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PIECEWISE PRINCIPAL COMODULE ALGEBRAS

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ABSTRACT. A comodule algebra P over a Hopf algebra H with bijective antipode is called principal if the coaction of H is Galois and P is H -equivariantly projective (faithfully flat) over the coaction-invariant subalgebra P^{coH} . We prove that principality is a piecewise property: given N comodule-algebra surjections $P \rightarrow P_i$ whose kernels intersect to zero, P is principal if and only if all P_i 's are principal. Furthermore, assuming the principality of P , we show that the lattice these kernels generate is distributive if and only if so is the lattice obtained by intersection with P^{coH} . Finally, assuming the above distributivity property, we obtain a flabby sheaf of principal comodule algebras over a certain space that is universal for all such N -families of surjections $P \rightarrow P_i$ and such that the comodule algebra of global sections is P .

1. INTRODUCTION

We are motivated by studying equivariant pullbacks of C^* -algebras exemplified, for instance, by a join construction for compact quantum groups. Insisting on equivariance to be given by “free actions” leads to Galois extensions of C^* -algebras by Hopf algebras, and generalising pullbacks to “multi-pullbacks” brings up distributive lattices as a fundamental language. As a by-product of our considerations, we obtain an equivalence between the distributive lattices generated by N ideals intersecting to zero and flabby sheaves of algebras over a certain $(N - 1)$ -projective space.

Comodule algebras provide a natural noncommutative geometry generalisation of spaces equipped with group actions. Less evidently, principal extensions [8] appear to be a proper analogue of principal bundles in this context (see Section 2 for precise definitions). Principal extensions can be considered as functors from the category of finite-dimensional corepresentations of the Hopf algebra (replacing the structure group) to the category of finitely generated projective modules over the coaction-invariant subalgebra (playing the role of the base space).

The aim of this article is to establish a viable concept of locality of comodule algebras and analyse its relationship with principality. The notion of locality we use herein results from decomposing algebras into “pieces”, meaning expressing them as multiple fibre products (called multirestricted direct sums in [26, p. 264]). If X is a compact Hausdorff space and X_1, \dots, X_N form a finite closed covering, then $C(X)$ can be expressed as such a multiple fibre product of its quotient C^* -algebras $C(X_i)$. This leads to a C^* -algebraic notion of a “covering of a quantum space” given by a finite family of algebra surjections $\pi_i : P \rightarrow P_i$ with $\bigcap_i \ker \pi_i = 0$ (see [10, 11], cf. [15]).

Recall that not all properties of group actions are local in nature: there is a natural example of a locally proper action of \mathbb{R} on \mathbb{R}^2 that is not proper (see [21, p.298], cf. [3, Example 1.14] for more details). On the other hand, a group action is

free if and only if it is locally free. Therefore, since for compact groups all actions are proper, the principal (i.e., free and proper) actions of compact groups are local in nature. Our main result (Theorem 3.3) is a noncommutative analogue of this statement: a comodule algebra P which is covered by “pieces” P_i is principal if and only if so are the pieces. In particular, a smash product of an H -module algebra B with the Hopf algebra H (with bijective antipode) is principal, so that gluing together smash products is a way of constructing principal comodule algebras.

However, it was pointed out in [11, p.369] that the aforementioned coverings can show a certain incompleteness when going beyond the C^* -setting. This is related to the fact that the lattice of ideals generated by the $\ker \pi_i$ ’s is in general not distributive. (This problem does not arise for C^* -algebras.) Hence we analyse a stronger notion of covering that includes this nontrivial assumption as part of the definition (see Definition 3.7). If all P_i ’s are smash products, we arrive at a concept of piecewise trivial comodule algebras. They appear to be a good noncommutative replacement of locally trivial compact principal bundles.

Our motivation for going beyond C^* -algebras comes from the way we consider compact principal bundles (the Hausdorff property assumed). We aim to use at the same time algebraic techniques of Hopf-Galois theory and analytic tools coming with C^* -algebras. To this end, we look at the total space of such a bundle in terms of the algebra of functions continuous along the base and polynomial along the fibres [3]. Then the base space algebra is always a C^* -algebra, but, unless the group is finite, the total space algebra is not C^* .

The data of a covering by N pieces can be equivalently encoded into a flabby sheaf of algebras over $\mathbb{P}^{N-1}(\mathbb{Z}/2)$. This is the 2-element field $(N-1)$ -projective space whose topology subbasis is its affine covering. It is a finite space encoding the “combinatorics” of an N -covering, and is non-Hausdorff unless $N=1$. The lattice of all open subsets of $\mathbb{P}^{N-1}(\mathbb{Z}/2)$ turns out to be isomorphic to a certain lattice of antichains in the set of all subsets of an N -element set. Combining this with the Chinese Remainder Theorem for distributive lattices of ideals in an arbitrary ring, we prove that distributive lattices generated by N ideals are equivalent to flabby sheaves over $\mathbb{P}^{N-1}(\mathbb{Z}/2)$.

In particular, consider a compact Hausdorff space X with a covering by N closed subsets X_1, \dots, X_N . Then we have the soft sheaf of continuous functions with N distinguished C^* -algebras $C(X_1), \dots, C(X_N)$. However, the soft sheaf of complex-valued continuous functions on X is not a sheaf of C^* -algebras. Therefore, there seems to be no evident way to use soft sheaves in the noncommutative setting. To overcome this difficulty, we declare the closed sets open and consider X with the new topology generated by these open sets. This leads us to flabby sheaves over $\mathbb{P}^{N-1}(\mathbb{Z}/2)$.

This way we obtain a covering version of Gelfand’s theorem: there is an equivalence between the category of compact Hausdorff spaces with N -coverings by closed subsets and the category of flabby sheaves of unital commutative C^* -algebras over $\mathbb{P}^{N-1}(\mathbb{Z}/2)$. In the noncommutative setting, this sheaf-theoretic reformulation of coverings allows us to view piecewise trivial comodule algebras as introduced in this paper as what is called “locally trivial quantum principal bundles” in [22].

Throughout, we work over a field k and all considered algebras, coalgebras etc. are over k . An unadorned \otimes denotes the tensor product of k -vector spaces. For coproduct and coactions we adopt the Sweedler notation with the summation sign suppressed: $\Delta(h) = h_{(1)} \otimes h_{(2)} \in H \otimes H$, $\Delta_P(p) = p_{(0)} \otimes p_{(1)} \in P \otimes H$.

2. BACKGROUND

2.1. Fibre products. We recall here elementary facts concerning pullback diagrams that will be used in what follows. To focus attention, we consider the category of vector spaces, which suffices for our applications. Let $\pi_1 : V_1 \rightarrow V_{12}$ and $\pi_2 : V_2 \rightarrow V_{12}$ be linear maps of vector spaces. The pullback (fibre product) $V_1 \times_{\pi_1, \pi_2} V_2$ of π_1 and π_2 is defined by a universal property, and turns out to be isomorphic to

$$(1) \quad \ker(\pi_1 - \pi_2 : V_1 \times V_2 \longrightarrow V_{12}) = \{(p, q) \in V_1 \times V_2 \mid \pi_1(p) = \pi_2(q)\}.$$

As a consequence of this description, we obtain:

$$(2) \quad (V_1 \times V_2)_{\pi_1, \pi_2}^{\otimes 2} = \ker((\pi_1 - \pi_2) \otimes \text{id}) \cap \ker(\text{id} \otimes (\pi_1 - \pi_2)).$$

This is because for any linear map $f : V \rightarrow W$ of vector spaces one has $\ker f \otimes \ker f = (\ker f \otimes V) \cap (V \otimes \ker f)$.

Next, let us consider the following commutative diagram of linear maps

$$(3) \quad \begin{array}{ccccc} & & V & & \\ & \phi_1 \swarrow & \downarrow \eta & \searrow \phi_2 & \\ & V_1 \times_{\pi_1, \pi_2} V_2 & & & \\ & \swarrow p_1 & & \searrow p_2 & \\ V_1 & & & & V_2 \\ & \searrow \pi_1 & & \swarrow \pi_2 & \\ & V_{12} & & & \end{array},$$

and show:

Lemma 2.1. *Assume that the ϕ_i 's and π_i 's in the diagram (3) are surjective. Then η is surjective if and only if $\ker(\pi_i \circ \phi_i) = \ker \phi_1 + \ker \phi_2$.*

Proof. Assume first that η is surjective, and $v \in \ker \pi_i \circ \phi_i$. Then both $(\phi_1(v), 0)$ and $(0, \phi_2(v))$ belong to $V_1 \times_{\pi_1, \pi_2} V_2$, and there exist v_1 and v_2 such that $\eta(v_1) = (\phi_1(v), 0)$ and $\eta(v_2) = (0, \phi_2(v))$. Clearly, $v - (v_1 + v_2) \in \ker \eta$. Therefore, as $v_1 \in \ker \phi_2$, $v_2 \in \ker \phi_1$, and

$$(4) \quad \ker \eta = \ker \phi_1 \cap \ker \phi_2,$$

we conclude that $v \in \ker \phi_1 + \ker \phi_2$.

