CONNECTED (GRADED) HOPF ALGEBRAS

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Abstract. We study algebraic and homological properties of two classes of infinite dimensional Hopf algebras over an algebraically closed field k of characteristic zero. The first class consists of those Hopf k-algebras that are connected graded as algebras, and the second class are those Hopf k-algebras that are connected as coalgebras. For many but not all of the results presented here, the Hopf algebras are assumed to have finite Gel'fand-Kirillov dimension. It is shown that if the Hopf algebra H is a connected graded Hopf algebra of finite Gel'fand-Kirillov dimension n, then H is a noetherian domain which is Cohen-Macaulay, Artin-Schelter regular and Auslander regular of global dimension n. It has $S^2 = \mathrm{Id}_H$, and is Calabi-Yau. Detailed information is also provided about the Hilbert series of H. Our results leave open the possibility that the first class of algebras is (properly) contained in the second. For this second class, the Hopf k-algebras of finite Gel'fand-Kirillov dimension n with connected coalgebra, the underlying coalgebra is shown to be Artin-Schelter regular of global dimension n. Both these classes of Hopf algebra share many features in common with enveloping algebras of finite dimensional Lie algebras. For example, an algebra in either of these classes satisfies a polynomial identity only if it is a commutative polynomial algebra. Nevertheless, we construct, as one of our main results, an example of a Hopf k-algebra H of Gel'fand-Kirillov dimension 5, which is connected graded as an algebra and connected as a coalgebra, but is not isomorphic as an algebra to $U(\mathfrak{g})$ for any Lie algebra $\mathfrak{g}$.

Introduction

Throughout k is an algebraically closed field of characteristic 0. Unless otherwise stated, all vector spaces and tensor products are over k.

0.1. **Connectedness.** The adjective "connected" can of course be applied to a multitude of mathematical structures, with a corresponding multiplicity of meanings. We are concerned here with two such structures, connected graded associative k-algebras, and connected k-coalgebras. On the one hand, a *connected graded* k-algebra A is one which is $\mathbb{N}$-graded, with $A = k \bigoplus_{i \geq 1} A(i)$; on the other hand, a *connected coalgebra* C is a coalgebra whose coradical C_0 is one-dimensional [1]. It is in this second sense that the adjective is traditionally applied to a Hopf algebra H: that is, the Hopf algebra H (whether graded or not) is said to be *connected* if its coradical H_0 is k, [Mo, Definition 5.1.5]. One main purpose of this paper is to study how these two versions of connectedness interact in the setting of Hopf k-algebras, where one, both or neither of them may hold for a given such algebra.

[1] The term "connected coalgebra" was used by Milnor and Milnor-Moore in their seminal work [Mil, MM] to refer to an $\mathbb{N}$-*graded* coalgebra C with $C_0 = k$, thus matching the usage in the algebra setting. But the standard modern usage for Hopf algebras is the one described here, that is, with no graded hypothesis on the coalgebra.

Connected Hopf k-algebras H of finite Gel'fand-Kirillov dimension (denoted by $\operatorname{GKdim} H < \infty$) have been the subject of a number of recent papers, for example [EG, Zh, Wa1, BO], and in part this paper continues that programme. We shall also consider the question: If H is a Hopf algebra with finite Gel'fand-Kirillov dimension, what is the effect on its structure of assuming that H *is connected graded as an algebra*? As will become clear, these two hypotheses - connected as a graded algebra, and connected as a coalgebra - are intimately linked.

0.2. **The cocommutative and commutative cases.** The *cocommutative* connected Hopf k-algebras were completely described half a century ago: by the celebrated theorem of Cartier-Kostant [Mo, Theorem 5.6.5], such Hopf algebras H are precisely the universal enveloping algebras $U(\mathfrak{g})$ of the k-Lie algebras $\mathfrak{g}$, with $\mathfrak{g}$ constituting the space of primitive elements of H. Such an enveloping algebra $U(\mathfrak{g})$ has $\operatorname{GKdim} H = n < \infty$ if and only if $\dim_k \mathfrak{g} = n$. Suppose on the other hand that a cocommutative Hopf k-algebra H is connected graded as an algebra. These conditions trivially ensure that the only group-like element of H is the identity element. Thus, since H is pointed as the ground field k is algebraically closed [Mo, §5.6], it is connected as a coalgebra and the Cartier-Kostant theorem again applies, yielding $H \cong U(\mathfrak{g})$ once more. Moreover, this time the connected graded hypothesis implies that, if $\operatorname{GKdim} H < \infty$, then $\mathfrak{g}$ is nilpotent.

There is an equally beautiful story in the other "classical" setting, where H is *commutative*. For a commutative Hopf k-algebra H the two finiteness conditions - being affine and being noetherian - coincide by Molnar's theorem [Mol], and this hypothesis implies finiteness of Gel'fand-Kirillov dimension. Writing $\dim U$ for the dimension of the variety U and $\operatorname{gldim} H$ for the global dimension of H, we have:

Theorem 0.1. *Let H be an affine commutative Hopf k-algebra. Then the following are equivalent:*

(1) *H is a connected Hopf algebra.*
(2) *H is a connected graded algebra.*
(3) *For some $n \geq 0$, $H \cong k[x_1, \ldots, x_n]$ as an algebra.*
(4) *There is a unipotent k-group U such that H is the algebra of regular functions $\mathcal{O}(U)$.*

When the above hold,

$$n = \operatorname{GKdim} H = \dim U = \operatorname{gldim} H.$$

Here (2) $\Rightarrow$ (3) follows since H has finite global dimension by [Wat, combining §11.4 and §11.6], so a theorem of Serre applies [Be, Theorem 6.2.2]; (3) $\Rightarrow$ (4) is a theorem of Lazard [Laz], (3) $\Rightarrow$ (2) is trivial, and the remaining implications (1) $\Leftrightarrow$ (4) and (4) $\Rightarrow$ (3) are basic facts concerning algebraic k-groups, see for example [BO, Example 3.1]. [2]

Summarising this classical picture: if H is a cocommutative or commutative Hopf algebra with finite Gel'fand-Kirillov dimension n and is connected graded as an algebra, then it is a connected coalgebra, (and hence, as an algebra, it is

[2]Connected graded Hopf algebras have been studied by Milnor-Moore in [MM] and Félix-Halperin-Thomas in [FHT1, FHT2] (assuming that both the coalgebra *and* the algebra structure are graded). Dualizing complexes and homological properties of locally finite $\mathbb{N}$-graded Hopf algebras have been studied in [WZ1].

the enveloping algebra of an n-dimensional nilpotent Lie algebra); and the converse (connected coalgebra implies connected graded as an algebra) holds in the commutative case but not the cocommutative one.

0.3. **Connected graded algebras.** The first part of the paper, §§ 1 and 2, concerns a Hopf algebra that is connected graded as an algebra. Our first main result gives structural information for connected graded Hopf algebras of finite GK-dimension. It shows that, while we cannot replicate the cocommutative conclusions, many of the algebraic and homological consequences of being isomorphic to the enveloping algebra of a nilpotent Lie algebra do survive:

Theorem 0.2. *Let H be a Hopf k-algebra with* $\mathrm{GKdim} H = n < \infty$ *which is connected graded as an algebra. Then H is a noetherian domain with $S^2 = \mathrm{Id}_H$. Furthermore, it is Cohen-Macaulay, Artin-Schelter regular and Auslander regular of global dimension n, and is Calabi-Yau. As a consequence, H satisfies Poincaré duality.*

Theorem 0.2 is Theorem 2.2 below. There is a result similar to Theorem 0.2 for connected Hopf algebras, see [Zh, Theorem 6.9 and Corollary 6.10]. In [Br1, Question C] the first-named author asks: *Is there an easy method to recognize when a noetherian Hopf algebra is regular, namely, has finite global dimension?* Theorem 0.2 answers [Br1, Question C] (as well as [Br2, Question E] and [Go2, Question 3.5]) in the case of connected graded algebras. Faced with the above list of ring-theoretic and homological properties and considering the cocommutative and commutative cases discussed in §0.2, one might speculate that every Hopf algebra H as in Theorem 0.2 is in fact isomorphic *as an algebra* to the enveloping algebra of a nilpotent Lie algebra. But this is not so - see Theorem 0.5(2) below.

Notwithstanding this example, under some additional hypotheses only enveloping algebras can occur in the circumstances of Theorem 0.2. Two such cases appear as parts (2) and (4) of the following result, which is also Theorem 2.4. The graded Hopf algebras studied by Félix-Halperin-Thomas in [FHT1, FHT2] are in the super sense, which means that one needs to use the Koszul sign rule when commuting two homogeneous symbols. We expect that there is a version of Theorem 0.2 for such Hopf algebras.

Theorem 0.3. *Let H be a Hopf k-algebra which is connected graded and locally finite as an algebra.*

(1) *The Hilbert series of H is*

$$\frac{1}{\prod_{i=1}^{\infty}(1-t^i)^{n_i}}$$

for some non-negative integers n_i. Further, the sequence $\{n_i\}_{i\geq 1}$ is uniquely determined by the Hilbert series of H.

(2) *Suppose H is generated in degree 1. Then H is isomorphic as an algebra to $U(\mathfrak{g})$ for a graded Lie algebra $\mathfrak{g}$ generated in degree 1. Hence, for all $i \geq 1$,*

$$\dim_k \mathfrak{g}_i = n_i,$$

where $\mathfrak{g}_i$ is the i^{th} degree component of $\mathfrak{g}$.

(3) *Suppose again that H is generated in degree 1, $H \neq k$. Then*

$$\{j : n_j \neq 0\} = [1, \ell) \cap \mathbb{N}, \text{ for some } \ell \in [2, \infty].$$

(4) *If the Hilbert series of* H *is* $\frac{1}{(1-t)^d}$, *then* H *is commutative, and hence* $H \cong k[x_1, \dots, x_d]$ *as algebras.*

(5) $\operatorname{GKdim} H < \infty$ *if and only if* $\sum_{i=1}^{\infty} n_i < \infty$. *In this case,*

$$\operatorname{GKdim} H = \sum_{i=1}^{\infty} n_i.$$

Both Theorems 0.2 and 0.3 can be used to show that a number of large classes of connected graded k-algebras *do not* admit any Hopf algebra structure - see Corollaries 2.3 and 2.8.

0.4. **Connected Hopf algebras.** A central underlying theme of this paper is the following. Let H be a Hopf k-algebra, say of finite Gel'fand-Kirillov dimension. What is the relationship between the following two hypotheses on H?

(CGA) : *Hopf algebra* H *is connected graded as an algebra.*

(CCA) : *Hopf algebra* H *is connected as a coalgebra.*

It is immediately clear that (CCA)$\not\Rightarrow$(CGA) - consider any enveloping algebra H of a finite dimensional non-nilpotent Lie algebra, with its standard cocommutative coalgebra structure. It remains feasible, however, that (CGA)$\Rightarrow$(CCA); indeed, all the results discussed in §§0.2 and 0.3 are consistent with this possibility, which we therefore leave as an open question:

Question 0.4. Does (CGA) $\Rightarrow$ (CCA)?

Note that if H satisfies (CGA), then its graded dual is a Hopf algebra satisfying (CCA).

A further possibility suggested by the results of §§0.2 and 0.3, already mentioned in §0.3, is that *if* H *satisfies* (CGA) *and/or* (CCA) *then* H *is isomorphic as an algebra to the enveloping algebra of a finite dimensional Lie algebra.* However, the second part of the following theorem rules this out. It is proved as Theorem 5.6, the main result of §5. Recall that a *graded Hopf algebra* H is by definition simultaneously graded as both an algebra and as a coalgebra: $H = \oplus_{i \geq 0} H(i)$ as a graded algebra, with $\Delta(x) \in \sum_{i=1}^{m} H(i) \otimes H(m-i)$ for all $x \in H(m)$, for all $m \geq 0$ and the antipode being a graded map [Zh, Definition 1, p.878]. Such a graded Hopf algebra is then *connected*, in *both* the coalgebra and the algebra sense of the word, if $H(0) = k$.

Theorem 0.5. *Let* H *be a Hopf* k*-algebra with* $\operatorname{GKdim} H = n < \infty$.

(1) *If* H *satisfies* (CCA) *and* $n \leq 4$ *then* H *is isomorphic as an algebra to the enveloping algebra of an* n*-dimensional Lie algebra.*

(2) *There exists an algebra* H *with* $n = 5$, *satisfying both* (CCA) *and* (CGA), *which is not isomorphic to the enveloping algebra of a Lie algebra.*

(3) *The algebra* H *of part* (2) *is a connected graded Hopf algebra.*

Theorem 0.5(1) follows easily from the classification in [Wa1] of all H with $\operatorname{GKdim} H \leq 4$ satisfying (CCA). The analogue of Theorem 0.5(1) for the (CGA) hypothesis remains open; obviously it would follow from a positive answer to Question 0.4 for $n \leq 4$. Note that Theorem 0.5(2) gives a negative answer to [BG1, Question L]. Theorem 0.5(2),(3) appear as Theorem 5.6 (and Lemma 5.5) below.

Let $\operatorname{gr}_c H$ denote the associated graded algebra of H with respect to the coradical filtration of H. The *signature* featuring in Corollary 0.6(4) below is a list of the degrees of the homogeneous generators of $\mathrm{gr}_c H$. It is defined and discussed in [BG2, Definition 5.3].

Corollary 0.6. *Let H be as in Theorem 0.5(2),(3). Then the following hold.*

(1) *There is a connected locally finite $\mathbb{N}$-filtration of H, namely, the coradical filtration c, such that the associated graded ring $\mathrm{gr}_c H$ is isomorphic to $k[x_1, \cdots, x_5]$ where $\deg x_i \geq 1$ (but not all equal 1).*
(2) *There does not exist a connected locally finite $\mathbb{N}$-filtration $\mathcal{F}$ such that $\mathrm{gr}_{\mathcal{F}} H \cong k[x_1, \cdots, x_5]$ with $\deg x_i = 1$ for all i.*
(3) *There is a negative $\mathbb{N}$-filtration $\mathcal{F}$ of H such that the associated graded ring $\mathrm{gr}_{\mathcal{F}} H \cong U(\mathfrak{g})$ for some 5-dimensional graded Lie algebra $\mathfrak{g}$. However H itself is not isomorphic to $U(\mathfrak{g})$ for any Lie algebra $\mathfrak{g}$.*
(4) *The signature of H is $(1,1,1,2,2)$.*
(5) *Let D be the graded k-linear dual of the Hopf algebra H with respect to the graded Hopf algebra structure of Theorem 0.5(3). Thus D is a connected graded Hopf algebra, affine noetherian of Gel'fand-Kirillov dimension 5. However, D is not isomorphic, as a coalgebra, to $\mathrm{gr}_c K$ for any connected Hopf algebra K. In particular, D is not isomorphic to $\mathrm{gr}_c D$ as coalgebras.*

Corollary 0.6 is proved in §5.5. Combining Theorem 0.5(2) with Corollary 0.6(4), one sees that a connected Hopf algebra with signature $(1,1,1,2,2)$ need not be isomorphic to an enveloping algebra of an anti-cocommutative coassociative Lie algebra. This answers negatively a question posed rather imprecisely in [Wa1, Remark 2.10(d1)].

