Bargaining, Conditional Consistency, and Weighted Lexicographic Kalai-Smorodinsky Solutions

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Abstract

We reconsider the class of weighted Kalai-Smorodinsky solutions of Dubra (2001), and using methods of Imai (1983), extend their characterization to the domain of multilateral bargaining problems. Aside from standard axioms in the literature, this result involves a new property that weakens the axiom Bilateral Consistency (Lensberg, 1988), by making the notion of consistency dependent on how ideal values in a reduced problem change relative to the original problem.

JEL-Classification: C78, D60, D70

Keywords: Bargaining, conditional consistency, leximin solutions.

1 Introduction

In this article we adopt the view that bargaining problems (Nash, 1950) reflect actual negotiations, and that bargaining solutions encapsulate all the strategic interactions that take place between negotiating agents. We further adopt Luce and Raiffa's (1957) view that the aspiration levels these agents hold – summarized in their ideal values – make up an important determining factor for the outcome they ultimately agree on.

Following Harsanyi (1959), Lensberg (1988) introduced an axiom named *Bilateral Consistency* (BCON),[1] which states that if an agent accepts a certain

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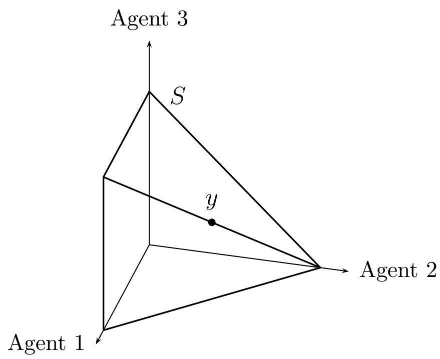
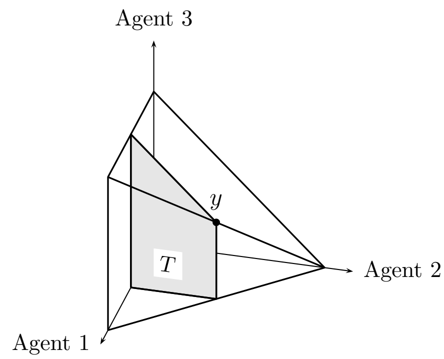
[1] Lensberg calls this axiom *Bilateral Stability*.

outcome, then he also accepts it in any two-person reduced problem that involves him.

Example. Consider three players – 1, 2 and 3 – facing the problem S, as depicted in Figure 1, and suppose the compromise they reach is given by $y := (\alpha, \alpha, \alpha)$ for some $\alpha > 0$. Suppose next that agent 1 leaves the negotiations



(a) The problem S



(b) The problem T

Figure 1: The problems S and T.

with his payoff $y_1 = \alpha$, and agents 2 and 3 are left to renegotiate their outcome (i.e., face reduced problem T). The axiom BCON states that they reach the same mutual agreement in T as they did in S. ‖

Thomson and Lensberg (1989, p. 100) motivate BCON as follows: "[A] rational agent i will not accept a tentative agreement for the bargaining problem [S] if he has reason to believe that he could successfully force or convince some other agent j to make a concession in his favor. If simultaneous challenges against more than one agent are not permitted, then i can base his beliefs about j's willingness to concede only on principles that would guide them in solving two-person problems involving just the two of them."

The underlying assumption is that the situation i faces in the two-person problem is comparable to the situation he faces in S. However, if a solution is meant to reflect the outcome of actual negotiations, and ideal points are pertinent to the outcomes they lead to, this assumption may not always hold. As the above example shows, the ideal values of two agents – and therefore also their bargaining attitudes with respect to one another – may be very different in the two-person reduced problem than in the original situation. As a result, what agent i can get from j in the reduced problem may not accurately reflect

j's willingness to concede in S. For instance, in the above example, agent 2's position with respect to agent 3 is weaker in T than in S; then a concession in his favor in T may not be enough for agent 3 to exclude y as a possible solution outcome of S. On the other hand, if the operation of problem reduction reduces both agents' ideal values in the same proportion, then their relative positions with respect to one another remain unchanged. What an agent j would then be willing to concede to i in this reduced problem, *is* representative of what he is willing to concede in the original situation.[2] Hence, agent i *can* form accurate beliefs about what j is entitled to in S, and *will* reject the tentative agreement if he believes it gives j too much. We propose an axiom, named *Conditional Consistency* (CCON), that captures exactly this idea: it imposes BCON under the condition that the two agents' ideal values in the reduced problem change *proportionally* with respect to the original problem.

A further implication of our interpretation of the bargaining problem concerns *Anonymity* (AN). The fact that a problem is the summation of an actual negotiation, is in line with the Nash program for bargaining; several results in this literature reveal that asymmetric solutions may arise quite naturally from asymmetries in the underlying bargaining protocol (see e.g. Laruelle and Valenciano (2008), Britz et al. (2010), and Anbarci and Sun (2013)).[3] As such, Anonymity may be too strong a requirement, and we do not impose it.

Our main result is that CCON, combined with several standard (and uncontroversial) properties, characterizes a class of weighted lexicographic extensions of the Kalai-Smorodinsky solution (Imai, 1983; Kalai and Smorodinsky, 1975). The Kalai-Smorodinsky – or KS – solution is known to be inefficient in a large class of problems (Roth, 1979); Imai's lexicographic maxmin – or leximin – solution is a lexicographic version of the KS solution that yields efficient outcomes on the full domain. Dubra (2001) defined and characterized a class of *weighted* lexicographic KS solutions on the specific domain of two-person problems. The family of solutions considered in this paper, and the associated characterization, may be seen as multilateral generalizations of Dubra's results.

This paper fits into the broader literature on lexicographic solutions. Examples of this literature, aside from other work on the (symmetric) lexicographic KS solution (Chang and Liang, 1998; Driesen, 2012), include lexicographic versions of the egalitarian solution (Chun and Peters, 1989; Thomson and Lensberg,

[2]This claim is more carefully motivated in Section 2.3.1.

[3]These papers provide non-cooperative support for the asymmetric Nash solution (Nash, 1950; Harsanyi and Selten, 1972; Kalai, 1977a).

1989; Nieto, 1992; Chang and Hwang, 1999; Chen, 2000) and the equal-loss solution (Chun, 1989; Chun and Peters, 1991). For problems with claims, lexicographic extensions of the proportional solution (Chun and Thomson, 1992) and the extended claims-egalitarian solution (Bossert, 1993) were studied by del Carmen Marco Gil (1995).

The article proceeds as follows. Section 2 introduces relevant definitions and notations. Section 3 contains the main result, a characterization of the family of weighted lexicographic KS solutions described above, and Section 4 discusses the independence of the axioms. Section 5 concludes.

2 Preliminaries

2.1 The Bargaining Problem

There is an infinite, countable population of agents, indexed by the set of natural numbers $\mathbb{N}$. The collection of all non-empty, finite subsets N of $\mathbb{N}$, is denoted $\mathcal{N}$. For each $N \in \mathcal{N}$, let $|N|$ be the number of agents contained in N, and let $\mathbb{R}^N$ be the Cartesian product of $|N|$ copies of $\mathbb{R}$, indexed by the members of N. Denoting the zero vector as $\bar{0}$, the positive and strictly positive orthant are given by $\mathbb{R}^N_+ := \{x \in \mathbb{R}^N \mid x \geqq \bar{0}\}$ and $\mathbb{R}^N_{++} := \{x \in \mathbb{R}^N \mid x > \bar{0}\}$.[4]

Given $N \in \mathcal{N}$, a *bargaining problem* – in short, a *problem* – is defined by a subset S of $\mathbb{R}^N$, that is non-empty, closed, convex and comprehensive (i.e., for all $x, y \in \mathbb{R}^N$, if $x \in S$ and $x \geqq y$, then $y \in S$), contains a point $z > \bar{0}$, and is further such that $S \cap \mathbb{R}^N_+$ is bounded.

The interpretation is as follows. An *outcome* or *point* $x \in \mathbb{R}^N$ represents a payoff profile for the agents in N, in the sense that each x_i, $i \in N$, specifies the utility realized by player i. The *feasible set* S represents all outcomes attainable by the players in N. The outcome $\bar{0}$ – the *disagreement point* – is the outcome that obtains if agents fail to find a compromise. Note that normalizing this point to $\bar{0}$ is without loss of generality. The condition that the feasible set holds outcomes that strictly dominate the disagreement point, represents the notion that all participating agents have some stake in the negotiations.

For each $N \in \mathcal{N}$, let Σ^N be the family of all problems for N, and let $\Sigma := \bigcup_{N\in\mathcal{N}} \Sigma^N$ be the family of *all* such problems. A *bargaining solution* – in short, a *solution* – is a real-valued function φ defined on Σ, that assigns for each $N \in \mathcal{N}$, and to each $S \in \Sigma^N$, a single outcome $\varphi(S)$ in S. This outcome, also

[4]Vector inequalities: $\geqq$, $\geq$, $>$.

called the *solution outcome*, represents the agreement the agents in N reach in the problem S.

2.2 Standard Axioms

This section restates a number of standard axioms for bargaining solutions. This requires some additional notation.

Given $N \in \mathcal{N}$, and given $S \in \Sigma^N$, the set of Pareto-optimal points in S is defined as $P(S) := \{x \in S \mid y \geq x \text{ implies } y \notin S\}$. For $x, y \in \mathbb{R}^N$, xy is a vector in $\mathbb{R}^N$ with $(xy)_i := x_i y_i$ for all $i \in N$. If there is a real number β such that $y_i = \beta$ for all $i \in N$, then xy is also denoted βx. Furthermore, $xS := \{xy \mid y \in S\}$. Given a permutation π of N, and a vector $x \in \mathbb{R}^N$, $\pi(x)$ is the vector in $\mathbb{R}^N$ with $(\pi(x))_i := x_{\pi(i)}$ for all $i \in N$. Furthermore, $\pi(S) := \{\pi(x) \mid x \in S\}$. The *ideal point* of S is a vector $u(S)$ in $\mathbb{R}^N$ with $u_i(S) := \max\{x_i \mid x \in S \cap \mathbb{R}^N_+\}$ for each $i \in N$. For $Q \in \mathcal{N}$ with $Q \subset N$, and $x \in \mathbb{R}^N$, the vector y in $\mathbb{R}^Q$ with $y_i = x_i$ for all $i \in Q$, is denoted x_Q.[5] Then S_{-i} denotes the closure of the set $\{x_{N\setminus\{i\}} \mid x \in S \text{ and } x \leqq u(S)\}$. Finally, if two points $x, y \in \mathbb{R}^N$ are proportional – i.e., $x = \beta y$ for some $\beta > 0$ – then we write $x \propto y$.

In our statement of the axioms, we omit the phrase 'For all $N \in \mathcal{N}$ and $S \in \Sigma^N$'.

Strong Individual Rationality (SIR). $\varphi(S) > \bar{0}$.

Pareto Optimality (PO). $\varphi(S) \in P(S)$.

Homogeneity (HOM). For all real $\beta \geqq 0$, $\varphi(\beta S) = \beta\varphi(S)$.

Scale Invariance (SI). For all $a \in \mathbb{R}^N_+$, $\varphi(aS) = a\varphi(S)$.

Anonymity (AN). For all permutations π of N, $\varphi(\pi(S)) = \pi(\varphi(S))$.

Independence of Irrelevant Alternatives (IIA). For all $T \in \Sigma^N$ with $\varphi(S) \in T \subseteq S$, $\varphi(T) = \varphi(S)$.

IIA other than Ideal Point (IIIA). For all $T \in \Sigma^N$ with $\varphi(S) \in T \subseteq S$ and $u(T) = u(S)$, $\varphi(T) = \varphi(S)$.

Restricted IIA (RIIA). For all $T \in \Sigma^N$ with $\varphi(S) \in T \subseteq S$ and $u(T) \propto u(S)$, $\varphi(T) = \varphi(S)$.

[5] Inclusion is denoted $\subseteq$, and *strict* inclusion $\subset$.

Individual Monotonicity (IM). For all $T \in \Sigma^N$ with $T \subseteq S$ and $S_{-i} = T_{-i}$ for some $i \in N$, $\varphi_i(S) \geqq \varphi_i(T)$.

Roth (1977) introduced and discussed IIIA. Dubra (2001) introduced the weaker axiom RIIA. The property IM was introduced by Kalai and Smorodinsky (1975), but the version presented here is due to Imai (1983). The other properties are well known, and require no further elaboration.

