On a question of Bumagin and Wise

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ABSTRACT. Motivated by a question of Bumagin and Wise, we construct a continuum of finitely generated, residually finite groups whose outer automorphism groups are pairwise non-isomorphic finitely generated, non-recursively-presentable groups. These are the first examples of such residually finite groups.

1. Introduction

In this paper we construct the first examples of finitely generated, residually finite groups G whose outer automorphism groups are finitely generated and not recursively presentable. Indeed, our main result, Theorem B, is the construction of a continuum, so $2^{\aleph_0}$, of such groups G with pairwise non-isomorphic outer automorphism groups. This construction is motivated by a question of Bumagin and Wise, who asked if every countable group Q could be realised as the outer automorphism group of a finitely generated, residually finite group G_Q. Bumagin and Wise solved the question for Q finitely presented [BW05], while in previous work the author solved the question for Q finitely generated and recursively presentable [Log15b, Theorem B][1]. Theorem B proves that these two partial solutions do not entirely resolve the question of Bumagin and Wise.

Theorem B follows from Theorem A, which solves a finite-index version of Bumagin and Wise's question for Q finitely generated and residually finite.

Residually finite groups. A group G is residually finite if for all $g \in G \setminus \{1\}$ there exists a homomorphism $\phi_g : G \to \mathcal{A}_g$ where $\mathcal{A}_g$ is finite and where $\phi_g(g) \neq 1$. Residual finiteness is a strong finiteness property. For example, finitely presentable, residually finite groups have solvable word problem, while finitely generated, residually finite groups are Hopfian [Mal40]. Our main result, which is Theorem B, contrasts with these "nice" properties as it implies that finitely generated, residually finite groups can have very complicated symmetries.

Fundamental to this paper is the existence of finitely generated, residually finite groups which are not recursively presentable. Bridson-Wilton [BW15, Section 2]

[1]This result is dependent on a positive solution to an open problem of Osin. Sapir has remarked that this open problem has a positive solution, and that this will be proven in his next paper [Sap14]. A slightly weaker result holds which is independent of Osin's problem [Log15b, Theorem A].

point out that the existence of such groups follows from work of Slobodskoĭ[Slo81]. The "continuum" statement in the main result, Theorem B, relies on the fact that there is a continuum of such groups [Gri85] (see also [MOS09]). To see that the existence of such groups is fundamental to our argument, suppose that every finitely generated, residually finite group is recursively presentable, and let G be a finitely generated, residually finite group with finitely generated outer automorphism group. Then $\mathrm{Aut}(G)$ is finitely generated and residually finite [Bau63], and hence is recursively presentable. Therefore, as the kernel of $\mathrm{Aut}(G) \to \mathrm{Out}(G)$ is finitely generated (because $\mathrm{Inn}(G) \cong G/Z(G)$), $\mathrm{Out}(G)$ is also recursively presentable. Hence, the existence of finitely generated, residually finite groups which are not recursively presentable is necessary for our argument.

The main construction. The main result of this paper, the result stated in the abstract, is Theorem B. This theorem follows from a more general construction, Theorem A. A group H has *Serre's property FA* if every action of H on any tree has a global fixed point [Ser80]. By F_n we mean the free group of rank n.

Theorem A. *Fix a group H such that H is*

> *(1) word-hyperbolic,*
> *(2) residually finite, and*
> *(3) large, (that is, H contains a finite index subgroup V which surjects onto F_2),*

and such that H has

> *(4) Serre's property FA.*

Then every finitely-generated group Q can be embedded as a finite index subgroup of the outer automorphism group of an HNN-extension G_Q of H, where G_Q is residually finite if Q is residually finite.

Note that it is a famous open problem whether (1) implies (2) or not [nib93] [KW00]. On the other hand, (3) and (4) and not recursively recognisable in the class of word-hyperbolic groups [BW15] [BO08].

Theorem A yields the following two corollaries, each of which individually solves Bumagin and Wise's question up to finite index for Q finitely generated and residually finite. A triangle group $T_{i,j,k} := \langle a, b; a^i, b^j, (ab)^k \rangle$ is called hyperbolic if $i^{-1} + j^{-1} + k^{-1} < 1$. Such triangle groups are well-known to possess the properties required by Theorem A [BMS87] [Ser80].