Despite the example from Theorem 0.5(2), the evidence so far assembled, in §§1 and 2 of the present paper for (CGA) and in previous work [EG, Zh, Wa1, BO, BG2] for (CCA), supports the intuition that Hopf algebras of finite Gel'fand-Kirillov dimension satisfying either of these hypotheses share many properties with enveloping algebras of finite dimensional k-Lie algebras $\mathfrak{g}$, (with $\mathfrak{g}$ graded and hence nilpotent in the case of (CGA)). We provide some further evidence in support of this philosophy, by generalising in Theorem 4.5 a well-known result on enveloping algebras of Latysev and Passman, [Lat, Pas], and also providing a version of it for connected graded algebras, Theorem 2.9:

Theorem 0.7. *Let H be a Hopf k-algebra (but don't assume that $\mathrm{GKdim}\, H$ is finite). If (CGA) or (CCA) holds for H, and H satisfies a polynomial identity, then H is commutative, (so Theorem 0.1 applies).*

0.5. **Further results.** It turns out that the arguments employed to prove Theorem 0.7 can be used, without any connectedness hypothesis, to give insight into another type of weak commutativity hypothesis on a Hopf algebra. We take a brief detour to follow this line, proving, as Proposition 3.1:

Theorem 0.8. *Let H be a Hopf k-algebra with $\mathrm{GKdim}\, H = n < \infty$, and let $\mathfrak{m}$ be the kernel of the counit of H.*

(1) $\dim \mathfrak{m}/\mathfrak{m}^2 \leq \mathrm{GKdim}\, H$.
(2) *Suppose that H is affine or noetherian. Then the following statements are equivalent:*
    (a) $\dim \mathfrak{m}/\mathfrak{m}^2 = \mathrm{GKdim}\, H$.
    (b) *$\mathfrak{m}$ contains a unique minimal prime ideal, P, and $H/P \cong \mathcal{O}(G)$ for a (connected) algebraic group G of dimension n.*

(c) *H has a commutative factor algebra A with* $\mathrm{GKdim}\, A = n$.

In another brief digression we follow up some work on Artin-Schelter regular coalgebras in [HT]. Unexplained terminology in the following theorem is defined in §4.1.

Theorem 0.9. *Let H be a connected Hopf algebra of GK-dimension $n < \infty$. Then the underlying coalgebra of H is Artin-Schelter regular of global dimension n.*

0.6. **Theorem of Etingof and Gelaki.** Connected Hopf algebras are studied under the name *coconnected Hopf algebras* in [EG]. In particular, [EG, Theorem 4.2] states that every connected Hopf $\mathbb{C}$-algebra is a deformation of a prounipotent proalgebraic groups with a Poisson structure, and that this data classifies these objects. This leaves, of course, the problems of understanding the ring-theoretic and homological properties of these Hopf algebras, and of investigating the range of possibilities generated by this recipe. This paper is independent of the work in [EG], and it should be expected that further progress will follow from combining their result with the methods discussed here.

0.7. The paper is organized as follows. §1 contains some preliminaries. Hopf algebras that are connected graded as an algebra are studied in §2 and Theorems 0.2 and 0.3 are proved there. Hopf algebras that are connected as a coalgebra are studied in §4 and Theorem 0.9 is proved there. The (CGA) [respectively, (CCA)] parts of Theorem 0.7 are proved in §2.6 [resp. §4.2]. The discussion of the almost commutative Hopf algebras of Theorem 0.8 is in §3. §5 is devoted to the proofs of Theorem 0.5(2) and Corollary 0.6. Some computational details of the proof of Theorem 5.6 are relegated to the Appendix, §6.

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1. Preliminaries

1.1. **General setup.** Throughout let k denote a base field, which we assume to be algebraically closed and of characteristic 0. (The algebraically closed assumption can probably be dropped for most of our results, but the restriction to characteristic 0 is undoubtedly needed for most of them.) All algebraic structures and unadorned tensor products $\otimes$ will be assumed to be over the base field k. We omit the definitions of some standard homological terms; these can be found in, for example, [BZ, Le, LP, RRZ, WZ2]. The Gel'fand-Kirillov dimension of a module M over a k-algebra A is denoted $\mathrm{GKdim}\, M$; we use [KL] as the standard reference for this.

1.2. **Hopf algebra notation.** For a general Hopf algebra H, the usual notation Δ, ϵ and S is used to denote the coproduct, counit and antipode respectively. Our standard reference for Hopf algebra material is [Mo]; details of the following, for instance, can be found in [Mo, Chapter 5]. The *coradical* filtration of H is denoted by $\{H_i\}_{i=0}^{\infty}$, where H_0 is the coradical of H, that is the sum of all simple subcoalgebras of H, and we define inductively, for all $i \geq 1$,

$$H_i := \Delta^{-1}(H \otimes H_{i-1} + H_0 \otimes H).$$

As explained already in §0.1, the Hopf algebra H is *connected* if $H_0 = k$.

1.3. **Filtrations in the setting of graded algebras.** We start with some material which is a slight variation of parts of [LvO1, LvO2]. Let $A = A(0) \oplus A(1) \oplus A(2) \oplus \cdots$ be an $\mathbb{N}$-graded k-algebra, which we shall assume unless otherwise stated to be *connected* - that is, $A(0) = k$. We shall also frequently assume that the grading of A is *locally finite* - that is, $\dim_k A(i) < \infty$ for all i. We shall call this given grading of A its *Adams grading*, with the degree of a homogeneous element w of A denoted by $\deg_a w$. Then the *Hilbert series* of A is defined and denoted by

$$H_A(t) = \sum_{i=0}^{\infty} \dim_k A(i)\, t^i.$$

We shall always denote the graded maximal ideal $\oplus_{i\geq 1} A(i)$ by $\mathfrak{m}$, and write $d(A)$ for the minimal number of generators of A as an algebra, $d(A) \in \mathbb{N} \cup \{\infty\}$. The symbol A_{ab} is used for the factor of A by the ideal $\langle [A, A] \rangle$. Note that A_{ab} inherits a grading from A. We recall the following elementary facts.

Lemma 1.1. *Let $A = \oplus_{i\geq 0} A(i)$ be a connected graded locally finite algebra, with $\mathfrak{m} = \sum_{i\geq 1} A(i)$.*

(1) *A is an affine algebra $\Longleftrightarrow \dim_k \mathfrak{m}/\mathfrak{m}^2 =: d < \infty$. In this case, $d(A) = d$.*
(2) *$d(A) = d(A_{\mathrm{ab}})$.*

The definition of a *Zariskian filtration* is omitted; it can be found in many places, for example [LvO1, LvO2]. We consider a special type of "graded Zariskian" filtration of a graded algebra A as specified in Lemma 1.1. A family of graded vector subspaces $\mathcal{F} := \{F_i A \subseteq A \mid i \in \mathbb{Z}\}$ of A is called a *graded filtration* of A if the following hold:

(F0) $1 \in F_0 A$ and $F_{-1} A \subseteq \mathfrak{m}$;
(F1) $F_i A \subseteq F_{i+1} A$ for all i;
(F2) $\bigcap_i F_i A = \{0\}$ and $\bigcup_i F_i A = A$;
(F3) $F_i A F_j A \subseteq F_{i+j} A$ for all i, j.

Define the graded ring associated to $\mathcal{F}$ to be

$$\mathrm{gr}_{\mathcal{F}} A := \bigoplus_{i=-\infty}^{\infty} F_i A / F_{i-1} A$$

and the Rees algebra associated to $\mathcal{F}$ to be

$$\mathrm{Rees}_{\mathcal{F}} A := \bigoplus_{i=-\infty}^{\infty} r^i F_i A,$$

where r is an indeterminate. If $x \in F_n A \setminus F_{n-1} A$, then the associated element in $(\mathrm{gr}_{\mathcal{F}} A)_n$ is denoted by $\overline{x}$. Note that both $\mathrm{gr}_{\mathcal{F}} A$ and $\mathrm{Rees}_{\mathcal{F}} A$ are $\mathbb{Z}^2$-graded where the first component is reserved for the Adams grading and the second component is for the grading coming from the filtration, denoted by $\deg_{\mathcal{F}}$. We define $\deg_a r := 0$ and $\deg_{\mathcal{F}} r := 1$, or $\deg r := (0,1)$. Now $\operatorname{gr}_{\mathcal{F}} A$ and A have the same Hilbert series when we consider the grading on $\operatorname{gr}_{\mathcal{F}} A$ induced by the Adams grading of A. In particular, $\operatorname{gr}_{\mathcal{F}} A$ is connected graded with respect to the Adams grading.

The *standard graded filtration* of A is defined by

$$F_i A = \begin{cases} A & i \geq 0 \\ \mathfrak{m}^{-i} & i < 0. \end{cases} \tag{E1.1.1}$$

In this case the associated graded ring $\operatorname{gr}_{\mathcal{F}} A$ is isomorphic to $\bigoplus_{i=0}^{\infty} \mathfrak{m}^i/\mathfrak{m}^{i+1}$. We record for future use the following obvious lemma:

Lemma 1.2. *Let A be a connected graded algebra generated in degree 1. Let $\mathfrak{m} = \oplus_{i\geq 1} A(i)$ and let $\mathcal{F}$ be the standard filtration. Then $\operatorname{gr}_{\mathcal{F}} A := \bigoplus \mathfrak{m}^i/\mathfrak{m}^{i+1}$ is isomorphic to A.*

Continuing with the above notation, define $\deg_a F_i A$ be the the lowest Adams degree of a nonzero homogeneous element in $F_i A$. A filtration $\mathcal{F} = \{F_i A : i \in \mathbb{Z}\}$ is called *strict* if

(F4) $\quad \underline{\lim}_{n\to\infty} \frac{\deg_a F_{-n} A}{n} > 0.$

It is clear that the standard graded filtration is strict since $\deg_a F_{-n} A \geq n$.

Lemma 1.3. *Let A be a connected graded algebra with a graded filtration $\mathcal{F} = \{F_i A : i \in \mathbb{Z}\}$. Let B be the associated graded algebra $\operatorname{gr}_{\mathcal{F}} A$.*

1. *If B is a domain, so is A.*
2. *If B is affine (=finitely generated) as an algebra, then $\mathcal{F}$ is strict.*

Proof. (1) This is standard.

(2) Assume that B is generated by a finite set of homogeneous elements $\{\overline{f_i}\}_{i=1}^{w}$ where $\{f_i\}$ are homogeneous elements in A with $\deg_a f_i = a_i > 0$ and $f_i \in F_{b_i} A \setminus F_{b_i - 1} A$. Let $\beta = \max\{|b_i| \mid i = 1, \cdots, w\}$ and $\alpha = \min\{a_i \mid i = 1, \cdots, w\}$. We claim that $\underline{\lim}_{n\to\infty} \frac{\deg_a F_{-n} A}{n} \geq \alpha/\beta > 0$. As a consequence, $\mathcal{F}$ is strict.

Suppose $\deg_a F_{-n} A \neq \infty$ for some $n > 0$. Let $x \in F_{-n} A$ be a nonzero homogeneous element such that $\deg_a x = \deg_a F_{-n} A$. We consider the following two cases.

Case 1: $x \notin F_{-n-1} A = F_{-(n+1)} A$. Then $\overline{x} \in B$, so we can write $\overline{x} = p(\overline{f_1}, \cdots, \overline{f_w})$ where p is a noncommutative polynomial in w variables. Recalling that deg is the bi-degree, this implies that $\deg \overline{x} = \deg(\overline{f_{i_1}} \cdots \overline{f_{i_m}})$ for some integers $1 \leq i_s \leq w$ where $s = 1, \cdots, m$. Thus we have

$$\deg_{\mathcal{F}} \overline{x} = -n = \sum_{s=1}^{m} \deg_{\mathcal{F}} \overline{f_{i_s}}, \tag{E1.3.1}$$

and

$$\deg_a \overline{x} = \sum_{s=1}^{m} \deg_a \overline{f_{i_s}}. \tag{E1.3.2}$$

By (E1.3.1), $n \leq \sum_{s=1}^{m} |\deg_{\mathcal{F}} \overline{f_{i_s}}| \leq m\beta$; that is, $m \geq n/\beta$. By (E1.3.2), we have

$$\deg_a x = \deg_a \overline{x} \geq m\alpha \geq \frac{n}{\beta}\alpha = (\alpha/\beta)n.$$

Case 2: $x \in F_{-(n+1)}A$. By (F2), $\bigcap_{n>0} F_{-n}A = \{0\}$, so there is an $n_0 > 0$ such that $x \in F_{-(n+n_0)}A \setminus F_{-(n+n_0+1)}A$. Note that $\deg_a x = \deg_a F_{-(n+n_0)}A$. By Case 1, we have

$$\deg_a x \geq (\alpha/\beta)(n+n_0) \geq (\alpha/\beta)n.$$

Combining these two cases, we have $\deg_a F_{-n}A = \deg_a x \geq (\alpha/\beta)n$ for all $n > 0$, proving the claim. □

Proposition 1.4. *Let A be a connected graded algebra with a strict graded filtration $\mathcal{F}$. Let B be the associated graded algebra $\operatorname{gr}_{\mathcal{F}} A$.*

(1) *B is (left) noetherian if and only if $\operatorname{Rees}_{\mathcal{F}} A$ is.*
(2) *If B is (left) noetherian, then so is A. In this case A is locally finite and affine.*
(3) *If B is noetherian Auslander Gorenstein and GK-Cohen-Macaulay, then so is A.*
(4) *If B as in (3) has finite global dimension, then so does A. In this case*

$$\operatorname{gldim} A = \operatorname{GKdim} A = \operatorname{GKdim} B = \operatorname{gldim} B. \tag{E1.4.1}$$

Proof. (1) By (F4), there is a positive integer d such that $\deg_a F_{-n}A > (n+1)/d$ for all $n > 0$. For the rest of the proof we change the Adams grading of A by setting a new Adams grading

$$\widetilde{\deg_a} x := d \deg_a x$$

for a homogeneous element x. Under the new Adams grading, we have $\deg_a F_{-n}A > n$ for all $n > 0$. Consider the Rees ring $R := \bigoplus_{i=-\infty}^{\infty} r^i F_i A$ with the total grading. In particular, one has $\deg r = 1$. Then R is a connected graded algebra and r is a central regular element in R of degree 1. It is clear that $R/rR \cong B$. The assertion follows from [Le, Proposition 3.5(a)].