2.3 Bilateral Consistency and Conditional Consistency

Lensberg (1988) introduced the axiom *Bilateral Consistency*, discussed in the introduction. Given $Q, N \in \mathcal{N}$ with $Q \subset N$, and $y \in S \cap \mathbb{R}^N_+$, let $m^y_Q(S)$ be the slice of S through y, parallel to $\mathbb{R}^Q$ – i.e., $m^y_Q(S) := \{x \in \mathbb{R}^Q \mid (x, y_{N\setminus Q}) \in S\}$. Note that $m^y_Q(S)$ is a well-defined problem in Σ^Q. Then Bilateral Consistency is as follows (again, stated for all $N \in \mathcal{N}$ and $S \in \Sigma^N$).

Bilateral Consistency (BCON). If $\varphi(S) \geqq \bar{0}$, then for all $Q \subset N$ with $|Q| = 2$ and for all $T \in \Sigma^Q$ with $T := m^{\varphi(S)}_Q(S)$, $\varphi(T) = (\varphi(S))_Q$.

We propose *Conditional Consistency*, an axiom that imposes BCON under the added condition that the two considered agents' ideal values change proportionally.

Conditional Consistency (CCON). If $\varphi(S) \geqq \bar{0}$, then for all $Q \subset N$ with $|Q| = 2$ and $T \in \Sigma^Q$ with $T := m^{\varphi(S)}_Q(S)$ and $u(T) \propto (u(S))_Q$, $\varphi(T) = (\varphi(S))_Q$.

The axiom CCON is weaker than BCON, and thus satisfied by solutions such as the Nash solution (Nash, 1950), the proportional solutions (Kalai, 1977b), and the lexicographic egalitarian solution (Thomson and Lensberg, 1989). It is further satisfied by the KS solution (Kalai and Smorodinsky, 1975). The Raiffa solution (Raiffa, 1953) violates CCON.[6]

Lensberg also introduced a stronger version of BCON, that does not restrict consistency to two-person reduced problems.

[6] For $N \in \mathcal{N}$ and $S \in \Sigma^N$, the *Nash solution* (Nash, 1950) is defined as the unique maximizer of $\prod_{i \in N} x_i$ on $S \cap \mathbb{R}^N_+$. A proportional solution is defined as $\beta^* w$ where w is some vector in $\mathbb{R}^N_{++}$, and $\beta^* := \max\{\beta \mid \beta w \in S\}$. The Kalai-Smorodinsky solution (Kalai and Smorodinsky, 1975) is defined as $K(S) := \beta^* u(S)$, where $\beta^* := \max\{\beta \mid \beta u(S) \in S\}$. The Raiffa solution is defined as the (possibly infinite) sum $\frac{1}{|N|}u(S) + \frac{1}{|N|}u(S - \frac{1}{|N|}u(S)) + \frac{1}{|N|}u(S - \frac{1}{|N|}u(S) - \frac{1}{|N|}u(S - \frac{1}{|N|}u(S))) + \ldots$.

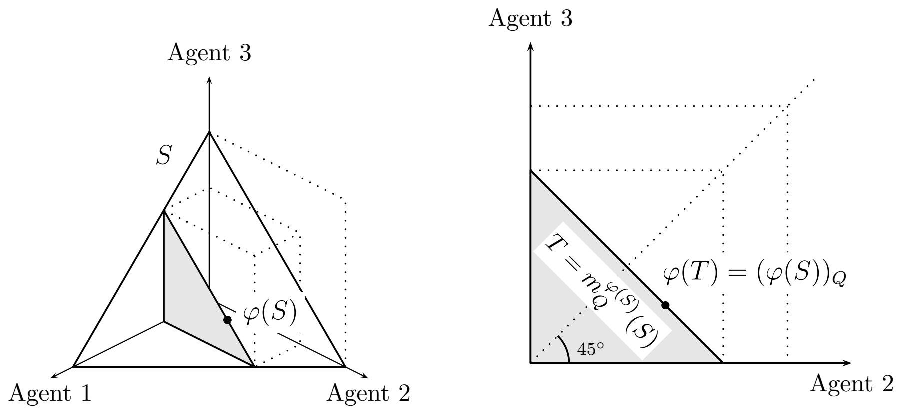


Figure 2: An illustration of CCON with $Q = \{2, 3\}$ and $N = \{1, 2, 3\}$.

Multilateral Consistency (MCON). If $\varphi(S) \geqq \bar{0}$, then for all non-empty $Q \subset N$ and $T := m_Q^{\varphi(S)}(S)$, $\varphi(T) = (\varphi(S))_Q$.

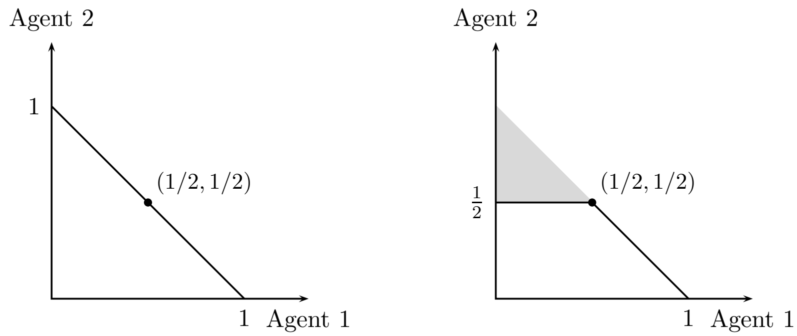
In similar fashion, we may also define a stronger version of CCON that applies to multilateral or single-agent reduced problems. However, such a strengthening of CCON turns out to be unnecessary for the purpose of this paper.

2.3.1 Discussion

Luce and Raiffa (1957) formulated the well-known criticism on Nash's (1950) IIA that it makes solutions too unresponsive to the geometry of the feasible set. The specific aspect they focused on was the ideal point: since ideal values represent the utilities agents may aspire to when engaging in the negotiations, they are an important psychological component to the attitudes these agents hold, and may therefore have an important influence on the outcome that is ultimately agreed on. This is illustrated by the example below:[7] whatever outcome agents 1 and 2 may ultimately agree on in problem A, if their ideal values matter in these negotiations, it is plausible that agent 2 will settle for a lower payoff in problem B.

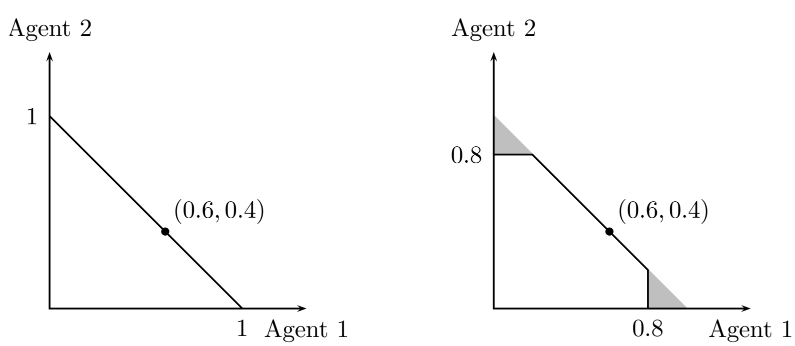
The intuition that agent 2 will not be able to secure as high a payoff in problem B as in A is not just driven by the fact that his own ideal point is lower in the former than in the latter, but also by the fact that for agent 1 the two problems are identical in this respect. The change in 2's solution payoff

[7] See also Luce and Raiffa (1957, p. 133).



(a) Problem A (b) Problem B

is not so much driven by the change in his ideal value per se, but rather by the change in his ideal value *relative to that of agent 1.* To further illustrate this point, consider the problems C and D below, and observe that while they have different ideal points, agents' *relative* ideal values in these problems are the same. Suppose now that agents have come to some agreement in problem C, and subsequently face the problem D. It is true that the ideal point then changes, but since agents' ideal values have decreased in the same proportion, the ability of both agents to stake out higher claims is curtailed in exactly the same way. Intuitively, there is then no obvious reason why this should lead either agent to demand a higher payoff. Both agents would have to hold a more timid attitude, but since their relative positions have not changed, it is not unreasonable to assume that what they deem a "fair" compromise in problem C remains a "fair" compromise in D. Note that this is exactly the reasoning that underpins RIIA (Dubra, 2001).



(a) Problem C (b) Problem D

Let us now turn our attention to BCON and its motivation. To this end, consider again the example given in the introduction. As agent 1 leaves the negotiations with his solution payoff α, agents 2 and 3 are renegotiating their payoffs (α, α) in the reduced problem T that results from agent 1's departure. The property BCON says that they should then realize the same payoffs (α, α) in the reduced problem T as they did in the original problem S. The motivation for this type of robustness against renegotiation lies in the idea that agents evaluate the worth of tentative agreements on the basis of a particular thought exercise: when confronted with the potential agreement y, an agent – say, agent 3 – will consider the hypothetical renegotiation of that agreement with each of his opponents. If any such renegotiation leads to a higher payoff than α, then agent 3 would conclude that he could do better than α, and that the corresponding outcome y should thus be rejected. A feasible outcome x can only be sustained as the solution outcome if no agent has a reason to oppose it on the basis of such reasoning.

While BCON seems sensible, if we accept Luce and Raiffa's argument that ideal points matter for the outcomes negotiations lead to, it becomes problematic. To see this, consider again the example from the introduction, and note that in the reduced problem T, agent 3 has a higher ideal value relative to agent 2 than in S itself. This means that in T, agent 2 would have a relatively more modest attitude towards agent 3 than in the original problem. Then does theorizing what he could get in T really help agent 3 in deciding whether y is a good offer in S? Arguably, it does not. Acknowledging that agents' ideal values play an important role in the bargaining process leads to the recognition that in problems S and T, agents 2 and 3 may stake out very different positions with respect to one another. As a result, the fact that agent 3 can hypothetically improve on the tentative agreement y by renegotiating it with agent 2, may not say very much about whether he should find that proposal acceptable in the current circumstance.

Consider next the example outlined in Figure 2, and note that the ideal values of agents 2 and 3 in the reduced problem T are proportional to the corresponding ideal values in S. In this case, agents 2 and 3 do have a comparable mutual position with respect to one another in the two problems, and the outcome agent 3 could secure in T does tell him something about 2's willingness to concede in the original problem S. In particular, if agent 3 were able to secure a higher payoff in T than he is realizing under the proposed outcome in S, it would now serve as a clear indication that the concession agent 2 is making in

S is too low. Agent 3 would then have a legitimate reason to oppose it. This reasoning is captured in CCON: a feasible outcome x can only be sustained as the solution outcome if no agent has a *legitimate* reason to oppose it.

It should be noted that the above discussion is only meant to motivate CCON, and should not be read as a criticism on BCON. The argument that BCON might be too strong a requirement hinges crucially on two assumptions we have made: that bargaining problems represent actual negotiations, and that in those negotiations, agents' ideal values play a crucial role. It may however be a perfectly acceptable property under other interpretations of the bargaining problem or other assumptions on the psychology of the agents.

2.3.2 The Connection Between CCON and WRGP

Peters et al. (1994) introduced an axiom named the *Weak Reduced-Game Property* (WRGP), that is reminiscent of CCON.[8] Specifically, given $Q, N \in \mathcal{N}$ with $Q \subset N$ and $|Q| = 2$, $S \in \Sigma^N$, and a point $y \in S \cap \mathbb{R}^N_+$ with $y_Q \in \mathbb{R}^Q_{++}$, they define a reduced problem $S^y_Q \in \Sigma^Q$ as the homogeneous transformation of the projection of $S \cap \mathbb{R}^N_+$ onto $\mathbb{R}^Q$, that is such that the projection of y onto $\mathbb{R}^Q$ is contained in its boundary. More specifically, given $y \in S \cap \mathbb{R}^N_+$ with $y_Q \in \mathbb{R}^Q_{++}$,

$$S^y_Q := \beta S_Q$$

where

$$S_Q := \{x \in \mathbb{R}^Q \mid x \leqq z_Q \text{ for some } z \in S \cap \mathbb{R}^N_+\}$$

and

$$\beta := \min\{\beta' \mid y_Q \in \beta' S_Q\}.$$

Then the axiom is as follows (for all $N \in \mathcal{N}$ and $S \in \Sigma^N$).

Weak Reduced-Game Property (WRGP). For $Q \in \mathcal{N}$ with $Q \subset N$ and $|Q| = 2$, if $(\varphi(S))_Q > \bar{0}$ then $\varphi(S^{\varphi(S)}_Q) = (\varphi(S))_Q$.

