THE OUTER AUTOMORPHISM GROUPS OF TWO-GENERATOR, ONE-RELATOR GROUPS WITH TORSION

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Abstract. The main result of this paper is a complete classification of the outer automorphism groups of two-generator, one-relator groups with torsion. To this classification we apply recent algorithmic results of Dahmani–Guirardel, which yields an algorithm to compute the isomorphism class of the outer automorphism group of a given two-generator, one-relator group with torsion.

1. Introduction

Let R denote a non-trivial freely and cyclically reduced word in $F(a, b)$ which is not a proper power of any other word (we maintain this convention throughout the paper). The group $G = \langle a, b; R^n\rangle$ has torsion if and only if $n > 1$ [MKS04, Theorem 4.12], and we call such a group G a *two-generator, one-relator group with torsion*.

In this paper we classify the outer automorphism groups of two-generator, one-relator groups with torsion. A *primitive element* of $F(a, b)$ is a word R which is contained in a basis of $F(a, b)$, and this paper studies the case when R is not primitive. This focus is because R is primitive if and only if $G = \langle a, b; R^n\rangle \cong \mathbb{Z} * C_n$ [Pri77a], and if and only if $G = \langle a, b; R^n\rangle$ is not one-ended [FKS72].

One-relator groups. One-relator groups are classically studied (see, for example, the classic texts in combinatorial group theory [MKS04] [LS77] or the early work of Magnus [Mag30] [Mag32]), and these groups continue to have applications to this day, for example in 3-dimensional topology and knot theory [IT15] [FT15]. One-relator groups with torsion are hyperbolic and as such they serve as important test cases for this larger class of groups. For example, the isomorphism problem for two-generator, one-relator groups with torsion was shown to be soluble [Pri77a] long before Dahmani–Guirardel's recent resolution of the isomorphism problem for all hyperbolic groups [DG11]. As another example, Wise recently resolved the classical conjecture of G. Baumslag that all one-relator groups with torsion are residually finite [Wis12], while it is still an open question as to whether all hyperbolic groups are residually finite.

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Two-generator, one-relator groups with torsion are particularly important within the class of one-relator groups with torsion for two reasons. Firstly, every one-relator group with torsion can be embedded into a two-generator, one-relator group with tosion. Secondly, every two-generator subgroup of a one-relator group with torsion is either a free group or a two-generator, one-relator group with torsion [Pri77b].

Main theorem. Writing D_n for the dihedral group of order $2n$, the main result of this paper is as follows. Recall that R denotes a non-trivial word in $F(a, b)$ which is not a proper power.

Theorem A. *Let G be a two-generator, one-relator group with torsion, so $G \cong \langle a, b; R^n \rangle$ with $n > 1$.*

1. *If $G \cong \langle a, b; [a, b]^n \rangle$ then* $\mathrm{Out}(G) \cong \mathrm{GL}_2(\mathbb{Z})$.
2. *Suppose that G is one-ended and $G \ncong \langle a, b; [a, b]^n \rangle$.*
   (a) *If* $\mathrm{Out}(G)$ *is infinite then it is isomorphic to $D_\infty \times C_2$, D_∞, $\mathbb{Z} \times C_2$ or $\mathbb{Z}$.*
   (b) *If* $\mathrm{Out}(G)$ *is finite then it embeds into either D_6 or D_4.*
3. *If G is infinitely ended, so $G \cong \mathbb{Z} * C_n$, then* $\mathrm{Out}(G) \cong D_n \rtimes \mathrm{Aut}(C_n)$.

Moreover, all the possibilities from (2a) and from (2b) occur.

Explicit examples of each of the groups occurring in (2a) are given in Lemma 5.7, and examples of each of the groups occurring in (2b) are given in Lemmas 6.1 and 6.2. If G is one-ended then our methods show that the isomorphism class of $\mathrm{Out}(G)$ depends only on the root R of the relator R^n.

A general theory exists for the outer automorphism groups of hyperbolic groups, based on JSJ-decompositions. However, this theory is limited as it describes only a finite-index subgroup of the outer automorphism group [Lev05]. In Theorem 3.1 we prove the relevant coarse description (our proof uses fixed points of automorphisms in free groups rather than JSJ-decompositions). The proof of Theorem A improves upon the coarse description of Theorem 3.1 by combining it with faithful linear representations over $\mathbb{Z}$ of the outer automorphism groups, which we prove exist in Theorem 4.2.

In Section 8 we give two applications of Theorem A. Our first application combines Theorem A with recent work of Dahmani–Guirardel on algorithms in hyperbolic groups [DG11] to prove the following corollary of Theorem A.

Corollary B. *There exists an algorithm which takes as input a presentation $\langle a, b; R^n \rangle$, $n > 1$, defining a group G and gives as output the isomorphism class of* $\mathrm{Out}(G)$.

The algorithm of Corollary B does not simply find the isomorphism class of $\mathrm{Out}(G)$ but also finds automorphisms of $G = \langle a, b; R^n \rangle$, given in terms of their action on the generators a and b, whose cosets generate $\mathrm{Out}(G)$.

Our second application in Section 8 is an explanation of how to use Theorem A to write down a presentation for the full automorphism group of a two-generator, one-relator group with torsion.

Previous results. Much of the previous work on the outer automorphism groups of one-relator groups with torsion is based around residual finiteness. For example, Kim–Tang proved certain specific two-generator, one-relator groups with torsion have residually finite outer automorphism groups [KT09,KT10], while it is a recent result of Carette that an (arbitrarily-generated) one-relator group with torsion has residually finite outer automorphism group [Car15].[1] Note that, restricting to the two-generator case, Theorem A is much stronger than these previous results.

It is worth mentioning that Theorem A shows that the outer automorphism groups of two-generator, one-relator groups with torsion are very similar to the outer automorphism groups of one-relator groups with non-trivial center [GHMR00] and to the outer automorphism groups of Baumslag–Solitar groups $\mathrm{BS}(m,n) = \langle b, s; s^{-1}b^m s = b^n\rangle$ with $|m| \geq |n|$ and n is not a proper divisor of m [Cla06]. In each of these two cases, the base groups are non-hyperbolic two-generator, one-relator groups *without* torsion, but their outer automorphism groups are found in the list given in Theorem A.

Overview of the paper. Our proof of the main result, Theorem A, is built around two technical theorems: Theorem 3.1, which states that if G is one-ended and $G \not\cong \langle a, b; [a,b]^n\rangle$ then $\mathrm{Out}(G)$ is virtually-cyclic, and Theorem 4.2, which embeds $\mathrm{Out}(G)$ into $\mathrm{GL}_2(\mathbb{Z})$ when G is one-ended. Another key result is Lemma 5.1, which is, essentially, a rewriting result in the spirit of Moldovanskii rewriting. Lemma 5.1 is notable because of its proof, which combines technical results and methods of Kapovich–Weidmann [KW99a] [KW99b] with results of the author on the structure of the JSJ-decompositions of one-relator groups with torsion [Log15].

In Section 2 we state a key result due to Pride on Nielsen equivalence. In Section 3 we apply results on fixed points of automorpisms in free groups to prove our first main technical theorem, Theorem 3.1. In Section 4 we prove our second main technical theorem, Theorem 4.2, and we prove that if $G \cong \langle a, b; [a,b]^n\rangle$ then $\mathrm{Out}(G) \cong \mathrm{GL}_2(\mathbb{Z})$. In Section 5 we determine the possibilities for $\mathrm{Out}(G)$ if $\mathrm{Out}(G)$ is infinite, G is one-ended and $G \not\cong \langle a, b; [a,b]^n\rangle$. In Section 6 we determine the possibilities for $\mathrm{Out}(G)$ if $\mathrm{Out}(G)$ is finite and G is one-ended. In Section 7 we assemble the proof of Theorem A from the previous sections. In Section 8 we prove Corollary B and we explain how to obtain a presentation for $\mathrm{Aut}(G)$.