(2) Suppose that B is left noetherian. As noted in the proof of (1), R is connected graded with respect to the total grading. By part (1), R is left noetherian. Since A is isomorphic to $R/(r-1)R$, A is left noetherian. Noetherian connected graded algebras are locally finite and affine, and hence the second assertion follows.

(3) Suppose that B is as stated. As in the proofs of parts (1) and (2), $B \cong R/rR$ and $A \cong R/(r-1)R$, where r is the central regular element of degree 1 in R introduced in (1). Thus R is Auslander Gorenstein and GK-Cohen-Macaulay, by [Le, Theorem 5.10]. By the Rees Lemma [Le, Proposition 3.4(b) and Remarks 3.4(3)], A is Auslander Gorenstein and GK-Cohen-Macaulay.

(4) Suppose that $\operatorname{gldim} B < \infty$, with B noetherian, Auslander-Gorenstein and GK-Cohen-Macaulay. By parts (2) and (3), A is noetherian, Auslander Gorenstein, and GK-Cohen-Macaulay. Moreover B is a domain by [Le, Theorem 4.8], so A is a domain by Lemma 1.3(1). It remains to prove (E1.4.1). Since $B = R/rR$ with r a central non-zero-divisor of degree 1, [LP, Lemma 7.6] implies that R has global dimension equal to $\operatorname{gldim} B + 1$. Let C be the localization $R[r^{-1}]$. Then

$$\operatorname{gldim} C \leq \operatorname{gldim} R = \operatorname{gldim} B + 1. \tag{E1.4.2}$$

Note that $C = A[r^{\pm 1}]$, so that $\operatorname{gldim} A + 1 = \operatorname{gldim} C$. Therefore, by (E1.4.2), $\operatorname{gldim} A \leq \operatorname{gldim} B$. Since both A and B are affine, one can use their Hilbert series to compute their GK-dimensions, [KL, Lemma 6.1(b)]; hence $\operatorname{GKdim} A \geq \operatorname{GKdim} B$. By the GK-Cohen-Macaulay condition for connected graded algebras, $\operatorname{GKdim} A = \operatorname{gldim} A$ and $\operatorname{GKdim} B = \operatorname{gldim} B$. Therefore (E1.4.1) follows. □

2. Hopf algebras whose algebra is connected graded

2.1. **Shifting the augmentation.**

Lemma 2.1. *Let H be a Hopf algebra.*

(1) *Let I be the kernel of a character $\chi : H \to k$. Then there is an algebra isomorphism $\sigma \in \mathrm{Aut}(H)$ such that $\sigma(I) = \ker \epsilon$.*
(2) *If H is connected graded as an algebra, then there is a grading of H such that $H = \bigoplus_{i=0}^{\infty} H(i)$ is connected graded and $\ker \epsilon = \bigoplus_{i\geq 1} H(i)$.*

Proof. (1) Take σ to be the left winding automorphism $h \mapsto \sum \chi(h_1)h_2$.

(2) Let $H = k \oplus \bigoplus_{i=1}^{\infty} B(i)$ be the given grading of H and let $I = \bigoplus_{i=1}^{\infty} B(i)$. By part (1), there is an algebra automorphism σ of A such that $\sigma(I) = \ker \epsilon$. Let $H(i) = \sigma(B(i))$. Then $H = \bigoplus_{i=0}^{\infty} H(i)$ is a connected graded algebra and $\ker \epsilon = \bigoplus_{i\geq 1} H(i)$. □

2.2. **Structure of connected graded Hopf algebras.** We now prove Theorem 0.2.

Let A be an algebra and M an A-bimodule. For any $i \geq 0$, the i^{th} Hochschild homology (respectively, Hochschild cohomology) of A with coefficients in M is denoted by $H_i(A, M)$ (respectively, $H^i(A, M)$). We say that *Poincaré duality* holds for Hochschild (co)homology over A if there is a non-negative integer n such that

$$H^{n-i}(A, M) \cong H_i(A, M)$$

for all i, $0 \leq i \leq n$, and for all A-bimodules M.

Theorem 2.2. *Let H be a Hopf k-algebra which is connected graded as an algebra. In parts (2,3,4,5), we further assume that $\mathrm{GKdim}\, H = n < \infty$.*

(1) *H is a domain.*
(2) *H is affine as an algebra, and is noetherian, Cohen-Macaulay, Auslander regular and Artin-Schelter regular, with $\mathrm{gldim}\, H = n$.*
(3) *H is Calabi-Yau; that is, the Nakayama automorphism of H is the identity.*
(4) *$S^2 = Id_H$.*
(5) *Poincaré duality holds for Hochschild (co)homology over H.*

Proof. (1) By Lemma 2.1(2), we may assume that $\mathfrak{m} := \bigoplus_{i\geq 1} H(i) = \ker \epsilon$. Let $\mathcal{F}$ be the standard graded filtration as in (E1.1.1), and write $\mathrm{gr}_{\mathcal{F}} H$ as $\mathrm{gr}\, H$. Then $\mathrm{gr}\, H \cong \bigoplus_{i=0}^{\infty} \mathfrak{m}^i/\mathfrak{m}^{i+1}$. By [GZ, Proposition 3.4(a)], as Hopf algebras,

$$\mathrm{gr}\, H \quad \cong \quad U(\mathfrak{g}) \tag{E2.2.1}$$

for some positively graded Lie algebra $\mathfrak{g}$, where, in (E2.2.1), $U(\mathfrak{g})$ has its standard cocommutative coalgebra structure. Since $U(\mathfrak{g})$ is a domain, so is H by Lemma 1.3(1).

(2) By [GZ, Proposition 3.4(b)], $\dim \mathfrak{g} \leq \mathrm{GKdim}\, H < \infty$. So $\mathfrak{g}$ is finite dimensional. Then $\mathrm{gr}\, H$ is affine. Since $\mathfrak{g}$ is finite dimensional, $\mathrm{gr}\, H$ is a noetherian Auslander regular and Cohen-Macaulay domain. The same list of properties for H, and with them the fact that $\mathrm{gldim}\, H = n$, follows from Proposition 1.4(3) and (4). Since H is a noetherian connected graded algebra, it is easily seen to be affine. Finally, H is Artin-Schelter regular by [BZ, Lemma 6.1].

(3,4) By (2) and [BZ, Corollary 0.4], H is skew Calabi-Yau, so it remains for (3) to show that the Nakayama automorphism is trivial. Since the bimodule $\int_H^{\ell} :=$

$\operatorname{Ext}^n_H(H/\mathfrak{m}, H)$ is graded and has dimension one by Artin-Schelter regularity (2), it is the trivial module (on both right and left). Hence, by [BZ, Theorem 0.3], the Nakayama automorphism of H is S^2. Note that, since H is a connected graded domain, it has no non-trivial inner automorphisms. Hence, by [BZ, Theorem 0.6], $S^4 = Id_H$. It remains to show that $S^2 = Id_H$.

By (E2.2.1), $S^2_{\operatorname{gr} H} = Id_{\operatorname{gr} H}$. Moreover, since $\operatorname{gr} H$ is the enveloping algebra of a nilpotent Lie algebra, the Nakayama automorphism $\mu_{\operatorname{gr} H}$ is the identity. Let μ_R be the Nakayama automorphism of $R := \operatorname{Rees}_{\mathcal{F}} H$. Keeping the notation introduced for R in §1.3, since r is central,

$$\mu_{\operatorname{gr} H} = \mu_R \otimes_R \operatorname{gr} H = \mu_R \otimes_R R/\langle r\rangle, \tag{E2.2.2}$$

by [RRZ, Lemma 1.5]. Similarly, since [RRZ, Lemma 1.5] holds in an ungraded setting (with the same proof),

$$\mu_H = \mu_R \otimes_R H = \mu_R \otimes_R R/\langle r-1\rangle. \tag{E2.2.3}$$

Since H, $\operatorname{gr} H$ and R are all graded, their Nakayama automorphisms preserve their gradings. Since r is central in R, $\mu_R(r) = r$. Suppose μ_H is not the identity. Nevertheless $\mu_H^2 = Id_H$ by the previous paragraph. Pick a nonzero homogeneous element $x \in H$ of smallest degree such that $\mu_H(x) = -x$. Fix i such that $x \in \mathfrak{m}^i \setminus \mathfrak{m}^{i+1}$. Since μ_R preserves the grading, $y := \mu_R(x) \in H$ has the same degree as x. So $\mu_R(xr^{-i}) = yr^{-i}$ where $y \in \mathfrak{m}^i \setminus \mathfrak{m}^{i+1}$. Write $w' = \operatorname{gr}_{\mathcal{F}} w$ for every $w \in H$. Then (E2.2.2) implies that

$$\begin{aligned} x' &= \mu_{\operatorname{gr} H}(x') \\ &= \mu_R(xr^{-i}) \mod (r) \\ &= yr^{-i} \mod (r) \\ &= y'. \end{aligned}$$

Hence $x - y \in \mathfrak{m}^{i+1}$, so $y = x + z$ for some $z \in \mathfrak{m}^{i+1}$. Similarly, (E2.2.3) implies that $y = -x$ as $\mu_H(x) = -y$. Therefore, $-x = x + z$, so that $x = -\frac{1}{2}z \in \mathfrak{m}^{i+1}$. This is a contradiction, and so $\mu_H = S^2 = Id_H$ as required.

(5) This follows from [BZ, Corollary 0.4] and the fact (3) that the Nakayama automorphism μ_H is the identity. □

Theorem 2.2(2) answers [Br1, Question C], [Br2, Question E] and [Go2, Question 3.5] affirmatively when H is connected graded of finite Gel'fand-Kirillov dimension as an algebra. Theorem 0.2 follows from the above theorem. The following are convenient reformulations of parts of Theorem 2.2.

Corollary 2.3. *Let A be an algebra of finite GK-dimension that is not Calabi-Yau.*

1. *If A is connected graded, then A does not possess a Hopf algebra structure.*
2. *If A has a Hopf algebra structure, then A cannot be connected graded as an algebra.*

2.3. **Hilbert series.** The following is Theorem 0.3.

Theorem 2.4. *Let H be a Hopf algebra that is connected graded and locally finite as an algebra.*

1. *The Hilbert series of H is*

$$\frac{1}{\prod_{i=1}^{\infty}(1-t^i)^{n_i}}$$

for some non-negative integers n_i. Further, the sequence $\{n_i\}_{i\geq 1}$ is uniquely determined by the Hilbert series of H.

(2) *Suppose H is generated in degree 1. Then H is isomorphic as an algebra to $U(\mathfrak{g})$ for a graded Lie algebra $\mathfrak{g}$ generated in degree 1. Hence, for all $i \geq 1$,*

$$\dim_k \mathfrak{g}_i \quad = \quad n_i,$$

where $\mathfrak{g}_i$ is the i^{th} degree component of $\mathfrak{g}$.

(3) *Suppose again that H is generated in degree 1, $H \neq k$. Then*

$$\{j : n_j \neq 0\} \quad = \quad [1, \ell) \cap \mathbb{N}, \text{ for some } \ell \in [2, \infty].$$

(4) *If the Hilbert series of H is $\frac{1}{(1-t)^n}$ for some $n \geq 1$, then H is commutative, and hence $H \cong k[x_1, \ldots, x_n]$.*

(5) GKdim$H < \infty$ *if and only if $\sum_{i=1}^{\infty} n_i < \infty$. In this case,*

$$\text{GKdim}\, H = \sum_{i=1}^{\infty} n_i.$$

Proof. (1),(2): Let $\mathcal{F}$ be the standard graded filtration. Since H is locally finite, $\text{gr}_{\mathcal{F}} H$ and H have the same Hilbert series with respect to their Adams grading. Thus, in proving (1), we may assume that $H = \text{gr}_{\mathcal{F}} H$. By [GZ, Proposition 3.4(a)], $\text{gr}_{\mathcal{F}} H$ is isomorphic to $U(\mathfrak{g})$ for a graded Lie algebra $\mathfrak{g}$, with $\mathfrak{g}$ generated by $\mathfrak{g}_1 := \mathfrak{m}/\mathfrak{m}^2$. Since both $\mathfrak{m}$ and $\mathfrak{m}^2$ are Adams graded, so is $\mathfrak{g}_1$. Therefore $\mathfrak{g}$ is Adams graded. Since H is locally finite, so is $\mathfrak{g}$. Now, by the PBW theorem,

$$H_H(t) = H_{U(\mathfrak{g})}(t) = \frac{1}{\prod_{i=1}^{\infty}(1-t^i)^{n_i}}$$

where n_i is the dimension of the degree i component $\mathfrak{g}_i$ of $\mathfrak{g}$. It is clear that the sequence $\{n_i\}_{i\geq 1}$ is uniquely determined by the Hilbert series of H.

Now (2) follows from the above together with Lemma 1.2.

(3) Suppose that H is generated in degree 1. By (2), so is $\mathfrak{g}$. Now (3) is a consequence of (2) and the fact that, for all $i > 1$, $\mathfrak{g}_i = [\mathfrak{g}_1, \mathfrak{g}_{i-1}]$.

(4) Suppose that the Hilbert series of H is $(1-t)^n$. Then (1) implies that $n = n_1$ and $n_i = 0$ for all $i > 1$. Hence, by the proof of (1), $\mathfrak{g}$ is concentrated in degree 1. Thus $\mathfrak{g}$ is abelian, and $\text{gr}_{\mathcal{F}} H$ is generated in degree 1. As a consequence, H also is generated in degree 1. By Lemma 1.2, $H \cong \text{gr}_{\mathcal{F}} H = U(\mathfrak{g})$, which is commutative.

(5) Suppose that $\sum_{i=1}^{\infty} n_i < \infty$. By (1), $\text{GKdim}\, H < \infty$. By Theorem 2.2, H is an affine domain, so that $\text{GKdim}\, H$ can be computed by its Hilbert series, by [KL, Lemma 6.1(b)]. Hence, $\text{GKdim}\, H = \sum_{i=1}^{\infty} n_i$ by [KL, Theorem 12.6.2].

Suppose conversely that GKdim$H < \infty$. Then $\text{GKdim}\, \text{gr}_{\mathcal{F}} H < \infty$ by [KL, Lemma 6.5]. Since $\text{gr}_{\mathcal{F}} H \cong U(\mathfrak{g})$ as noted in the proof of (1), $\dim_k \mathfrak{g} = \sum_i n_i < \infty$ by [KL, Example 6.9]. □

Remarks 2.5. (1) It is trivial but nevertheless perhaps relevant to observe that the hypothesis of degree 1 generation in parts (2) and (3) of the theorem is not always valid: for example, the coordinate ring of the 3-dimensional Heisenberg group U is a graded Hopf algebra $\mathcal{O}(U) \cong k[X, Y, Z]$, where the generators have degrees 1,1 and 2.

(2) By part (2) of the above theorem, when H is generated in degree 1, then H is isomorphic as an algebra to the enveloping algebra of a nilpotent Lie algebra $\mathfrak{g}$.

This is not in general true, however, when the hypothesis of degree 1 generation is dropped - see Theorem 0.5(2) for an example.