The difference between WRGP and CCON lies in the definition of reduced problems. In the axiom of Peters et al., the reduced problem S^y_Q is based on the *projection* of the individually rational part of S (and that of the point y) onto the subspace $\mathbb{R}^Q$. In our axiom the reduced problem $m^y_Q(S)$ is the *slice* of S, through the point y and *parallel* to $\mathbb{R}^Q$. The distinction is important: CCON does not imply WRGP, or vice versa.

[8] Peters et al. refer to reduced *problems* as reduced *games*.

To see that CCON does not imply WRGP, consider the Nash solution. Lensberg (1988) showed that it satisfies BCON; it therefore also satisfies CCON. To see that it violates WRGP, consider the following example.[9]

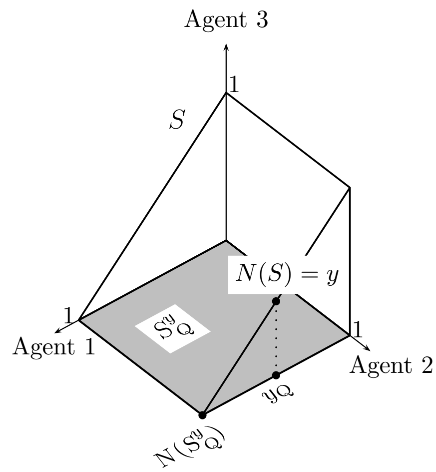


Figure 3: The Nash solution violates WRGP.

Example. Consider $N := \{1, 2, 3\}$ and $S \in \Sigma^N$ with $S := cch\{(1, 1, 0), (0, 1, 1)\}$, and note that $N(S) = (0.5, 1, 0.5)$. Let $y := N(S)$ and note that for $Q := \{1, 2\}$, $S^y_Q = cch\{(1, 1)\}$. Then $N(S^y_Q) = (1, 1) \neq (0.5, 1) = y_Q$, a violation of WRGP. ‖

To see that WRGP does not imply CCON, consider a solution F that for all $N \in \mathcal{N}$ and $S \in \Sigma^N$ yields $K(S)$ whenever $|N| < 3$, and $\frac{1}{2}K(S)$ otherwise.

Observation 2.1 *F satisfies WRGP.*

Proof. Let $S \in \Sigma^N$ with $|N| \geqq 3$, and without loss of generality, assume $u_i(S) = u_j(S)$ for all $i, j \in N$. The KS solution outcome is then the maximal point in S with all entries equal. Therefore, $y := F(S)$ is a point in $\mathbb{R}^N$ with all entries equal. Take any $Q \subset N$ with $|Q| = 2$. By definition of the reduced problem S^y_Q, y_Q is the maximal point in S^y_Q with both entries equal. Furthermore, $u_i(S^y_Q) = u_j(S^y_Q)$ for $i, j \in Q$. It follows that $F(S^y_Q) = K(S^y_Q) = y_Q$. ■

The following example shows that F violates CCON.

[9]Given $N \in \mathcal{N}$ and $V \subset \mathbb{R}^N$, $cch\, V$ denotes the *convex comprehensive hull* of (the points in) V. It is defined as the intersection of all convex and comprehensive sets in $\mathbb{R}^N$ that contain (the points in) V.

Example. Let $N := \{1,2,3\}$ and $S \in \Sigma^N$ with $S := cch\{(6,0,0),(0,6,0),(0,0,6)\}$, and note that $F(S) = (1,1,1)$. Let $y := F(S)$ and $Q := \{1,2\}$, and observe that $T := m_Q^y(S) = cch\{(4,0),(0,4)\}$. Then $F(T) = (2,2) \neq (1,1) = y_Q$, a violation of CCON. $\|$

2.4 A Family of Weighted Lexicographic KS Solutions

To formally define our solutions of interest, it is useful to first introduce the 'lexicographic maxmin ordering'.

Definition. Given $N \in \mathcal{N}$ and $x \in \mathbb{R}^N$, let $\bar{N} := \{1,\ldots,|N|\}$, and let $\mu(x)$ be the vector in $\mathbb{R}^{\bar{N}}$, that is obtained by relabeling the coordinates of x such that $\mu_1(x) \leqq \ldots \leqq \mu_{|N|}(x)$. The *lexicographic maxmin ordering for* N, denoted $\succeq_N$, is defined as follows: For $x, y \in \mathbb{R}^N$, $x \succ_N y$ if and only if there is a $j \in \bar{N}$ such that $\mu_j(x) > \mu_j(y)$, and $\mu_i(x) = \mu_i(y)$ for all $i \in \bar{N}$ with $i < j$. Furthermore, $x \sim_N y$ if and only if $x = y$. Let $\succeq := \{\succeq_N \mid N \in \mathcal{N}\}$.

It is further useful to introduce the *lexicographic egalitarian solution* (Thomson and Lensberg, 1989). It is a well-defined solution by Lemmas 3 and 4 of Imai (1983, p. 395).

Definition. For $S \in \Sigma$, the *lexicographic egalitarian solution* ξ is defined as the unique maximum in S with respect to $\succeq$. That is, $\xi(S) := \{y \in S \mid y \succeq x \text{ for all } x \in S\}$.

Definition. For $x \in \mathbb{R}^N_{++}$, define $x^{-1} := (1/x_i)_{i \in N}$. Then, given $w \in \mathbb{R}^{\mathbb{N}}_{++}$, $N \in \mathcal{N}$, and $S \in \Sigma^N$, the *weighted lexicographic KS solution* L^w is defined as

$$L^w(S) := b\xi(b^{-1}S), \tag{1}$$

where $b = w_N u(S)$. The family of all such solutions is denoted by $\mathcal{L}$, i.e., $\mathcal{L} := \{L^w \mid w \in \mathbb{R}^{\mathbb{N}}_{++}\}$.

Consider $N \in \mathcal{N}$ and $S \in \Sigma^N$ with $u_i(S) = 1$ for all $i \in N$. Given $w \in \mathbb{R}^{\mathbb{N}}_{++}$, $L^w(S)$ is obtained by the following procedure. Starting from the disagreement point $\bar{0}$, increase the utilities of all the agents in $N(\equiv Q^1)$ simultaneously in the direction w_{Q^1}, until the boundary of S is reached, say in the point x^1. There is a number of agents for whom a further improvement would result in an infeasible alternative. Fix the payoffs of these agents at their x^1-levels, and continue

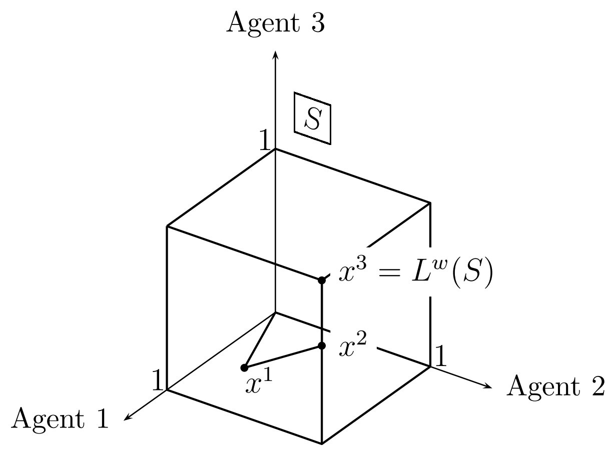


Figure 4: An illustration of L^w (with $w_1 > w_2 > w_3$).

increasing the utilities of the remaining agents (call this set of remaining agents Q^2) in the direction w_{Q^2}. This leads again to a point – say x^2 – from which further increase of utilities means stepping out of S. Since the total number of agents in N is finite, and since at each iteration at least one is excluded from further improvement, this procedure terminates in a finite number of steps. The resulting outcome corresponds with $L^w(S)$.

The *lexicographic KS solution* L (Imai, 1983) is the unique symmetric solution in $\mathcal{L}$.

Theorem 2.2 (Imai, 1983) *A solution φ satisfies PO, SI, AN, IM and IIIA, if and only if $\varphi = L$.*

Dubra (2001) defined a class $\mathcal{K}$ of *weighted Kalai-Smorodinsky solutions*, for the specific domain of two-person problems (i.e., problems for N with $|N| = 2$). Except for the two corner solutions where one player's weight is zero, the class of solutions $\mathcal{L}$ is a multilateral generalization of $\mathcal{K}$.

Theorem 2.3 (Dubra, 2001) *A solution $\varphi : \bigcup_{N \in \mathcal{N}:|N|=2} \to \bigcup_{N \in \mathcal{N}:|N|=2} \mathbb{R}^N$ satisfies PO, SI, IM and RIIA if and only if $\varphi \in \mathcal{K}$.*

3 Main Result

The aim of this section is to obtain a characterization result for the above-defined solution class.

Theorem 3.1 *A solution φ satisfies SIR, PO, SI, IM, IIIA and CCON if and only if $\varphi \in \mathcal{L}$.*

Note the connection with the previously mentioned results. Imai's (1983) axiom set does not include SIR, since it is implied by the combination of PO, AN, IM, and IIIA. Dubra's (2001) axiom set contains RIIA, rather than IIIA. If the set of agents exceeds two – as is the case in our framework – then RIIA is implied by the other axioms of Theorem 3.1. Theorem 2.3 does not involve SIR. Aside from the restriction to two-player problems, $\mathcal{K}$ is somewhat broader than $\mathcal{L}$. In particular, it allows for a specific type of lexicographic dictatorial solution.[10]

We first prove that solutions $\varphi \in \mathcal{L}$ satisfy the properties of Theorem 3.1. To this end, the following Lemma is useful.

Lemma 3.2 *The lexicographic egalitarian solution satisfies SIR, PO, IIA, HOM and BCON.*

Proof. It is obvious from its definition that the lexicographic egalitarian solution satisfies SIR, PO, IIA, and HOM. Proposition 9.1 of Thomson and Lensberg (1989, p. 132–133) shows that it satisfies MCON, and therefore also BCON. ■

Proposition 3.3 *If $\varphi \in \mathcal{L}$, then it satisfies SIR, PO, SI, IM, IIIA and CCON.*

Proof. Consider $L^w \in \mathcal{L}$. It follows directly from the definition that this solution satisfies SI. It further follows from Lemma 3.2 that it satisfies SIR, PO and IIIA. By the same reasoning as in Proposition 1 of Imai (1983, p. 397), it satisfies IM.[11] To establish CCON, let $N \in \mathcal{N}$ and $S \in \Sigma^N$, and assume that $u(S) = w_N^{-1}$, such that $L^w(S) = \xi(S)$. This is without loss of generality by SI. Let $Q \subset N$ with $|Q| = 2$, let $T := m_Q^y(S)$ where $y := \xi(S)$, and assume $u(T) \propto (u(S))_Q$. The latter means there exists a strictly positive real number β such that $u(T) = \beta w_Q^{-1}$. Then since ξ satisfies HOM and BCON, $L^w(T) = \beta\xi\left(\frac{1}{\beta}T\right) = \xi(T) = y_Q$. ■

The next step is to demonstrate that solutions in $\mathcal{L}$ are the *only* solutions satisfying the properties of Theorem 3.1.

Proposition 3.4 *If φ satisfies SIR, PO, SI, IM, IIIA and CCON, then $\varphi \in \mathcal{L}$.*

[10] See the solution D, defined in Section 4.1.
[11] This proof is included in the Appendix.

To establish this result, it must be demonstrated that for any solution $\varphi \in \Sigma$ satisfying the axioms of Theorem 3.1, there exists, up to a multiplicative constant, a unique weights vector $w \in \mathbb{R}^{\mathbb{N}}_{++}$ such that $\varphi = L^w$. More specifically, fixing a set of players $N \in \mathcal{N}$ and a problem $S \in \Sigma^N$, it must be shown that there exists a weights vector $w \in \mathbb{R}^{\mathbb{N}}_{++}$ – unique up to its restriction to the coordinates in N and a multiplicative constant – such that $\varphi(S) = L^w(S)$. The argument is similar to Proposition 2 of Imai (1983), and it is thus useful to recall some of his notations.