Conversely, if $(\phi_1(v_1), \phi_2(v_2))$ is in the fibre product, then $v_1 - v_2 \in \ker \phi_1 + \ker \phi_2$, so that $v_1 - v_2 = k_1 + k_2$ for some $k_1 \in \ker \phi_1$ and $k_2 \in \ker \phi_2$. Hence, for $v := v_1 - k_1 = v_2 + k_2$, we have $(\phi_1(v_1), \phi_2(v_2)) = (\phi_1(v), \phi_2(v)) = \eta(v)$. \square

2.2. Distributive lattices. We need a method yielding a presentation of elements in finitely generated distributive lattices. We denote by \mathcal{A}_N the set of antichains in $2^{\{1, \dots, N\}}$. For any $l \subset 2^{\{1, \dots, N\}}$ we define

$$(5) \quad \min l \equiv \{u \in l \mid \forall v \subsetneq u : v \notin l\},$$

$$(6) \quad \mathbf{u}(l) \equiv \{u \subset \{1, \dots, N\} \mid \exists v \in l : v \subseteq u\}.$$

It is easy to see that $\min l \in \mathcal{A}_N$. If (Λ, \vee, \wedge) is a lattice generated by $\lambda_1, \dots, \lambda_N$, then we define map $L^\Lambda : \Lambda \rightarrow \mathcal{A}_N$,

$$(7) \quad L^\Lambda(\lambda) = \min\{\{i_1, \dots, i_k\} \subset \{1, \dots, N\} \mid \lambda_{i_1} \wedge \dots \wedge \lambda_{i_k} \leq \lambda\}$$

Conversely, we define map $R^\Lambda : 2^{\{1, \dots, N\}} \rightarrow \Lambda$,

$$(8) \quad R^\Lambda(l) = \bigvee_{(i_1, \dots, i_k) \in l} (\lambda_{i_1} \wedge \dots \wedge \lambda_{i_k}).$$

By the definition of L^Λ it follows that for any $\lambda \in \Lambda$, $R^\Lambda(L^\Lambda(\lambda)) \leq \lambda$. On the other hand, as Λ is a distributive lattice generated by the λ_i 's, there exists $l \subset 2^{\{1, \dots, N\}}$ such that $\lambda = R^\Lambda(l)$. Note that for any $\{i_1, \dots, i_k\} \in l$, $\lambda_{i_1} \wedge \dots \wedge \lambda_{i_k} \leq \lambda = R^\Lambda(l)$, and hence there exists $\{j_1, \dots, j_n\} \in L^\Lambda(\lambda)$ such that $\{j_1, \dots, j_n\} \subseteq \{i_1, \dots, i_k\}$. It follows that $\lambda_{i_1} \wedge \dots \wedge \lambda_{i_k} \leq R^\Lambda(L^\Lambda(\lambda))$ and therefore $\lambda \leq R^\Lambda(L^\Lambda(\lambda))$. Thus, we have proven that for any finitely generated distributive lattice Λ

$$(9) \quad R^\Lambda \circ L^\Lambda = \text{id}_\Lambda.$$

Let us define two binary operations $\wedge, \vee : \mathcal{A}_N \times \mathcal{A}_N \rightarrow \mathcal{A}_N$:

$$(10) \quad l_1 \wedge l_2 = \min\{u_1 \cup u_2 \mid u_1 \in l_1, u_2 \in l_2\}, \quad l_1 \vee l_2 = \min\{l_1 \cup l_2\}.$$

It is immediate by the distributivity of Λ that for all $l_1, l_2 \in \mathcal{A}_N$,

$$(11) \quad R^\Lambda(l_1 \wedge l_2) = R^\Lambda(l_1) \wedge R^\Lambda(l_2), \quad R^\Lambda(l_1 \vee l_2) = R^\Lambda(l_1) \vee R^\Lambda(l_2)$$

2.3. Principal extensions. Let $(H, \Delta, \varepsilon, S)$ be a Hopf algebra with bijective antipode. A right H -comodule algebra P is a unital associative algebra equipped with an H -coaction $\Delta_P : P \rightarrow P \otimes H$ that is an algebra map. For a comodule algebra P , we call

$$(12) \quad P^{\text{co}H} := \{p \in P \mid \Delta_P(p) = p \otimes 1\}$$

the subalgebra of coaction-invariant elements in P . The assumed existence of the inverse of the antipode allows us to define a left coaction ${}_P\Delta : P \rightarrow H \otimes P$ by the formula $p \mapsto S^{-1}(p_{(1)}) \otimes p_{(0)}$. This makes P a left H -comodule and a left H^{op} -comodule algebra.

Definition 2.2. *Let P be a right comodule algebra over a Hopf algebra H with bijective antipode, and let $B := P^{\text{co}H}$ be the coaction-invariant subalgebra. The comodule algebra P is called principal if the following conditions are satisfied:*

(1) *the coaction of H is Galois, that is, the map*

$$\text{can} : P \otimes_B P \longrightarrow P \otimes H, \quad p \otimes q \longmapsto pq_{(0)} \otimes q_{(1)},$$

(called the canonical map) is bijective,

- (2) the comodule algebra P is right H -equivariantly projective as a left B -module, i.e., there exists a right H -colinear and left B -linear splitting of the multiplication map $B \otimes P \rightarrow P$.

This splitting can always be chosen to be unital [8, 9]. Also, in this setting, one can show that the H -equivariant projectivity of P over B is equivalent to the faithful flatness of P as a B -module [27, 28]. If P is a principal comodule algebra, then the extension of algebras $B \subseteq P$ is a special case of principal extensions defined in [8].

A cleft Hopf-Galois extension (e.g., a smash product $B \# H$ of an H -module algebra B by H) is always principal. Indeed, by [4, p.42], cleft Hopf-Galois extensions always enjoy the normal basis property, and the latter can be viewed as equivariant freeness, a special case of equivariant projectivity. For more details and an introduction to Hopf-Galois theory, see, e.g., [20, 29].

2.4. Strong connections. The inverse of the canonical map defines a monomorphism $H \rightarrow P \otimes_B P$, $h \mapsto \text{can}^{-1}(1 \otimes h)$ called the translation map. It turns out that lifts of this map to $P \otimes P$ that are both right and left H -colinear yield an equivalent approach to principality [8] :

Definition 2.3. Let H be a Hopf algebra with bijective antipode. Then a strong connection (cf. [16, 13]) on a right H -comodule algebra P is a unital linear map $\ell : H \rightarrow P \otimes P$ satisfying

$$(\text{id}_P \otimes \Delta_P) \circ \ell = (\ell \otimes \text{id}_H) \circ \Delta, \quad ({}_P\Delta \otimes \text{id}_P) \circ \ell = (\text{id}_H \otimes \ell) \circ \Delta, \quad \widetilde{\text{can}} \circ \ell = 1 \otimes \text{id}.$$

Here $\widetilde{\text{can}} : P \otimes P \rightarrow P \otimes H$ is the lift of can to $P \otimes P$.

The last property of the strong connection (splitting of $\widetilde{\text{can}}$) gives rise to the commutative diagram

$$(13) \quad \begin{array}{ccc} H & \xrightarrow{\ell} & P \otimes P \\ \text{1} \otimes \text{id} \downarrow & \swarrow \widetilde{\text{can}} & \downarrow \text{canonical surjection} \\ P \otimes H & \xleftarrow{\text{can}} & P \otimes_B P. \end{array}$$

Using the Sweedler-type notation $h \mapsto \ell(h)^{(1)} \otimes \ell(h)^{(2)}$ (summation suppressed), we can write the bilinearity and splitting property of a strong connection as follows:

$$(14) \quad \ell(h)^{(1)} \otimes \ell(h)^{(2)}_{(0)} \otimes \ell(h)^{(2)}_{(1)} = \ell(h_{(1)})^{(1)} \otimes \ell(h_{(1)})^{(2)} \otimes h_{(2)},$$

$$(15) \quad \ell(h)^{(1)}_{(0)} \otimes \ell(h)^{(1)}_{(1)} \otimes \ell(h)^{(2)} = \ell(h_{(2)})^{(1)} \otimes S(h_{(1)}) \otimes \ell(h_{(2)})^{(2)},$$

$$(16) \quad \ell(h)^{(1)} \ell(h)^{(2)}_{(0)} \otimes \ell(h)^{(2)}_{(1)} = 1 \otimes h.$$

Applying $\text{id} \otimes \varepsilon$ to the last equation yields a very useful formula:

$$(17) \quad \ell(h)^{(1)} \ell(h)^{(2)} = \varepsilon(h).$$

One can prove that an H -comodule algebra P is principal if and only if it admits a strong connection [8, 9, 6]. Given a strong connection ℓ , one can show that the formula

$$(18) \quad P \otimes H \longrightarrow P \otimes_B P, \quad p \otimes h \longmapsto p \ell(h)^{(1)} \otimes \ell(h)^{(2)},$$

defines the inverse of the canonical map can , so that the coaction of H is Galois. Next, one can also show that

$$(19) \quad s : P \ni p \longmapsto p_{(0)} \ell(p_{(1)})^{(1)} \otimes \ell(p_{(1)})^{(2)} \in B \otimes P.$$

is a splitting whose existence proves the equivariant projectivity. Much as above, one argues that the formula

$$(20) \quad s' : P \ni p \longmapsto \ell(S^{-1}(p_{(1)}))^{(1)} \otimes \ell(S^{-1}(p_{(1)}))^{(2)} p_{(0)} \in P \otimes B$$

provides a left H -colinear and right B -linear splitting of the multiplication map $P \otimes B \rightarrow P$.