2.4. **Another proof of Theorem 0.3(4).** We offer in this subsection an alternative proof of Theorem 0.3(4), which will motivate the generalisation in §3. We use here the notation introduced in §1.3.

Lemma 2.6. *Let H be an affine Hopf algebra. Suppose H is connected graded as an algebra.*

(1) *The abelianization H_{ab} is isomorphic to a commutative polynomial ring.*
(2) $d(H) \leq \mathrm{GKdim} H$ *and* $d(H) \leq \mathrm{Kdim} H$.
(3) *Suppose that* $d(H) = \mathrm{GKdim} H$ *or that* $d(H) = \mathrm{Kdim} H$. *Then* $H = H_{ab}$.

Proof. (1) This follows from Theorem 0.1(3) since H_{ab} is commutative and connected graded.

(2) As noted in Lemma 1.1, since H is connected graded,

$$d(H_{ab}) = d(H). \tag{E2.6.1}$$

Since H_{ab} is a polynomial ring by part (1),

$$d(H_{ab}) = \mathrm{GKdim} H_{ab} = \mathrm{Kdim} H_{ab}. \tag{E2.6.2}$$

The assertion follows from (E2.6.1) and (E2.6.2).

(3) Suppose that $d(H) = \mathrm{GKdim}\, H$. Then $\mathrm{GKdim}\, H_{ab} = \mathrm{GKdim}\, H$ by (E2.6.1) and (E2.6.2). Since H is a domain by Theorem 2.2(1), $H = H_{ab}$ by [KL, Proposition 3.15]. The argument for Krull dimension is similar. □

Proof of Theorem 0.3(4). Replace H by A. Suppose that $H_A(t) = \frac{1}{(1-t)^n}$. Then

$$d(A) \geq \dim_k A_1 = n = \mathrm{GKdim} A. \tag{E2.6.4}$$

Thus $A = A_{ab}$ by Lemmas 2.6(2),(3), and so A is a polynomial algebra by Lemma 2.6(1). □

2.5. **Consequences.** We give two "no Hopf structure" results. The first is a straightforward consequence of [GZ, Proposition 3.4(a)], which we record here since it fits the present context. The second assembles some immediate consequences of Theorem 0.3.

Proposition 2.7. *Let A be a connected graded algebra and let x, y be nonzero elements in A such that $xy = qyx$ for some $q \in k \setminus \{1\}$. Then there is no Hopf algebra structure on A.*

Proof. Suppose A is a Hopf algebra. Passing to the associated graded ring associated to the standard filtration, we may assume that $A = U(\mathfrak{g})$ for a graded Lie algebra $\mathfrak{g}$, by [GZ, Proposition 3.4(a)]. Taking a Hopf subalgebra, we might assume that $\mathfrak{g}$ is locally finite, and then, factoring by a suitable Lie ideal, we may assume that $\mathfrak{g}$ is finite dimensional. Now there is a connected $\mathbb{N}$-filtration $\mathcal{F}$ associated to the Lie algebra such that $\mathrm{gr}_{\mathcal{F}} A$ is isomorphic to $S(\mathfrak{g})$, a commutative polynomial ring. But this is impossible if there are nonzero elements $x, y \in A$ and $q \neq 1$, such that $xy = qyx$. □

Corollary 2.8. *There is no Hopf algebra structure on the following Koszul Artin-Schelter regular algebras:*

(1) *Sklyanin algebras of any dimension;*

(2) *skew polynomial rings* $k_{p_{ij}}[x_1, \cdots, x_n]$ *(except for the case* $p_{ij} = 1$ *for all* i, j*);*
(3) *quantum matrix algebras* $\mathcal{O}_q(M_{n\times n})$ *(except for* $q = 1$*).*
(4) *noncommutative Koszul Artin-Schelter regular algebras of dimension* ≤ 4*.*

Proof. We claim that the algebras listed above have Hilbert series $(1-t)^{-d}$ for some d. For parts (1,2,3), this is clear. For part (4), this follows from [LP, Proposition 1.4]. Now the assertion follows from Theorem 0.3(4). □

2.6. **The theorem of Latysev-Passman for connected graded algebras.** Latysev [Lat] proved that the enveloping algebra $U(\mathfrak{g})$ of a finite dimensional Lie algebra $\mathfrak{g}$ over a field of characteristic 0 satisfies a polynomial identity (or we say $U(\mathfrak{g})$ is PI) if and only if $\mathfrak{g}$ is abelian. The hypothesis of finite dimensionality was removed by Passman [Pas]. The aim of this subsection is to show that the same result applies to Hopf algebras which are connected graded as algebras.

Theorem 2.9. *Let* H *be a Hopf algebra. Suppose that* H *is connected graded or that* $\bigcap_{i=1}^{\infty} \mathfrak{m}^i = \{0\}$ *where* $\mathfrak{m} = \ker \epsilon$*. Then* H *is PI if and only if* H *is commutative.*

Proof. Note that, in the light of Lemma 2.1, if H is connected graded as an algebra then $\bigcap_{i=1}^{\infty} \mathfrak{m}^i = \{0\}$. Hence we assume that $\bigcap_{i=1}^{\infty} \mathfrak{m}^i = \{0\}$.

Let $\mathcal{F}$ be the standard filtration, as defined in (E1.1.1) in §1.3. If H is PI, then so is $\operatorname{gr}_{\mathcal{F}} H$. By [GZ, Proposition 3.4(a)], $\operatorname{gr}_{\mathcal{F}} H \cong U(\mathfrak{g})$. Hence, by [Pas], $\operatorname{gr}_{\mathcal{F}} H$ is commutative. By [GZ, Lemma 3.5], H is commutative, as required. □

Remark 2.10. The above theorem fails when k has positive characteristic. Consider, for example, the enveloping algebra H of the 3-dimensional Heisenberg Lie algebra over a field of positive characteristic. Then H is connected graded as an algebra, and is a finite module over its centre and is therefore PI. However H is not commutative.

3. Almost commutative Hopf algebras

At this point we take a small diversion to record a variation of the strand initiated in Theorem 2.9, using similar ideas to those employed above, and making further use of [GZ, §3]. The outcome can be viewed as an un-graded version of Lemma 2.6. Unlike in most of the rest of the paper, there is no connectedness hypothesis on H in the following results; however we continue to assume that k is algebraically closed of characteristic 0.

Proposition 3.1. *Let* H *be a Hopf* k*-algebra with* $\operatorname{GKdim} H = n < \infty$*, and let* $\mathfrak{m} = \ker\epsilon$*.*

(1) $\dim \mathfrak{m}/\mathfrak{m}^2 \leq \operatorname{GKdim} H$.
(2) *Suppose that* H *is affine or noetherian. Then the following statements are equivalent:*
   (a) $\dim \mathfrak{m}/\mathfrak{m}^2 = \operatorname{GKdim} H$.
   (b) $\mathfrak{m}$ *contains a unique minimal prime* P *of* H*, with* $H/P \cong \mathcal{O}(G)$ *for a connected algebraic group* G *of dimension* n*.*
   (c) H *has a commutative factor algebra* A *with* $\operatorname{GKdim} A = n$*.*

Proof. (1) Let $\mathcal{F}$ be the filtration $\{\mathfrak{m}^i : i \geq 0\}$ of H, so that $\operatorname{gr}_{\mathcal{F}} H = \oplus_{i\geq 0}\mathfrak{m}^i/\mathfrak{m}^{i+1}$. Then $\operatorname{gr}_{\mathcal{F}} H \cong U(\mathfrak{g})$ for some Lie algebra $\mathfrak{g}$ by [GZ, Proposition 3.4(a)]. Moreover,

$$\begin{aligned} \operatorname{GKdim} H \ &\geq\ \dim_k \mathfrak{g} && \text{[GZ, Proposition 3.4(b)]} \\ &\geq \dim_k \mathfrak{m}/\mathfrak{m}^2, && \text{since } \mathfrak{m}/\mathfrak{m}^2 \hookrightarrow \mathfrak{g}, \end{aligned}$$

where the inclusion in the last line is given by [GZ, Lemma 3.3(d)].

(2) $(b) \Rightarrow (c)$: Trivial.

$(a) \Rightarrow (b)$: Suppose that $\dim_k \mathfrak{m}/\mathfrak{m}^2 = n$. Set $J_{\mathfrak{m}} := \bigcap_{i\geq 0} \mathfrak{m}^i$. Then

$$\operatorname{gr}_{\mathcal{F}} H \cong U(\mathfrak{g}) \cong \operatorname{gr}_{\mathcal{F}}(H/J_{\mathfrak{m}}), \tag{E3.1.1}$$

and

$$\dim_k \mathfrak{g} = \operatorname{GKdim} \operatorname{gr}_{\mathcal{F}} H \leq \operatorname{GKdim} H = n, \tag{E3.1.2}$$

by [KL, Lemma 6.5 and Example 6.9]. By [GZ, Lemma 3.3(d)], $\mathfrak{m}/\mathfrak{m}^2 \subseteq \mathfrak{g}$, so that, from (a) and (E3.1.2),

$$\operatorname{GKdim} \operatorname{gr}_{\mathcal{F}} H = n.$$

Since $\operatorname{GKdim} H/J_{\mathfrak{m}} \geq \operatorname{GKdim} \operatorname{gr}_{\mathcal{F}} H$ by (E3.1.1) and [KL, Lemma 6.5], we deduce that

$$\operatorname{GKdim} H = \operatorname{GKdim} H/J_{\mathfrak{m}}. \tag{E3.1.3}$$

Moreover, $H/J_{\mathfrak{m}}$ is a domain, since its associated graded algebra $\operatorname{gr}_{\mathcal{F}} H/J_{\mathfrak{m}}$ is a domain and $\mathcal{F}$ is separating on $H/J_{\mathfrak{m}}$. Hence, given (E3.1.3), $J_{\mathfrak{m}}$ is a minimal prime ideal of H.

Now $H/J_{\mathfrak{m}}$ is a Hopf algebra by [LWZ, Lemma 4.7], so that $(H/J_{\mathfrak{m}})_{ab}$ is also a Hopf algebra by [GZ, Lemma 3.7]. If H is noetherian, then so is $(H/J_{\mathfrak{m}})_{ab}$, and hence this commutative Hopf algebra is affine by Molnar's theorem [Mol]. Thus, since k has characteristic 0, and whether H is affine or noetherian, $(H/J_{\mathfrak{m}})_{ab}$ is the coordinate ring of an algebraic group G, [Wat]; and as such, $(H/J_{\mathfrak{m}})_{ab}$ has finite global dimension, [Wat, §11.4 and §11.6]. More precisely, by, for example, [Hu, Theorem 5.2],

$$\dim G = \operatorname{GKdim}(H/J_{\mathfrak{m}})_{ab} = \operatorname{gldim}(H/J_{\mathfrak{m}})_{ab} = \dim_k \mathfrak{n}/\mathfrak{n}^2, \tag{E3.1.4}$$

where $\mathfrak{n}$ denotes the augmentation ideal of $(H/J_{\mathfrak{m}})_{ab}$. But, by the definition of $(H/J_{\mathfrak{m}})_{ab}$, we have

$$\mathfrak{n}/\mathfrak{n}^2 = \mathfrak{m}/\mathfrak{m}^2.$$

Therefore, invoking hypothesis (a), all the dimensions in (E3.1.4) are equal to n. Now $J_{\mathfrak{m}}$ is a prime ideal of H and $\operatorname{GKdim} H/J_{\mathfrak{m}} = n$ by (a) and (E3.1.3), so $H/J_{\mathfrak{m}}$ is an Ore domain by [KL, Theorem 4.12]. Hence, the proper factors of $H/J_{\mathfrak{m}}$ have Gel'fand-Kirillov dimension strictly less than n by [KL, Proposition 3.15], and we thus deduce that

$$H/J_{\mathfrak{m}} \ =\ (H/J_{\mathfrak{m}})_{ab} \ =\ \mathcal{O}(G).$$

That is, G is connected, and (b) is proved, with $P = J_{\mathfrak{m}}$.

$(c) \Rightarrow (a)$: Suppose that H has a commutative factor algebra C with $\operatorname{GKdim} C = n$. By (1), it is enough to prove that $\dim_k \mathfrak{m}/\mathfrak{m}^2 \geq n$. Thus we only need to prove the result for the factor H_{ab} of H, which is a Hopf algebra by [GZ, Lemma 3.7]. For commutative Hopf algebras the noetherian and affine hypotheses coincide, by the Hilbert basis theorem and Molnar's theorem [Mol], so we can assume that $H = \mathcal{O}(G)$ is a commutative affine Hopf algebra with $\operatorname{GKdim} H = n < \infty$. Now $J_{\mathfrak{m}} = 0$ without loss of generality since we can pass to a suitable prime Hopf factor of H (namely, the coordinate ring of the connected component of 1_G), and in this factor domain Krull's intersection theorem applies. So by standard commutative algebra (or see [GZ, Proposition 3.6]), $\dim_k \mathfrak{m}/\mathfrak{m}^2 = n$, as required. □

As an immediate consequence of the above proposition we have

Corollary 3.2. *Let H be an affine prime Hopf algebra of finite GK-dimension. Suppose that* $\dim \mathfrak{m}/\mathfrak{m}^2 \geq \operatorname{GKdim} H$*. Then H is commutative.*

Remark 3.3. Proposition 3.1 prompts an obvious problem, namely: to completely describe the Hopf algebras featuring in part (2) of the proposition. We leave this for the future.

4. Hopf algebras that are connected as a coalgebra

In this section we study a connected Hopf algebra H of finite Gel'fand-Kirillov dimension n, writing $\operatorname{GKdim} H = n$. The associated graded ring with respect to the coradical filtration of H, whose definition was recalled in §1.2, is denoted by $\operatorname{gr}_c H$. By [Zh, Theorem 6.9 and Corollary 6.10],

(1) H and $\operatorname{gr}_c H$ are affine noetherian domains;
(2) $\operatorname{gr}_c H$ is isomorphic to a commutative polynomial ring in n indeterminates;
(3) H is Auslander regular of global dimension n;
(4) H is GK-Cohen-Macaulay.

From (1),(3) and [BZ, Lemma 6.1], H is Artin-Schelter regular. Therefore, by [BZ, Theorem 0.3], H is skew Calabi-Yau (CY), with Nakayama automorphism given by $S^2 \circ \Xi^l_\chi$, where Ξ^l_χ denotes the left winding automorphism of H arising from the right character χ of the left integral $\int^l_H$ of H.

4.1. **Artin-Schelter regularity of the coalgebra structure.**

The main result of this subsection concerns the Artin-Schelter regularity (or skew CY property) of the coalgebra structure of H. The Artin-Schelter regularity of an artinian coalgebra is introduced in [HT, Definition 2.1 and Remark 2.2]. We now prove Theorem 0.9. Recall the definition of a *graded Hopf algebra* from §0.4. If A is a locally finite graded Hopf algebra, then the graded k-linear dual of A, denoted by A°, is also a graded Hopf algebra.