- For $N \in \mathcal{N}$, for $x, y \in \mathbb{R}^N$, let $x \cdot y$ denote the inner product $\sum_{i \in N} x_i y_i$. For $p \in \mathbb{R}^N$ and $\beta \in \mathbb{R}$, let $H(p, \beta) := \left\{x \in \mathbb{R}^N \mid p \cdot x \leqq \beta\right\}$.
- Given $N \in \mathcal{N}$, let e^i be the vector in $\mathbb{R}^N$ for which entry i is 1, and all others 0. For non-empty $Q \subseteq N$, we write $\sum_{i \in Q} e^i$ as $e(Q)$.
- Given $Q, N \in \mathcal{N}$ with $Q \subseteq N$, a problem $S \in \Sigma^N$ is *Q-symmetric* if for any permutation π of N with $\pi(i) = i$ for all $i \in N \setminus Q$, $\pi(S) = S$.
- For $N \in \mathcal{N}$, $S \subset \mathbb{R}^N$ and $y \in S$, let $Q(S, y) := \{i \in N \mid y + \varepsilon e^i \in S \text{ for some } \varepsilon > 0\}$. For $y \in S$ with $Q(S, y) \neq \emptyset$, define

  $$z(S, y) := y + a(S, y)\ e(Q(S, y))$$

  where

  $$a(S, y) := \max\{a \in \mathbb{R} \mid y + a\ e(Q(S, y)) \in S\}.$$

  For $y \in S$ with $Q(S, y) = \emptyset$, $a(S, y) := 0$ and $z(S, y) := y$, by convention. For $S \in \Sigma^N$, let $z^0 := \bar{0}$ and $z^j := z(S, z^{j-1})$ for $j \geqq 1$. Let k be the smallest integer such that $z^j = z^{j+1}$. Then for $j = 1, \ldots, k$, define

  $$Q^j := Q(S, z^{j-1}), \quad \text{and} \quad a^j := a(S, z^{j-1}).$$

  The sequences $\{z^j\}_{j=1}^k$, $\{Q^j\}_{j=1}^k$ and $\{a^j\}_{j=1}^k$ are referred to as the *defining sequences* of $\xi(S)$.

Imai further proved the following lemma.[12]

Lemma 3.5 (Imai, 1983) *For $N \in \mathcal{N}$ and $S \in \Sigma^N$,*

i) and for defining sequence $\{z^j\}_{j=1}^k$ of $\xi(S)$, $\xi(S) = z^k$;

[12] Parts i) and ii) of Lemma 3.5 respectively correspond with lemmas 3 and 8 of Imai (1983, p. 395, 397).

ii) and for $T := S \cap H(p, \beta)$ with $p > \bar{0}$ and $\beta > 0$ such that $u(S) = u(T)$, $S_{-i} = T_{-i}$ for all $i \in N$.

Before going into the details of the proof of Proposition 3.4, it is useful to present the argument in a more informal manner. First, the relevant weights vector – i.e., the vector $w_N \in \mathbb{R}^N_{++}$ – is determined. It is obtained as the solution outcome of a generic problem $\underline{S}^0$ in Σ^N.

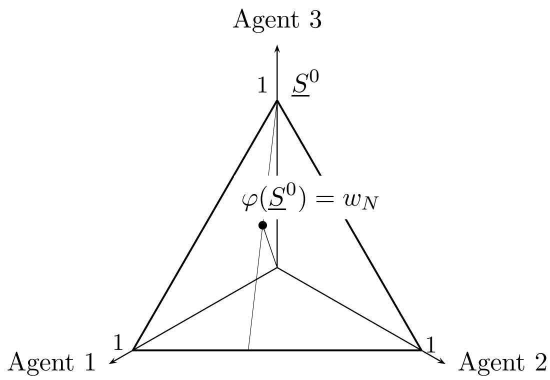


Figure 5: Weights are determined by the solution in a generic problem.

By SI, we may without loss of generality assume that the considered problem S is normalized, in the sense that $u(S) = e(N)$. Given the defining sequence $\{z^j\}_{j=1}^k$ of $\xi(w_N^{-1}S)$, the sequence $\{x^j\}_{j=0}^k$ is constructed, where $x^0 := w_N$, and $x^j := w_N z^j$ for each $j \geqq 1$. By part i) of Lemma 3.5 and the definition of L^w, it then follows that $L^w(S) = x^k$. Hence, it is sufficient to show that $\varphi(S) = x^k$.

As in Imai's proof, this is established by induction. In particular, a set of auxiliary problems is constructed for each $j = 1, \ldots, k$, and it is subsequently shown that x^j is the common solution outcome of all stage-j auxiliary problems. This implies that x^k is the common solution outcome of the final-stage auxiliary problems. The observation that x^k is efficient in S is then sufficient to conclude that $\varphi(S) = x^k$, the desired result.

The main difference between the present argument and that of Imai, lies in the induction step. By combining AN with (Q-)symmetry of the auxiliary problems, Imai asserts that agents' utilities are always updated in an egalitarian direction. Since AN is not in the axiom set of Theorem 3.1, this approach is not available to us. Instead we rely on the property CCON. Consider two-person problems H and T as in Figure 6.

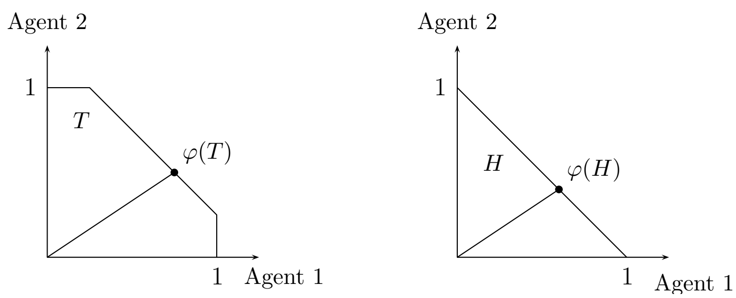


Figure 6: If $\varphi(T) < (1,1)$, $\varphi(T)$ is proportional to $\varphi(H)$.

Using the axioms of Theorem 3.1, we show that the solution outcomes of H and T are proportional. More specifically, we show that if the solution outcome of T is strictly dominated by the ideal point $(1,1)$, then it is proportional to the solution outcome of H. The usefulness of this observation is illustrated in Figure 7. The three-person problem $\overline{S}^1$ is $\{1,2\}$-symmetric, so CCON may be applied whatever the solution outcome may be. Thus, by CCON and the above reasoning, the solution payoffs of agents 1 and 2 must be proportional to their initial weights (w_1, w_2), regardless of the exact position of the solution outcome.

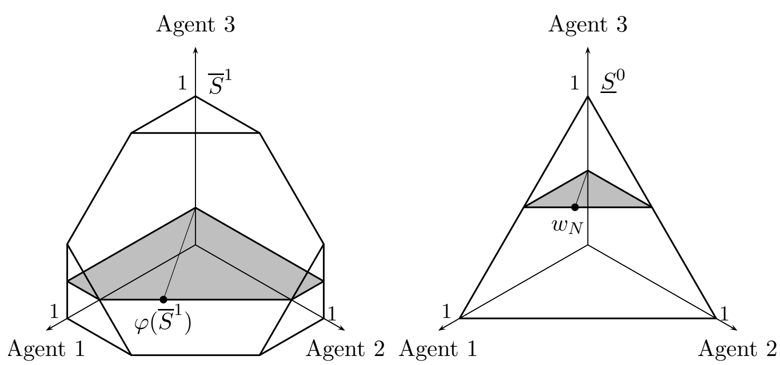


Figure 7: If $(\varphi_1(\overline{S}^1), \varphi_2(\overline{S}^1)) < (1,1)$, it is proportional to (w_1, w_2).

Agents 1 and 2 were chosen without loss of generality; repeating this analysis for any other pair, establishes that $\varphi(\overline{S}^1)$ is proportional to w_N. By PO, this pins down the solution outcome of $\overline{S}^1$.

Lemma 3.6 *Let φ satisfy the properties of Theorem 3.1. For $N := \{1,2\}$ and* *for $H, T \in \Sigma^N$ defined as*

$$H := \{x \in \mathbb{R}^N \mid x_1 + x_2 \leqq 1\}, \text{ and}$$
$$T := \{x \in \mathbb{R}^N \mid x_1 + x_2 \leqq 1 \text{ and } x \leqq (\beta, \beta)\}, \quad (\beta > 1/2),$$

if $\varphi(T) < (\beta, \beta)$, then $\varphi(H) = \varphi(T)$.

Proof. If $\beta \geqq 1$, then by SIR, $\varphi(H) \in T \subset H$. $\beta \geqq 1$ further implies $u(T) = u(H) = (1, 1)$. Then $\varphi(H) = \varphi(T)$ follows immediately from IIIA. Thus, assume $\beta < 1$. We first prove the following result.

$$\text{If } \varphi(H) \in T, \text{ then } \varphi(T) = \varphi(H). \tag{2}$$

Assume that $\varphi(H) \in T$, and define $\tilde{H} := \{x \in \mathbb{R}^N \mid x_1 + x_2 \leqq 1 \text{ and } x \leqq (1, 1)\}$. By PO and a two-fold application of IM, $\varphi(\tilde{H}) = \varphi(H)$. Let $k \in \mathbb{N} \setminus N$, define $Q := \{1, 2, k\}$, and for $x \in \mathbb{R}^Q$, represent the utility of agent k by the third coordinate (i.e., $x = (x_1, x_2, x_k)$). Let $H', T' \in \Sigma^Q$ with

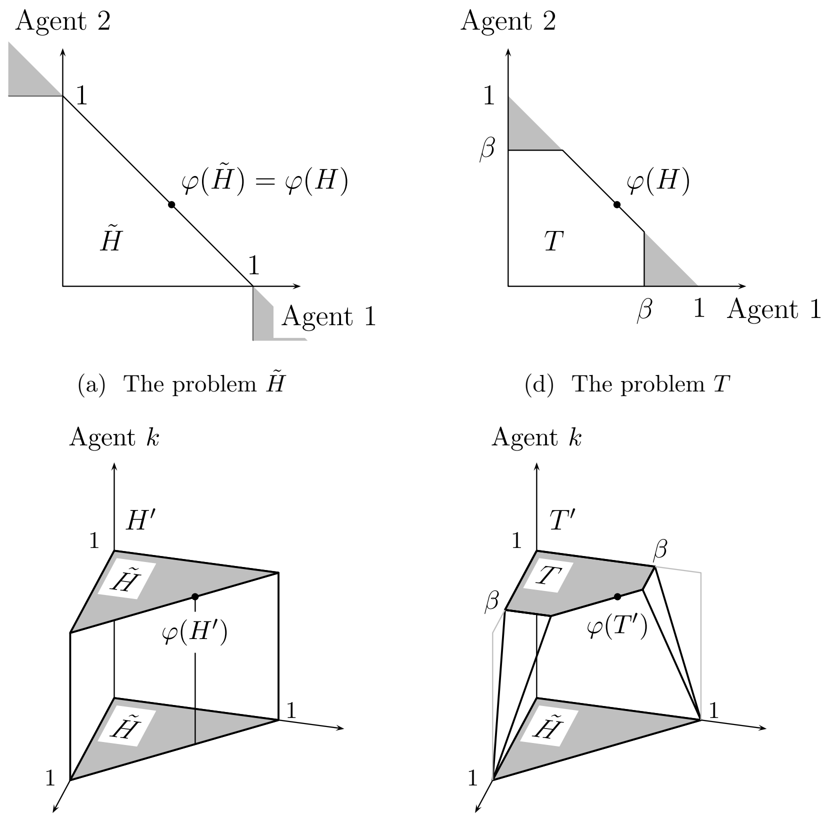
$$H' := cch\,\{(1, 0, 1), (0, 1, 1)\} \text{ and}$$
$$T' := cch\,\{(\beta, 1 - \beta, 1), (1 - \beta, \beta, 1), (1, 0, 0), (0, 1, 0)\}.$$

Then H' is the convex comprehensive hull of all $(x, 1) \in \mathbb{R}^Q$ with $x \in \tilde{H}$. Similarly, T' is the convex comprehensive hull of all $(x, 0), (y, 1) \in \mathbb{R}^Q$ with $x \in \tilde{H}$ and $y \in T$. By PO and CCON, $\varphi(H') = (\varphi(\tilde{H}), 1)$. Since $\varphi(\tilde{H}) \in T$, $\varphi(H') \in T'$. Moreover, $u(H') = u(T') = e(Q)$, so by IIIA, $\varphi(T') = \varphi(H')$. Then $\varphi(T) = \varphi(\tilde{H})$ by CCON. This establishes (2).