2. Nielsen equivalence of generating pairs

In this section we state Proposition 2.1, which classifies the Nielsen equivalence classes of generating pairs in a two-generator, one-relator group with torsion. We begin by motivating our use of Proposition 2.1.

The inducing homomorphism. Proposition 2.1 is due to Pride, who used it to solve the isomorphism problem for two-generator, one-relator groups with torsion [Pri77a]. Proposition 2.1 implies that the automorphisms of a one-ended group $G = \langle a, b; R^n\rangle$ are induced by automorphisms of the ambient free group $F(a,b)$, so there exists a homomorphism, the *inducing homomorphism*, from a subgroup H of $\mathrm{Out}(F(a,b))$ to $\mathrm{Out}(G)$, $\theta : H \twoheadrightarrow \mathrm{Out}(G)$. The two main technical results of this paper, Theorems 3.1 and 4.2, both apply this inducing homomorphism in fundamental ways.

Nielsen equivalence. A *Nielsen transformation* ϕ_t *of the pair* (a,b), or *of* $F(a,b)$,

[1]Note that Carette uses the deep result of Wise that one-relator groups with torsion are residually finite [Wis12].

is an automorphism of the free group $F(a, b)$. Two generating pairs (y_1, y_2) and (z_1, z_2) of a group $G = \langle x_1, x_2; \mathbf{r}\rangle$ are *Nielsen equivalent* if there exists some Nielsen transformation ϕ_t of the pair (x_1, x_2) such that if $\phi_t(x_1) = w_1(x_1, x_2)$ and $\phi_t(x_2) = w_2(x_1, x_2)$ then the word $w_i(y_1, y_2)$ is equal in G to the word z_i, for $i = 1, 2$.

$$(w_1(y_1, y_2), w_2(y_1, y_2)) =_G (z_1, z_2)$$

The equivalence classes of this equivalence relation are called *Nielsen equivalence classes (of generating pairs).*

Proposition 2.1 (Pride, 1977). *Let $G = \langle a, b; R^n\rangle$ with $n > 1$ and R not a proper power. Suppose that R is not a primitive element of $F(a, b)$, or R is a primitive element and $n = 2$. Then G has only one Nielsen equivalence class of generating pairs. Suppose, on the other hand, that R is primitive and $n > 2$. Then G has $\frac{1}{2}\varphi(n)$ Nielsen equivalence classes (where φ is the Euler totient function).*

3. Out(G) is virtually cyclic

In this section we prove Theorem 3.1. Our proof applies results of Bogopolski on fixed points of automorphisms of free groups [Bog00]. We apply Theorem 3.1 in Section 5, where we combine Theorems 3.1 and 4.2 to completely classify the possibilities for $\mathrm{Out}(G)$ when $\mathrm{Out}(G)$ is infinite and $G \not\cong \langle a, b; [a, b]^n\rangle$.

We shall write γ_g to mean conjugation by g, so $\gamma_g : h \mapsto h^g$, while ϵ denotes 1 or -1, $\epsilon = \pm 1$.

Theorem 3.1. *Let $G = \langle a, b; R^n\rangle$, with $n > 1$. Suppose that G is one-ended. Then either* $\mathrm{Out}(G)$ *is virtually-cyclic or $G \cong \langle a, b; [a, b]^n\rangle$.*

Proof. Every automorphism ϕ of $G = \langle a, b; R^n\rangle$ is induced by a Nielsen transformation ϕ_t of the ambient free group $F(a, b)$, by Proposition 2.1, and such a map ϕ_t sends R to a conjugate of R or of R^{-1} in $F(a, b)$ [MKS04, Theorem N5].

Suppose R is not a conjugate of $[a, b]$ or $[a, b]^{-1}$ (equivalently, $G \not\cong \langle a, b; [a, b]^n\rangle$ [Pri77a]). Define the subgroup $\mathrm{Stab}_0(R)$ of $\mathrm{Out}(F(a, b))$ as follows.

$$\mathrm{Stab}_0(W) := \{\phi_t \in \mathrm{Aut}(F(a, b)) : \exists g \in G \text{ s.t. } \phi_t(R) = \gamma_g(R^\epsilon)\}$$

By the above observation, $\mathrm{Out}(G)$ is a homomorphic image of the following group.

$$H_0(R) := \mathrm{Stab}_0(R)/(\mathrm{Stab}_0(R) \cap \mathrm{Inn}(F(a, b)))$$

We prove that $H_0(R)$ is virtually cyclic. Hence $\mathrm{Out}(G)$ is virtually cyclic. Define

$$\mathrm{Stab}(R) := \{\phi \in \mathrm{Aut}(F(a, b)) : \exists g \in G \text{ s.t. } \phi(R) = \gamma_g(R)\}$$

and note that the group $H(R) := \mathrm{Stab}(R)/(\mathrm{Stab}(R) \cap \mathrm{Inn}(F(a, b)))$ is virtually cyclic [Bog00]. Then $\mathrm{Stab}_0(R) \cap \mathrm{Inn}(F(a, b) = \mathrm{Stab}(R) \cap \mathrm{Inn}(F(a, b)$, as $g^{-1}Rg \not\equiv R^{-1}$ unless R is the empty word. Therefore, $H_0(R)/H(R) \cong \mathrm{Stab}_0(R)/\mathrm{Stab}(R)$. Then $[\mathrm{Stab}_0(R) : \mathrm{Stab}(R)] = 2$ and so $[H_0(R) : H(R)] = 2$, and we conclude that $H_0(R)$ is virtually cyclic, as required. □

Theorem 3.1 can also be proven using JSJ-decompositions [Log14, Section 3.1]. From this viewpoint, the omission of $G \cong \langle a, b; [a, b]^n\rangle$ is natural because this corresponds to when G is Fuchsian [FR93], and JSJ-decompositions yield no information about $\mathrm{Out}(G)$ when G is Fuchsian [Lev05].

4. Out(G) embeds into Out($F(a,b)$)

In this section we prove Theorem 4.2, which states that if G is a one-ended two-generator, one-relator group with torsion then $\mathrm{Out}(G)$ embeds into $\mathrm{GL}_2(\mathbb{Z})$. We do this by proving that the inducing homomorphism, as defined in Section 2, is always an isomorphism. In Sections 5 and 6 we apply this embedding, along with Theorem 3.1, to determine the possibilities for $\mathrm{Out}(G)$.

We begin with Proposition 4.1. This is a variant of a classical result of Nielsen [MKS04, Corollary $N4$], and allows us to view $\mathrm{Out}(F(a,b))$ as a group of matrices. This view is useful in the proof of Theorem 4.2. After proving Theorem 4.2, Proposition 4.1 further allows us to view the groups $\mathrm{Out}(G)$ as groups of matrices.