Theorem 4.1. *Let H be a connected Hopf algebra with GK-dimension $n < \infty$. Then, as a coalgebra, H is artinian and Artin-Schelter regular of global dimension n.*

Proof. First we assume that $H = \operatorname{gr}_c H$, which is a graded Hopf algebra and is connected graded as an algebra. Let H° be the graded dual of H. By [Zh, Theorem 6.9], H is commutative, so H° is a cocommutative Hopf algebra which is connected graded as an algebra. Consequently, H° is isomorphic to $U(\mathfrak{g})$ for a Lie algebra $\mathfrak{g}$ of dimension n, which is positively graded, since H° is generated in degree 1 by [AS2, Lemma 5.5]. (Also see the discussion in subsection §0.2). Thus, for example, by the discussion in the opening paragraph of §3, H° is a noetherian algebra of global dimension n satisfying the Artin-Schelter property. By duality, or a graded version of [HT, Propositions 1.2 and 2.3], it follows that H is an artinian coalgebra with global dimension n, satisfying the following Artin-Schelter condition, see [HT, Definition 2.1 and Remark 2.2] and Definition 4.3 below:

$$\text{(E4.1.1)} \qquad \operatorname{Ext}^i_{H-comod}(H,k)=\begin{cases}0 & i\neq n,\\ k & i=n.\end{cases}$$

Here $H-comod$ (respectively, $comod-H$) denotes the category of left (respectively, right) H-comodules.

Now, for general H as in the theorem, let $K=\operatorname{gr}_c H$. By the first paragraph, the coalgebra K is artinian of global dimension n. Hence [3], so is H. Since K has global dimension n, the primitive cohomology dimension of H is no more than n by [Wa2, Lemma 8.4(1)]. Since H is connected and artinian, the primitive cohomology dimension of H equals the global dimension of H by [NTZ, Corollary 3].

It remains to show that H satisfies the Artin-Schelter condition (E4.1.1). Let H^* be the complete ring $\lim_{i\to\infty}(H_i)^*$, where $\{H_i : i\geq 0\}$ is the coradical filtration of H. Then H^* is a local noetherian algebra [HT, Proposition 1.1]. Further, H^* is a filtered algebra with $\mathfrak{m}$-adic filtration where $\mathfrak{m}$ is the maximal ideal of H^*. It is easy to check that $\operatorname{gr}_{\mathfrak{m}} H^*$ is the graded dual of $K(=\operatorname{gr}_c H)$, which is denoted by K°. Since K° is noetherian, the $\mathfrak{m}$-filtration of H^* is a Zariskian filtration [LvO1, §II.2.2, Proposition 1]. By the first paragraph, $K^\circ\cong U(\mathfrak{g})$, which is Auslander regular. Hence H^* is Auslander regular [LvO1, §III.2.2, Theorem 5]. Since $\operatorname{Ext}^n_{K^\circ}(k,K^\circ)=k$, which is one-dimensional, $\operatorname{Ext}^n_{H^*}(k,H^*)$ is one-dimensional by [LvO2, Theorem 4.7]. By a local version of [Le, Theorem 6.3] (the proof of [Le, Theorem 6.3] works for the local version), local Auslander regular algebras are Artin-Schelter regular. So H^* is Artin-Schelter regular of global dimension n. By [HT, Proposition 2.3], H satisfies the Artin-Schelter condition (E4.1.1), as required. □

The Calabi-Yau property of a coalgebra is defined and studied in [HT], with the Nakayama automorphism of a coalgebra being defined there also. Every connected graded Artin-Schelter regular algebra is skew CY as an algebra by [RRZ, Lemma 1.2]. Following [HT, Theorem 3.2], every Artin-Schelter regular connected coalgebra is also called skew Calabi-Yau. In this connection, we propose a conjecture dual to [BZ, Theorem 0.3].

Conjecture 4.2. *Let H be a connected Hopf algebra of finite GK-dimension, then the Nakayama automorphism of the coalgebra H is given by S^2.*

If H is a Hopf algebra that is Artin-Schelter Gorenstein as a coalgebra, then we can define the co-integral as follows.

Definition 4.3. Let H be a Hopf algebra satisfying the following Artin-Schelter condition [HT, Definition 2.1] for the coalgebra structure of H: there is an integer n such that

$$\text{(E4.3.1)} \qquad \operatorname{Ext}^i_{comod-H}(H,k)=\begin{cases}0 & i\neq n,\\ S & i=n,\end{cases}$$

and

$$\text{(E4.3.2)} \qquad \operatorname{Ext}^i_{H-comod}(H,k)=\begin{cases}0 & i\neq n,\\ T & i=n\end{cases}$$

[3] This is dual to the following fact in the algebra setting: if A is complete and $\operatorname{gr}_{\mathfrak{m}} A$ is noetherian and has finite global dimension, then A is noetherian and has finite global dimension [LvO1, Theorem I.5.7 and §I.7.2, Corollary 2].

where S and T are 1-dimensional H-bi-comodules. Then the *right co-integral* of H is defined to be the H-bi-comodule S in (E4.3.1) and the *left co-integral* of H is defined to be the H-bi-comodule T in (E4.3.2). We say the co-integrals are trivial if S and T are isomorphic to the trivial bi-comodule k.

In the connected case, H has trivial co-integrals by (E4.1.1). Conjecture 4.2 should be a special case of a more general result about the Nakayama automorphism of a coalgebra H when H is a Hopf algebra.

4.2. **The theorem of Latysev-Passman for connected Hopf algebras.** As discussed in §2.6, Latysev [Lat] and Passman [Pas] proved that the enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a Lie k-algebra $\mathfrak{g}$ satisfies a polynomial identity if and only if $\mathfrak{g}$ is abelian. In this subsection we prove an extension of this result to all connected Hopf algebras. Note that not every connected Hopf algebra is isomorphic as an algebra to an enveloping algebra, by Theorem 0.5(2).

Let $\{H_i : i \geq 0\}$ be the coradical filtration of the connected Hopf algebra H, so $H = \bigcup_{i\geq 0} H_i$. Then $H_1 = k \oplus P(H)$ where $P(H)$ denotes the space of primitive elements of H. Note that $P(H)$ is a Lie algebra with bracket $[x, y] := xy - yx$ for all $x, y \in P(H)$, and that $U(P(H))$ is a Hopf subalgebra of H by [Mo, Corollary 5.4.7]. For the following lemma, define $\delta(t) = \Delta(t) - 1 \otimes t - t \otimes 1$ for any $t \in H$.

Lemma 4.4. *Let H be a connected Hopf algebra and suppose that the Lie algebra $P(H)$ is abelian. Fix $i \geq 1$.*

(1) *For any $x \in P(H)$ and $y \in H_i$, $[x, y] \in H_{i-1}$.*
(2) *Let $x \in P(H)$ and $y \in H_i$. Then, for any integer j with $j \geq i$, $x^j y = y_{x,j} x$ for some $y_{x,j} \in H$.*
(3) *The multiplicatively closed set of monomials $\langle P(H) \setminus \{0\}\rangle$ is an Ore set of regular elements in H.*

Proof. (1) We argue by induction on i. Since $P(H)$ is abelian, there is nothing to prove for $i = 1$. Suppose now that $i > 1$ and that the result is proved for $y \in H_{i-1}$. Let $x \in P(H)$ and $y \in H_i$. By definition and [Mo, Theorems 5.2.2 and 5.4.1(2)], $\delta(y) \in \sum_{s=1}^{i-1} H_s \otimes H_{i-s}$. Then

$$\delta([x, y]) = [x \otimes 1 + 1 \otimes x, \delta(y)] \in \sum_{s=1}^{i-1} [x, H_s] \otimes H_{i-s} + H_s \otimes [x, H_{i-s}].$$

By induction, $[x, H_s] \subset H_{s-1}$ for all $s \leq i-1$, and $[x, H_1] = 0$ since $P(H)$ is abelian. Hence

$$\delta([x, y]) \in \sum_{t=1}^{i-1} H_t \otimes H_{i-1-t}.$$

Thus $[x, y] \in H_{i-1}$, proving the induction step.

(2) We again use induction on i. Since $P(H)$ is abelian, the result holds for $i = 1$. Assume now that $i > 1$ and that the result is proved for all $x \in P(H)$ and all $y \in H_{i-1}$. Let $x \in P(H)$ and $y \in H_i$. By part (1), $xy = yx + z$ where $z \in H_{i-1}$. By induction, for all $j \geq 0$,

$$x^j y = yx^j + \sum_{s=0}^{j-1} x^s z x^{j-1-s}.$$

Since $z \in H_{i-1}$, for each $j \geq i-1$ there exists $z_{x,j} \in H$ such that $x^j z = z_{x,j} x$. Then, for all $j \geq i$, for some $y' \in H$,

$$x^j y = yx^j + \sum_{s=0}^{j-1} x^s z x^{j-1-s} = x^{j-1} z + y'x = z_{x,j-1} x + y'x = y_{x,j} x,$$

and the induction step is proved.

(3) It follows routinely from part (2) that $\langle P(H) \setminus \{0\} \rangle$ is a left Ore set in H; the argument appears as [KL, Lemma 4.1], for example. By symmetry, it is also a right Ore set. Since H is a domain by [Zh, Theorem 6.6], $\langle P(H) \setminus \{0\} \rangle$ consists of regular elements. □

The argument in Lemma 4.4 to deduce parts (2) and (3) from (1) is essentially the one due to Borho and Kraft which is reproduced as [KL, Theorem 4.9]. It is given here for the reader's convenience.

In the setting of Lemma 4.4, let $Q(H)$ be the localization of H obtained by inverting $\langle P(H) \setminus \{0\} \rangle$. Note that H is PI if and only if $Q(H)$ is PI, since if H is PI then $Q(H)$ is a subalgebra of the central simple quotient division algebra Q of H, which satisfies the same identities as H. With minimal extra effort, we can prove a local version of the result stated as the (CCA) case of Theorem 0.7 in the Introduction. We say that a ring R is *locally PI* if every finite set of elements of R is contained in a subring of R which satisfies a polynomial identity.

Theorem 4.5. *Let H be a connected Hopf algebra. Then H is locally PI if and only if it is commutative.*

Proof. If H is commutative, then it is trivially PI.

Conversely, assume that H is locally PI. Every finite set of elements of H is contained in an affine Hopf subalgebra of H, by [Zh, Corollary 3.4]. Thus, in proving that H is commutative, we may assume that H is affine, and hence that H satisfies a polynomial identity. Since $H = \bigcup_i H_i$, it suffices to show that elements in H_i, for all i, are central in H. For this, we use induction on i.

Initial step: Since $H_1 = k \oplus P(H)$, the subalgebra of H generated by H_1 is $\mathcal{U}(P(H))$. By [Pas, Theorem 1.3], $P(H)$ is abelian, so Lemma 4.4 applies. Let $x \in H_1 \setminus k$. We claim, for each $j \geq 1$, that x commutes with every element in H_j. There is nothing to be proved when $j = 1$. Now assume $j > 1$, and that

(E4.5.1) $$[x, H_{j-1}] = 0.$$

Let $y \in H_j$, and write $z = [x, y]$. By the definition of the coradical filtration, $\delta(y) \in H_{j-1} \otimes H_{j-1}$. By the induction hypothesis (E4.5.1),

$$\delta(z) = [1 \otimes x + x \otimes 1, \delta(y)] = 0.$$

That is, $z \in P(H)$, and in particular, z commutes with x. If $z \neq 0$, then the equation $z = [x, y]$ implies that, in the localization $Q(H)$ of H,

$$x(z^{-1}y) - (z^{-1}y)x = 1.$$

So $Q(H)$ contains a copy of the first Weyl algebra, which is not PI since we are in characteristic 0. This yields a contradiction. Therefore $z = 0$ and x commutes with y. So the induction step is proved, and with it the claim that H_1 is central in H.

Induction step: Now assume that $i > 1$ and that elements in H_{i-1} are central in H, and let $x \in H_i \setminus H_{i-1}$. We claim that x commutes with every element in H_j for all j. Nothing needs to be proved when $j \leq i-1$. So assume that $j \geq i$

and that $[x, H_{j-1}] = 0$. Let $y \in H_j$, so $\delta(y) \in H_{j-1} \otimes H_{j-1}$. Set $z = [x, y]$. Then $\delta(z) = [1 \otimes x + x \otimes 1, \delta(y)] = 0$. So $z \in P(H)$, and in particular, z commutes with x. By using the same argument as in the initial step, we obtain $z = 0$. This proves the induction step, as required. □

Remarks 4.6. (1) The enveloping algebra of a finite dimensional Lie algebra over a field of positive characteristic is a finite module over its centre, [Za], so Theorem 4.5 requires the characteristic 0 hypothesis on k.

(2) Since every domain of finite Gel'fand-Kirillov dimension is an Ore domain by the result of Borho-Kraft [KL, Theorem 4.12], a small recasting of the above argument yields the following dichotomy: *Let H be a connected Hopf algebra with* $\operatorname{GKdim} H < \infty$. *Then either*

(i) *H is commutative, or*
(ii) *its quotient division ring contains a copy of the first Weyl algebra.*

The hypothesis of finite Gel'fand-Kirillov dimension in the above reformulation is only introduced to guarantee the Ore condition, prompting

Question 4.7. Let H be a connected Hopf k-algebra. Under what circumstances is H an Ore domain?

Note that, by a well-known result due to Jategaonkar, [KL, Proposition 4.13], H, being a domain by [Zh, Theorem 6.6], satisfies the Ore condition if it does not contain a free algebra on two generators.

The above reformulation of Theorem 4.5 also immediately brings to mind variants of the Gel'fand-Kirillov conjecture, [AOV]. For example:

Question 4.8. Let H be a connected Hopf algebra with $\operatorname{GKdim} H < \infty$. What conditions on H ensure that its quotient division ring is a Weyl skew field?

5. Examples

The main result of this section is Theorem 5.6, which answers a number of open questions by giving an example of a connected Hopf algebra L with $\operatorname{GKdim} L = 5$, which is also connected graded as a (Hopf) algebra, but which is not isomorphic to the enveloping algebra of a Lie algebra.

5.1. **Algebra H.** First, take two copies of the Heisenberg Lie algebra of dimension 3. Thus, let

$$\mathfrak{h}_1 = ka \oplus kb \oplus kc$$

be the Lie algebra with $[a, b] = c$ and $c \in Z(\mathfrak{h}_1)$. Let

$$\mathfrak{h}_2 = kz \oplus kw \oplus kd$$

where $[z, w] = d$ and $d \in Z(\mathfrak{h}_2)$. Set $H := U(\mathfrak{g})$, where $\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$. By the Poincaré-Birkhoff-Witt theorem H has basis $\mathcal{B}$ consisting of the ordered monomials in

$$\{a, b, c, z, w, d\},$$

which we call the *PBW-generators* of H. Since H is an enveloping algebra it comes equipped with a natural cocommutative coproduct. We show next that there also exists a non-cocommutative Hopf structure on the algebra H.