Corollary 1.1. *Fix a hyperbolic triangle group $H := T_{i,j,k}$. Then every finitely-generated group Q can be embedded as a finite index subgroup of the outer automorphism group of an HNN-extension G_Q of H, where G_Q is residually finite if Q is residually finite.*

The next corollary follows from a result of Agol [Ago13]. Note that, for example, a random group, in the sense of Gromov, at density $< 1/6$ satisfies the conditions of the corollary [DGP11] [OW11].

Corollary 1.2. *Fix a word-hyperbolic group H which has Serre's property FA and which acts properly and cocompactly on a $\mathrm{CAT}(0)$ cube complex. Then every finitely-generated group Q can be embedded as a finite index subgroup of the outer automorphism group of an HNN-extension G_Q of H, where G_Q is residually finite if Q is residually finite.*

The main result of the paper is the following. By a *continuum* we mean a set of cardinality $2^{\aleph_0}$, so of cardinality equal to that of the real numbers $\mathbb{R}$.

Theorem B. *There exists a continuum of finitely generated, residually finite groups whose outer automorphism groups are pairwise non-isomorphic finitely generated, non-recursively-presentable groups.*

We prove Theorem B by noting the existence of a continuum of finitely generated, residually finite groups which are not recursively presentable, and then apply Theorem A (or rather, either of the above corollaries) to these groups.

Outline of the paper. In Section 2 we give two preliminary results on "special" HNN-extensions. These are Theorem 2.1, which describes a certain subgroup of the outer automorphism group of a special HNN-extension, and Proposition 2.2, which classifies the residual finiteness of a certain class of special HNN-extensions. In Section 3 we prove Theorems A and B. In Section 4 we prove a related result for finitely presented (rather than finitely generated) residually finite groups.

2. Two preliminary results

Our construction of Theorem A, which leads to the main result, applies two preliminary results on *special HNN-extensions*, which are HNN-extensions where the action of the stable letter on the associated subgroup(s) is an inner automorphism of the base group. Such an HNN-extension G has the following form (up to isomorphism).

$$G \cong \langle H, t; k^t = k, k \in K \rangle$$

The first result of this section, Theorem 2.1, relates to the outer automorphism groups of special HNN-extensions, while the second result, Proposition 2.2, relates to their residual finiteness.

Outer automorphism groups. The first preliminary result, Theorem 2.1, describes a subgroup of the outer automorphism group of an special HNN-extension. This subgroup, denoted $\mathrm{Out}^H(G)$, is the subgroup which consists of those outer automorphisms Φ with a representative $\phi \in \Phi$ which fixes H setwise, $\phi(H) = H$.

$$\mathrm{Out}^H(G) = \{\Phi \in \mathrm{Out}(H) : \text{ there exists } \phi \in \Phi \text{ such that } \phi(H) = H\}$$

Theorem 2.1 gives, under certain conditions, the isomorphism class of $\mathrm{Out}^H(G)$ up to finite index. We write $A \leq_f B$ to mean that A is a finite index subgroup of B.

Theorem 2.1. *Let G be a special HNN-extension of H with associated subgroup $K \lneq H$. If V is a subgroup of H such that $K \leq V \leq N_H(K)$ and such that $V \cap Z(H) = 1$ then V/K embeds into $\mathrm{Out}^H(G)$. In addition, if $V \leq_f N_H(K)$ and if both $\mathrm{Out}(H)$ and $C_H(K)$ are finite then this embedding is with finite index.*

Proof. Let $\mathrm{Out}_H(G)$ denote the subgroup of $\mathrm{Out}(G)$ consisting of those outer automorphisms Φ with a representative ϕ which fixes H setwise and which sends t to a word containing precisely one t-term. Note that $\mathrm{Out}_H(G) \leq \mathrm{Out}^H(G)$. The result holds for $\mathrm{Out}_H(G)$ in place of $\mathrm{Out}^H(G)$ [Log15a, Theorem B & Lemma 5.2]. Then $\mathrm{Out}_H(G) = \mathrm{Out}^H(G)$ by a result of M. Pettet [Pet99, Lemma 2.6]. $\square$

Residual finiteness. The second preliminary result is a criterion for residual finiteness of special HNN-extensions. Ateş-Logan-Pride actually prove a more general version of the result proven here [ALP16]. We use the fact that a finite index subgroup J of a group G is residually finite if and only if G is residually finite implicitly throughout the proof of this theorem. To prove this equivalence, note that subgroups of residually finite groups are clearly residually finite, while for the other direction re-write the definition of a residually finite group using normal subgroups (corresponding to the kernels of the homomorphisms ϕ_g), and note that every finite index subgroup of J contains a finite index subgroup which is normal in G.