Now assume $\varphi(T) < (\beta, \beta)$. If $\varphi(H) \leqq (\beta, \beta)$, then $\varphi(H) \in T$, which by (2) implies $\varphi(H) = \varphi(T)$. Thus, assume there is an agent $i \in N$ such that $\varphi_i(H) > \beta$, and without loss of generality, say $i = 2$. Let $\gamma := \varphi_2(H)$, and define

$$\bar{T} := \{x \in \mathbb{R}^N \mid x_1 + x_2 \leqq 1 \text{ and } x \leqq (\gamma, \gamma)\}, \text{ and}$$
$$\hat{T} := \frac{\beta}{\gamma}\bar{T}.$$

By similar reasoning as above, $\varphi(\bar{T}) = \varphi(H)$, that is, $\varphi(\bar{T}) = (1 - \gamma, \gamma)$. By SI, $\varphi(\hat{T}) = (\frac{\beta}{\gamma}(1 - \gamma), \beta)$. Observe that $\hat{T}_{-i} = T_{-i}$ for $i = 1, 2$ and $\hat{T} \subset T$. Thus, by a two-fold application of IM, $\varphi(T) \geqq \varphi(\hat{T}) = (\frac{\beta}{\gamma}(1 - \gamma), \beta)$. Then by PO,



(a) The problem $\tilde{H}$

(d) The problem T

(b) The problem H'

(c) The problem T'

Figure 8: If $\varphi(H) \in T$, then $\varphi(T) = \varphi(H)$.

$\varphi(T) = (1 - \beta, \beta)$, contradicting the initial assumption that $\varphi(T) < (\beta, \beta)$. ■

Consider again the problem $\overline{S}^1$, introduced above. It is clear that Lemma 3.6 only has bite whenever $(\varphi_1(\overline{S}^1), \varphi_2(\overline{S}^1)) < (1, 1)$. Since $\varphi(\overline{S}^1)$ is not known, the possibility that this condition is violated cannot be excluded. Therefore, it is necessary to *approximate* this solution outcome. In particular, we construct intermediate problems for which the condition *is* satisfied, and show that the solution outcomes of these problems converge to the desired outcome.

Proof of Proposition 3.4. Let φ be a solution satisfying SIR, PO, SI, IM, IIIA and CCON. Fix some $N \in \mathcal{N}$, consider the problem $\underline{S}^0 \in \Sigma^N$ defined as $\underline{S}^0 := H(e(N), 1)$, and define $w_N := \varphi(\underline{S}^0)$. By SIR and PO, $w_N > \bar{0}$ and

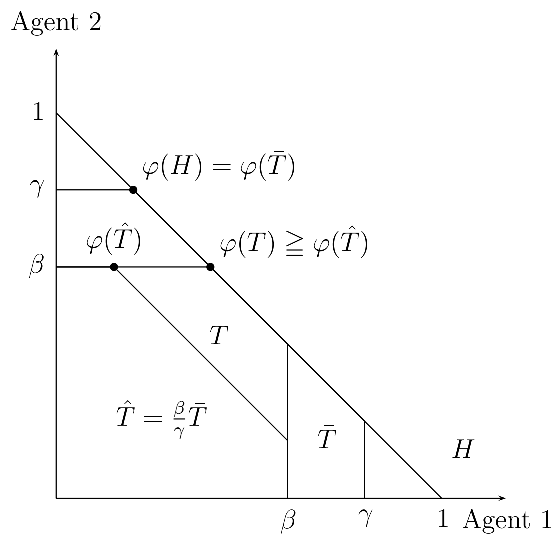


Figure 9: If $\varphi_2(H) > \beta$, then $\varphi_2(T) = \beta$.

$\sum_{i \in N} w_i = 1$.

Consider a problem $S \in \Sigma^N$ with $u(S) = e(N)$ and $S = S \cap (e(N) - \mathbb{R}^N_+)$. By SIR, SI and IIIA, this choice is without loss of generality. Let $\{z^j\}_{j=1}^k$, $\{Q^j\}_{j=1}^k$ and $\{a^j\}_{j=1}^k$ be the defining sequences of $\xi(w_N^{-1} S)$. Then define $\{x^j\}_{j=0}^k$ by $x^0 := w_N$ and $x^j := w_N z^j$ for each $j = 1, \ldots, k$. Furthermore, define $\{\alpha^j\}_{j=1}^k$ by $\alpha^j := \sum_{i \in N} x^j$ for each j. By part i) of Lemma 3.5 and the definition of L^w, it is sufficient to show $\varphi(S) = x^k$. This is achieved by induction on a set of auxiliary problems.

Let $p^1 := \bar{0}$ and for $j = 2, \ldots, k$, let $p^j := e(N \setminus Q^j)$. Then

$$\overline{S}^j := H(e(N), \alpha^j) \cap \left(\bigcap_{j'=1}^{j} H(p^{j'},\ p^{j'} \cdot x^{j'}) \right) \cap (e(N) - \mathbb{R}^N_+) \qquad j = 1, \ldots, k;$$

$$\underline{S}^j := \overline{S}^j \cap H(p^{j+1},\ p^{j+1} \cdot x^{j+1}) \qquad j = 1, \ldots, k-1;$$

$$S^j := H(e(N), \alpha^j) \cap S \qquad j = 1, \ldots, k;$$

$$S'^j := \overline{S}^j \cap S \qquad j = 1, \ldots, k.$$

These problems are exactly as in Proposition 2 of Imai. Consider the problem $\overline{S}^j$. The set $H(e(N), \alpha^j) = \{x \in \mathbb{R}^N \mid e(N) \cdot x \leqq \alpha^j\}$ is a halfspace, and since $\alpha^j = \sum_{i \in N} x_i^j = e(N) \cdot x^j$, it has x^j in its boundary. Furthermore, since $\xi(w_N^{-1} S) \leqq w_N^{-1}$, $x^j = w_N \xi(w_N^{-1} S) \leqq e(N)$; therefore, x^j is a Pareto optimal point in the set $H(e(N), \alpha^j) \cap (e(N) - \mathbb{R}_+^N)$. Take some $1 \leqq j' \leqq j$, and observe that

$$\begin{aligned} H(p^{j'}, p^{j'} \cdot x^j) &= \{x \in \mathbb{R}^N \mid p^{j'} \cdot x \leqq p^{j'} \cdot x^{j'}\} \\ &= \{x \in \mathbb{R}^N \mid e(N \setminus Q^{j'}) \cdot x \leqq e(N \setminus Q^{j'}) \cdot x^{j'}\} \\ &= \{x \in \mathbb{R}^N \mid \textstyle\sum_{i \in N \setminus Q^{j'}} x_i \leqq \sum_{i \in N \setminus Q^{j'}} x_i^{j'}\}. \end{aligned}$$

It is a $N \setminus Q^{j'}$-symmetric half-space through the point $x^{j'}$ that does not restrict the utilities of agents in $Q^{j'}$. Since $Q^j \subseteq Q^{j'}$, it thus also leaves the utilities of agents in Q^j free. Furthermore, since $x^{j'}_{N \setminus Q^{j'}} = x^j_{N \setminus Q^{j'}}$, it has x^j in its boundary. Thus, $\overline{S}^j$ is a problem in Σ^N that is $Q^j \setminus Q^{j-1}$-symmetric, $Q^{j-1} \setminus Q^{j-2}$-symmetric, etc., and furthermore, such that $x^j \in P(\overline{S}^j)$. The problem $\underline{S}^j$ (with $j \geqq 1$) is the intersection of $\overline{S}^j$ with a half-space that now puts a restriction on the utilities of agents in $Q^j \setminus Q^{j+1}$. The problems S^j and S'^j are clear. Further illustration of these auxiliary problems by means of an example, can be found in the Appendix.

The following claim says that these auxiliary problems are all normalized in the sense that their ideal points are equal to the unit vector. The argument is similar to Imai's, and thus relegated to the Appendix.

Claim 1 $u(\overline{S}^j) = u(\underline{S}^j) = u(S^j) = u(S'^j) = e(N)$ *for each* $j = 1, \ldots, k$.

Observe that $\varphi(\underline{S}^0) = x^0$. Thus, assume $\varphi(\underline{S}^{j-1}) = x^{j-1}$. The aim is to show that this implies $\varphi(\overline{S}^j) = x^j$. To this end it is useful to define the class of intermediate problems between $\underline{S}^{j-1}$ and $\overline{S}^j$:

Definition. For $\alpha \in [\alpha^{j-1}, \alpha^j]$, let $T^\alpha := \overline{S}^j \cap H(e(N), \alpha)$.

Note that for all $\alpha \in [\alpha^{j-1}, \alpha^j]$, T^α is Q^j-symmetric and $u(T^\alpha) = e(N)$.

Claim 2 *If* $j > 1$, *then for all* $\alpha \in [\alpha^{j-1}, \alpha^j]$, $(\varphi(T^\alpha))_{N \setminus Q^j} = x^j_{N \setminus Q^j}$.

Proof. Since $\varphi(\underline{S}^{j-1}) = x^{j-1}$ by assumption, and $x^{j-1}_{N \setminus Q^j} = x^j_{N \setminus Q^j}$, the claim is trivially true when $\alpha = \alpha^{j-1}$. Thus, take $\alpha \in (\alpha^{j-1}, \alpha^j]$. Since

$H(e(N), \alpha^{j-1}) \subset H(e(N), \alpha)$,

$$\underline{S}^{j-1} = T^{\alpha} \cap H(e(N), \alpha^{j-1}).$$

By part ii) of Lemma 3.5 this implies $\underline{S}^{j-1}_{-i} = T^{\alpha}_{-i}$ for all $i \in N$. Hence, by an $|N|$-fold application of IM, $\varphi(T^{\alpha}) \geqq x^{j-1}$. By the definition, if for some x, $x^{j-1} \leqq x \in \overline{S}^j$, then $x_i = x^{j-1}_i$ for all $i \in N \setminus Q^j$. Since $x^{j-1} \leqq \varphi(T^{\alpha}) \in \overline{S}^j$ and $x^{j-1}_{N\setminus Q^j} = x^j_{N\setminus Q^j}$, the claim follows. ■

Claim 3 *There exists an* $\bar{\alpha} \in (\alpha^{j-1}, \alpha^j]$ *such that*

- $(\varphi(T^{\alpha}))_{Q^j} < x^j_{Q^j}$ *for all* $\alpha \in [\alpha^{j-1}, \bar{\alpha})$;
- $(\varphi(T^{\alpha}))_{Q^j} \not< x^j_{Q^j}$ *for all* $\alpha \in [\bar{\alpha}, \alpha^j]$.

Proof. Define $h : [\alpha^{j-1}, \alpha^j] \to \mathbb{R}^{Q^j}$ as $h(\alpha) := (\varphi(T^{\alpha}))_{Q^j}$. By an $|N|$-fold application of IM, PO, and part ii) of Lemma 3.5, h is continuous and monotonically increasing. Furthermore, $h(\alpha^{j-1}) = x^{j-1}_{Q^j}$ and $h(\alpha^j) = (\varphi(\overline{S}^j))_{Q^j}$.

- Since $x^{j-1}_{Q^j} < x^j_{Q^j}$, continuity of h implies that there exists an $\alpha \in (\alpha^{j-1}, \alpha^j]$ such that $h(\alpha) < x^j_{Q^j}$.
- By PO, $\varphi(\overline{S}^j) \in P(\overline{S}^j)$. Then either $\varphi(\overline{S}^j) = x^j$, or there exists an $i \in N$ – and thus by Claim 2, an $i \in Q^j$ – such that $\varphi_i(\overline{S}^j) > x^j_i$. Hence, there exists an $\alpha \in (\alpha^{j-1}, \alpha^j]$ such that $h(\alpha) \not< x^j_{Q^j}$.

By continuity and monotonicity of h, these observations imply that $\bar{\alpha} := \sup\{\alpha \mid h(\alpha) < x^j_{Q^j}\}$ is well-defined, and $\bar{\alpha} \in (\alpha^{j-1}, \alpha^j]$. ■

If $|Q^j| = 1$, then Claim 2 is sufficient to conclude that $\varphi(\overline{S}^j) = x^j$, since φ satisfies PO. Thus, assume $|Q^j| \geqq 2$. Take $Q \subseteq Q^j$ with $|Q| = 2$, and without loss of generality, assume $Q = \{1, 2\}$. Furthermore, fix some $\alpha \in [\alpha^{j-1}, \bar{\alpha})$.

Claim 4 *Let* $y \in \mathbb{R}^Q$ *with* $y := (\varphi(T^{\alpha}))_Q$, *and define problems* $H, T \in \Sigma^Q$ *by*

$$H := \frac{1}{w_1 + w_2} m^{w_N}_Q(\underline{S}^0) \quad \textit{and} \quad T := \frac{1}{y_1 + y_2} m^{\varphi(T^{\alpha})}_Q(T^{\alpha}).$$

Then

$$H = \{x \in \mathbb{R}^Q \mid x_1 + x_2 \leqq 1\}, \quad \textit{and} \tag{3}$$

$$T = \{x \in \mathbb{R}^Q \mid x_1 + x_2 \leqq 1 \textit{ and } x \leqq (\beta, \beta)\}, \tag{4}$$

where $\beta := 1/(y_1 + y_2)$.