5.2. **Algebra J: defining a new coproduct.** Let J be a second copy of the algebra H. We now define a noncocommutative bialgebra structure (J, Δ, ϵ).

Lemma 5.1. *Retain the above notation. Then there exist algebra homomorphisms $\Delta : J \to J \otimes J$ and $\epsilon : J \to k$ such that*

(E5.1.1) $$\Delta(a) = 1 \otimes a + a \otimes 1; \quad \Delta(b) = 1 \otimes b + b \otimes 1; \quad \Delta(c) = 1 \otimes c + c \otimes 1;$$

(E5.1.2) $$\Delta(z) = 1 \otimes z + a \otimes c - c \otimes a + z \otimes 1;$$

(E5.1.3) $$\Delta(w) = 1 \otimes w + b \otimes c - c \otimes b + w \otimes 1;$$

(E5.1.4) $$\Delta(d) = 1 \otimes d + c \otimes c^2 + c^2 \otimes c + d \otimes 1;$$

and

(E5.1.5) $$\epsilon(a) = \epsilon(b) = \epsilon(c) = \epsilon(z) = \epsilon(w) = \epsilon(d) = 0.$$

With these definitions (J, Δ, ϵ) is a bialgebra.

The proof of the lemma is a straightforward but long computation. The interested reader can read it in the Appendix, §6.

Proposition 5.2. *Retain the above notation. Then J is a noncommutative, noncocommutative connected Hopf algebra with* $\operatorname{GKdim} J = 6$.

Proof. From §5.1, J, as an algebra, is H, which has a PBW-basis $\mathcal{B}$. Define the degree of a PBW-monomial $p = a^{n_1} b^{n_2} c^{n_3} z^{n_4} w^{n_5} d^{n_6} \in \mathcal{B}$ to be

$$N(p) := \sum_{i=1}^{3} n_i + 2(n_4 + n_5) + 3n_6.$$

Define $F_0 = k$, and for each $n \geq 1$ let F_n be the vector space spanned by all monomials in $\mathcal{B}$ of degree at most n. In particular, $z, w \in F_2, d \in F_3$, and

(E5.2.2) $$\Delta(z) = 1 \otimes z + z \otimes 1 + (a \otimes c - c \otimes a) \in F_0 \otimes F_2 + F_1 \otimes F_1 + F_0 \otimes F_2;$$

(E5.2.3) $$\Delta(w) = 1 \otimes w + w \otimes 1 + (b \otimes c - c \otimes b) \in F_0 \otimes F_2 + F_1 \otimes F_1 + F_0 \otimes F_2;$$

(E5.2.4) $$\Delta(d) = 1 \otimes d + d \otimes 1 + c^2 \otimes c + c \otimes c^2 \in \sum_{i=0}^{3} F_i \otimes F_{3-i}.$$

Claim: $\{F_n\}$ is an algebra and a coalgebra filtration of J.

That $\{F_n\}$ is an exhaustive vector space filtration of J is immediate. That it is then an algebra filtration is clear from the defining relations of J. Thus it suffices to prove $\{F_n\}$ is a coalgebra filtration. To this end, let $p \in \mathcal{B}$ be a PBW monomial with $N(p) = n$. We claim

(E5.2.5) $$\Delta(p) \in \sum_{i=0}^{n} F_i \otimes F_{n-i}.$$

Indeed, noting (E5.2.2), (E5.2.3) and (E5.2.4), we have

$$\begin{aligned}\Delta(p) &= \Delta(a)^{n_1}\Delta(b)^{n_2}\Delta(c)^{n_3}\Delta(z)^{n_4}\Delta(w)^{n_5}\Delta(d)^{n_6}\\ &\subseteq \left(\sum_{j_1=0}^{1} F_{j_1}\otimes F_{1-j_1}\right)^{n_1}\cdots\left(\sum_{j_6=0}^{1} F_1\otimes F_{1-j_3}\right)^{n_3}\\ &\qquad \left(\sum_{j_4=0}^{2} F_{j_4}\otimes F_{2-j_4}\right)^{n_4}\left(\sum_{j_5=0}^{2} F_{j_5}\otimes F_{2-j_5}\right)^{n_5}\left(\sum_{j_6=0}^{3} F_{j_6}\otimes F_{3-j_6}\right)^{n_6}\\ &\subseteq \left(\sum_{j_1=0}^{n_1} F_{j_1}\otimes F_{n_1-j_1}\right)\cdots\left(\sum_{j_3=0}^{n_3} F_{j_3}\otimes F_{n_3-j_3}\right)\\ &\qquad \left(\sum_{j_4=0}^{2n_4} F_{j_4}\otimes F_{2n_4-j_4}\right)\left(\sum_{j_5=0}^{2n_5} F_{j_5}\otimes F_{2n_5-j_5}\right)\left(\sum_{j_6=0}^{3n_6} F_{j_6}\otimes F_{3n_6-j_6}\right)\\ &\subseteq \sum_{i=0}^{n} F_i\otimes F_{n-i}.\end{aligned}$$

where the last two inclusions follow from the fact that $\{F_n\}$ is an algebra filtration. This proves (E5.2.5). By [Mo, Lemma 5.3.4], it follows that $J_0 \subset F_0 = k$, and hence J is a connected bialgebra. By [Mo, Lemma 5.2.10], J is thus a connected Hopf algebra. Moreover, since J is, as an algebra, the enveloping algebra of a Lie algebra of dimension 6, $\operatorname{GKdim} J = \operatorname{GKdim} U(\mathfrak{g}) = 6$ by [Mo, Example 6.9]. □

5.3. **Definition of L.** Let J be the Hopf algebra defined in §5.2. Using the defining relations of J and the definition of its coproduct, a straightforward calculation shows that $d - \frac{1}{3}c^3$ is a primitive element central element of J. Let I be the principal ideal $(d - \frac{1}{3}c^3)J$ of J, and define $L := J/I$. Recall that an ideal P in a ring R is called *completely prime* if R/P is a domain.

Proposition 5.3. *Retain the above notation. Then L is a connected Hopf algebra with* $\operatorname{GKdim} L = 5$*, and I is a completely prime Hopf ideal of J.*

Proof. From the above comments and since $\epsilon(I) = 0$, I is a Hopf ideal of J and hence L is a Hopf algebra, and is connected by [Mo, Corollary 5.3.5] because it is a factor of J, which is connected by Proposition 5.2. By [Zh, Corollary 6.11], L, being connected, is a domain. Hence I is a completely prime ideal.

It remains to show that $\operatorname{GKdim} L = 5$. It is easy to check that $I' = \langle d, c\rangle$ is a ideal of J and that the factor ring $L' := J/(d, c)$ is isomorphic as an algebra to the enveloping algebra of the Lie algebra $\mathfrak{g}/(kd + kc)$. By [KL, Example 6.9], $\operatorname{GKdim} L' = 6-2 = 4$. We have natural algebra surjections of domains $J \to L \to L'$, so that

$$6 = \operatorname{GKdim} J \geq \operatorname{GKdim} L \geq \operatorname{GKdim} L' = 4. \tag{E5.3.1}$$

Moreover, since L is a proper factor of the domain J, the first inequality above is strict. Similarly, L' is a proper factor of L, since a short exercise with the PBW monomials $\mathcal{B}$ shows that $c \notin \langle d - \frac{1}{3}c^3\rangle$. Thus the second inequality in (E5.3.1) is also strict. As $\operatorname{GKdim} L \in \mathbb{Z}$ by [Zh, Theorem 6.9], it follows that $\operatorname{GKdim} L = 5$. □

Of course, the fact that I is completely prime can easily be proved by a direct ring theoretic argument, or, alternatively, one could use the fact that $\mathfrak{h}_1 \oplus \mathfrak{h}_2$ is a nilpotent Lie algebra, and apply [MR, Theorem 14.2.11].

Since $L' = L/cL$ and L' is an enveloping algebra of a Lie algebra with basis the images of a, b, z, w, it is easy to deduce that L also has a PBW-basis, namely the ordered monomials in $\{a, b, c, z, w\}$. Here and below, we are abusing notation by omitting "bars" above these elements

For the reader's convenience, we give an explicit presentation for the Hopf algebra L in the remark below. To calculate the effect of the antipode on its algebra generators, we use the fact that $(S_L \otimes \mathrm{id})\Delta = \epsilon$.

Remark 5.4. Retain the above notation. Then L is the connected Hopf algebra with algebra generators a, b, c, z, w subject to the relations

$$[a, c] = [a, z] = [a, w] = [b, c] = [b, z] = [b, w] = [c, z] = [c, w] = 0.$$

and

$$[a, b] = c, \quad [z, w] = \frac{1}{3}c^3.$$

with coproduct $\Delta : L \to L \otimes L$, counit $\epsilon : L \to k$ and antipode $S : L \to L$ defined on generators as follows:

$$\Delta(a) = 1 \otimes a + a \otimes 1, \quad \Delta(b) = 1 \otimes b + b \otimes 1, \quad \Delta(c) = 1 \otimes c + c \otimes 1,$$
$$\Delta(z) = 1 \otimes z + a \otimes c - c \otimes a + z \otimes 1,$$
$$\Delta(w) = 1 \otimes w + b \otimes c - c \otimes b + w \otimes 1,$$

and

$$S(x) = -x; \quad \epsilon(x) = 0$$

for $x \in \{a, b, c, z, w\}$.

5.4. **Properties of L.** For the definition and basic properties of the *signature* of a connected Hopf algebra of finite Gel'fand-Kirillov dimension, see [BG2, Definition 5.3]; the signature records the degrees of the homogeneous generators of the commutative polynomial algebra $\mathrm{gr}_c H$, the associated graded algebra of H with respect to its coradical filtration. An *iterated Hopf Ore extension* (*IHOE*) is a Hopf algebra constructed as a finite ascending sequence of Hopf subalgebras, each an Ore extension of the preceding one - see [BO, Definition 3.1].

Lemma 5.5. *Keep the above notation.*

(1) *L is an IHOE.*
(2) $\mathrm{GKdim}\, L = 5$.
(3) *The signature $\sigma(L)$ of L is* $(1, 1, 1, 2, 2)$.
(4) *The centre $Z(L)$ of L is $k[c]$.*
(5) *L is connected graded as an algebra and is a graded Hopf algebra with respect to the same grading.*

Proof. (1) Adjoin the generators of L in the order c, a, b, z, w to produce an iterated Ore extension

$$L(0) = k, \quad L(1) = L(0)[c], \quad L(2) = L(1)[a],$$
$$L(3) = L(2)[b; -c\frac{\partial}{\partial a}], \quad L(4) = L(3)[z], \quad L(5) = L(4)[w; -\frac{1}{3}c^3\frac{\partial}{\partial z}].$$

It is clear from the definition of the coproduct in §5.2 that each step in this construction gives a Hopf subalgebra of L; that is, it yields an iterated Hopf Ore extension in the sense of [BO, Definition 3.1].

(2) This follows from part (1) and [BO, Theorem 2.6].

(3) Since a, b, c are primitive, the signature $\sigma(L)$ contains $(1,1,1)$. By the definitions of $\Delta(z)$ and $\Delta(w)$, the images of the elements z and w are linearly independent in $P_2(L)/(P(L))$ (see the definitions given in [Wa1, Definition 2.4]). Thus, by [Wa1, Lemma 2.6(e)], the images of z and w are linearly independent in $P_2(\operatorname{gr} L)/(P(\operatorname{gr} L))$. Hence the signature $\sigma(L)$ contains $(2,2)$. Since $\operatorname{GKdim} L = 5$, the assertion follows.

(4) Clearly, $k[c] \subseteq Z(L)$. For the reverse inclusion, from part (1), we can write L as a free (left, say) $k[c]$-module, with basis the ordered monomials in a, b, z, w. By definition,

$$[a, -] = c\frac{\partial}{\partial b}, \quad [b, -] = -c\frac{\partial}{\partial a}, \quad [z, -] = \frac{1}{3}c^3\frac{\partial}{\partial w}, \quad [w, -] = -\frac{1}{3}c^3\frac{\partial}{\partial z}.$$

Each of the inner derivations listed above reduces the total degree of a, b, z, w by 1. Suppose that $\alpha \in Z(L) \setminus k[c]$. Obtain a contradiction by writing α in terms of the above $k[c]$-basis and applying an appropriate inner derivation from the collection $[a, -], [b, -], [z, -], [w, -]$ to α.

(5) Assign degrees to the generators as follows:

$$\deg\ a = \deg\ b = 1; \quad \deg\ c = 2; \quad \deg\ z = \deg w = 3.$$

It is easy to check from the relations that L is then connected graded as an algebra. Checking the definition of Δ on the generators, one finds that L is a graded Hopf algebra with these assigned degrees. □

We can now prove the main result of this section. For convenience of reference, we restate in it parts of Lemma 5.5.

Theorem 5.6. *Let L be the algebra defined in* §5.3.

(1) *L is a connected Hopf algebra.*
(2) *L is connected graded as an algebra.*
(3) *As an algebra, L is not isomorphic to $U(\mathfrak{n})$ for any finite dimensional Lie algebra $\mathfrak{n}$.*
(4) *L is an IHOE.*
(5) $\operatorname{GKdim} L = 5$.
(6) *The signature $\sigma(L)$ of L is $(1,1,1,2,2)$.*

Proof. Everything except (3) is proved in Lemma 5.5. If $L \cong U(\mathfrak{n})$ for a Lie algebra $\mathfrak{n}$, then

$$\dim_k \mathfrak{n} = \operatorname{GKdim} U(\mathfrak{n}) = \operatorname{GKdim} L = 5.$$

So, let $\mathfrak{n}$ be a five-dimensional Lie algebra, and write $U := U(\mathfrak{n})$. Suppose that, as algebras,

$$L \cong U \tag{E5.6.1}$$

via the map $\theta : L \to U$. Easy calculations show that the commutator ideal $\langle [L, L] \rangle$ of L is the principal ideal cL, and hence that $L/\langle [L, L] \rangle$ is a commutative polynomial algebra in 4 variables. By (E5.6.1),

$$\dim_k(\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]) = 4. \tag{E5.6.2}$$

So $\dim_k([\mathfrak{n},\mathfrak{n}]) = 1$. Write $[\mathfrak{n},\mathfrak{n}] = kx$ for some $x \in \mathfrak{n}$. Then the isomorphism (E5.6.1) takes cL to xU. Since L and U are domains with only scalars as units, we may assume without loss that $\theta(c) = x$. In particular, $[\mathfrak{n},\mathfrak{n}]$ is a 1-dimensional space contained within $Z(\mathfrak{n})$. More precisely, from Lemma 5.5(4), $Z(\mathfrak{n}) = kx$. That is,

(E5.6.3) $\mathfrak{n}$ *is nilpotent of class 2, with* $Z(\mathfrak{n}) = kx$.