2.5. Actions of compact quantum groups. The functions continuous along the base and polynomial along the fibre on a principal bundle with compact structure group have an analogue in the noncommutative setting: Let \bar{H} be the C^* -algebra of a compact quantum group in the sense of Woronowicz [30, 31] and H its dense Hopf $*$ -subalgebra spanned by the matrix coefficients of the irreducible unitary corepresentations. Let \bar{P} be a unital C^* -algebra and let $\delta : \bar{P} \rightarrow \bar{P} \otimes_{\min} \bar{H}$ be a C^* -algebraic right coaction of \bar{H} on \bar{P} . (See [1, Definition 0.2] for a general definition and [5, Definition 1] for the special case of compact quantum groups.) Then the subalgebra $P \subset \bar{P}$ of elements for which the coaction lands in $\bar{P} \otimes H$ (algebraic tensor product),

$$(21) \quad P := \{p \in \bar{P} \mid \delta(p) \in \bar{P} \otimes H\},$$

is an H -comodule algebra. It follows from results of [5] and [24] that P is dense in \bar{P} . We call P the comodule algebra associated to the C^* -algebra \bar{P} . It is straightforward to verify that the operation $\bar{P} \mapsto P$ commutes with taking fibre products. Note also that $\bar{P}^{co\bar{H}} = P^{coH}$.

3. PIECEWISE PRINCIPALITY

To show the piecewise nature of principality, we begin by proving lemmas concerning quotients and fibre products of principal comodule algebras.

Lemma 3.1. *Let $\pi : P \rightarrow Q$ be a surjection of right H -comodule algebras (bijective antipode assumed). If P is principal, then:*

- (1) *The induced map $\pi^{coH} : P^{coH} \rightarrow Q^{coH}$ is surjective.*
- (2) *There exists a unital H -colinear splitting of π .*

Proof. It follows from the colinearity of π that $\pi(P^{coH}) \subseteq Q^{coH}$. To prove the converse inclusion, we take advantage of the left P^{coH} -linear retraction of the inclusion $P^{coH} \subseteq P$ that was used to prove [8, Theorem 2.5(3)]:

$$(22) \quad \sigma : P \longrightarrow P^{coH}, \quad \sigma(p) := p_{(0)} \ell(p_{(1)})^{(1)} \varphi(\ell(p_{(1)})^{(2)}).$$

Here ℓ is a strong connection on P and φ is any unital linear functional on P . If $\pi(p) \in Q^{coH}$, then $\sigma(p)$ is a desired element of P^{coH} that is mapped by π to $\pi(p)$. Indeed, since $\pi(p_{(0)}) \otimes p_{(1)} = \pi(p) \otimes 1$, using the unitality of ℓ , π and φ , we compute

$$(23) \quad \pi(\sigma(p)) = \pi(p_{(0)}) \pi(\ell(p_{(1)})^{(1)}) \varphi(\ell(p_{(1)})^{(2)}) = \pi(p).$$

Concerning the second assertion, one can readily verify that the formula

$$(24) \quad \varsigma(q) := \alpha^{coH}(q_{(0)} \pi(\ell(q_{(1)})^{(1)})) \ell(q_{(1)})^{(2)}$$

defines a unital colinear splitting of π . Here $\alpha^{\text{co}H}$ is any k -linear unital splitting of $\pi^{\text{co}H}$, e.g., $\alpha^{\text{co}H} = \varsigma^{\text{co}H}$, and ℓ is again a strong connection on P . \square

Lemma 3.2. *Let P be a fibre product in the category of right H -comodule algebras:*

$$\begin{array}{ccc} & P & \\ \swarrow & & \searrow \\ P_1 & & P_2 \\ \searrow \pi_2^1 & & \swarrow \pi_1^2 \\ & P_{12} & \end{array}$$

Then, if P_1 and P_2 are principal and π_2^1 and π_1^2 are surjective, P is a principal comodule algebra.

Proof. Given strong connections ℓ_1 and ℓ_2 on P_1 and P_2 , respectively, we want to construct a strong connection on P . A first approximation for such a strong connection is as follows:

$$(25) \quad \lambda : H \longrightarrow P \otimes P, \quad \lambda(h) := (\ell_1(h)^{\langle 1 \rangle}, f_2^1(\ell_1(h)^{\langle 1 \rangle})) \otimes (\ell_1(h)^{\langle 2 \rangle}, f_2^1(\ell_1(h)^{\langle 2 \rangle})).$$

Here $f_2^1 := \sigma_2 \circ \pi_2^1$, and σ_2 is a unital colinear splitting of π_2^1 , which exists by Lemma 3.1(2). The map λ is unital and bilinear, but it does not split the lifted canonical map:

$$(1, 1) \otimes h - \widetilde{\text{can}}(\lambda(h)) = (0, 1) \otimes h - (0, f_2^1(\ell_1(h_{(1)})^{\langle 1 \rangle}) f_2^1(\ell_1(h_{(1)})^{\langle 2 \rangle})) \otimes h_{(2)} \in P_2 \otimes H.$$

Now, let $\widetilde{\text{can}}_2$ be the lifted canonical map on $P_2 \otimes P_2$. Applying the splitting of $\widetilde{\text{can}}_2$ given by ℓ_2 (cf. (18)) to the right hand side of the above equation, gives a correction term for λ :

$$(26) \quad \begin{aligned} T(h) &:= \ell_2(h) - f_2^1(\ell_1(h_{(1)})^{\langle 1 \rangle}) f_2^1(\ell_1(h_{(1)})^{\langle 2 \rangle}) \ell_2(h_{(2)})^{\langle 1 \rangle} \otimes \ell_2(h_{(2)})^{\langle 2 \rangle} \\ &= (\varepsilon(h_{(1)}) - f_2^1(\ell_1(h_{(1)})^{\langle 1 \rangle}) f_2^1(\ell_1(h_{(1)})^{\langle 2 \rangle})) \ell_2(h_{(2)})^{\langle 1 \rangle} \otimes \ell_2(h_{(2)})^{\langle 2 \rangle}. \end{aligned}$$

This defines a bilinear map into $P_2 \otimes P_2$ which annihilates 1. Considering λ as a map into $(P_1 \oplus P_2)^{\otimes 2}$, we can add these two maps. The map $\lambda + T$ is still unital and bilinear and splits the lifted canonical map on $(P_1 \oplus P_2)^{\otimes 2}$. Remembering (2) and (17), it is clear from the formula for T that to make it taking values in $P \otimes P$ we only need to add the term

$$(27) \quad T'(h) := (\varepsilon(h_{(1)}) - f_2^1(\ell_1(h_{(1)})^{\langle 1 \rangle}) f_2^1(\ell_1(h_{(1)})^{\langle 2 \rangle})) \ell_2(h_{(2)})^{\langle 1 \rangle} \otimes f_1^2(\ell_2(h_{(2)})^{\langle 2 \rangle}).$$

Much as above, here $f_1^2 := \sigma_1 \circ \pi_1^2$ and σ_1 is a unital colinear splitting of π_1^2 , which exists by Lemma 3.1(2). The formula (27) defines a bilinear map into $P_2 \otimes P_1$ which annihilates 1. Since the lifted canonical map on $(P_1 \oplus P_2)^{\otimes 2}$ vanishes on $P_2 \otimes P_1$, the sum $\lambda + T + T'$ splits the lifted canonical map, takes values in $P \otimes P$ and is unital and bilinear. Thus it is as desired a strong connection on P . \square

Let us now consider a family $\pi_i : P \rightarrow P_i$, $i \in \{1, \dots, N\}$, of surjections of right H -comodule algebras with $\bigcap_{i=1}^N \ker \pi_i = 0$. Denote $J_i := \ker \pi_i$. By Lemma 2.1

and formula 4, for any $k = 1, \dots, N - 1$ there is a fibre-product diagram of right H -comodule algebras

$$(28) \quad \begin{array}{ccc} & P/(J_1 \cap \dots \cap J_{k+1}) & \\ \swarrow & & \searrow \\ P/(J_1 \cap \dots \cap J_k) & & P/J_{k+1} \\ \searrow^{\pi_2} & & \swarrow_{\pi_1^2} \\ & P/((J_1 \cap \dots \cap J_k) + J_{k+1}) & \end{array} .$$

