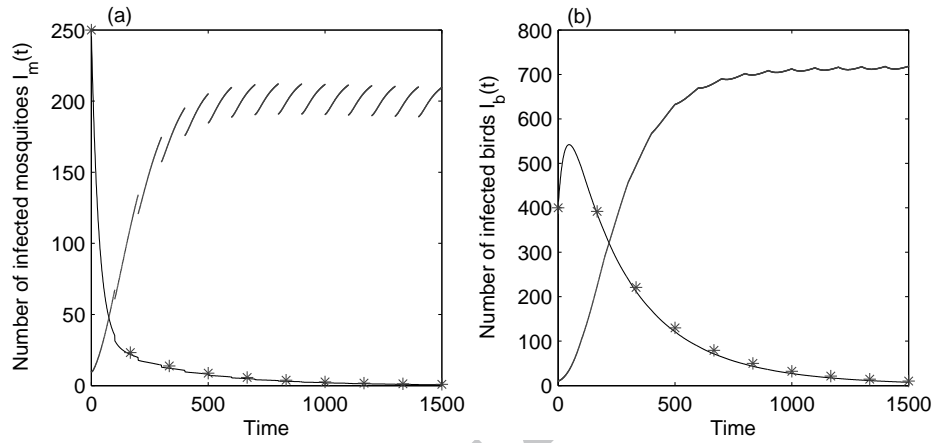
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Figure 7: Existence of the bifurcated periodic solution and disease-free periodic solution. Parameter values are $\lambda_m = 50, \lambda_b = 5, c = 0.05, \beta_{mb} = 0.16, \beta_{bm} = 0.88, \mu_m = 0.029, \mu_b = 0.001, d_b = 0.005, T = 100$ and $p = 0.1$. The threshold gives $R_0 = 0.8316$. When the initial values are $S_m(0) = 10^3, I_m(0) = 250, S_b(0) = 10^4, I_b(0) = 400$, the number of infected mosquitoes (the solid line with stars in (a)) and birds (the solid line with stars in (b)) goes to zero. While the initial condition equals $S_m(0) = 10, I_m(0) = 10, S_b(0) = 10, I_b(0) = 10$, the disease finally persists, see the solid lines in (a) and (b).
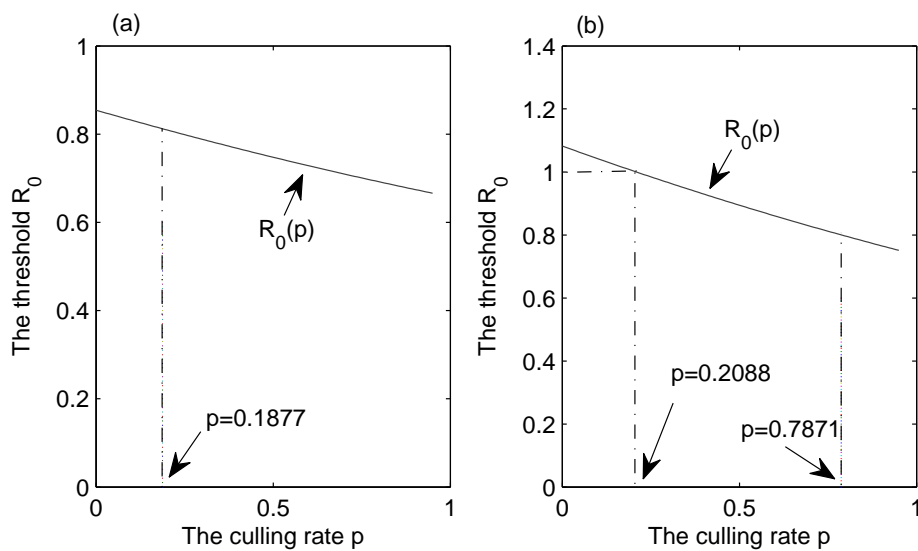
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Figure 8: Dependence of R_0 for system (1) on the culling rate. (a) $\lambda_m = 50, c = 0.05, \beta_{mb} = 0.16, \beta_{bm} = 0.88, \mu_m = 0.029, \mu_b = 0.001, d_b = 0.005, T = 100, \lambda_b = 5$. (b) $\lambda_m = 50, c = 0.05, \beta_{mb} = 0.16, \beta_{bm} = 0.88, \mu_m = 0.029, \mu_b = 0.001, d_b = 0.005, T = 100, \lambda_b = 3$.