By [Go1] (or use linear algebra), there is up to isomorphism precisely one Lie algebra of dimension 5 satisfying (E5.6.3). Namely, $\mathfrak{n}$ has a basis $\{x, x_1, x_2, x_3, x_4\}$ such that

$$[x_1, x_2] = x = [x_3, x_4],$$

and all other brackets are 0.

Let L^+ denote the augmentation ideal of L with respect to its given Hopf algebra structure. Since the winding automorphisms of U (and similarly those of L) transitively permute the character ideals of U (and, respectively, of L), we can follow the map θ of (E5.6.1) by an appropriate winding automorphism of U, and thus assume without loss of generality that

(E5.6.4) $$\theta(L^+) = \mathfrak{n}U = \langle x, x_1, x_2, x_3, x_4\rangle =: I.$$

Define now

(E5.6.5) $$B := L^+/(L^+)^3 \text{ and } S := I/I^3.$$

By (E5.6.1) and (E5.6.4), θ induces an isomorphism (of algebras without identity) of B with S. We shall show however that

(E5.6.6) $$\dim_k(Z(B)) = 13, \text{ and } \dim_k(Z(S)) = 11.$$

This is manifestly a contradiction, so (E5.6.1) must be false, and the proof is complete. It remains therefore to prove (E5.6.6). This is done in Proposition 5.7 below. □

Proposition 5.7. *Retain the notation introduced in Theorem* 5.6.

(1) $\dim_k(B) = \dim_k(S) = 15$.
(2) $\dim_k(Z(B)) = 13$.
(3) $\dim_k(Z(S)) = 11$.

Proof. (1) To prove that

(E5.7.1) $$\dim_k(B) = 15,$$

we show that

(E5.7.2) B *has basis (the images of)* $\mathcal{B}_B$,

where

$$\mathcal{B}_B := \{a, b, c, z, w, ab, az, aw, a^2, b^2, bz, bw, z^2, zw, w^2\}.$$

Since (abusing notation regarding images), $L/cL \cong k[a, b, z, w] =: T$, it follows that

$$\begin{aligned} T^+/(T^+)^3 &= L^+/((L^+)^3 + cL) \\ &= L^+/((L^+)^3 + ck) \\ &= B/c\bar{L}, \end{aligned}$$

where $\bar{cL}$ is the image of cL in B, and where the second equality holds since c multiplied by any element of L^+ is in $(L^+)^3$. Now $T^+/(T^+)^3$ is clearly a vector space of dimension 14 with basis the image of $\mathcal{B}_B \setminus \{c\}$.

Moreover,

$$\bar{cL} = ((L^+)^3 + cL)/(L^+)^3 = ((L^+)^3 + ck)/(L^+)^3,$$

so that $\dim_k(cB) \leq 1$. Thus, to prove (E5.7.1) it remains to show that [4]

$$\text{(E5.7.3)} \qquad c \notin (L^+)^3.$$

By definition, $(L^+)^3$ is the k-span of the words of length at least 3 in $\{a, b, c, z, w\}$. By a routine straightening argument using the defining relations for L, using for definiteness the ordering $\{a, b, c, z, w\}$ of the PBW-generators, we calculate that

$$\text{(E5.7.4)} \qquad (L^+)^3 = \mathcal{S} + \mathcal{C},$$

where

$$\mathcal{S} := k - \textit{span of ordered words in } a, b, z, w \textit{ of length at least } 3,$$

and

$$\mathcal{C} := c \times \{k - \textit{span of non-empty ordered words in } a, b, c, z, w\}.$$

By the linear independence part of the PBW-theorem for L, which holds because L is an IHOE, Lemma 5.5(1), (E5.7.3) now follows from (E5.7.4).

The argument for S is similar to the one for B. In particular, one shows that S has basis

$$\mathcal{B}_S := \{x, x_1, x_2, x_3, x_4\} \cup \{\textit{ordered words of length 2 in } x_i : 1 \leq i \leq 4\}.$$

(2) By inspection,

$$k - \textit{span}\{\mathcal{B}_B \setminus \{a, b\}\} \subset Z(B).$$

Let $f = \lambda a + \mu b + d \in B$, where $d \in k - \textit{span}\{\mathcal{B} \setminus \{a, b, \}\}$. Then, if $f \in Z(B)$, $0 = [a, f] = \mu c$. Hence, $\mu = 0$. Similarly, $\lambda = 0$, and (2) is proved.

(3) By an argument very similar to (2), one shows that $Z(S)$ has k-basis

$$\mathcal{B}_S \setminus \{x_1, x_2, x_3, x_4\}.$$

This proves (3). □

Remarks 5.8. Let L be the connected Hopf algebra in Theorem 5.6.

1. (1) L provides a negative answer to Question L of [BG1].
2. (2) L is an example of a connected Hopf algebra with signature $(1, 1, 1, 2, 2)$ which is not isomorphic as a Hopf algebra to the enveloping algebra of any coassociative Lie algebra. By contrast, every connected Hopf algebra with signature $(1, 1, \ldots, 1, 2)$ *is* isomorphic as an algebra to such an enveloping algebra. See [Wa3] for the relevant definition and theorem.

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[4] It seems this may be more or less contained in [RU, Corollary 3.3], but in any case we sketch here a direct proof.

5.5. **Proof of Corollary 0.6.** In this subsection we will discuss the dual of L and prove Corollary 0.6. We begin with the proof of part (2) of the corollary, which is immediate from the following lemma, a variant of a result of Duflo [KL, Theorem 7.2].

Lemma 5.9. *Let R be an augmented k-algebra with augmentation $\epsilon : R \longrightarrow k$ and let n be a positive integer. Suppose that R is a filtered algebra, $R = \cup_{i\geq 0} R_i$, with $R_0 = k$ and $\operatorname{gr} R$ a commutative polynomial algebra in n variables, generated in degree 1. Then $R \cong U(\mathfrak{g})$, where $\mathfrak{g}$ is the n-dimensional vector space R_1/R_0, which is a Lie algebra with respect to the bracket induced by the commutator in R.*

Proof. Since $\operatorname{gr} R = k[x_1, \dots, x_n]$ is generated in degree 1, R is also affine, generated by elements of R_1 which are lifts of the x_i. In particular, R_1 is finite dimensional, so we can fix a k-basis $y_1, \dots, y_n$ of $R_1 \cap \ker \epsilon$ with $R = k\langle y_1, \dots, y_n\rangle$. Commutativity of $\operatorname{gr} R$ and the fact that $\ker \epsilon$ is an ideal of R ensure that $\mathfrak{g} := \sum_{i=1}^n k y_i$ is a Lie subalgebra of R. The universal property of the enveloping algebra $U(\mathfrak{g})$ implies that $R \cong U(\mathfrak{g})/I$ for some ideal I of $U(\mathfrak{g})$. But $\operatorname{GKdim} R = n = \operatorname{GKdim} U(\mathfrak{g})$ by [KL, Proposition 6.6, Example 6.9], so $I = \{0\}$ by [KL, Proposition 3.15]. □

We recall some facts about completion in the algebraic setting. Let A be a k-algebra with an ideal $\mathfrak{m}$ with $\dim_k A/\mathfrak{m} = 1$. In practice, $\mathfrak{m}$ is canonically given. Assume that $\bigcap_{i=0}^\infty \mathfrak{m}^i = \{0\}$, so that the $\mathfrak{m}$-filtration $\{\mathfrak{m}^i\}_{i=0}^\infty$ is separated. The *completion* of A with respect to the $\mathfrak{m}$-adic topology is defined to be

$$\widehat{A} = \varprojlim_n A/\mathfrak{m}^n.$$

The following lemma is well-known.

Lemma 5.10. *Let $A := \bigoplus_{i=0}^\infty A(i)$ be an affine connected graded algebra with $\mathfrak{m} = A_{\geq 1} := \bigoplus_{i=1}^\infty A(i)$. Then*

$$\widehat{A} = \varprojlim_n A/A_{\geq n} = \{\prod_{i=0}^{\infty} a_i \mid a_i \in A(i)\}.$$

As a consequence, a k-linear basis of A consisting of homogeneous elements serves as a topological basis of $\widehat{A}$.

Proof. The second equality is obvious. The first follows from the fact that $\{A_{\geq n}\}_{n=0}^\infty$ is cofinite with $\{\mathfrak{m}^n\}_{n=0}^\infty$. The consequence is clear. □

We also need an easy lemma about the Gel'fand-Kirillov dimension of a connected coalgebra. Let C be a connected coalgebra with coradical filtration $\{C_n\}_{n=0}^\infty$. Assume that each C_n is finite dimensional over k. Define the *Gel'fand-Kirillov dimension* of C to be

(E5.10.1) $$\operatorname{GKdim}_c C := \overline{\lim_{n\to\infty}} \log_n(\dim_k C_n).$$

Lemma 5.11. *Let H be a connected Hopf algebra. Assume that k has characteristic zero as always.*

(1) *If H_1 is infinite dimensional, then* $\operatorname{GKdim} H = \infty$.
(2) *Suppose H_1 is finite dimensional. Then* $\operatorname{GKdim}_c H = \operatorname{GKdim} H$.

Proof. (1) In this case, by [Mo, Lemma 5.3.3] H contains $U(P(H))$ where $P(H)$ is an infinite dimensional Lie algebra. The assertion follows from [KL, Lemma 6.5].

(2) When H_1 is finite dimensional, each H_n is finite dimensional [Zh, Lemma 5.3]. Passing from H to $\mathrm{gr}_c H$ does not change the k-dimension sequence of the coradical filtration, as noted in [AS1, Definition 1.13]. Hence $\mathrm{GKdim}_c H = \mathrm{GKdim}_c \mathrm{gr}_c H$. On the other hand, by [Zh, Theorem 6.9], $\mathrm{GKdim}\, H = \mathrm{GKdim}\,\mathrm{gr}_c H$. Hence we can assume that H is $\mathrm{gr}_c H$ which is both connected graded and coradically graded [Zh, Definition 2.1]. By definition, $\mathrm{GKdim}_c H$ is defined by using the Hilbert series of H, see (E5.10.1). By Theorem 0.3(1,5), $\mathrm{GKdim}\, H$ can be computed by using the Hilbert series of H, in fact, by the same formula as (E5.10.1). Therefore $\mathrm{GKdim}_c H = \mathrm{GKdim}\, H$. □

Now let L be the graded Hopf algebra defined in §5.3. Let $\mathfrak{m} = \ker\epsilon = L_{\geq 1}$ and let $\widehat{L}$ be the completion of L with respect to the $\mathfrak{m}$-adic topology. We study $\widehat{L}$ in the next proposition, part (6) of which is needed to prove Corollary 0.6(5). We show in particular that $\widehat{L}$ is an iterated skew power series algebra of the type whose basic properties are described in [SV]. In the interest of brevity we leave the routine checking of some of the details here to the reader.

Proposition 5.12. *Let $B := \widehat{L}$ be defined as above.*

(1) $\{c^i a^j b^k z^l w^m \mid i,j,k,l,m \geq 0\}$ *is a topological basis of B.*
(2) *B is an iterated Ore extension of a similar pattern to that given in the proof of Lemma 5.5, but now in the setting of formal power series.*
(3) *The centre $Z(B)$ is the formal power series ring $k[[c]]$.*
(4) *The commutator ideal $\langle [B,B]\rangle$ of B is the principal ideal cB.*
(5) *The factor ring $B/\langle [B,B]\rangle$ is a formal power series ring in four commuting variables.*
(6) *There is no positively graded (hence nilpotent) Lie algebra $\mathfrak{g}$ such that B is isomorphic to $\widehat{U(\mathfrak{g})}$, where the completion is with respect to the maximal ideal $\mathfrak{b} := \ker \epsilon_{U(\mathfrak{g})}$.*

Proof. (1) This follows from Lemma 5.10.

(2,3,4,5) These all follow by a straightforward imitation of the proofs of Lemma 5.5 and Theorem 5.6. Note in particular that the skew formal power series extensions involve only derivations, and that these derivations are all id-*nilpotent* in the sense of [SV, Definition, page 354], thus ensuring that the extensions are well-defined.

(6) Note that if $\mathfrak{g}$ is a nilpotent Lie algebra, then $\bigcap_{i=0}^{\infty} \mathfrak{b}^i = \{0\}$. When $\mathfrak{g}$ is positively graded, the PBW basis of $U(\mathfrak{g})$ serves as a topological basis of $\widehat{U(\mathfrak{g})}$ by Lemma 5.10. Now, the key idea of comparison of finite dimensional factors, used to prove Theorem 5.6(3), transfers to this complete setting; details are omitted. Therefore the assertion holds. □

For the proof of Corollary 0.6(5) we introduce some further notation. Let C be a connected coalgebra with coradical filtration $\{C_n\}$, and let C^* denote the complete algebra

$$C^* \;:=\; \lim_{\longleftarrow n} (C_n)^*.$$

On the other hand, let $D = \bigoplus_{i=0}^{\infty} D(i)$ be a connected graded coalgebra, and define a second type of dual algebra, the *graded dual* of D,

$$D^{\mathrm{gr}\,*} := \bigoplus_{i=0}^{\infty} (D(i))^*,$$

so, of course, $D^{\mathrm{gr}\,*}$ is a connected graded algebra. Now consider again a connected coalgebra C with coradical filtration $\{C_n\}$, which is also a connected graded coalgebra,

$$C := \bigoplus_{i=0}^{\infty} C(i),$$

so $C(0) = k = C_0$ and $\Delta(C(i)) \subseteq \bigoplus_{j=0}^{i} C(j) \otimes C(i-j)$ for $i \geq 0$. Since, by definition, $C_i = \Delta^{-1}(C \otimes C_{i-1} + C_0 \otimes C)$, [Mo, (5.2.1)], an easy induction on i shows that $\oplus_{j=0}^{i} C(j) \subseteq C_i$ for all i. Assume that C is an artinian coalgebra, so in particular $\dim_k C(i) < \infty$ for all i. Then it is straightforward to show that

$$\text{(E5.12.1)} \qquad C^* = \lim_{\longleftarrow n} (C_n)^* \cong \lim_{\longleftarrow n} (\bigoplus_{i=0}^{n} C(i))^* \cong \lim_{\longleftarrow n} C^{\mathrm{gr}\,*}/(C^{\mathrm{gr}\,*})_{>n} = \widehat{C^{\mathrm{gr}\,*}}.$$

Now we are ready to prove Corollary 0.6.

Proof of Corollary 0.6. The algebra L defined in §5.3 is the algebra H of Corollary 0.6.

(1) Since H is connected, this follows from [Zh, Theorem 1.2].

(2) If a filtration of H exists such that $\mathrm{gr}\, H$ is a polynomial algebra generated in degree 1, then Lemma 5.9 and Theorem 5.6(3) yield a contradiction.