Assume that all the $P_i \cong P/J_i$ are principal. Then Lemma 3.2 implies by an obvious induction that $P/(J_1 \cap \dots \cap J_k)$ is principal for all $k = 1, \dots, N$. In particular ($k = N$), P is principal. On the other hand, if P is principal then all the P_i 's are principal: If $\ell : H \rightarrow P \otimes P$ is a strong connection on P , then

$$(29) \quad (\pi_i \otimes \pi_i) \circ \ell : H \longrightarrow P_i \otimes P_i$$

is a strong connection on P_i . Thus we have proved the following:

Theorem 3.3. *Let $\pi_i : P \rightarrow P_i$, $i \in \{1, \dots, N\}$, be surjections of right H -comodule algebras such that $\bigcap_{i=1}^N \ker \pi_i = 0$. Then P is principal if and only if all the P_i 's are principal.*

Our next step is a statement about a relation between the ideals of a principal comodule algebra P that are also subcomodules, and ideals in the subalgebra B of coaction-invariant elements. Both sets are obviously lattices with respect to the operations $+$ and \cap .

Proposition 3.4. *Let P be a principal right H -comodule algebra and $B := P^{\text{co}H}$ the coaction-invariant subalgebra. Denote by Ξ_B the lattice of all ideals in B and by Ξ_P the lattice of all ideals in P which are simultaneously subcomodules. Then the map*

$$\mathcal{L} : \Xi_P \longrightarrow \Xi_B, \quad \mathcal{L}(J) := J \cap B$$

is a monomorphism of lattices.

Proof. The only non-trivial step in proving that \mathcal{L} is a homomorphism of lattices is establishing the inclusion $(B \cap J) + (B \cap J') \supseteq B \cap (J + J')$. To this end, we proceed along the lines of the proof of Lemma 3.1(1). Since J is a comodule and an ideal, from the formula (19) we obtain:

$$(30) \quad p \in J \implies s(p) = p_{(0)} \ell(p_{(1)})^{(1)} \otimes \ell(p_{(1)})^{(2)} \in (J \cap B) \otimes P.$$

Now, let $p \in J$, $q \in J'$, $p + q \in B$. Then (30) implies that

$$(31) \quad (p + q) \otimes 1 = s(p) + s(q) \in (B \cap J) \otimes P + (B \cap J') \otimes P.$$

Applying any unital linear functional $P \rightarrow k$ to the second tensor component implies $p + q \in (B \cap J) + (B \cap J')$. Finally, since s is a splitting of the multiplication map, it follows from (30) that $J = (J \cap B)P$. This, in turn, proves the injectivity of \mathcal{L} . \square

Remark 3.5. Much as the formula (19) implies (30), the formula (20) implies

$$(32) \quad p \in J \implies s'(p) = \ell(S^{-1}(p_{(1)}))^{(1)} \otimes \ell(S^{-1}(p_{(1)}))^{(2)} p_{(0)} \in P \otimes (J \cap B).$$

Therefore, since s' is a splitting of the right multiplication map, $J = P(J \cap B)$. Combining this with the above discussed left-sided version yields

$$(33) \quad P(J \cap B) = J = (J \cap B)P.$$

Remark 3.6. Note that the homomorphism \mathcal{L} is not surjective in general. A counterexample is given by a smash product (trivial principal comodule algebra) of the Laurent polynomials $B = k[u, u^{-1}]$ with the Hopf algebra $H = k[v, v^{-1}]$ of Laurent polynomials ($\Delta(v) = v \otimes v$). The action is defined by $v \triangleright u = qu$, $q \in k \setminus \{0, 1\}$. Viewing u and v as generators of P , it is clear that $vu = quv$. It is straightforward to verify that, if I is the two-sided ideal in B generated by $u - 1$, then the right ideal IP is not a two-sided ideal of P . Hence the map \mathcal{L} cannot be surjective by (33).

In [11], families $\pi_i : P \rightarrow P_i$ of algebra homomorphisms as in Theorem 3.3 were called coverings. However, it was explained therein that such coverings are well-behaved when the kernels $\ker \pi_i$ generate a distributive lattice of ideals (with respect to $+$ and \cap as lattice operations). Hence we adopt in the present paper the following terminology:

Definition 3.7. A finite family $\pi_i : P \rightarrow P_i$, $i = 1, \dots, N$, of surjective algebra homomorphisms is called a weak covering if $\bigcap_{i=1, \dots, N} \ker \pi_i = \{0\}$. A weak covering is called a covering if the lattice of ideals generated by the $\ker \pi_i$'s is distributive.

The above definition can obviously be extended to the case when the π_i 's are algebra and H -comodule morphisms. Then the $\ker \pi_i$'s are ideals and H -subcomodules.

The next claim is concerned with the distributivity condition from Definition 3.7 for coverings of principal comodule algebras. It follows from Proposition 3.4 and Lemma 3.1(1).

Corollary 3.8. Let $\pi_i : P \rightarrow P_i$, $i = 1, \dots, N$, be surjective homomorphisms of right H -comodule algebras. Assume that P is principal. Then $\{\pi_i : P \rightarrow P_i\}_i$ is a covering of P if and only if $\{\pi_i^{\text{co}H} : P^{\text{co}H} \rightarrow P_i^{\text{co}H}\}$ is a covering of $P^{\text{co}H}$.

The above corollary is particularly helpful when $P^{\text{co}H}$ is a C^* -algebra because lattices of closed ideals in a C^* -algebra are always distributive (due to the property $I \cap J = IJ$).

We are now ready to propose a noncommutative-geometric replacement of the concept of local triviality of principal bundles. Since these are the closed rather than open subsets of a compact Hausdorff space that admit a natural translation into the language of C^* -algebras, we use finite closed rather than open coverings to trivialize bundles. As is explained in [3, Example 1.24], there is a difference between these two approaches. We reserve the term “locally trivial” for bundles trivializable over an open cover, and call bundles trivializable over a finite closed cover “piecewise trivial”. It is the latter (slightly more general) property that we generalize to the noncommutative setting.

Definition 3.9. An H -comodule algebra P is called piecewise principal (trivial) if there exist comodule algebra surjections $\pi_i : P \rightarrow P_i$, $i = 1, \dots, N$, such that:

- (1) The restrictions $\pi_i|_{P^{\text{co}H}} : P^{\text{co}H} \rightarrow P_i^{\text{co}H}$ form a covering.

- (2) *The P_i 's are principal. (The P_i 's are isomorphic as H -comodule algebras to a smashed product $P_i^{\text{co}H} \#_i H$.)*

While not every compact principal bundle is piecewise (or locally) trivial ([3, Example 1.22]), every piecewise principal compact G -space (i.e., covered by finitely many compact principal G -bundles) is clearly a compact principal G -bundle. The second statement becomes non-trivial when we replace compact G -spaces by comodule algebras. However, it is an immediate consequence of Theorem 3.3 and Corollary 3.8:

Corollary 3.10. *Let H be a Hopf algebra with bijective antipode and P be an H -comodule algebra that is piecewise principal with respect to $\{\pi_i : P \rightarrow P_i\}_i$. Then P is principal and $\{\pi_i : P \rightarrow P_i\}_i$ is a covering of P .*

Finally, let us consider the relationship between piecewise triviality and a similar concept referred to as “local triviality” in [22]. Therein, sheaves \mathcal{P} of comodule algebras were viewed as quantum analogues of principal bundles. They were called locally trivial provided that the space X on which \mathcal{P} is defined admits an open covering $\{U_i\}_i$ such that all $\mathcal{P}(U_i)$'s are smash products. If we assume such a sheaf to be flabby (that is, for all open subsets of $V, U, V \subseteq U$, the restriction maps $\pi_{U,V} : \mathcal{P}(U) \rightarrow \mathcal{P}(V)$ are surjective), then we can use Theorem 3.3 to deduce the principality of all $\mathcal{P}(U)$'s:

Corollary 3.11. *Let H be a Hopf algebra with bijective antipode and \mathcal{P} be a flabby sheaf of H -comodule algebras over a topological space X . If $\{U_i\}_i$ is a finite open covering such that all $\mathcal{P}(U_i)$'s are principal, then $\mathcal{P}(U)$ is principal for any open subset $U \subseteq X$.*