Proposition 4.1. *(Nielsen, 1924) Let $\phi_t : a \mapsto A$, $b \mapsto B$ be an arbitrary Nielsen transformation of $F(a,b)$. Then the map,*

$$\begin{aligned} \xi : \mathrm{Aut}(F(a,b)) &\to GL(2,\mathbb{Z}) \\ \phi_t &\mapsto \begin{pmatrix} \sigma_a(A) & \sigma_b(A) \\ \sigma_a(B) & \sigma_b(B) \end{pmatrix} \end{aligned}$$

is an epimorphism, and $\mathrm{Ker}(\xi) = \mathrm{Inn}(F(a,b))$.

We now state Theorem 4.2. The remainder of this section proves this theorem. We shall use $\sigma_x(W)$ to denote the exponent sum of the letter x in the word W, and, for $\phi \in \mathrm{Aut}(H)$ with H some group, we shall use $\widehat{\phi}$ to denote the element of $\mathrm{Out}(H)$ with representative ϕ. Finally, note that if $G = \langle a, b; R^n \rangle$ is a one-relator group with torsion then, using Moldovanskii rewriting [FR94], the relator R^n can be re-written as a cyclically reduced word S^n such that $\sigma_a(S) = 0$ and $G \cong \langle a, b; S^n \rangle$.

Theorem 4.2. *Let $G = \langle a, b; R^n \rangle$ with $n > 1$. Suppose that G is one-ended. Then $\mathrm{Out}(G)$ embeds in $\mathrm{Out}(F(a,b))$, and the embedding is as follows. Rewrite the relator R such that $\sigma_a(R) = 0$ and such that R is cyclically reduced, and let $\widehat{\phi}$ be an element of $\mathrm{Out}(G)$ with a representative Nielsen transformation $\phi_t : a \mapsto A$, $b \mapsto B$. Then the following map gives the embedding.*

$$\begin{aligned} \Theta : \mathrm{Out}(G) &\to \mathrm{GL}(2,\mathbb{Z}) \\ \widehat{\phi} &\mapsto \begin{pmatrix} \sigma_a(A) & \sigma_b(A) \\ \sigma_a(B) & \sigma_b(B) \end{pmatrix} \end{aligned}$$

The following corollary is obtained by applying the fact that every automorphism of $F(a,b)$ maps $[a,b]$ to a conjugate of $[a,b]^{\pm 1}$ [MKS04, Theorem 3.9].

Corollary 4.3. *Let $G \cong \langle a, b; [a,b]^n \rangle$ with $n > 1$. Then $\mathrm{Out}(G) \cong \mathrm{GL}_2(\mathbb{Z})$.*

The embedding Θ in Theorem 4.2 is the composition of the map $\theta_1 : \mathrm{Out}(G) \to \mathrm{Out}(F(a,b)), \widehat{\phi} \mapsto \widehat{\phi}_t$ from Proposition 2.1, which is the reverse of the inducing homomorphism θ, with the isomorphism $\overline{\xi} : \mathrm{Out}(F(a,b)) \to \mathrm{GL}_2(\mathbb{Z})$ induced by the map ξ from Proposition 4.1. Therefore, to prove Theorem 4.2 it is sufficient to prove that the inducing homomorphism $\theta : H \twoheadrightarrow \mathrm{Out}(G)$ is injective.

The proof of Theorem 4.2 is split into two cases: $R \in F(a,b)'$ (Lemma 4.4) and $R \notin F(a,b)'$ (Lemma 4.6), where $F(a,b)'$ denotes the derived subgroup of $F(a,b)$.

Lemma 4.4. *Let $G = \langle a, b; R^n \rangle$ with $n > 1$ and $R \in F(a,b)'$. Then $\mathrm{Out}(G)$ embeds into $\mathrm{Out}(F(a,b))$ by the map given in Theorem 4.2.*

Proof. It is sufficient to prove that if ϕ_t is a Nielsen transformation with $\theta(\phi_t) \in \mathrm{Inn}(G)$, where θ is the inducing homomorphism, then $\phi \in \mathrm{Inn}(F(a,b))$. So, let ϕ_t be a Nielsen transformation of the pair (a,b) with $\phi_t(a) := A$ and $\phi_t(b) := B$ and such that there exists $W \in F(a,b)$ with $a^W =_G A$ and $b^W =_G B$, and we prove that $\phi_t \in \mathrm{Inn}(F(a,b))$. As $a^W = A$ and $b^W = B$ in G it must hold that $a^W = A \mod G'$ and $b^W = B \mod G'$. However, $G^{ab} = \langle a, b; [a,b]\rangle \cong \mathbb{Z}\times\mathbb{Z}$, as $R \in F(a,b)'$. Therefore, it must hold that $\sigma_a(A) = 1$ and $\sigma_b(A) = 0$, and that $\sigma_a(B) = 0$ and $\sigma_b(B) = 1$. Proposition 4.1 then implies $\phi_t \in \mathrm{Inn}(F(a,b))$, as required. □

We now prove Theorem 4.2 in the case when $G = \langle a, b; R^n\rangle$ is one-ended and $R \notin F(a,b)'$. We begin by proving Lemma 4.5, which gives a description of the automorphisms of such a group G. The statement of Lemma 4.5 assumes that the relator R^n in $G = \langle a, b; R^n\rangle$ is such that $\sigma_a(R) = 0$, $\sigma_b(R) \neq 0$ and $R \neq b^\epsilon$. These assumptions are valid by Moldovanskii rewriting [FR94]. Recall that if $\psi \in \mathrm{Aut}(G)$ then $\widehat{\psi}$ denotes the element of $\mathrm{Out}(G)$ with representative ψ.

Lemma 4.5. *Let $G = \langle a, b; R^n\rangle$ with $n > 1$, $\sigma_a(R) = 0$, $\sigma_b(R) \neq 0$, and $R \neq b^\epsilon$ cyclically reduced. Suppose that ψ is an arbitrary automorphism of G. Then $\widehat{\psi} \in$* $\mathrm{Out}(G)$ *has a representative $\phi \in \widehat{\psi}$ of the following form.*

$$\begin{aligned} \phi : a &\mapsto a^{\epsilon_0} b^k \\ b &\mapsto b^{\epsilon_1} \end{aligned}$$

Proof. Note that if $\phi : a \mapsto a^{\epsilon_0} b^k, b \mapsto b^{\epsilon_1}$ is a homomorphism then it is also an automorphism, as G is Hopfian. By Proposition 2.1, the automorphism ψ can be realised as a Nielsen transformation ψ_t of the pair (a,b), where $\psi_t(a) :\equiv A$ and $\psi_t(b) :\equiv B$.

Let $\pi : G \to G^{ab}$ be the abelianisation map. The abelianisation has presentation $G^{ab} = \langle x, y; y^m, [x,y]\rangle$, where $x := \pi(a)$ and $y := \pi(b)$ while $m := \sigma_b(R)$. Let $x^i y^\alpha := \pi(A)$ and let $x^j y^\beta := \pi(B)$. Then automorphisms of G induce automorphisms of G^{ab}, so $\pi(B)$ has order $m \neq 0$. We therefore have the following.

$$(x^j y^\beta)^m = 1 \Rightarrow x^{mj} y^{m\beta} = 1 \Rightarrow x^{mj} = 1$$

Then $j = 0$ as x has infinite order in G^{ab}, and so $\pi(B) = b^\beta$. Hence, $\sigma_a(B) = 0$. Therefore, applying Proposition 4.1, the Nielsen transformation ψ_t corresponds to the following matrix of $\mathrm{GL}_2(\mathbb{Z})$.

$$\begin{pmatrix} \sigma_a(A) & \sigma_b(A) \\ 0 & \sigma_b(B) \end{pmatrix}$$

Hence, $|\sigma_a(A)| = 1 = |\sigma_b(B)|$. Taking $k := \sigma_b(A)$, $\epsilon_0 := \sigma_a(A)$ and $\epsilon_1 := \sigma_b(B)$, the Nielsen transformation $\phi_t : a \mapsto a^{\epsilon_0} b^k, b \mapsto b^{\epsilon_1}$ also corresponds to the matrix M. If two Nielsen transformations are equal modulo $\mathrm{Inn}(F(a,b))$ then they are equal modulo $\mathrm{Inn}(G)$, and so taking $\phi := \phi_t$ we are done. □

We now apply Lemma 4.5 to prove of the case of $R \notin F(a,b)'$ in Theorem 4.2.