Proposition 2.2 (Ateş-Logan-Pride [ALP16])**.** *Let G be a special HNN-extension of a group H with non-trivial associated subgroup $K \lneqq H$. Suppose H is finitely generated and residually finite, and suppose that $N_H(K)$ has finite index in H. Then G is residually finite if and only if $N_H(K)/K$ is residually finite.*

Our application of Proposition 2.2 only uses the "if" direction, and not the "only if" direction.

Proof. Firstly, $N_H(K)/K$ embeds into $\mathrm{Aut}(G)$ [Log15a, Proposition 5.3], hence G is residually finite only if $N_H(K)/K$ is residually finite [Bau63].

For the other direction, note that the HNN-extension G is residually finite if for all finite sets $\{g_1, \ldots, g_n\}$ with $g_i \in H \setminus K$ there exists some finite index normal subgroup N of H, $N \trianglelefteq_f H$, such that $g_i K \cap N$ is empty for all $i \in \{1, \ldots, n\}$ [BT78, Lemma 4.4]. We prove that this condition holds under the conditions of this lemma. To do this, we find for each such g_i a normal subgroup N_i of finite index in H such that $g_i K \cap N_i$ is empty. Then, the finite-index subgroup $N := \cap N_i$ has the required properties. There are two cases: $g_i \notin N_H(K)$, and $g_i \in N_H(K)$.

Suppose $g_i \notin N_H(K)$. Take the normal subgroup N_i to be the intersection of the (finitely many) conjugates of $N_H(K)$. Then $hK \cap N_i$ is non-empty if and only if $h \in N_H(K)$, and hence $g_i K \cap N_i$ is empty.

Suppose $g_i \in N_H(K)$. Then $g_i K \neq K$ and because $N_H(K)/K$ is residually finite there exists a map $\psi_i : N_H(K)/K \to \mathcal{A}_i$, such that $\mathcal{A}_i$ is finite and $g_i K$ is not contained in the kernel of ψ_i. Therefore, there exists a map $\widetilde{\psi_i} : N_H(K) \to N_H(K)/K \xrightarrow{\psi_i} \mathcal{A}_i$ such that g_i is not contained in the kernel of $\widetilde{\psi_i}$, and take N_i to be the kernel of the map $\widetilde{\psi_i}$. Then, $g_i K \cap N_i$ is empty by construction. $\square$

3. The proof of the main result

We now prove our main results, Theorems A and B, as stated in the introduction.

Proof of Theorem A. Firstly, note that H contains a torsion-free subgroup U of finite index. This follows from conditions (1) and (2) in the statement of the theorem, because word-hyperbolic groups have finitely many conjugacy classes of elements of finite order [BH99, Theorem III.Γ.3.2].

We give the construction, and then we prove that the required properties hold.

The group G_Q is a special HNN-extension, $G_Q = \langle H, t; k^t = k, k \in K \rangle$. Specifying the associated subgroup K completes the construction. Let N be a subgroup of H such that $V/N \cong F_2$, with V as in the statement of the theorem. Note that we can assume V is torsion-free, as for U the torsion-free subgroup of finite

index the image of $V \cap U$ under the map induced by N is free and non-abelian, so rewrite $V := V \cap U$. Then, for every natural number n it holds that H contains a torsion-free finite-index subgroup V_n which maps onto F_n, which can be seen by applying the correspondence theorem to the fact that the free group on two-generators contain finite-index free subgroups of arbitrary rank.

Let Q be a finitely generated group. Then take a presentation $\langle X; \mathbf{r} \rangle$ of Q with $2 \leq |X| < \infty$ and $\mathbf{r}$ non-empty, and so V_n maps onto Q with $n := |X|$. Take K to be the kernel of this map, so $K < H$ and $V_n/K \cong Q$. Note that because V_n has finite index in H, we have that $V_n \leq_f N_H(K) \leq_f H$. Recall that V_n is torsion-free, so if K is virtually-cyclic it must be cyclic.