4. COVERINGS AND FLABBY SHEAVES

In this section, we focus entirely on a flabby-sheaf interpretation of distributive lattices of ideals defining coverings of algebras (see Definition 3.7). We will explain that for a flabby sheaf in Corollary 3.11, the underlying topological space plays only a secondary role and can be replaced by a certain space that is universal for all N -element coverings. This space is the 2-element field $N - 1$ -projective space

$$(34) \quad \mathbb{P}^{N-1}(\mathbb{Z}/2) := \{0, 1\}^N \setminus \{(0, \dots, 0)\}$$

whose topology subbasis is its affine covering, i.e., it is the topology generated by the subsets

$$(35) \quad A_i := \{(z_1, \dots, z_N) \in \mathbb{P}^{N-1}(\mathbb{Z}/2) \mid z_i \neq 0\}.$$

Consider now an arbitrary space X with a finite covering $\{U_1, \dots, U_N\}$. Define on X the topology generated by the U_i 's (considered as open sets) and pass to the quotient by the equivalence relation

$$(36) \quad x \sim y \Leftrightarrow (\forall i : x \in U_i \Leftrightarrow y \in U_i).$$

Obviously, X/\sim depends on the specific features of the covering $\{U_i\}_i$. However, for a fixed N , it can always be embedded into $\mathbb{P}^{N-1}(\mathbb{Z}/2)$:

Proposition 4.1. *Let $X = U_1 \cup \dots \cup U_N$ be any set equipped with the topology generated by the U_i 's. Let $p : X \rightarrow X/\sim$ be the quotient map defined by the equivalence relation (36). Then*

$$\xi : X/\sim \longrightarrow \mathbb{P}^{N-1}(\mathbb{Z}/2), \quad p(x) \longmapsto (z_1, \dots, z_N), \quad z_i = 1 \Leftrightarrow x \in U_i, \quad \forall i,$$

is an embedding of topological spaces.

Proof. It is immediate that ξ is well defined and injective. Next, since $p^{-1}(\xi^{-1}(A_i)) = U_i$ is open for each i , all $\xi^{-1}(A_i)$'s are open in the quotient topology on X/\sim . Now the continuity of ξ follows from the fact that A_i 's form a subbasis of the topology of $\mathbb{P}^{N-1}(\mathbb{Z}/2)$.

The key step is to show that images of open sets in X/\sim are open in $\xi(X/\sim)$. First note that by the definition of the relation (36),

$$(37) \quad p^{-1}(p(U_{i_1} \cap \dots \cap U_{i_n})) = U_{i_1} \cap \dots \cap U_{i_n}.$$

Therefore, as preimages and images preserve unions and any open set in X is the union of intersections of U_i 's, p is an open map. On the other hand, by the surjectivity of p , we have $p(p^{-1}(V)) = V$ for any subset $V \subset X/\sim$. Hence it follows that a set in X/\sim is open if and only if it is an image under p of an open set in X . Finally, by the definition of ξ ,

$$(38) \quad \xi(p(U_{i_1} \cap \dots \cap U_{i_n})) = A_{i_1} \cap \dots \cap A_{i_n} \cap \text{Im}(\xi),$$

and the claim follows from the distributivity of \cap with respect to \cup . \square

Note that the map ξ is a homeomorphism precisely when the U_i 's are in a generic position, that is when all intersections $U_{i_1} \cap \dots \cap U_{i_k} \cap (X \setminus U_{j_1}) \cap \dots \cap (X \setminus U_{j_l})$ such that $\{i_1, \dots, i_k\} \cap \{j_1, \dots, j_l\} = \emptyset$ are non-empty.

Thus we have shown that if we consider X and the U_i 's as in Corollary 3.11, then the composition $\xi \circ p : X \rightarrow \mathbb{P}^{N-1}(\mathbb{Z}/2)$ is continuous. Hence we can produce flabby sheaves over $\mathbb{P}^{N-1}(\mathbb{Z}/2)$ by taking direct images of flabby sheaves over X . They will have the same sections globally and on the covering sets. In this sense, they carry an essential part of the data encoded in the original sheaf.

Example 4.2. Let $X = \mathbb{P}^{N-1}(\mathbb{C})$. Denote by $[x_1 : \dots : x_N]$ the class of $(x_1, \dots, x_N) \in \mathbb{C}^N$ in $\mathbb{P}^{N-1}(\mathbb{C})$. Define a family of closed sets

$$X_i = \{[x_1 : \dots : x_N] \in \mathbb{P}^{N-1}(\mathbb{C}) \mid |x_i| = \max(\{|x_1|, \dots, |x_N|\})\}, \quad i = 1, \dots, N.$$

The X_i 's cover X , and moreover, for all $\Lambda, \Gamma \subset \{1, \dots, N\}$ such that $\Lambda \neq \emptyset$, $\Lambda \cap \Gamma = \emptyset$, the element

$$[x_1 : \dots : x_N] \in X, \text{ where } x_i = \begin{cases} 1 & \text{if } i \in \Lambda \\ 0 & \text{otherwise} \end{cases}$$

belongs to $\bigcap_{i \in \Lambda} X_i \cap \bigcap_{j \in \Gamma} (X \setminus X_j)$. It follows that the X_i 's are in generic position and the map $\xi : X/\sim \rightarrow \mathbb{P}^{N-1}(\mathbb{Z}/2)$ (Proposition 4.1) is a homeomorphism (if X/\sim is considered with finite topology as in Proposition 4.1). Let

$$V_i = \{[x_1 : \dots : x_N] \in \mathbb{P}^{N-1}(\mathbb{C}) \mid x_i \neq 0\}, \quad i = 1, \dots, N$$

be the standard (open) affine cover of $\mathbb{P}^{N-1}(\mathbb{C})$ and Ψ_i be the homeomorphism

$$\Psi_i : V_i \rightarrow \mathbb{C}^{N-1}, \quad [x_1 : \dots : x_N] \mapsto (x_1/x_i, \dots, \widehat{x_i/x_i}, \dots, x_N/x_i).$$

Observe that for all $i \in \{1, \dots, N\}$, $X_i \subset V_i$ and

$$\Psi_i(X_i) = \{(y_1, \dots, y_{N-1}) \in \mathbb{C}^{N-1} \mid \forall_j |y_j| \leq 1\}.$$

In particular, X_i 's are indeed closed sets.

Our next aim is to demonstrate that flabby sheaves over $\mathbb{P}^{N-1}(\mathbb{Z}/2)$ are just a reformulation of the notion of covering introduced in Definition 3.7. It turns out that the distributivity condition discussed in the previous section is the key property needed to reconcile the results from [10, 12] with those from [22]. In particular, our results imply the principality of Pflaum's noncommutative instanton bundle (see the last section for details).

Let us consider the lattice Γ_N of open subsets of $\mathbb{P}^{N-1}(\mathbb{Z}/2)$ generated by the A_i 's (35).

Lemma 4.3. *If $A_{i_1} \cap \dots \cap A_{i_k} \subseteq R^{\Gamma_N}(l)$, for some $l \subset 2^{\{1, \dots, N\}}$ then $\{i_1, \dots, i_k\} \in \mathbf{u}(l)$.*

Proof. Suppose that $\{i_1, \dots, i_k\} \notin \mathbf{u}(l)$. Therefore, for all $u \in l$ there exists $j(u) \in u$, such that $j(u) \notin \{i_1, \dots, i_k\}$. Then

$$\begin{aligned} A_{i_1} \cap \dots \cap A_{i_k} \cap (\mathbb{P}^{N-1}(\mathbb{Z}/2) \setminus R^{\Gamma_N}(l)) &= A_{i_1} \cap \dots \cap A_{i_k} \cap \bigcap_{u \in l} \bigcup_{j \in u} (\mathbb{P}^{N-1}(\mathbb{Z}/2) \setminus A_j) \\ &\supseteq A_{i_1} \cap \dots \cap A_{i_k} \cap \bigcap_{u \in l} (\mathbb{P}^{N-1}(\mathbb{Z}/2) \setminus A_{j(u)}) \neq \emptyset, \end{aligned}$$

as A_i 's are in generic position (see the remark after Proposition 4.1). Hence $A_{i_1} \cap \dots \cap A_{i_k}$ cannot be contained in $R^{\Gamma_N}(l)$. \square

By Lemma 4.3, it is clear that

$$(39) \quad \{\{i_1, \dots, i_k\} \subset \{1, \dots, N\} \mid A_{i_1} \cap \dots \cap A_{i_k} \subseteq R^{\Gamma_N}(l)\} = \mathbf{u}(l).$$

As $\min \mathbf{u}(l) = \min l$, it follows that for any $l \subset 2^{\{1, \dots, N\}}$,

$$(40) \quad L^{\Gamma_N}(R^{\Gamma_N}(l)) = \min l.$$

Thus, for any $U, U' \subset \mathbb{P}^{N-1}(\mathbb{Z}/2)$, using (40), (9) and (11),

$$\begin{aligned} (41) \quad L^{\Gamma_N}(U \cap U') &= L^{\Gamma_N}(R^{\Gamma_N}(L^{\Gamma_N}(U)) \cap R^{\Gamma_N}(L^{\Gamma_N}(U'))) \\ &= L^{\Gamma_N}(R^{\Gamma_N}(L^{\Gamma_N}(U) \wedge L^{\Gamma_N}(U'))) = \min(L^{\Gamma_N}(U) \wedge L^{\Gamma_N}(U')) = L^{\Gamma_N}(U) \wedge L^{\Gamma_N}(U'). \end{aligned}$$