Proof. Since $\sum_{i\in N} w_i = 1$, $w_1 + w_2 = 1 - \sum_{i\in N\setminus Q} w_i$. Hence,

$$\begin{aligned} m_Q^{w_N}(\underline{S}^0) &= \{x \in \mathbb{R}^Q \mid x_1 + x_2 \leqq 1 - \textstyle\sum_{i\in N\setminus Q} w_i\} \\ &= \{x \in \mathbb{R}^Q \mid x_1 + x_2 \leqq w_1 + w_2\} \end{aligned}$$

This establishes (3). To prove (4), note first that

$$T^\alpha = H(e(N), \alpha) \cap \left(\bigcap_{j'=1}^{j} H(p^{j'},\ p^{j'} \cdot x^{j'}) \right) \cap (e(N) - \mathbb{R}^N_+)$$

Since $\varphi(T^\alpha) \in P(T^\alpha) \subset P(H(e(N), \alpha))$, $\sum_{i\in N} \varphi_i(T^\alpha) = \alpha$. Thus, $y_1 + y_2 = \alpha - \sum_{i\in N\setminus Q} \varphi_i(T^\alpha)$. Then similar to the above,

$$\begin{aligned} m_Q^{\varphi(T^\alpha)}(H(e(N), \alpha)) &= \{x \in \mathbb{R}^Q \mid x_1 + x_2 \leqq \alpha - \textstyle\sum_{i\in N\setminus Q} \varphi_i(T^\alpha)\} \\ &= \{x \in \mathbb{R}^Q \mid x_1 + x_2 \leqq y_1 + y_2\}. \end{aligned}$$

Furthermore,

$$\begin{aligned} m_Q^{\varphi(T^\alpha)}(e(N) - \mathbb{R}^N_+) &= \{x \in \mathbb{R}^Q \mid (x, (\varphi(T^\alpha))_{N\setminus Q}) \leqq e(N)\} \\ &= \{x \in \mathbb{R}^Q \mid x \leqq (1, 1)\}. \end{aligned}$$

Finally, the set $\bigcap_{j'=1}^{j} H(p^{j'},\ p^{j'} \cdot x^{j'})$ does not restrict the utilities of the agents in Q, and may thus be ignored. Combining these three observations, we obtain

$$m_Q^{\varphi(T^\alpha)}(T^\alpha) = \{x \in \mathbb{R}^Q \mid x_1 + x_2 \leqq y_1 + y_2 \text{ and } x \leqq (1, 1)\}.$$

This establishes (4). ■

Claim 5 *Outcomes y and x^j_Q are proportional.*

Proof. By SI and CCON,

$$\varphi(H) = \left(\frac{w_1}{w_1 + w_2}, \frac{w_2}{w_1 + w_2} \right) \quad \text{and} \quad \varphi(T) = \left(\frac{y_1}{y_1 + y_2}, \frac{y_2}{y_1 + y_2} \right).$$

Moreover, by the choice of α, $y < x^j_Q \leqq (1, 1)$, and thus $\varphi(T) < (\beta, \beta)$. Then by Claim 4 and Lemma 3.6, $\varphi(T) = \varphi(H)$. Since $x^j_{Q^j} = a^j w_{Q^j}$, this implies $y_1/y_2 = w_1/w_2 = x^j_1/x^j_2$, as desired. ■

Claim 6 $\varphi(\overline{S}^j) = x^j$.

Proof. We may repeat Claims 4 and 5 for all pairs of agents in Q^j. This leads to the conclusion that $(\varphi(T^\alpha))_{Q^j}$ and $x^j_{Q^j}$ are proportional. By continuity of h, this implies that there exists a $\gamma > 0$ such that

$$h(\bar{\alpha}) = \lim_{\alpha \to \bar{\alpha}} h(\alpha) = \gamma x^j_{Q^j}.$$

By definition of $\bar{\alpha}$, $\gamma x^j_{Q^j} \not< x^j_{Q^j}$. Hence, there must be an $i \in Q^j$ such that $\gamma x^j_i \geqq x^j_i$. Since $x^j_i > 0$, this implies $\gamma \geqq 1$. Thus, assume $\gamma > 1$. Since then $h(\bar{\alpha}) > x^j_{Q^j}$, it follows by continuity of h that there exists an $\alpha < \bar{\alpha}$ such that $h(\alpha) > x^j_{Q^j}$. This contradicts the definition of $\bar{\alpha}$. Thus, $\gamma = 1$.

Hence, $h(\bar{\alpha}) = x^j_{Q^j}$. Since $\alpha^j \geqq \bar{\alpha}$, this implies $h(\alpha^j) \geqq x^j_{Q^j}$. Then by Claim 2, $\varphi(\overline{S}^j) \geqq x^j$. Since $x^j \in P(\overline{S}^j)$, this implies $\varphi(\overline{S}^j) = x^j$. ■

The rest of the proof is similar to Proposition 2 of Imai, and thus relegated to the Appendix.

Remark. Without CCON, the axioms of Dubra (2001) (i.e., Theorem 2.3) do not characterize a family of single-valued solutions. They pin down the first iteration of the lexicographic optimization procedure, but have no bite when this procedure does not terminate in one step. For instance, they admit for any solution F^w, $w \in \mathbb{R}^{\mathbb{N}}_{++}$, of the following type: for $N \in \mathcal{N}$ and $S \in \Sigma^N$, F^w picks a unique outcome from the set $P(S) \cap \{x \mid x \geqq \alpha^* w_N u(S)\}$, where $\alpha^* = \max\{\alpha \in \mathbb{R} \mid \alpha w_N u(S) \in S\}$.

Further weakening RIIA to IIIA admits solutions, akin to lexicographic monotone path solutions (Chun and Peters, 1989). Let Λ be the class of all continuous, strictly increasing functions $\lambda : [0,1] \to \mathbb{R}^{\mathbb{N}}_{+}$ with $\lambda_i(0) = 0$ and $\lambda_i(1) = 1$ for all $i \in \mathbb{N}$. Then for $N \in \mathcal{N}$, $S \in \Sigma^N$ and $\lambda \in \Lambda$, a solution G^λ is admitted that picks a unique outcome from the set $P(S) \cap \{x \mid x \geqq (\lambda(\alpha^*))_N u(S)\}$, where $\alpha^* = \max\{\alpha \in [0,1] \mid (\lambda(\alpha))_N u(S) \in S\}$. Adding AN, we obtain the axiom set of Imai (i.e., Theorem 2.2), and thus the lexicographic KS solution L. ‖

4 Independence of the Axioms

The purpose of this Section is to investigate the independence of the axioms of Theorem 3.1.

4.1 Strong Individual Rationality

To show that SIR is not implied by the other axioms of Theorem 3.1, consider the following solution.

Definition. For $N \in \mathcal{N}$ and $x, y \in \mathbb{R}^N$, $x \succ^* y$ if and only if there is a $j \in N$ such that $x_j > y_j$ and $x_i = y_i$ for all $i \in N$ with $i < j$. Furthermore, $x \sim^* y$ if and only if $x = y$. For $S \in \Sigma^N$ define $D(S)$ as the outcome that picks the unique maximum from $S \cap \mathbb{R}^N_+$ with respect to the ordering $\succeq^*$, i.e., $D(S) := \{y \in S \cap \mathbb{R}^N_+ \mid y \succeq^* x \text{ for all } x \in S \cap \mathbb{R}^N_+\}$.

It is immediately clear that D violates SIR. For instance, if $N := \{1, 2\}$ and $S := cch\{e^1, e^2\}$, then $D(S) = (1, 0)$. We next show that D satisfies all other axioms of Theorem 3.1.

Observation 4.1 *D satisfies PO, SI, IM, IIIA, and CCON.*

Proof. Consider $N \in \mathcal{N}$ and $S \in \Sigma^N$, and define $y := D(S)$. To see that D satisfies PO, assume $y \notin P(S)$. Then there is an $i \in N$ and a $z \in S \cap \mathbb{R}^N_+$ such that $z_j = y_j$ for all $j \in N \setminus i$, and $z_i > y_i$. But then $z \succ^* y$, in contradiction with the definition of D.

To see that D satisfies IM, let $T \in \Sigma^N$ with $T \subseteq S$ and $T_{-i} = S_{-i}$ for some $i \in N$. By Lemma A.1, $Q(T, y) = Q(S, y)$ for any $y \in T$ with $i \in Q(T, y)$. Hence, $D_i(S) \geqq D_i(T)$.

To see that D satisfies IIIA, let $T \in \Sigma^N$ such that $y \in T \subseteq S$. Since $y \succeq^* x$ for all $x \in S \cap \mathbb{R}^N_+$ and $T \subseteq S$, also $y \succeq^* x$ for all $x \in T \cap \mathbb{R}^N_+$. Since $y \in T \cap \mathbb{R}^N_+$ and $y \succeq^* x$ for all $x \in T \cap \mathbb{R}^N_+$, $D(T) = y$. Hence, D satisfies IIA, and thus also IIIA.

To see that D satisfies CCON, we prove that D satisfies MCON. Assume $|N| \geqq 3$, and let $Q \in \mathcal{N}$ with $Q \subset N$. Without loss of generality, assume $Q := \{1, 2, \ldots, k\}$. Define $T := m^y_Q(S)$ and $z := D(T)$. Since $y \succeq^* (z, y_{N \setminus Q})$, we must have $y_1 \geqq z_1$. Suppose that $y_1 > z_1$. Since $y_Q \in T$, this would imply $y_Q \succ^* z$, contradicting $z = D(T)$. Hence, $z_1 = y_1$. Let $k' \in \{1, \ldots, k-1\}$, and suppose $z_{k''} = y_{k''}$ for all $k'' \leqq k'$. Then $y \succeq^* (z, y_{N \setminus Q})$ implies $y_{k'+1} \geqq z_{k'+1}$. However, if $y_{k'+1} > z_{k'+1}$, then $y_Q \succ^* z$, contradicting $z = D(T)$. Hence, $z = y_Q$, that is, D satisfies MCON.

Finally, to see that D satisfies SI, consider $v, w \in \mathbb{R}^N$, and assume that $v \succ^* w$. Then there is a $j \in N$ such that $v_j > w_j$, and $v_i = w_i$ for all $i \in N$ with $i < j$. Then for $a \in \mathbb{R}^N_{++}$, also $a_j v_j > a_j w_j$ and $a_i v_i = a_i w_i$ for $i \in N$ with

$i < j$. In other words, $av \succ^* aw$. Hence, for $S \in \Sigma^N$, a point v is maximal in $S \cap \mathbb{R}^N_+$ with respect to the ordering $\succeq^*$, if and only if av is maximal in $a(S \cap \mathbb{R}^N_+)$ with respect to $\succeq^*$. Since $a(S \cap \mathbb{R}^N_+) = aS \cap \mathbb{R}^N_+$, this implies $D(aS) = aD(S)$. Consider next an $a \in \mathbb{R}^N_+$, and let $Q \subset N$ be the set of agents $i \in N$ for whom $a_i > 0$. Since the utilities of agents in $N \setminus Q$ cannot be increased within $aS \cap \mathbb{R}^N_+$ from their zero level, $(D(aS))_{N\setminus Q} = a_{N\setminus Q} y_{N\setminus Q} = \bar{0}_{N\setminus Q}$. Define $T := m^y_Q(S)$, and observe that $D(T) = y_Q$ by MCON. Furthermore, observe that $aT = m^{ay}_Q(aS)$. Since $(D(aS))_{N\setminus Q} = (ay)_{N\setminus Q}$, this implies $aT = m^{D(aS)}_Q(aS)$, and thus, since D satisfies MCON, $D(aT) = (D(aS))_Q$. Then by the above, $(D(aS))_Q = D(aT) = aD(T) = ay_Q$. In conclusion, $D(aS) = aD(S)$. ■

4.2 Pareto Optimality

Roth (1979) showed that the Kalai-Smorodinsky solution K violates PO. On the other hand, it is immediate that K satisfies SIR and SI. K further satisfies IM, IIIA, and CCON.