(3) This follows from [GZ, Proposition 3.4(a)].

(4) This is Lemma 5.5(3).

(5) Let D be the graded k-linear dual of H, and suppose for a contradiction that K is a connected Hopf algebra such that D is isomorphic, as a coalgebra, to $\mathrm{gr}_c\, K$. Then the space of primitive elements of K is, like that of D, finite dimensional. By [Zh, Theorem 6.9] and Lemma 5.11(2),

$$\mathrm{GKdim}\, \mathrm{gr}_c\, K = \mathrm{GKdim}\, K = \mathrm{GKdim}_c\, K = \mathrm{GKdim}_c\, D = \mathrm{GKdim}\, D = 5.$$

By Theorem 4.1, $\mathrm{gr}_c\, K$, K, D and $\mathrm{gr}_c\, D$ are artinian as coalgebras. So we can use (E5.12.1). Let T be the graded dual of $\mathrm{gr}_c\, K$. Then T is a Hopf algebra that is connected and cocommutative as a coalgebra and connected graded, and generated in degree 1, as an algebra. By the discussion in §0.2 or Theorem 0.3(2), $T \cong U(\mathfrak{g})$ as algebras where $\mathfrak{g}$ is a positively graded Lie algebra of dimension 5. Since D is the graded dual of H, by (E5.12.1) we have algebra isomorphisms,

$$D^* \cong \widehat{D^{\mathrm{gr}\,*}} = \widehat{H}.$$

Similarly, by (E5.12.1), we have algebra isomorphisms,

$$(\mathrm{gr}_c\, K)^* \cong \widehat{(\mathrm{gr}_c\, K)^{\mathrm{gr}\,*}} \cong \widehat{T} \cong \widehat{U(\mathfrak{g})}.$$

Since we assumed that $D \cong \mathrm{gr}_c\, K$ as coalgebras, we obtain that $\widehat{H} \cong \widehat{U(\mathfrak{g})}$, which contradicts Proposition 5.12(6). □

Remark 5.13. It is not hard to calculate the Lie algebra $\mathfrak{g}$ appearing in Corollary 0.6(3) - it is the direct sum of a 3-dimensional Heisenberg algebra with a 2-dimensional abelian algebra.

6. Appendix: Proof of Lemma 5.1

Here we give a complete proof of Lemma 5.1, as follows.

Lemma 6.1. *Let J be the algebra defined as in §4.1. Let $\Delta : J \to J \otimes J$ and $\epsilon : J \to k$ be the maps such that*

$$\Delta(a) = 1 \otimes a + a \otimes 1; \quad \Delta(b) = 1 \otimes b + b \otimes 1; \quad \Delta(c) = 1 \otimes c + c \otimes 1; \tag{6.1}$$

$$\Delta(z) = 1 \otimes z + a \otimes c - c \otimes a + z \otimes 1; \tag{6.2}$$

$$\Delta(w) = 1 \otimes w + b \otimes c - c \otimes b + w \otimes 1; \tag{6.3}$$

$$\Delta(d) = 1 \otimes d + c \otimes c^2 + c^2 \otimes c + d \otimes 1; \tag{6.4}$$

and

$$\epsilon(a) = \epsilon(b) = \epsilon(c) = \epsilon(z) = \epsilon(w) = \epsilon(d) = 0. \tag{6.5}$$

Then these maps satisfy the following properties:

(1) *Δ defines an algebra homomorphism.*
(2) *ϵ defines an algebra homomorphism.*
(3) *Δ is coassociative.*
(4) *$(1 \otimes \epsilon) \circ \Delta = 1 = (\epsilon \otimes 1) \circ \Delta$, where 1 denotes the identity map on J.*

As a result, (J, Δ, ϵ) is a bialgebra.

Proof. (1) Define a free algebra F on generators

$$\{a, b, c, z, w, d\}$$

and a map $\Delta' : F \to F \otimes F$ defined on these generators as in equations (6.1) – (6.4). (Here we are abusing notation by using the same letters for generators of F and their images in J.) Since F is a free algebra on these generators, we can extend Δ multiplicatively so that it defines an algebra homomorphism $F \to F \otimes F$. Setting I to be the ideal generated by the relations defining the Lie algebra $\mathfrak{g}$, the algebra J (or H) is the factor algebra F/I. Therefore, to check that there exists an algebra homomorphism $\Delta : J \to J \otimes J$ defined on generators as in equations (6.1) – (6.4), it suffices to prove that the algebra homomorphism $\Delta' : F \to F \otimes F$ defined above satisfies

$$\Delta'(I) \subseteq I \otimes F + F \otimes I. \tag{6.6}$$

Since $\Delta' : F \to F \otimes F$ is an algebra homomorphism, it suffices to check this on the generators of I, that is, on the relations defining $\mathfrak{g}$. We list these relations below

$$[a, b] - c, \quad [a, c], \quad [b, c],$$
$$[a, z], \quad [b, z], \quad [c, z], \quad [a, w], \quad [b, w], \quad [c, w],$$
$$[z, w] - d, \quad [a, d], \quad [b, d], \quad [c, d], \quad [z, d], \quad [w, d].$$

This is already immediately clear for the ideal generators which involve only PBW-generators set to be primitive under Δ', so it suffices to check only those involving z, w or d,. For each $t \in F$, define $\delta'(t) = \Delta'(t) - t \otimes 1 - 1 \otimes t$. It is trivial that, for $t \in I$, $\Delta'(t) \in I \otimes F + F \otimes I$ if and only if $\delta'(t) \in I \otimes F + F \otimes I$. It is easy to check that

$$\delta'([t_1, t_2]) = [t_1 \otimes 1 + 1 \otimes t_1, \delta'(t_2)] + [\delta'(t_1), t_2 \otimes 1 + 1 + \otimes t_2] + [\delta'(t_1), \delta'(t_2)].$$

Next we check that $\delta'(t) \in I \otimes F + F \otimes I$ for all t in the third and fourth row of ideal generators.

(1) Generator: $[a, z]$.

$$\begin{aligned}\delta'([a,z]) =&[1 \otimes a + a \otimes 1, \delta'(z)] = [1 \otimes a + a \otimes 1, a \otimes c - c \otimes a] \\ =&a \otimes [a,c] + [a,c] \otimes a \quad \in \quad I \otimes F + F \otimes I.\end{aligned}$$

(2) Generator: $[b, z]$.

$$\begin{aligned}\delta'([b,z]) =&[1 \otimes b + b \otimes 1, \delta'(z)] = [1 \otimes b + b \otimes 1, (a \otimes c - c \otimes a)] \\ =&a \otimes [b,c] - c \otimes [b,a] + [b,a] \otimes c - [b,c] \otimes a \\ =&a \otimes [b,c] - [b,c] \otimes a - c \otimes ([b,a] + c) + ([b,a] + c) \otimes c \\ &\in I \otimes F + F \otimes I.\end{aligned}$$

(3) Generator: $[c, z]$.

$$\begin{aligned}\delta'([c,z]) =&[1 \otimes c + c \otimes 1, (a \otimes c - c \otimes a)] \\ =&- c \otimes [c,a] + [c,a] \otimes c \quad \in \quad I \otimes F + F \otimes I.\end{aligned}$$

(4) Generator: $[a, w]$

$$\begin{aligned}\delta'([a,w]) =&[1 \otimes a + a \otimes 1, (b \otimes c - c \otimes b)] \\ =&b \otimes [a,c] - c \otimes [a,b] + [a,b] \otimes c - [a,c] \otimes b \\ =&b \otimes [a,c] - c \otimes ([a,b] - c) + ([a,b] - c) \otimes c - [a,c] \otimes b \\ &\in I \otimes F + F \otimes I.\end{aligned}$$

(5) Generator: $[b, w]$

$$\begin{aligned}\delta'([b,w]) =&[1 \otimes b + b \otimes 1, (b \otimes c - c \otimes b)] \\ &\in I \otimes F + F \otimes I.\end{aligned}$$

(6) Generator: $[c, w]$

$$\begin{aligned}\delta'([c,w]) =&[1 \otimes c + c \otimes 1, (b \otimes c - c \otimes b)] \\ &\in I \otimes F + F \otimes I.\end{aligned}$$

(7) Generator: $[z, w] - d$.

$$\begin{aligned}\delta'([z,w] - d) =&([1 \otimes z + z \otimes 1, (b \otimes c - c \otimes b)] + [(a \otimes c - c \otimes a), 1 \otimes w + w \otimes 1] \\ &+ [(a \otimes c - c \otimes a), (b \otimes c - c \otimes b)] - \delta'(d) \\ =&(b \otimes [z,c] - c \otimes [z,b] + [z,b] \otimes c - [z,c] \otimes b) \\ &+ (a \otimes [c,w] + [a,w] \otimes c - c \otimes [a,w] - [c,w] \otimes a) \\ &- ac \otimes cb + ca \otimes bc - cb \otimes ac + bc \otimes ac \\ &+ ([a,b] - c) \otimes c^2 + c^2 \otimes ([a,b] - c)) \\ =&(b \otimes [z,c] - c \otimes [z,b] + [z,b] \otimes c - [z,c] \otimes b) \\ &+ (a \otimes [c,w] + [a,w] \otimes c - c \otimes [a,w] - [c,w] \otimes a) \\ &+ [c,a] \otimes cb - ca \otimes [c,b] + [b,c] \otimes ac - bc \otimes [a,c] \\ &+ ([a,b] - c) \otimes c^2 + c^2 \otimes ([a,b] - c)) \\ &\in I \otimes F + F \otimes I.\end{aligned}$$

(8) Generator: $[a,d]$.

$$\begin{aligned}\delta'([a,d]) =& c\otimes [a,c^2] + c^2\otimes [a,c] + [a,c]\otimes c^2 + [a,c^2]\otimes c\\ &\in I\otimes F + F\otimes I.\end{aligned}$$

(9) Generator: $[b,d]$.

$$\begin{aligned}\delta'([b,d]) =& c\otimes [b,c^2] + c^2\otimes [b,c] + [b,c]\otimes c^2 + [b,c^2]\otimes c\\ &\in I\otimes F + F\otimes I.\end{aligned}$$

(10) Generator: $[c,d]$.

$$\begin{aligned}\delta'([c,d]) =& c\otimes [c,c^2] + c^2\otimes [c,c] + [c,c]\otimes c^2 + [c,c^2]\otimes c\\ =&0 \quad \in \quad I\otimes F + F\otimes I.\end{aligned}$$

(11) Generator: $[z,d]$

$$\begin{aligned}\delta'([z,d]) =&[1\otimes z + z\otimes 1, c\otimes c^2 + c^2\otimes c] + [b\otimes c - c\otimes b, 1\otimes d + d\otimes 1]\\ &+ [b\otimes c - c\otimes b, c\otimes c^2 + c^2\otimes c]\\ =&[z,c]\otimes c^2 + [z,c^2]\otimes c + c\otimes [z,c^2] + c^2\otimes [z,c]\\ &+ b\otimes [c,d] + [b,d]\otimes c - c\otimes [b,d] - [c,d]\otimes b\\ &+ [b,c]\otimes c^3 + [b,c^2]\otimes c^2 - c^2\otimes [b,c^2] - c^3\otimes [b,c]\\ &\in I\otimes F + F\otimes I.\end{aligned}$$

(12) Generator: $[w,d]$

$$\begin{aligned}\delta'([w,d]) =&[1\otimes w + w\otimes 1, c\otimes c^2 + c^2\otimes c] + [a\otimes c - c\otimes a, 1\otimes d + d\otimes 1]\\ &[a\otimes c - c\otimes a, c\otimes c^2 + c^2\otimes c]\\ =&[w,c]\otimes c^2 + [w,c^2]\otimes c + c\otimes [w,c^2] + c^2\otimes [w,c]\\ &+ [a,d]\otimes c - c\otimes [a,d] + a\otimes [c,d] - [c,d]\otimes a\\ &+ [a,c]\otimes c^3 + [a,c^2]\otimes c^2 - c^2\otimes [a,c^2] - c^3\otimes [a,c]\\ &\in I\otimes F + F\otimes I.\end{aligned}$$

Thus there exists an algebra homomorphism $\Delta : J \to J\otimes J$ defined on the generators $\{a,b,c,z,w,d\}$ as in equations (6.1) – (6.4).

(2) Retain the notation of part (1). Define an algebra homomorphism $\epsilon' : F \to k$ which takes the value 0 on all generators $\{a,b,c,z,w,d\}$ of F. To check that there exists an algebra homomorphism $\epsilon : J \to k$ defined on generators as in (6.5), it suffices to check that $\epsilon'(I) = 0$. This is clear.

(3) As noted in [GZ, §1], to check that the algebra homomorphism $\Delta : J \to J\otimes J$ is coassociative, it suffices to check this on the algebra generators

$$\{a,b,c,z,w,d\}$$

of J. This is already clear for each of the generators a,b,c, so we are left to check for the generators z,w and d. Let $\delta(t) = \Delta(t) - t\otimes 1 - 1\otimes t$ for any $t\in J$. It is routine to check that Δ is coassociative if and only if δ is. So it suffices to show that δ is coassociative on z, w and d. We calculate

$$\begin{aligned}(\delta\otimes 1)\circ\delta(z) &= (\delta\otimes 1)(a\otimes c - c\otimes a)\\ &= \delta(a)\otimes c - \delta(c)\otimes a = 0,\end{aligned}$$

on the other hand,

$$\begin{aligned}(1\otimes\delta)\circ\delta(z) &= (1\otimes\delta)(a\otimes c - c\otimes a)\\ &= a\otimes\delta(c) - c\otimes\delta(a) = 0.\end{aligned}$$

So $(\delta\otimes 1)\circ\delta(z) = (\delta\otimes 1)\circ\delta(z)$. Similarly, $(\delta\otimes 1)\circ\delta(w) = 0 = (1\otimes\delta)\circ\delta(w)$.

Note that $\delta(c^2) = 2c\otimes c$. Then

$$\begin{aligned}(\delta\otimes 1)\circ\delta(d) =&(\delta\otimes 1)(c\otimes c^2 + c^2\otimes c)\\ =&\delta(c)\otimes c^2 + \delta(c^2)\otimes c\\ =&2c\otimes c\otimes c.\end{aligned}$$

By symmetry, $(1\otimes\delta)\circ\delta(d) = 2c\otimes c\otimes c$. Therefore $(\delta\otimes 1)\circ\delta(d) = (1\otimes\delta)\circ\delta(d)$. Therefore δ (and then Δ) is coassociative.

(4) Again, as noted in [GZ, §1], it suffices to check that

$$(\epsilon\otimes 1)\circ\Delta(g) = g = (1\otimes\epsilon)\otimes\Delta(g)$$

for each algebra generator g of J. Noting that $\epsilon(1) = 1_k$ and that $\epsilon(g) = 0$ for all $g\in\{a,b,c,z,w,d\}$, this is clear. □

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