Similarly one can show that

$$(42) \quad L^{\Gamma_N}(U \cup U') = L^{\Gamma_N}(U) \vee L^{\Gamma_N}(U').$$

Hence, we have proven

Lemma 4.4. *The map $L^{\Gamma_N} : \Gamma_N \rightarrow \mathcal{A}_N$ is an isomorphism of distributive lattices (with $(L^{\Gamma_N})^{-1} = R^{\Gamma_N}|_{\mathcal{A}_N}$) where \mathcal{A}_N is a distributive lattice with meet and join operations defined in (10).*

In the following proposition, we use the above results for the lattice Γ_N of open subsets of $\mathbb{P}^{N-1}(\mathbb{Z}/2)$ and the lattice $\Lambda_{(\ker \pi_i)_i}$ of ideals generated by the kernels of surjections forming an N -covering. For the first lattice, we consider the category of flabby sheaves over $\mathbb{P}^{N-1}(\mathbb{Z}/2)$, and for the second, the category of N -coverings of

algebras. Here “ N -covering” means a covering by N surjections, and a morphism between N -coverings $\{\pi_i : P \rightarrow P_i\}_i$ and $\{\eta_i : Q \rightarrow Q_i\}_i$ consists of morphisms $\xi : P \rightarrow Q$ and $\xi_i : P_i \rightarrow Q_i$ such that $\eta_i \circ \xi = \xi_i \circ \pi_i, \forall i \in \{1, \dots, N\}$.

Theorem 4.5. *Let \mathbf{C}_N be the category of N -coverings of algebras, and \mathbf{F}_N be the category of flabby sheaves of algebras over $\mathbb{P}^{N-1}(\mathbb{Z}/2)$. Then the following assignments*

$$(43) \quad \mathbf{C}_N \ni \{\pi_i : P \rightarrow P_i\}_i \longmapsto \{\mathcal{P} : U \mapsto P/R^{\Lambda(\ker \pi_i)_i}(L^{\Gamma_N}(U))\}_U \in \mathbf{F}_N,$$

$$(44) \quad \mathbf{F}_N \ni \mathcal{P} \longmapsto \{\mathcal{P}(\mathbb{P}^{N-1}(\mathbb{Z}/2)) \rightarrow \mathcal{P}(A_i)\}_i \in \mathbf{C}_N$$

yield an equivalence of categories.

Proof. Suppose we are given a flabby sheaf \mathcal{P} of algebras over $\mathbb{P}^{N-1}(\mathbb{Z}/2)$. Let $\pi_{V,U} : \mathcal{P}(V) \rightarrow \mathcal{P}(U)$ denote the restriction map for any open $U, V, U \subseteq V$. For brevity, we write π_U instead of $\pi_{V,U}$, if $V = \mathbb{P}^{N-1}(\mathbb{Z}/2)$. By the flabbiness of \mathcal{P} , the morphisms π_{A_i} (see (35)) are surjective. The property $\bigcap_{i=1}^N \ker \pi_{A_i} = \{0\}$ follows from the sheaf condition.

It remains to prove the distributivity of the lattice generated by the kernels of π_{A_i} ’s. Lattices of sets are always distributive, so that it is enough to show that the assignment $U \mapsto \ker \pi_U$ defines a surjective morphism from the lattice of open subsets of $\mathbb{P}^{N-1}(\mathbb{Z}/2)$ onto the lattice of ideals generated by $\ker \pi_{A_i}$ ’s. Here by a morphism of lattices we mean a map that transforms the union and intersection of open subsets to the intersection and sum of ideals, respectively.

To show this, let U', U'' be open subsets of $\mathbb{P}^{N-1}(\mathbb{Z}/2)$. Since \mathcal{P} is a sheaf, we know that $\mathcal{P}(U' \cup U'')$ is the fibre product of $\mathcal{P}(U')$ and $\mathcal{P}(U'')$. Now it follows from (4) that $\ker \pi_{U' \cup U''} = \ker \pi_{U'} \cap \ker \pi_{U''}$, as needed. Similarly, since the sheaf \mathcal{P} is flabby, Lemma 2.1 implies that $\ker \pi_{U' \cap U''} = \ker \pi_{U'} + \ker \pi_{U''}$. Thus we have shown that (44) assigns coverings to flabby sheaves.

Conversely, assume that we are given a covering. For brevity, let us denote for any $U \in \Gamma_N$, $\widehat{L(U)} := R^{\Lambda(\ker \pi_i)_i}(L^{\Gamma_N}(U))$. Let U, U' be open subsets of $\mathbb{P}^{N-1}(\mathbb{Z}/2)$ such that $U \subseteq U'$. For all $\{i_1, \dots, i_k\} \in L^{\Gamma_N}(U)$, $A_{i_1} \cap \dots \cap A_{i_k} \subseteq U \subseteq U'$ which implies that there exists a subset $u \subset \{i_1, \dots, i_k\}$ such that $u \in L^{\Gamma_N}(U')$. It is then clear that

$$(45) \quad \widehat{L(U')} = \bigcap_{(j_1, \dots, j_n) \in L^{\Gamma_N}(U')} (\ker \pi_{j_1} + \dots + \ker \pi_{j_n}) \\ \subseteq \bigcap_{(i_1, \dots, i_k) \in L^{\Gamma_N}(U)} (\ker \pi_{i_1} + \dots + \ker \pi_{i_k}) = \widehat{L(U)},$$

and one can define restriction map

$$(46) \quad \pi_{U',U} : \mathcal{P}(U') \rightarrow \mathcal{P}(U), \quad p + \widehat{L(U')} \mapsto p + \widehat{L(U)}.$$

Hence \mathcal{P} is a presheaf.

Let U be an open subset of $\mathbb{P}^{N-1}(\mathbb{Z}/2)$ and let $(U_i)_i$ be an open covering of U (i.e. $\bigcup_i U_i = U$). Suppose that $(p_{U_i})_i$ is a family of elements, where for each i, j , $p_{U_i} \in \mathcal{P}(U_i)$, and $\pi_{U_i, U_i \cap U_j}(p_{U_i}) = \pi_{U_j, U_i \cap U_j}(p_{U_j})$. By the distributivity of lattice of ideals generated by $\ker \pi_i$ and generalised Chinese remainder theorem, (see e.g. [25], Theorem 18 on p. 280) there exists an element $p_U \in \mathcal{P}(U)$ such that, for

all i , $\pi_{U,U_i}(p_U) = p_{U_i}$. It is easy to see that if $\widehat{L(U \cup U')} = \widehat{L(U)} \cap \widehat{L(U')}$, for any open subsets U, U' of $\mathbb{P}^{N-1}(\mathbb{Z}/2)$ then this element is unique. But this follows by Lemma 4.4 and (11).

Denote assignment in (43) by F and functor defined by (44) by G . The functoriality of G is immediate. Let $\xi : P \rightarrow Q$, $(\xi_i : P_i \rightarrow Q_i)_i$ be the morphism of N -coverings $\{\pi_i : P \rightarrow P_i\}_i$ and $\{\eta_i : Q \rightarrow Q_i\}_i$. Note that from $\eta_i \circ \xi = \xi_i \circ \pi_i, \forall i \in \{1, \dots, N\}$ it follows that for all i , $\xi(\ker \pi_i) \subseteq \ker \eta_i$. Accordingly, for any open subset U of $\mathbb{P}^{N-1}(\mathbb{Z}/2)$, $\xi(R^{\Lambda(\ker \pi_i)_i}(L^{\Gamma_N}(U))) \subseteq R^{\Lambda(\ker \eta_i)_i}(L^{\Gamma_N}(U))$, and therefore for all open U , the following family defines a map of corresponding sheaves

$$(47) \quad \begin{aligned} P/R^{\Lambda(\ker \pi_i)_i}(L^{\Gamma_N}(U)) &\longrightarrow Q/R^{\Lambda(\ker \eta_i)_i}(L^{\Gamma_N}(U)) \\ p + R^{\Lambda(\ker \pi_i)_i}(L^{\Gamma_N}(U)) &\mapsto \xi(p) + R^{\Lambda(\ker \eta_i)_i}(L^{\Gamma_N}(U)) \end{aligned}$$

That $G \circ F$ is a functor naturally isomorphic to identity functor in the category of N -coverings is obvious, as $\widehat{L(A_i)} = \ker \pi_i$ and $P_i \simeq P/\ker \pi_i = \mathcal{P}(A_i)$.