Observation 4.2 *K satisfies IM, IIIA, and CCON.*

Proof. Let $N \in \mathcal{N}$ and $S \in \Sigma^N$, and let β^* be such that $K(S) = \beta^* u(S)$. To see that K satisfies IM, consider $T \subseteq S$ with $T_{-i} = S_{-i}$ for some $i \in N$, and define $\hat{\beta} := \max\{\beta \mid \beta u(T) \in S\}$. Since $u_j(T) = u_j(S)$ for all $j \in N \setminus i$ and $u_i(T) \leqq u_i(S)$, $\hat{\beta} \geqq \beta^*$, and thus $\hat{\beta} u_j(T) \geqq K_j(S)$ for all $j \in N \setminus i$. Since $\hat{\beta} u(T)$ and $K(S)$ are weakly Pareto optimal in S, $\hat{\beta} u_i(T) \leqq K_i(S)$. Finally, since $T \subseteq S$, $K(T) \leqq \hat{\beta} u(T)$. Hence, $K_i(T) \leqq K_i(S)$.

To see that K satisfies IIIA, consider $T \in \Sigma^N$ with $K(S) \in T \subseteq S$ and $u(T) = u(S)$. Then $K(T) = \beta u(S)$ for some β. If $\beta < \beta^*$, then $K(S) \notin T$, and if $\beta > \beta^*$, then $T \nsubseteq S$. Hence, $\beta = \beta^*$.

To see that K satisfies CCON, let $|N| \geqq 3$, and assume without loss of generality that $u(S) = e(N)$; then $K(S)$ is the maximal feasible point in egalitarian direction. Let $Q \subset N$ with $|Q| = 2$, let $T \in \Sigma^Q$ with $T := m^y_Q(S)$ and $y := K(S)$, and assume that $u(T) \propto (u(S))_Q$. Note that y_Q is the maximal point in T that lies in egalitarian direction. Since $u(T) \propto (u(S))_Q$, this implies $y_Q = K(T)$. ■

4.3 Scale Invariance

Consider the lexicographic egalitarian solution ξ, and note that by Lemma 3.2, it satisfies SIR, PO, IIIA, and CCON. By Lemma A.1 it further satisfies IM. To see that it violates SI, consider $N := \{1,2\}$ and the problems $S := cch\{e^1, e^2\}$ and $T := cch\{e^1, 2e^2\}$. Then $\xi(S) = (1/2, 1/2) \neq (2/3, 2/3) = \xi(T)$.

4.4 Individual Monotonicity

Consider the Nash solution, here denoted by φ^N. It is obvious that it satisfies SIR. Lensberg (1988) showed that it further satisfies PO, SI, IIIA, and CCON. To see that it violates IM, consider $N := \{1,2\}$ and $S, T \in \Sigma^N$ with $S := cch\{(1/2, 1), (1, 1/2)\}$ and $T := \{(1/2, 1), (1, 0)\}$. Then $T \subseteq S$ and $T_{-2} = S_{-2}$, but $\varphi_2^N(T) = 1 > 3/4 = \varphi_2^N(S)$.

4.5 Conditional Consistency

Consider the solution that for any $N \in \mathcal{N}$ is equal to $L^w \in \mathcal{L}$, with w such that the lowest-index agent in N has weight $|N|$, and all others weight 1. Since CCON is the only axiom that involves problem reduction it follows from Proposition 3.3 that this solution satisfies SIR, PO, SI, IM, and IIIA. To see that it violates CCON, consider $N := \{1,2,3\}$ and $Q := \{2,3\}$, and problems $S \in \Sigma^N$ and $T \in \Sigma^Q$ with $S := cch\{e^i \mid i \in N\}$ and $T := m_Q^y(S)$ where y is the solution outcome of S. Then $y = (3/5, 1/5, 1/5)$ and $T = cch\{(2/5, 0), (0, 2/5)\}$. Note that $u(T)$ is proportional to $(u(S))_Q$. However, the solution outcome of T is $(4/15, 2/15) \neq y_Q$.

4.6 IIA other than the Ideal Point

Whether IIIA is logically independent from the other axioms of Theorem 3.1 is an open question. Imai (1983, p. 392) described a solution that would satisfy SIR, PO, SI, IM, and AN, but not IIIA. In particular, this solution would obtain by updating agents' utilities in the direction of the utopia point until x^{k-1} is reached, and from there on updating utilities of the agents of Q^k in the direction of their *global* utopia values. While such a solution would indeed violate IIIA, it would also violate CCON.

In the rest of this section, we provide an alternative characterization of $\mathcal{L}$, that makes use of slightly weaker axioms than Theorem 3.1. We then demonstrate that for these weaker axioms, logical independence is unproblematic.

4.6.1 An Alternative Characterization of $\mathcal{L}$

Imai (1983) introduced the following axiom:

Combined Individual Monotonicity (CIM). For $N \in \mathcal{N}$ and $S, T \in \Sigma^N$ with $T \subseteq S$ and $S_{-i} = T_{-i}$ for all $i \in N$, $\varphi(S) \geqq \varphi(T)$.

CIM is directly implied by IM, and its interpretation is analogous: no individual agent benefits from a contraction of the feasible set, that leaves unaltered the maximally attainable alternatives of all agents.

We further introduce the following weaker version of CCON.[13]

Individually Rational CCON (iCCON). If $\varphi(S) \geqq \bar{0}$ and $S = cch\,(S \cap \mathbb{R}^N_+)$, then for all $T \in \Sigma^Q$ with $T := m_Q^{\varphi(S)}(S)$ and $u(T) \propto (u(S))_Q$, $\varphi(T) = (\varphi(S))_Q$.

Compared to CCON, the premise of iCCON includes the additional requirement that $S = cch\,(S \cap \mathbb{R}^N_+)$. Note that Lensberg's (1988) version of BCON was only defined for individually rational bargaining problems. Under this domain restriction, the condition $S = cch\,(S \cap \mathbb{R}^N_+)$ – as well as the condition that $\varphi(S) \geqq \bar{0}$ – is trivially satisfied.[14] In other words, on the domain of individually rational problems, CCON and iCCON coincide. We obtain the following characterization result.

Theorem 4.3 *$\varphi \in \mathcal{L}$ iff it satisfies SIR, PO, SI, CIM, IIIA and iCCON.*

Proof. Note first that CCON implies iCCON, and that IM implies CIM. Careful examination of the proof of Proposition 3.4 reveals that the argument also holds if CIM is imposed, rather than IM. In view of Theorem 3.1, it is thus sufficient to show that SIR, PO, SI, CIM, IIIA, and iCCON together imply CCON.

Let φ be a solution that satisfies SIR, PO, SI, CIM, IIIA, and iCCON. Let $N \in \mathcal{N}$ with $|N| \geqq 3$ and $Q \subset N$ with $|Q| = 2$, let $S \in \Sigma^N$, and let $T \in \Sigma^Q$ with $T := m_Q^{\varphi(S)}(S)$ and $u(T) \propto (u(S))_Q$. To show is that $\varphi(T) = (\varphi(S))_Q$. To this end, define

$$S' := cch\,(S \cap \mathbb{R}^N_+) \quad \text{and} \quad T' := m_Q^{\varphi(S')}(S').$$

By SIR and IIIA, $\varphi(S') = \varphi(S) =: y$. Using this, we now demonstrate that $T' = cch\,(T \cap \mathbb{R}^Q_+)$. Fix some $x_Q \in \mathbb{R}^Q$.

[13] For all $Q, N \in \mathcal{N}$ with $Q \subset N$ and $|Q| = 2$, and for all $S \in \Sigma^N$.

[14] It is not known in general whether Imai's (1983) lexicographic Kalai-Smorodinsky solution – and by extension, the weighted generalizations considered in this paper – can be characterized on this smaller domain.

1. Suppose $x_Q \in cch\,(T \cap \mathbb{R}^Q_+)$. Then there are $z_Q \in (T \cap \mathbb{R}^Q_+)$ such that $z_Q \geqq x_Q$. Then $(z_Q, y_{N\setminus Q}) \geqq (x_Q, y_{N\setminus Q})$. Hence, there are $z \in (S \cap \mathbb{R}^N_+)$ such that $z \geqq (x_Q, y_{N\setminus Q})$. That is, $(x_Q, y_{N\setminus Q}) \in cch\,(S \cap \mathbb{R}^N_+) = S'$.

2. Suppose $(x_Q, y_{N\setminus Q}) \in S'$. Then there are $z \in (S \cap \mathbb{R}^N_+)$ such that $(z_Q, z_{N\setminus Q}) \geqq (x_Q, y_{N\setminus Q})$. Take such a z, and construct $z' = (z_Q, y_{N\setminus Q}) \in (S \cap \mathbb{R}^N_+)$. By $z_{N\setminus Q} \geqq y_{N\setminus Q}$ and comprehensiveness it follows that $z' \geqq (x_Q, y_{N\setminus Q})$. Thus there are $z_Q \in (T \cap \mathbb{R}^Q_+)$ such that $z_Q \geqq x_Q$. In other words, $x_Q \in cch\,(T \cap \mathbb{R}^Q_+)$.

Hence, $(x_Q, y_{N\setminus Q}) \in S'$ if and only if $x_Q \in cch\,(T \cap \mathbb{R}^Q_+)$. Then

$$T' = \{x \in \mathbb{R}^Q \mid (x, y_{N\setminus Q}) \in S'\} = cch\,(T \cap \mathbb{R}^Q_+).$$

Since $u(T) = u(T')$, it follows by SIR and IIIA that $\varphi(T) = \varphi(T')$. Furthermore, it implies $u(T') \propto (u(S'))_Q$, and thus by iCCON, $\varphi(T') = (\varphi(S'))_Q$. Since $\varphi(S') = \varphi(S)$, this implies $\varphi(T) = (\varphi(S))_Q$, as desired. ■

4.6.2 Independence of the Axioms of Theorem 4.3

Since iCCON and CIM are weaker than CCON and IM, the same examples given above demonstrate that none of the axioms SIR, PO, SI, CIM, or iCCON is implied by the other axioms of Theorem 4.3. We next show a similar result for IIIA. To this end, consider the following solution.

Definition. For $N \in \mathcal{N}$ and $i \in N$, define $u^i(S) := u(S)e^i$. For $S \in \Sigma^N$, define $G(S) := L^w(S)$, where

$$w_i = \begin{cases} 2 & \text{if } |N| \geqq 2 \text{ and } S_{-i} = cch\{u^j(S) \mid j \in N \setminus i\}, \\ 1 & \text{otherwise} \end{cases}$$

for $i \in N$.

Observation 4.4 *G does not satisfy IIIA.*

Proof. Let $N := \{1, 2, 3\}$, and consider $S := H(e(N), 3/2) \cap (e(N) - \mathbb{R}^N_+)$ and $T := \{x \in S \mid x_1 + x_2 \leqq 1\}$. Since there is no $i \in N$ such that $S_{-i} = cch\,\{u^j(S) \mid j \in N \setminus i\}$, $G(S) = L(S)$. Then $G(S) \in T \subseteq S$ and $u(T) = u(S)$, i.e. the premise of IIIA is satisfied. However, since $T_{-3} = cch\,\{u^1(T), u^2(T)\}$, $G(T) = L^w(T)$ with $w_N = (1, 1, 2)$. Hence, $G(T) \neq G(S)$. ■

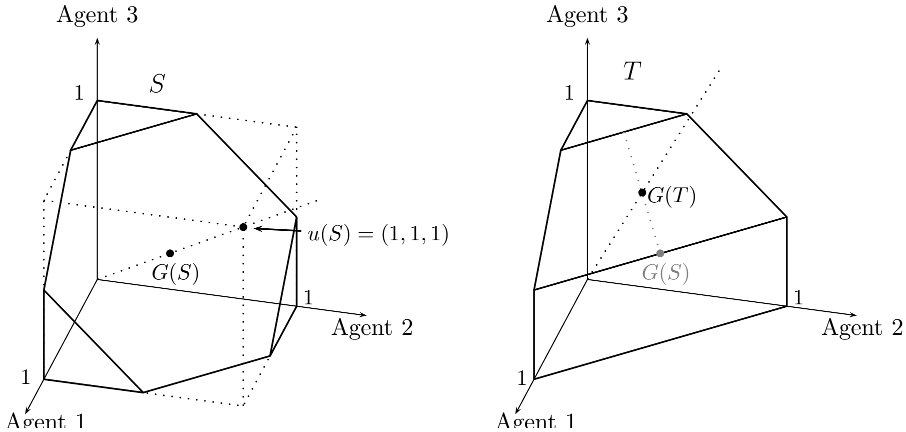


Figure 10: G violates IIIA.