Lemma 4.6. *Let $G = \langle a, b; R^n\rangle$ with $n > 1$, $\sigma_a(R) = 0$, $\sigma_b(R) \neq 0$ and $R \neq b^\epsilon$ cyclically reduced. Then* $\mathrm{Out}(G)$ *embeds in* $\mathrm{Out}(F(a,b))$ *by the map given in Theorem 4.2.*

Proof. It is sufficient to prove that if ϕ is an inner automorphism of G, $\phi \in \mathrm{Inn}(G)$, such that $\phi : a \mapsto a^{\epsilon_0} b^k$, $b \mapsto b^{\epsilon_1}$ then $a \equiv a^{\epsilon_0} b^k$ and $b \equiv b^{\epsilon_1}$, by Lemma 4.5. So, let $\phi : a \mapsto a^{\epsilon_0}b^k$, $b \mapsto b^{\epsilon_1}$ with either $\epsilon_0 \neq 1$, or $\epsilon_1 \neq 1$, or $k \neq 0$, and assume that ϕ is inner, $\phi \in \mathrm{Inn}(G)$. Therefore, there exists some word $W(a,b)$ such that $a^W =_G a^{\epsilon_0}b^k$ and $b^W =_G b^{\epsilon_1}$. By the malnormality of the subgroup $\langle b\rangle$ [New73], we can assume $W \equiv b^i$.

We shall now prove that $i \neq 0$ (so $W \neq_G 1$). Suppose that $i = 0$, so $a =_G a^{\epsilon_0}b^k$ and $b =_G b^{\epsilon_1}$. If $\epsilon_0 = -1$ then $a^2 = b^k$, but a has infinite order in the abelianisation while b has finite order, a contradiction. Therefore, if $i = 0$ then $\epsilon_0 = 1$ and so $b^{\epsilon_1} =_G b$ and $b^k =_G 1$. Now, b has infinite order [MKS04, Corollary 4.11], and so $\epsilon_1 = 1$ and $k = 0$, a contradiction. Thus, $i \neq 0$.

As $i \neq 0$ we have that $b^{-i}ab^i =_G a^{\epsilon_0}b^k$, so $a^{\epsilon_0}b^{k-i}a^{-1}b^i =_G 1$. We shall prove that $a^{\epsilon_0}b^{k-i}a^{-1}b^i$ cannot represent the trivial word in G, which is a contradiction and so proves the lemma. Note that if a word $U(a,b) =_G 1$ then $\sigma_a(U) = 0$, because the order of a under the abelianisation map is infinite. Thus, $\sigma_a(a^{\epsilon_0}b^{k-i}a^{-1}b^i) = 0$ and so $\epsilon_0 = 1$. Moreover, $i \neq k$ as the generator b has infinite order. Hence, $\epsilon_0 = 1$ and $i \neq k$.

So, writing $j := k - i$, we have that $ab^ja^{-1}b^i =_G 1$, for $i, j \neq 0$. However, by the Newman-Gurevich Spelling Theorem [HP84], a word of this form cannot represent the trivial word in a group $G = \langle a, b; R^n\rangle$ under the restrictions of this lemma. □

We now assemble the proof of Theorem 4.2.

Proof of Theorem 4.2. Let $G = \langle a, b; R^n\rangle$. By applying Moldovanskii rewriting, we can assume that $\sigma_a(R) = 0$ [FR94]. If $R \in F(a,b)$ then the result follows from Lemma 4.4. If $R \notin F(a,b)'$ then the result follows from Lemma 4.6. □

5. The possibilities for Out(G) when it is infinite

In this section we work under the assumption that G is a one-ended two-generator, one-relator group with torsion with Out(G) infinite and determine, in Theorem 5.6, the possible isomorphism classes for Out(G). We prove, in Lemma 5.5, that every possibility occurs. In this section we further assume that $G \ncong \langle a, b; [a,b]^n\rangle$, as this case was dealt with in Corollary 4.3.

Throughout this section we use Theorem 4.2 to view the relevant outer automorphism groups as matrix groups.

5.1. The form of outer automorphisms when Out(G) is infinite.

We begin by proving, in Lemma 5.1, that the relator R can be rewritten in a particularly nice way (in Section 8 we show that this rewriting can be made algorithmic). This new form for the relator will allow us to view automorphisms as Nielsen transformations: previously, by Proposition 2.1, we only knew that such a view existed. Note that under the assumptions of this section Out(G) is virtually-$\mathbb{Z}$, by Theorem 3.1.

The proof of Lemma 5.1 is based on arguments of Kapovich-Weidmann [KW99a].

Lemma 5.1. *Let $G = \langle a, b; R^n\rangle$ with $n > 1$. Suppose that $G \ncong \langle a, b; [a,b]^n\rangle$ and that G is one-ended. Then the following equivalence holds.*

$$\mathrm{Out}(G) \text{ is virtually-}\mathbb{Z} \iff G \cong \langle a, b; S^n(a^{-1}ba, b)\rangle$$

Proof. Suppose that $G \cong \langle a, b; S^n(a^{-1}ba, b)\rangle$. Then the map $a \mapsto ab$, $b \mapsto b$ is an automorphism of G, and it has infinite order by Theorem 4.2. As $G \ncong \langle a, b; [a,b]^n\rangle$, Theorem 3.1 implies that Out(G) is virtually-$\mathbb{Z}$.

Suppose that $\mathrm{Out}(G)$ is virtually-$\mathbb{Z}$. Then G splits as an HNN-extension or free product with amalgamation with vertex groups having finite center and edge groups virtually cyclic with infinite center [Lev05, Theorem 1.4]. The edge groups must be infinite cyclic [KS71b], so $G = H*_{C_X^t=C_Y}$ or $G = H *_C K$ with $C_X, C_Y, C \cong \mathbb{Z}$ and $H, K \ncong \mathbb{Z}$. We shall prove that G must split as an HNN-extension $H*_{C_X^t=C_Y}$ where the base group H is two-generated, which implies that the subgroup H is a two-generator, one-relator group with torsion [Pri77b]. This implies that G is isomorphic to a group of the required form.