On the other hand suppose that we are given a flabby sheaf \mathcal{P} . By the sheaf property, for any open sets U, U' , $\mathcal{P}(U \cup U')$ is a fibre product of $\mathcal{P}(U)$ and $\mathcal{P}(U')$ over $\mathcal{P}(U \cap U')$. Then using flabbiness, we can apply Lemma 2.1 and formula 4 to conclude that

$$(48) \quad \ker \pi_{U \cup U'} = \ker \pi_U \cap \ker \pi_{U'}, \quad \ker \pi_{U \cap U'} = \ker \pi_U + \ker \pi_{U'}.$$

Then by the property that assignment $U \mapsto \widehat{L(U)}$ is a morphism of lattices and again using flabbiness one sees that for all open U , $\mathcal{P}(U) \simeq \mathcal{P}(\mathbb{P}^{N-1}(\mathbb{Z}/2))/R^{\Lambda(\ker \pi_{A_i})_i}(L^{\Gamma_N}(U))$. But this immediatly shows that $F \circ G$ is naturally isomorphic to identity functor on \mathbf{F}_N , which ends the proof. \square

Since the intersection of closed ideals in a C^* -algebra equals their product, lattices of closed ideals in C^* -algebras are always distributive. Thus we immediately obtain:

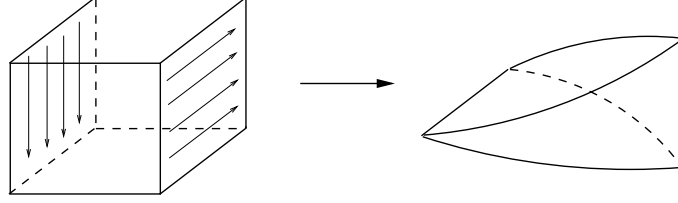
Corollary 4.6. *Compact Hausdorff spaces with a fixed covering by N closed subsets are equivalent to flabby sheaves of commutative unital C^* -algebras over $\mathbb{P}^{N-1}(\mathbb{Z}/2)$.*

Here, the zero algebra is allowed as a unital C^* -algebra. This is needed if the closed sets are not in generic position. The set of unital morphisms from the zero C^* algebra to any other is understood to be empty.

5. EXAMPLES

In this last section we recall from [14, 17, 2, 18] the construction of examples for the above concepts that illustrate possible areas of applications.

5.1. A noncommutative join construction. If G is a compact group, then the join $G * G$ becomes a G -principal fibre bundle over the unreduced suspension ΣG of G , see e.g. [7], Proposition VII.8.8 or [3]. For example, one can obtain the Hopf fibrations $S^7 \rightarrow S^4$ and $S^3 \rightarrow S^2$ in this way using $G = SU(2)$ and $G = U(1)$, respectively. Recall that $G * G$ is obtained from $[0, 1] \times G \times G$ by shrinking to a point one factor G at $0 \in [0, 1]$ and the other factor G at 1.



Alternatively, one can shrink $G \times G$ at 0 to the diagonal. This picture is generalised in [14]. Our aim in this first part of Section 5 is to describe a noncommutative analogue of this construction that nicely fits into our general concepts and will be studied in greater detail in [14].

To this end, let H be the Hopf algebra underlying a compact quantum group \bar{H} (see [30, 31] or Chapter 11 of [19] for details). We define

$$\begin{aligned} P_1 &:= \{f \in C([0, 1], \bar{H}) \otimes H \mid f(0) \in \Delta(H)\}, \\ P_2 &:= \{f \in C([0, 1], \bar{H}) \otimes H \mid f(1) \in \mathbb{C} \otimes H\} \end{aligned}$$

which will play the roles of the two trivial pieces of the principal extension. Here we identify elements of $C([0, 1], \bar{H}) \otimes H$ with functions $[0, 1] \rightarrow \bar{H} \otimes H$. The P_i 's become H -comodule algebras by applying the coproduct of H to H , $\Delta_{P_i} = \text{id}_{C([0, 1], \bar{H})} \otimes \Delta$, and the subalgebras of H -invariants can be identified with

$$\begin{aligned} B_1 &:= \{f \in C([0, 1], \bar{H}) \mid f(0) \in \mathbb{C}\}, \\ B_2 &:= \{f \in C([0, 1], \bar{H}) \mid f(1) \in \mathbb{C}\}. \end{aligned}$$

Furthermore, $P_1 \simeq B_1 \# H$, $P_2 \simeq B_2 \otimes H$, where H acts on B_1 via the adjoint action, $(a \triangleright f)(t) = a_{(1)} f(t) S(a_{(2)})$, $a \in H, f \in B_1, t \in [0, 1]$, see [14]. Now one can define P as a glueing of the two pieces along $P_{12} := \bar{H} \otimes H$, that is, as the pull-back

$$P := \{(p, q) \in P_1 \oplus P_2 \mid \pi_1^1(p) = \pi_1^2(q)\}$$

of the P_i 's along the evaluation maps

$$\pi_2^1 : P_1 \rightarrow P_{12}, \quad f \mapsto f(1), \quad \pi_1^2 : P_2 \rightarrow P_{12}, \quad f \mapsto f(0).$$

Theorem 3.3 implies that P is principal.

5.2. The Heegaard-type quantum 3-sphere. Based on the idea of a Heegaard splitting of S^3 into two solid tori, a noncommutative deformation of S^3 was proposed in [12, 17, 2]. On the level of C^* -algebras, it can be presented as a fibre product $C(S^3_{pq\theta})$ of two C^* -algebraic crossed products $\mathcal{T} \rtimes_{\theta} \mathbb{Z}$ and $\mathcal{T} \rtimes_{-\theta} \mathbb{Z}$ of the Toeplitz algebra \mathcal{T} by \mathbb{Z} . We denote the isometries generating \mathcal{T} in the two algebras by z_+, z_- . The \mathbb{Z} -actions are implemented by unitaries u_+, u_- , respectively, in the following way:

$$u_+ \triangleright_{\theta} z_+ = u_+ z_+ u_+^{-1} := e^{2\pi i \theta} z_+, \quad u_- \triangleright_{-\theta} z_- = u_- z_- u_-^{-1} := e^{-2\pi i \theta} z_-.$$

The fibre product is taken over $C(S^1) \rtimes_{\theta} \mathbb{Z}$ with action $U_+ \triangleright_{\theta} Z_+ := e^{2\pi i \theta} Z_+$, where Z_+ is the generator of $C(S^1)$ and U_+ is the unitary giving the \mathbb{Z} -action in this algebra. The corresponding surjections defining the fibre product are

$$\begin{aligned} \pi_2^1 : \mathcal{T} \rtimes_{\theta} \mathbb{Z} &\rightarrow C(S^1) \rtimes_{\theta} \mathbb{Z}, \quad z_+ \mapsto Z_+, \quad u_+ \mapsto U_+, \\ \pi_1^2 : \mathcal{T} \rtimes_{-\theta} \mathbb{Z} &\rightarrow C(S^1) \rtimes_{\theta} \mathbb{Z}, \quad z_- \mapsto U_+, \quad u_- \mapsto Z_+. \end{aligned}$$

There is a natural $U(1)$ -action on $C(S_{pq\theta}^3)$ corresponding classically to the action in the Hopf fibration, see [17]. Its restriction to the two crossed products is not the canonical action of $U(1)$ viewed as the Pontryagin dual of \mathbb{Z} . However, to obtain the canonical actions one can identify $C(S_{pq\theta}^3)$ with a fibre product of the same crossed products, but formed with respect to the surjections

$$\begin{aligned}\hat{\pi}_2^1 : \mathcal{T} \rtimes_{\theta} \mathbb{Z} &\rightarrow C(S^1) \rtimes_{\theta} \mathbb{Z}, & z_+ &\mapsto Z_+, & u_+ &\mapsto U_+, \\ \hat{\pi}_1^2 : \mathcal{T} \rtimes_{-\theta} \mathbb{Z} &\rightarrow C(S^1) \rtimes_{\theta} \mathbb{Z}, & z_- &\mapsto Z_+^{-1}, & u_- &\mapsto Z_+ U_+.\end{aligned}$$

The identification is given by

$$\begin{array}{ccccc}\mathcal{T} \rtimes_{\theta} \mathbb{Z} & & \mathcal{T} \rtimes_{-\theta} \mathbb{Z} & & \mathcal{T} \rtimes_{\theta} \mathbb{Z} & & \mathcal{T} \rtimes_{-\theta} \mathbb{Z} \\ & \searrow^{\phi_1} & & \searrow^{\phi_2} & & \searrow^{\phi_1} & & \searrow^{\phi_2} \\ \pi_2^1 \swarrow & & \pi_1^2 \swarrow & & \hat{\pi}_2^1 \swarrow & & \hat{\pi}_1^2 \swarrow \\ C(S^1) \rtimes_{\theta} \mathbb{Z} & \xrightarrow{\phi_{12}} & C(S^1) \rtimes_{\theta} \mathbb{Z} & .\end{array}$$

Here isomorphisms ϕ are given on respective generators by

$$z \mapsto zu, \quad u \mapsto u.$$

The C^* -subalgebra of $U(1)$ -invariants is the C^* -algebra of the mirror quantum 2-sphere from [18]. As mentioned in the introduction, we can pass from $C(S_{pq\theta}^3)$ to the associated principal extension, and this procedure commutes with taking fibre products. In this way, we obtain a subalgebra $P \subset C(S_{pq\theta}^3)$ which is a piecewise trivial $\mathbb{C}\mathbb{Z}$ -comodule algebra, so that it fits the setting of this paper. The invariant subalgebra $P^{\text{co}H}$ is again the C^* -algebra of the mirror quantum 2-sphere.