We now prove that the required properties hold. As $N_H(K)$ has finite index in H, Proposition 2.2 implies that G_Q is residually finite if Q is residually finite. We now prove that Q can be embedded as a finite index subgroup into $\mathrm{Out}(G_Q)$. We show that the conditions of Theorem 2.1 are satisfied, with $V := V_n$, and so Q embeds with finite index into $\mathrm{Out}^H(G_Q)$. The result then follows because H having Serre's property FA implies that $\mathrm{Out}^H(G_Q) = \mathrm{Out}(G_Q)$ [Log15a, Lemma 3.1]. So, $\mathrm{Out}(H)$ is finite as the base group H is a word-hyperbolic group with Serre's property FA [Lev05]. Now, K is non-cyclic because the map $V_n \to V_n/K$ factors through a non-cyclic free group (by assumption the set of relators $\mathbf{r}$ in the presentation for Q is non-empty), and so $C_H(K)$ is finite as H is word-hyperbolic. By construction we have $K \leq V_n \leq_f N_H(K)$, and finally $V_n \cap Z(H)$ as V_n is torsion-free by construction while $Z(H)$ is finite as H is word-hyperbolic. $\qquad\square$

We now prove the main result of this paper, Theorem B. Recall that by a *continuum* we mean a set of cardinality $2^{\aleph_0}(= |\mathbb{R}|)$.

Proof of Theorem B. Begin by noting that there exists a continuum of finitely generated, residually finite groups, and hence there is a set $\mathcal{Q}$, with cardinality the continuum, of such groups which are not recursively presentable [Gri85]. Applying Theorem A to the set $\mathcal{Q}$, we obtain a set $\mathcal{G} = \{G_Q : Q \in \mathcal{Q}\}$ which consists of finitely generated, residually finite groups whose outer automorphism groups are finitely generated but not recursively presentable. Moreover, for $G_Q \in \mathcal{G}$, $\mathrm{Out}(G_Q)$ has only countably many subgroups of finite index, and hence the set $\mathcal{G}$ contains a (subset consisting of a) continuum of groups with pairwise non-isomorphic outer automorphism groups. $\qquad\square$

All the outer automorphism groups in Theorem B are residually finite. This leads us to the following question.

Question 3.1. *Does there exist a finitely generated, non-recursively-presentable, non-residually-finite group Q which can be realised as the outer automorphism group of a finitely generated, residually finite group G_Q?*

4. When G_Q is finitely presented

We now prove a result on $\mathrm{Out}(G_Q)$ for G_Q finitely presented and residually finite.

Theorem 4.1. *For every finitely presented, residually finite group Q there exists a finitely presented, residually finite group G_Q such that Q embeds into $\mathrm{Out}(G_Q)$.*

Proof. A version of Rips' construction due to Wise [Wis03] gives a finitely presented, centerless, residually finite group H_Q with a three-generated subgroup $N = \langle a, b, c \rangle$ such that $H_Q/N \cong Q$.[2] Then the HNN-extension $G_Q = \langle H_Q, t; a^t = a, b^t = b, c^t = c \rangle$ is residually finite, by Theorem 2.1, while $Q \cong H_Q/K$ embeds into $\mathrm{Out}(G_Q)$ by Proposition 2.2, with $V := H_Q = N_{H_Q}(K)$. $\square$

Note that the groups Q in Theorem 4.1 can be taken to be any group which embeds into a finitely presentable, residually finite group.

We know nothing about the embedding $Q \hookrightarrow \mathrm{Out}(G_Q)$ in Theorem 4.1. Indeed, Theorem 4.1 is similar to a result of Wise, who proved the analogous theorem for finitely generated, residually finite groups G_Q by proving that G/N embeds into $\mathrm{Out}(N)$ [Wis03, Corollary 3.3]. Bumagin and Wise altered Rips' construction to make Wise's embedding an isomorphism [BW05]. It may be possible to similarly alter the construction of Theorem 4.1 to answer the following question, Question 4.2. Note that if Q is finitely generated and G_Q is finitely presented and residually finite then Q must be recursively presentable [Log15b, Proposition 3.4].

Question 4.2. *Can every finitely presented group Q be realised as the outer automorphism group of some finitely presented, residually finite group G_Q? And for Q finitely generated and recursively presentable?*

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