Suppose that G splits as a free product with amalgamation $G = H *_C K$ with $C \cong \mathbb{Z}$ and $H, K \ncong \mathbb{Z}$, and we shall prove that either H or K is infinite cyclic, a contradiction. Begin by noting that H and K are both hyperbolic [KM98, Theorem 6], and that the amalgamating subgroup C is malnormal in H or K [KM98, Corollary 2]. We suppose, without loss of generality, that C is malnormal in K. Now, C is contained in a malnormal infinite cyclic subgroup H_0 of G [Log15, Lemma 5.5]. Note that $H_0 \leq H$ because C is malnormal in K. As H is hyperbolic, the amalgamating subgroup C is contained in a unique maximal, virtually-cyclic subgroup of H, and this must be H_0. We shall now prove that $H_0 = H$, which is our required contradiction as $H_0 \cong \mathbb{Z}$ but $H \ncong \mathbb{Z}$. Assume otherwise, so $H_0 \lneq H$, and we shall look for a contradiction. Then G can be written $G = H *_{H_0} K_0$ where $K_0 = H_0 *_C K$ is the group generated by H_0 and K. Then H_0 is malnormal in H while its image is malnormal in K_0 [Log14, Lemma 3.1.10].[2] However, G is a one-ended, two-generated group and so cannot have the form $P *_Q R$ where Q is malnormal in both P and R [KS71a, Theorem 6], a contradiction. Hence, $H = H_0$, a contradiction. We conclude that G does not split as $H *_C K$ where $C \cong \mathbb{Z}$ and $H, K \ncong \mathbb{Z}$.

Therefore, $G \cong H*_{C_X^t=C_Y}$ with $C_X, C_Y \cong \mathbb{Z}$. It is now sufficient to prove that the base group H is two-generated. Begin by noting that the associated subgroups C_X and C_Y are subgroups of malnormal, infinite cyclic subgroups of G [Log15, Lemma 5.1], and indeed that one of C_X or C_Y is malnormal in H and $C_X \cap C_Y^g$ is trivial for all $g \in H$ [KM98, Corollary 1]. Therefore, $G = \langle H, t; t^{-1}x^m t = y\rangle$ where $\langle x\rangle$ and $\langle y\rangle$ are malnormal in H. This all implies that there exists some $h \in H$ such that $G = \langle th, x\rangle$ [KW99b, Corollary 3.1].

Write $d = h^{-1}yh$ and $s = th$, so $\langle s, x\rangle = G$ and we shall prove that $\langle x, d\rangle = H$. Note that $\langle x, d\rangle \leq H$, so assume that it is a proper subgroup and look for a contradiction. So, suppose that there exists $g \in H \setminus \langle x, d\rangle$. Then g can be written in terms of x and s, as these elements generate G, and, moreover, g can be written in terms of x, s and d. Write g as a word W in x, s and d such that the number of occurrences of s is minimal. That is, write g in the following way, where $h_j \in \langle x, d\rangle$ and k is minimal (note that h_i can be trivial).

$$g =_G W(x, s, d) = h_0 s^{\epsilon_1} h_1 s^{\epsilon_2} h_2 \dots s^{\epsilon_k} h_k$$

Note that $k > 0$ as $g \notin \langle x, d\rangle$. If W has a subword of the form $s^{-1}x^{lm}s$ then this can be replaced by d^l to gain a word with fewer s-terms. Similarly, if W has a subword of the form $sd^l s^{-1}$ then this can be replaced by x^{lm} to gain a word with fewer s-terms. Therefore, by the minimality of k, W is a reduced sequence for g

[2]Kapovich–Weidmann [KW99a] use this result without proof. The citation is for completeness, and the proof is a straightforward application of results on conjugation [MKS04, Theorem 4.6] and commutativity [MKS04, Theorem 4.5] in free products with amalgamation.

which contains $k > 0$ t-terms. Applying Britton's Lemma [LS77], we have that $g \notin H$, a contradiction. Therefore, $H = \langle x, d \rangle$, as required. $\square$

In Lemma 5.4, a key result in our proof of Theorem 5.6, we prove under certain assumptions that if $\psi \in \mathrm{Aut}(G)$ then $\widehat{\phi} \in \mathrm{Out}(G)$ has a representative $\psi \in \widehat{\phi}$ of one of the following four forms.

$$\begin{array}{cccc} \alpha_i : a \mapsto a^{-1}b^i & \beta_i : a \mapsto ab^i & \zeta_i : a \mapsto a^{-1}b^i & \delta_i : a \mapsto ab^i \\ b \mapsto b & b \mapsto b^{-1} & b \mapsto b^{-1} & b \mapsto b \end{array}$$

We shall use the labels α_i, β_i, ζ_i, and δ_i in the rest of Section 5.1 to refer to these forms, and we make no notational distinction between viewing these maps as Nielsen tranformations or as automorphisms of G. We later (Section 5.2) use $\alpha := \alpha_0$, $\beta := \beta_0$, $\zeta := \zeta_0$ and $\delta := \delta_1$.

Virtually cyclic subgroups. We can assume that δ_1 is an automorphism of G by Lemma 5.1. We shall now work out all possible virtually cyclic subgroups of $\mathrm{GL}_2(\mathbb{Z})$ which contain the following matrix Δ_1, which corresponds to the outer automorphism $\widehat{\delta_1}$ under the embedding given by Theorem 4.2.

$$\Delta_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Lemma 5.2 allows roots of the matrix $\Delta_i := \Delta_1^i$, $i \neq 0$, to be computed. Note that Δ_i corresponds to $\widehat{\delta_i}$. Lemma 5.2 is easily proven by induction on the power m, and so the proof is left to the reader.

Lemma 5.2. *Let A be a matrix from $\mathrm{GL}_2(\mathbb{Z})$:*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then A^m has the following form where x_m, y_m and z_m are such that $x_m - y_m = z_m(a - d)$.

$$A^m = \begin{pmatrix} x_m & bz_m \\ cz_m & y_m \end{pmatrix}$$

We apply lemma 5.2 to the following result, which tells us about certain virtually cyclic subgroups of $\mathrm{GL}_2(\mathbb{Z})$. Recall that if $\psi \in \mathrm{Aut}(H)$ for some group H then $\widehat{\psi}$ denotes the element of $\mathrm{Out}(H)$ with representative ψ.

Lemma 5.3. *Let $\psi_t \in \mathrm{Aut}(F(a, b))$ be such that $\langle \widehat{\psi_t}, \widehat{\delta_1} \rangle$ is a virtually cyclic subgroup of $\mathrm{Out}(F(a, b))$. Then $\widehat{\psi_t}$ corresponds to $\widehat{\alpha}_i$, $\widehat{\beta}_i$, $\widehat{\zeta}_i$, or $\widehat{\delta}_i$ in $\mathrm{Out}(F(a, b))$.*

Proof. Our proof uses the equivalence of $\mathrm{Out}(F(a, b))$ and $\mathrm{GL}_2(\mathbb{Z})$. Write $\Delta := \Delta_1$ for the matrix corresponding to $\widehat{\delta_1}$ and Ψ for the matrix corresponding to $\widehat{\psi}$. Using the notation of Lemma 5.2, take $A := \Psi$ and we shall use Lemma 5.2 to prove that $c = 0$. This is sufficient as then $|a| = 1 = |d|$, by looking at the determinant of Ψ, which implies that $\widehat{\psi}$ is of the required form.

Suppose that Ψ has infinite order. Then $\Psi^j = \Delta^k$ for some $j, k \neq 0$. Therefore, $x_j = 1 = y_j$, $cz_j = 0$ and $bz_j = k \neq 0$. Thus, $z_j \neq 0$ and so $c = 0$, as required.

Suppose that Ψ has finite order. Then, $\Psi\Delta\Psi^{-1}$ has infinite order and so $\Psi\Delta^j\Psi^{-1} = \Delta^k$ for some $j, k \neq 0$. We have the following, for $\epsilon := \det(\Psi) = \pm 1$.

$$\begin{aligned}\Psi\Delta^j\Psi^{-1} &= \epsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ &= \epsilon \begin{pmatrix} ad - bc - jac & ja^2 \\ -jc^2 & jac + ad - bc \end{pmatrix} \\ &= \Delta^k\end{aligned}$$

Then because $j \neq 0$ we have that $c = 0$, as required. □

Combining Lemmas 5.1 and 5.3 gives the following result.