On the other hand, there is a second natural Hopf-like $U(1)$ -action on $C(S_{pq\theta}^3)$ described in [18] (see also [9]). Again, its restriction to the two crossed products making up the fibre product $C(S_{pq\theta}^3)$ is not the canonical action of $U(1)$. This fibre product can be transformed into an isomorphic one (carrying the canonical $U(1)$ -action) constructed by gluing two copies of $\mathcal{T} \rtimes_{-\theta} \mathbb{Z}$ over $C(S^1) \rtimes_{-\theta} \mathbb{Z}$ (with generators Z_-, U_-) using the gluing maps

$$\begin{aligned}\tilde{\pi}_2^1 : \mathcal{T} \rtimes_{-\theta} \mathbb{Z} &\rightarrow C(S^1) \rtimes_{-\theta} \mathbb{Z}, & z_- &\mapsto Z_-, & u_- &\mapsto U_-, \\ \tilde{\pi}_1^2 : \mathcal{T} \rtimes_{-\theta} \mathbb{Z} &\rightarrow C(S^1) \rtimes_{-\theta} \mathbb{Z}, & z_- &\mapsto Z_-, & u_- &\mapsto Z_- U_-.\end{aligned}$$

The identifying maps are now given by

$$\begin{array}{ccccc}\mathcal{T} \rtimes_{\theta} \mathbb{Z} & & \mathcal{T} \rtimes_{-\theta} \mathbb{Z} & & \mathcal{T} \rtimes_{-\theta} \mathbb{Z} & & \mathcal{T} \rtimes_{-\theta} \mathbb{Z} \\ & \searrow^{\tilde{\phi}_1} & & \searrow^{\tilde{\phi}_2} & & \searrow^{\tilde{\phi}_1} & & \searrow^{\tilde{\phi}_2} \\ \pi_2^1 \swarrow & & \pi_1^2 \swarrow & & \tilde{\pi}_2^1 \swarrow & & \tilde{\pi}_1^2 \swarrow \\ C(S^1) \rtimes_{\theta} \mathbb{Z} & \xrightarrow{\tilde{\phi}_{12}} & C(S^1) \rtimes_{-\theta} \mathbb{Z} & .\end{array}$$

Here isomorphisms ϕ are given on generators by

$$\begin{aligned}\tilde{\phi}_1 &: z_+ \mapsto z_- u_-, & u_+ \mapsto u_-^{-1}, \\ \tilde{\phi}_2 &: z_- \mapsto u_-^{-1} z_-, & u_- \mapsto u_-, \\ \tilde{\phi}_{12} &: Z_+ \mapsto Z_- U_-, & U_+ \mapsto U_-^{-1}.\end{aligned}$$

The subalgebra of $U(1)$ -invariants is now the C^* -algebra of the generic Podleś quantum 2-sphere from [23]. However, note that it is not possible to obtain the algebraic Podleś sphere in this way by replacing $\mathcal{T} = P_i^{\text{co}H}$ by the coordinate algebra of a quantum disc with generator x satisfying $x^*x - qxx^* = 1 - q$ [11]. This is related to the fact that already in the commutative case the algebra of polynomial functions on a sphere has no covering corresponding to two hemispheres – there are no nontrivial polynomials vanishing on a hemisphere. Therefore to be in this setting of fibre products we use more complete algebras, e.g., C^* -algebras.

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REFERENCES

- [1] S. Baaĵ, G. Skandalis, *Unitaires multiplicatifs et dualité pour les produits croisés de C^* -algèbres*, Ann. Sci. École Norm. Sup. (4) 26 (1993), no. 4, 425–488
- [2] P.F. Baum, P.M. Hajac, R. Matthes, W. Szymański, *The K -theory of Heegaard-type quantum 3-spheres*, K-Theory 35 (2005), 159–186
- [3] P.F. Baum, P.M. Hajac, R. Matthes, W. Szymański, *Noncommutative geometry approach to principal and associated bundles*, math.DG/0701033
- [4] R.J. Blattner, S. Montgomery, *Crossed products and Galois extensions of Hopf algebras*, Pacific J. Math. 137 (1989), 37–54
- [5] F.P. Boca, *Ergodic actions of compact matrix pseudogroups on C^* -algebras*, Recent advances in operator algebras (Orléans, 1992). Astérisque No. 232 (1995), 93–109
- [6] G. Böhm, T. Brzeziński, *Strong connections and the relative Chern-Galois character for corings*, Int. Math. Res. Not. (2005), 2579–2625
- [7] G. Bredon, *Topology and geometry*, Springer, 1993
- [8] T. Brzeziński, P.M. Hajac, *The Chern-Galois character* C.R. Acad. Sci. Paris, Ser. I 338 (2004), 113–116
- [9] T. Brzeziński, P.M. Hajac, R. Matthes, W. Szymański, *The Chern character for principal extensions of noncommutative algebras*, in preparation, see www.impan.gov.pl/~pmh for a draft version
- [10] R.J. Budzyński, W. Kondracki, *Quantum principal fibre bundles: Topological aspects* Rep. Math. Phys. 37 (1996), 365–385
- [11] D. Calow, R. Matthes, *Covering and gluing of algebras and differential algebras*, J. Geom. Phys. 32 (2000), 364–396

- [12] D. Calow, R. Matthes, *Connections on locally trivial quantum principal fibre bundles*, J. Geom. Phys. 41 (2002), 114-165
- [13] L. DĄBROWSKI, H. GROSSE and P. M. HAJAC, *Strong connections and Chern-Connes pairing in the Hopf-Galois theory*, Comm. Math. Phys. 220 (2001), 301-331
- [14] L. Dąbrowski, T. Hadfield, P.M. Hajac, *Principal extensions from a noncommutative join construction*, in preparation
- [15] M. Durdević, *Quantum principal bundles and corresponding gauge theories* J. Phys. A 30 (1997), no. 6, 2027-2054
- [16] P.M. Hajac, *Strong connections on quantum principal bundles* Comm. Math. Phys. 182 (1996), 579-617
- [17] P.M. Hajac, R. Matthes, W. Szymański, *A locally trivial quantum Hopf fibration* Algebr. Represent. Theory 9 (2006), no. 2, 121-146
- [18] P.M. Hajac, R. Matthes, W. Szymański, *Noncommutative index theory for mirror quantum spheres* C.R. Acad. Sci. Paris, Ser. I 343 (2006), 731-736
- [19] A. Klimyk, K. Schmüdgen, *Quantum groups and their representations*, Springer, 1997
- [20] S. Montgomery, *Hopf algebras and their actions on rings*, CBMS Regional Conference Series in Mathematics, 82. Washington, DC; American Mathematical Society, Providence, RI, 1993.
- [21] R.S. Palais, *On the existence of slices for actions of non-compact Lie groups*, Ann. of Math. 73 (1961), 295-323
- [22] M. Pflaum, *Quantum groups on fibre bundles*, Commun. Math. Phys. 166 (1994), 279-315
- [23] P. Podleś, *Quantum Spheres*, Lett. Math. Phys. 14 (1987), 193-202
- [24] P. Podleś, *Symmetries of quantum spaces. Subgroups and quotient spaces of quantum SU(2) and SO(3) groups*, Comm. Math. Phys. 170 (1995), no.1, 1-20
- [25] P. Samuel, O. Zariski, *Commutative algebra, vol. I*, D. van Nostrand, 1958
- [26] G.K. Pedersen, *Pullback and pushout constructions in C^* -algebra theory*, J. Funct. Anal. 167 (1999), no. 2, 243-344
- [27] P. Schauenburg, H.J. Schneider, *On generalized Hopf-Galois extensions*, J. Pure Appl. Algebra 202 (2005), no. 1-3, 168-194
- [28] P. Schauenburg, H.J. Schneider, *Galois-type extensions and Hopf algebras*
- [29] H.-J. Schneider, *Hopf-Galois extensions, crossed products and Clifford theory*, in: *Advances in Hopf algebras*, Lecture Notes Pure Appl. Math. 158 (1994), 267-297
- [30] S.L. Woronowicz, *Compact matrix pseudogroups*, Comm. Math. Phys. 111 (1987), 613-665.
- [31] S.L. Woronowicz, *Compact quantum groups* in: *Symétries quantiques (Les Houches, 1995)*, North-Holland 1998, 845-884

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