Lemma 5.4. *Let $G \cong \langle a, b; S^n(a^{-1}ba, b)\rangle$ with $n > 1$. Suppose that $G \ncong \langle a, b; [a, b]^n\rangle$ and that G is one-ended. If $\psi \in \mathrm{Aut}(G)$ then $\widehat{\psi}$ corresponds to $\widehat{\alpha}_i$, $\widehat{\beta}_i$, $\widehat{\zeta}_i$, or $\widehat{\delta}_i$.*

5.2. **The possibilities.** We now prove Theorem 5.6, which gives the possible isomorphism classes for $\mathrm{Out}(G)$ under the restrictions of this section, and Lemma 5.7, which gives explicit examples for each of the possible generating sets in Theorem 5.6.

First we determine, in a certain sense, the possible generators for $\mathrm{Out}(G)$. We use the notation $\delta := \delta_1$, $\alpha := \alpha_0$, $\beta := \beta_0$ and $\zeta := \zeta_0$ for these generators.

Lemma 5.5. *Let $G = \langle a, b\rangle$ be an arbitrary two-generator group. Suppose that $\delta \in \mathrm{Aut}(G)$. Then the following hold for all $i \in \mathbb{Z}$.*

1. *If $\alpha_i \in \mathrm{Aut}(G)$ then $\alpha \in \mathrm{Aut}(G)$.*
2. *If $\beta_i \in \mathrm{Aut}(G)$ then $\beta \in \mathrm{Aut}(G)$.*
3. *If $\zeta_i \in \mathrm{Aut}(G)$ then $\zeta \in \mathrm{Aut}(G)$.*

Proof. The result holds as $\alpha_i = \delta^i\alpha$, $\beta_i = \delta^{-i}\beta$, and $\zeta_i = \delta^{-i}\zeta$. □

Theorem 5.6 gives the possible isomorphism classes when $\mathrm{Out}(G)$ is virtually-$\mathbb{Z}$.

Theorem 5.6. *Let $G = \langle a, b; R^n\rangle$, with $n > 1$. Suppose that $G \ncong \langle a, b; [a, b]^n\rangle$ and that G is one-ended. If $\mathrm{Out}(G)$ is infinite then $\mathrm{Out}(G)$ is isomorphic to one of $D_\infty \times C_2$, D_∞, $\mathbb{Z} \times C_2$ or $\mathbb{Z}$.*

Moreover, there are five choices of generating set for $\mathrm{Out}(G)$: we always have $\delta \in \mathrm{Aut}(G)$, and either none of, one of or all three of α, β and ζ are automorphisms of G. The following isomorphisms then hold.

1. *If $\alpha, \beta, \zeta \notin \mathrm{Aut}(G)$ then $\mathrm{Out}(G) \cong \mathbb{Z}$.*
2. *If $\alpha \in \mathrm{Aut}(G)$ but $\beta, \zeta \notin \mathrm{Aut}(G)$ then $\mathrm{Out}(G) \cong D_\infty$.*
3. *If $\beta \in \mathrm{Aut}(G)$ but $\alpha, \zeta \notin \mathrm{Aut}(G)$ then $\mathrm{Out}(G) \cong D_\infty$.*
4. *If $\zeta \in \mathrm{Aut}(G)$ but $\alpha, \beta \notin \mathrm{Aut}(G)$ then $\mathrm{Out}(G) \cong \mathbb{Z} \times C_2$.*
5. *If $\alpha, \beta, \zeta \in \mathrm{Aut}(G)$ then $\mathrm{Out}(G) \cong D_\infty \times C_2$.*

Proof. By Lemma 5.1, we can assume that $R \in \langle a^{-1}ba, b\rangle$, so $\delta \in \mathrm{Aut}(G)$. Lemmas 5.4 and 5.5 then give the generating sets (1)–(5). The isomorphisms are easily computed using the embedding of Theorem 4.2, which proves the theorem. □

The following lemma, Lemma 5.7, implies that each of the possible groups and generating sets in Theorem 5.6 occur. The examples in Lemma 5.7 can be verified by checking whether or not $\phi(R)$ is freely conjugate to $R^{\pm 1}$ for each $\phi \in \{\alpha, \beta, \zeta\}$. This works because none of the maps α, β or ζ change the length of the relator R so we can apply the Newman–Gurevich Spelling Theorem [HP84].

Lemma 5.7. *For each $Q \in \{D_\infty \times C_2, D_\infty, \mathbb{Z} \times C_2, \mathbb{Z}\}$ there exists a group $G = \langle a, b; R^n \rangle$, $n > 1$, with $\mathrm{Out}(G) \cong Q$. The following groups give explicit examples.*

1. *If $G = \langle a, b; (aba^{-1}b^2ab^3a^{-1}b^4)^n \rangle$ then $\alpha, \beta, \zeta \notin \mathrm{Aut}(G)$ and $\mathrm{Out}(G) \cong \mathbb{Z}$.*
2. *If $G = \langle a, b; (aba^{-1}b^2ab^3a^{-1}bab^2a^{-1}b^3)^n \rangle$ then $\alpha \in \mathrm{Aut}(G)$ but $\beta, \zeta \notin \mathrm{Aut}(G)$ and $\mathrm{Out}(G) \cong D_\infty$.*
3. *If $G = \langle a, b; (aba^{-1}b^2)^n \rangle$ then $\beta \in \mathrm{Aut}(G)$ but $\alpha, \zeta \notin \mathrm{Aut}(G)$ and $\mathrm{Out}(G) \cong D_\infty$.*
4. *If $G = \langle a, b; (aba^{-1}b^2ab^2a^{-1}bab^3a^{-1}b^3)^n \rangle$ then $\zeta \in \mathrm{Aut}(G)$ but $\alpha, \beta \notin \mathrm{Aut}(G)$ and $\mathrm{Out}(G) \cong \mathbb{Z} \times C_2$.*
5. *If $G = \langle a, b; (aba^{-1}b)^n \rangle$ then $\alpha, \beta, \zeta \in \mathrm{Aut}(G)$ and $\mathrm{Out}(G) \cong D_\infty \times C_2$.*

6. The possibilities for Out(G) when it is finite

In this section we work under the assumption that G is a one-ended two-generator, one-relator group with torsion with $\mathrm{Out}(G)$ finite and we determine the possible isomorphism classes for $\mathrm{Out}(G)$. We prove that every possibility occurs.

Suppose that G is one-ended with $\mathrm{Out}(G)$ finite. Then $\mathrm{Out}(G)$ is isomorphic to a finite subgroup of $\mathrm{GL}_2(\mathbb{Z})$, by Theorem 4.2, and hence to a subgroup of D_6 or of D_4 [Zim96], where D_n denotes the dihedral group of order $2n$. We now prove that six of these nine groups can be realised in this way. The remaining three groups are dealt with in Lemma 6.2, which we state but do not prove.

Lemma 6.1. *For each $Q \in \{D_6, D_4, D_3, C_6, C_4, C_3\}$ there exists a group $G = \langle a, b; R^n \rangle$, $n > 1$, with $\mathrm{Out}(G) \cong Q$. The following groups give explicit examples.*

1. *If $G = \langle a, b; (a^2bab^2a^{-2}b^{-1}a^{-1}b^{-2})^n \rangle$ then $\mathrm{Out}(G) \cong D_6$.*
2. *If $G = \langle a, b; [a^2, b^2]^n \rangle$ then $\mathrm{Out}(G) \cong D_4$.*
3. *If $G = \langle a, b; (a^2(ab)^{-2}b^2)^n \rangle$ then $\mathrm{Out}(G) \cong D_3$.*
4. *If $G = \langle a, b; R^n \rangle$ where R is the word*
$$a^2b^3aba^{-1}b^{-2}ababa^2ba^{-1}b^{-1}a^{-1}b^{-1}a^2b^{-1}ab$$
$$a^{-2}b^{-3}a^{-1}b^{-1}ab^2a^{-1}b^{-1}a^{-1}b^{-1}a^{-2}b^{-1}ababa^{-2}ba^{-1}b^{-1}$$
then $\mathrm{Out}(G) \cong C_6$.
5. *If $G = \langle a, b; (ab^2aba^{-2}ba^{-1}b^{-2}a^{-1}b^{-1}a^2b^{-1})^n \rangle$ then $\mathrm{Out}(G) \cong C_4$.*
6. *If $G = \langle a, b; (ab^{-1}a^2b^{-1}a^{-2}b^{-1}a^{-1}b^{-1}a^{-1}bab^3)^n \rangle$ then $\mathrm{Out}(G) \cong C_3$.*

Proof. We shall use the fact that if $\mathrm{Out}(G)$ is infinite and $G \not\cong \langle a, b; [a, b]^n \rangle$ then any finite order elements of $\mathrm{Out}(G)$ have order two, which follows from Lemma 5.6. We also use the fact that none of the groups G in the statement of the lemma are isomorphic to $\langle a, b; [a, b]^n \rangle$, as $\langle a, b; R^n \rangle \cong \langle a, b; [a, b]^n \rangle$ if and only if the word R is mapped to $[a, b]^\epsilon$ under some Nielsen transformation of $F(a, b)$ [Pri77a], if and only if R is freely conjugate to $[a, b]^{\pm 1}$ [MKS04, Theorem 3.9]. Note that by Theorem 4.2, all interactions between outer automorphisms can be verified by viewing them as elements of $\mathrm{GL}_2(\mathbb{Z})$.
The D_6 case: Note that the maps $\phi : a \mapsto b^{-1}, b \mapsto ab$ and $\psi : a \mapsto ab$, $b \mapsto b^{-1}$ define automorphisms of $G = \langle a, b; (a^2bab^2a^{-2}b^{-1}a^{-1}b^{-2})^n \rangle$. Now, $\mathrm{Out}(G)$ is finite because $\widehat{\phi}$ has order six but $G \not\cong \langle a, b; [a, b]^n \rangle$. Therefore, $\mathrm{Out}(G)$ is isomorphic to either D_6 or C_6. Then, as $\widehat{\phi}^3 \neq \widehat{\psi}$ but $\widehat{\psi}$ has order two, we conclude that $\mathrm{Out}(G) \cong D_6$, as required.
The D_4 case: This is similar to the D_6 case, using the maps $\phi : a \mapsto b, b \mapsto a^{-1}$ and $\psi : a \mapsto b$, $b \mapsto a$

The D_3 case: Note that the maps $\phi : a \mapsto a^{-1}b^{-1}, b \mapsto a$ and $(\beta_1 =)\psi : a \mapsto ab$, $b \mapsto b^{-1}$ define automorphisms of $G = \langle a, b; (a^2(ab)^{-2}b^2)^n\rangle$. As in the D_6 case, $\mathrm{Out}(G)$ is finite because $\widehat{\phi}$ has order three but $G \not\cong \langle a, b; [a, b]^n\rangle$. Noting that $\widehat{\psi}$ has order two, $\mathrm{Out}(G)$ contains an element of order three and an element of order two, and so is isomorphic to one of D_6, C_6 or D_3. Suppose $\mathrm{Out}(G)$ contains an element of order six, and consider the embedding of $\mathrm{Out}(G)$ in $\mathrm{GL}_2(\mathbb{Z})$. The matrices Ω_1 and Ω_2 are the only matrices which satisfy the relation $\Omega^2 = \Phi$, where the matrix Φ is the image of $\widehat{\phi}$ in $\mathrm{GL}_2(\mathbb{Z})$, and so one of these must have order six.

$$\Omega_1 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \quad \Omega_2 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

However, $\Omega_2 = \Phi^{-1}$ has order three while any Nielsen transformation which corresponds to Ω_1 does not preserve the relation of the group, $(a^2(ab)^{-2}b^2)^n$, and so does not correspond to an automorphism of G, a contradiction. Thus, $\mathrm{Out}(G)$ contains no element of order six and so $\mathrm{Out}(G) \cong D_3$, as required.
The C_6 case: This is similar to the D_3 case, using the map $\phi : a \mapsto b^{-1}, b \mapsto ab$. We suppose that $\mathrm{Out}(G) \cong D_6$, and obtain six matrices, $\Psi_1, \ldots, \Psi_6$, which are of order two and which satisfy the relator $(\Phi\Psi_i)^2$, where Φ is the image of $\widehat{\phi}$ in $\mathrm{GL}_2(\mathbb{Z})$. However, any Nielsen transformation which corresponds to one of these six matrices does not preserve the relator of the group, and so does not define an automorphism of G, a contradiction.
The C_4 case: This is similar to the D_3 case, using the map $\phi : a \mapsto b, b \mapsto a^{-1}$.
The C_3 case: This is similar to the D_3 case, using the map $\phi : a \mapsto a^{-1}b^{-1}, b \mapsto a$. Note that we have to eliminate each of D_6, C_6 and D_3. □

The remaining three finite subgroups of $\mathrm{GL}_2(\mathbb{Z})$ do each occur as $\mathrm{Out}(G)$ with G one-ended. Indeed, the following lemma, Lemma 6.2, gives explicit examples. We do not prove Lemma 6.2 here as the proof is disproportionately long. The issue is that the three remaining groups all embed into $D_\infty \times C_2$ (hence the methods used to prove Lemma 6.1 will not work). The result may be verified by applying an algorithm of Dahmani-Guirardel [DG11, Theorem 3], while the author has proven Lemma 6.2 using a different method in his PhD thesis [Log14, Section 3.4.1].

Lemma 6.2. *For Q each of the groups $C_2 \times C_2$, C_2 and the trivial group there exists a group $G = \langle a, b; R^n\rangle$, $n > 1$, with $\mathrm{Out}(G) \cong Q$. The following groups give explicit examples.*

1. *If $G = \langle a, b; (a^2ba^{-2}b)^n\rangle$ then $\mathrm{Out}(G) \cong C_2 \times C_2$.*
2. *If $G = \langle a, b; (a^2ba^{-3}b)^n\rangle$ then $\mathrm{Out}(G) \cong C_2$.*
3. *If $G = \langle a, b; (a^{-2}ba^4ba^{-3}ba^5b)^n\rangle$ then $\mathrm{Out}(G)$ is trivial.*

7. Assembling the proof of Theorem A

We now assemble the proof of Theorem A, as stated in the introduction. We have so far omitted the case of $G = \langle a, b; R^n\rangle$ with R primitive, which occurs precisely when G has more than one end, and precisely when $G \cong \mathbb{Z} * C_n$. Proposition 7.1 now deals with this case.

Proposition 7.1. *Let $G = \langle a, b; R^n\rangle$ with $n > 1$. Suppose that R is a primitive element of $F(a, b)$. Then the following isomorphism holds, where* $\mathrm{Aut}(C_n)$ *commutes* *with the flip generator of D_n and acts on the rotation generator in the natural way as automorphisms of C_n.*

$$\mathrm{Out}(G) \cong D_n \rtimes \mathrm{Aut}(C_n)$$

We do not prove Proposition 7.1, as here $G \cong \mathbb{Z} * C_n$. There are a number of ways to approach the outer automorphism group of such a free product, and indeed presentations for the automorphism groups of free products are known [FR40] [Gil87] (of course the presentation of an automorphism group does not necessarily yield the presentation of the corresponding outer automorphism group). A sketch proof of Proposition 7.1 can be found in the author's PhD thesis [Log14, Section 3.5].

Proof of Theorem A. Let $G = \langle a, b; R^n \rangle$ with $n > 1$ and R not a proper power.

Suppose that $G \cong \langle a, b; [a, b]^n \rangle$. Then $\mathrm{Out}(G) \cong \mathrm{GL}_2(\mathbb{Z})$ by Corollary 4.3.

Suppose that G is one-ended, not isomorphic to $\langle a, b; [a, b]^n \rangle$, and $\mathrm{Out}(G)$ is infinite. Then $\mathrm{Out}(G)$ is one of $D_\infty \times C_2$, D_∞, $\mathbb{Z} \times C_2$ or $\mathbb{Z}$, by Lemma 5.6. By Lemma 5.7, all these possibilities occur.

Suppose that G is one-ended, not isomorphic to $\langle a, b; [a, b]^n \rangle$ and $\mathrm{Out}(G)$ is finite. Then $\mathrm{Out}(G)$ is isomorphic to a finite subgroup of $\mathrm{GL}_2(\mathbb{Z})$, and the finite subgroups of $\mathrm{GL}_2(\mathbb{Z})$, up to isomorphism, are precisely the subgroups of D_4 and of D_6 [Zim96], as required. By Lemma 6.1 and Lemma 6.2, all these possibilities occur.

Suppose that G is not one-ended. Then R is a primitive element of $F(a, b)$ [Pri77a] [FKS72], and then $\mathrm{Out}(G) \cong D_n \rtimes \mathrm{Aut}(C_n)$ by Proposition 7.1. □

8. Two applications

We conclude this paper by proving Corollary B from the introduction and by describing how to give a presentation of $\mathrm{Aut}(G)$ for G one-ended.

Proof of Corollary B. Recall that Corollary B states the existence of an algorithm which takes as input a presentation $\langle a, b; R^n \rangle$, $n > 1$, defining a group G and gives as output the isomorphism class of $\mathrm{Out}(G)$.

Proof of Corollary B. To prove Corollary B we give the relevant algorithm. Our algorithm applies the facts that there exists an algorithm to rewrite a word U in $F(a, b)$ as V^n where V is not a proper power in $F(a, b)$, that it is decidable if a hyperbolic group splits over a virtually-cyclic group with infinite center [DG11, Proposition 6.1], and that there exists an algorithm to determine the generators of the outer automorphism group of a hyperbolic group [DG11, Theorem 3]. Note that if it is known that G splits over a virtually-$\mathbb{Z}$ group with infinite center then $G \cong \langle a, b; S^n(a^{-1}ba, b) \rangle$ for some word S, by Lemma 5.1, and the word S can be found by enumerating the Nielsen transformations of $F(a, b)$ and applying them to R [Pri77a]. Given $G = \langle a, b; R^n \rangle$, $n > 1$, our algorithm is as follows.

(1) Rewrite R^n such that R is freely and cyclically reduced and not a proper power in $F(a, b)$, and such that $\sigma_a(R) = 0$.
  (a) If $R = b^\epsilon$ then $\mathrm{Out}(G) \cong D_{2n} \rtimes \mathrm{Aut}(C_n)$.
  (b) If R is empty or a cyclic shift of $[a, b]^{\pm 1}$ then $\mathrm{Out}(G) \cong \mathrm{GL}_2(\mathbb{Z})$.
(2) Does the JSJ-decomposition of G split?
  (a) If *yes* then rewrite R as $S(a^{-1}ba, b)$ and apply Section 5 to find $\mathrm{Out}(G)$.
  (b) If *no* then obtain the generators for $\mathrm{Out}(G)$ and apply Section 6 to find $\mathrm{Out}(G)$.

□

Presentations for Aut(G). We now describe, with examples, how to write down a presentation for $\mathrm{Aut}(G)$ using $\mathrm{Out}(G)$, where $G = \langle a, b; R^n\rangle$, $n > 1$, is one-ended (the infinitely-ended case has appeared in print before [FR40] [Gil87]).

To begin, note that $\mathrm{Inn}(G) \cong G$ in the canonical way (as G has trivial center [BT67]), so we have the following relation.

$$\gamma_{R^n} = 1 \tag{1}$$

Next, take a transversal T for $\mathrm{Out}(G)$ which consists of Nielsen transformations and denote by $\mathcal{O}$ a subset of this transversal which will generate $\mathrm{Out}(G)$. This transversal T exists by Theorem 4.2. Now, we have that $\gamma_w^\psi = \gamma_{\psi(w)}$ for all $\psi \in \mathrm{Aut}(G)$. Therefore, we have the following relations for all $\psi \in \mathcal{O}$.

$$\gamma_a^\psi = \gamma_{\psi(a)},\ \ \gamma_b^\psi = \gamma_{\psi(b)} \tag{2}$$

We now have to ascertain how the elements of $\mathcal{O}$ multiply together. So, let $\langle X; \mathbf{r}\rangle$ be a presentation for $\mathrm{Out}(G)$ which corresponds to the generators $\mathcal{O}$, then if $S \in \mathbf{r}$ is a relator we have that

$$S = \gamma_w \tag{3}$$

is a relation in $\mathrm{Aut}(G)$ where γ_w is the appropriate inner automorphism. It is clear that these three kinds of relations (1)–(3) are all the relations.

We now give some examples. We write w for γ_w (so w represents the automorphism corresponding to conjugation by w).

Example 8.1*:* If $\mathrm{Out}(G) = \langle\widehat{\alpha}_i\rangle$ then $\mathrm{Aut}(G)$ is the following group.

$$\mathrm{Aut}(G) = \langle \alpha_i, a, b; R^n(a,b), \alpha_i^2 = b^i, a^{\alpha_i} = a^{-1}b^i, b^{\alpha_i} = b\rangle$$

Example 8.2*:* If $\mathrm{Out}(G) = \langle\widehat{\beta}_i\rangle$ then $\mathrm{Aut}(G)$ is the following group.

$$\begin{aligned}\mathrm{Aut}(G) &= \langle \beta_i, a, b; R^n(a,b), \beta_i^2 = 1, a^{\beta_i} = ab^i, b^{\beta_i} = b^{-1}\rangle \\ &\cong G \rtimes C_2\end{aligned}$$

Example 8.3*:* If $G \cong \langle a, b; [a,b]^n\rangle$ and writing $\mathrm{Aut}(F(a,b)) = \langle a, b, X; \mathbf{r}\rangle$ then we have the following group.

$$\mathrm{Aut}(G) = \langle a, b, X; \mathbf{r}, [a,b]^n\rangle$$

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