Proposition 4.5 *G satisfies SIR, PO, SI, CIM and iCCON.*

Proof. It is immediate that G satisfies SIR, PO and SI. Let $N \in \mathcal{N}$ with $|N| \geqq 2$ and $S \in \Sigma^N$ be given. To see that G satisfies CIM, consider $T \in \Sigma^N$ with $T \subseteq S$ and $T_{-i} = S_{-i}$ for all $i \in N$. To show is that $G(S) \geqq G(T)$. Since $T_{-i} = S_{-i}$ for all $i \in N$, the same weights vector w (up to its restriction to N) is used in S and T. Since L^w satisfies CIM, $G(S) = L^w(S) \geqq L^w(T) = G(T)$.

To see that G satisfies iCCON, assume that $S = cch\,(S \cap \mathbb{R}^N_+)$. If $|N| = 2$, then the condition $S_{-i} = cch\,\{u^j(S) \mid j \in N \setminus \{i\}\}$ is trivially satisfied for both $i \in N$, such that G coincides with L. Suppose $|N| > 2$, let $Q \subset N$ with $|Q| = 2$, and consider $T \in \Sigma^Q$ with $T := m^y_Q(S)$ and $y := G(S)$. Without loss of generality, assume $Q := \{1, 2\}$. We distinguish between two cases.

- $G(S) = L^w(S)$ where $w_1 = w_2$:

  Note that L^w satisfies iCCON. Hence, if $u(T) \propto (u(S))_Q$, then $L^w(T) = (L^w(S))_Q = y_Q$. Since $w_1 = w_2$, $L^w(T) = L(T)$, and thus, since G coincides with L in two-person problems, $L^w(T) = G(T)$. Hence, $G(T) = y_Q$.

- $G(S) = L^w(S)$ where $w_1 \neq w_2$:

  Since G satisfies SI, we may assume without loss of generality that $u(S) = e(N)$. Define $\gamma := 1 - \sum_{k \in N \setminus Q} y_k$. Since $y \in S$ it follows by convexity and comprehensiveness of S that $(\gamma, 0, y_{N\setminus Q})$ and $(0, \gamma, y_{N\setminus Q})$ are both in S. This implies $u_i(T) \geqq \gamma$ for $i \in Q$.

Assume without loss of generality that $w_1 = 1$ and $w_2 = 2$. The latter means that $S_{-2} = cch\,\{e^j \mid j \in N \setminus 2\}$. But then for all $(x_Q, y_{N\setminus Q}) \in S$, $x_1 \leqq \gamma$. Hence, $u_1(T) = \gamma$.

That $w_1 = 1$ implies $cch\,\{e^j \mid j \in N \setminus 1\} \subset S_{-1}$. By convexity of S it follows that for any v in $cch\,\{e^j \mid j \in N \setminus 1\} \cap \mathbb{R}^{N\setminus 1}_{++}$ there is an $\epsilon > 0$ such that $v + \epsilon e(N \setminus 1) \in S_{-1}$. This implies $u_2(T) > \gamma$.

That $u_2(T) > u_1(T)$ implies $u(T) \not\propto e(Q) = (u(S))_Q$, meaning iCCON is vacuously satisfied. ■

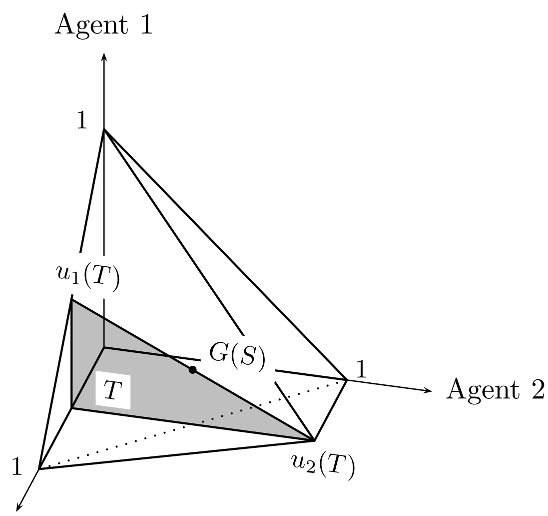


Figure 11: If $w_1 \neq w_2$, the premise of iCCON is violated.

5 Concluding Remarks

The framework in this article assumed an infinite population of agents. All results continue to hold in a finite-population environment, provided that this population counts at least three agents. If there are only two agents in the population, then there exist other solutions that satisfy the properties of Theorem 3.1. Dubra (2001) defines such a solution: For $N := \{1, 2\}$ and $S \in \Sigma^N$,

$$F(S) := \{x \in P(S) \mid x \geqq (\beta^*, u_2(S)/2)\},$$

where $\beta^* := \max\{\beta \mid (\beta, u_2(S)/2) \in S\}$.

We further made the assumption that problems are comprehensive. Other than making the domain closed under the operation of problem reduction, this restriction does not play any role in our result. To see this, consider solutions φ defined on the domain that extends Σ to the non-comprehensive problems. It is easily verified that any such solution, that further satisfies IIIA, yields the same outcome on a problem S as it does on the convex comprehensive hull of S.

Finally, since a lexicographic version of the proportional solutions (Kalai, 1977b) would satisfy both BCON and IIA, the characterization of $\mathcal{L}$ presented in this article, can be extended to this solution class. Note that such solutions would violate SI, a property used in the proofs of Lemma 3.6 and Claim 5; however, in both instances, SI may be replaced by HOM.

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A Appendix

This Appendix elaborates on several results of Imai (1983). In particular, Section A.1 corresponds with Imai's Lemmas 6 and 7, and part of his Proposition 1 (p. 396–397). Section A.2 is a visual illustration of Imai's auxiliary problems. Section A.3 is a modification of Imai's Lemma 5, and part of his Proposition 2 (p. 396, 398). Section A.4 repeats the final part of that same Proposition 2.

A.1 Solutions $\varphi \in \mathcal{L}$ satisfy IM

Consider some solution $L^w \in \mathcal{L}$ with $w \in \mathbb{R}^{\mathbb{N}}_{++}$. Let $N \in \mathcal{N}$, and let S and T be problems in Σ^N with $T \subseteq S$ and $T_{-i} = S_{-i}$ for some $i \in N$. Since L^w satisfies SI, we may assume without loss of generality that $u(S) = w_N^{-1}$, where w_N is the restriction of w to the agents in N. Then

$$L^w(S) = \xi(S). \tag{5}$$

Define $u := u(S)u(T)^{-1}$, and observe that

$$L^w(T) = u^{-1}\xi(uT). \tag{6}$$

By Equations (5) and (6) it is sufficient to show that $\xi_i(S) \geqq \frac{1}{u_i}\xi_i(uT)$. This is done in two steps. First it is established that $\xi_i(S) \geqq \xi_i(T)$, and subsequently, that $\xi_i(T) \geqq \frac{1}{u_i}\xi_i(uT)$.

Lemma A.1 *Let $y \in T$ with $i \in Q(T, y)$. Then $Q(T, y) = Q(S, y)$.*

Proof. Suppose there is some $i' \in Q(S, y) \setminus Q(T, y)$. Define $x := x(S, y)$ and $x' := x(T, y)$. Since $S_{-i} = T_{-i}$ and $x \in S$, there is a $z \in T$ such that $z_j = x_j$ for all $j \in N \setminus \{i\}$. By convexity of T it follows that $\lambda x' + (1 - \lambda)z \in T$ for all $\lambda \in [0, 1]$. We have $x, x' \geqq y$, $z_j = x_j$ for all $j \in N \setminus \{i\}$, and $x'_i > y_i$ (since $i \in Q(T, y)$). Hence, there exists a $\lambda \in (0, 1)$ such that $z^* := \lambda x' + (1 - \lambda)z \geqq y$ and $z^* \in T$. Since $i' \in Q(S, y) \setminus Q(T, y)$, $x_{i'} > y_{i'} = x'_{i'}$. Then $z^*_{i'} = \lambda x'_{i'} + (1 - \lambda)x_{i'} > y_{i'}$. This implies $i' \in Q(T, y)$, a contradiction. ■

In essence, as long as the i-th coordinate can be further increased in problem T, it can be further increased in S as well. It follows that $\xi_i(S) \geqq \xi_i(T)$.

To show that $\xi_i(T) \geqq \frac{1}{u_i}\xi_i(uT)$, define the following:

$$x := \xi(T) \qquad y := u\xi(T)$$
$$x' := u^{-1}\xi(uT) \quad y' := \xi(uT)$$

The aim is to show that $x_i \geqq x'_i$.

Observation A.2 *$x \succ_N x'$ and $y' \succ_N y$ (or $x = x'$ and $y = y'$).*

Proof. Observe that $\xi(T) \in T$, implying $u\xi(T) \in uT$. Since $\xi(uT) \succ_N z$ for all $z \in uT$ with $z \neq \xi(uT)$, $\xi(uT) \succ_N u\xi(T)$ (or $\xi(uT) = u\xi(T)$). In other words, $y' \succ_N y$ (or $y' = y$). Similarly, observe that $\xi(uT) \in uT$, and thus $u^{-1}\xi(uT) \in T$. Since $\xi(T) \succ_N z$ for all $z \in T$ with $z \neq \xi(T)$, this implies $\xi(T) \succ_N u^{-1}\xi(uT)$ (or $\xi(T) = u^{-1}\xi(uT)$). In other words, $x \succ_N x'$ (or $x = x'$). Since $x = x'$ if and only if $y = y'$, the observation follows. ■

Lemma A.3 *$x_i \geqq x'_i$.*

Proof. Clearly, if $x = x'$ and $y = y'$, then the inequality holds trivially, and we are done. Hence, assume $x \succ_N x'$ and $y' \succ_N y$. Since $y' \succ_N y$, there is an index $m \leqq |N|$ such that the first $m-1$ elements of $\mu(y)$ and $\mu(y')$ are equal, and $\mu_m(y') > \mu_m(y)$. Assume first that $x'_i \geqq \mu_m(y')$. Since x' and y' only differ in the i-th coordinate, $x'_i \geqq \mu_m(y')$ implies that the first m elements of $\mu(x')$ and $\mu(y')$ coincide. Then the first $m-1$ elements of $\mu(x')$ coincide with the first $m-1$ elements of $\mu(y)$. Observe that x and y only differ in the i-th coordinate and $y_i > x_i$, so there are two possibilities:

1. there is some $k \leqq m-1$ such that the first $k-1$ elements of $\mu(y)$ and $\mu(x)$ coincide, and $\mu_k(y) > \mu_k(x)$. Then the first $k-1$ elements of $\mu(x')$ and $\mu(x)$ coincide and $\mu_k(x') > \mu_k(x)$. Then $x' \succ_N x$, a contradiction.

2. the first m elements of $\mu(y)$ and $\mu(x)$ coincide. Since the first $m-1$ elements of $\mu(x')$ and $\mu(y)$ coincide, this implies that the first $m-1$ elements of $\mu(x)$ and $\mu(x')$ coincide. However, the m-th element of $\mu(x')$ coincides with the m-th element of $\mu(y')$. Since $\mu_m(y') > \mu_m(y)$ this implies $\mu_m(x') > \mu_m(y)$. Since $\mu_m(x) = \mu_m(y)$ this implies $\mu_m(x') > \mu_m(x)$, and thus $x' \succ_N x$. This is a contradiction.

It follows from the above that $x'_i < \mu_m(y')$. Hence, $x'_i = \mu_{m'}(x')$ for some $m' \in \{1, \dots, m-1\}$, and take the lowest m' in case of ties. Similarly, $x_i = \mu_k(x)$

for some $k \in \{1, \ldots, n\}$, and take the highest possible k in case of ties. There are two possibilities: $k \geqq m'$ or $k < m'$.

1. If $k < m'$, then $\mu_k(x') = \mu_k(y')$ (they only differ in the i-th coordinate; since $\mu_{m'}(x') = x_i$, $k < m'$ implies that $\mu_{k'}(x') = \mu_{k'}(y')$ for all $k' = 1, \ldots, k$). Since the first $m-1$ elements of $\mu(y)$ and $\mu(y')$ coincide, and $k < m' \leqq m-1$, the first k elements of $\mu(y)$ and $\mu(y')$ coincide. Since x and y only differ in the i-th coordinate and $x_i = \mu_k(x)$, the first $k-1$ elements of $\mu(x)$ and $\mu(y)$ coincide. This in turn implies that the first $k-1$ elements of $\mu(x)$ and $\mu(x')$ coincide. Since $y_i > x_i$ and by the choice of k (it was chosen such that $\mu_{k+1}(x) > \mu_k(x)$), $\mu_k(y) > \mu_k(x)$. Since the first k elements of $\mu(y)$ coincide with the first k elements of $\mu(x')$, this implies $\mu_k(x') > \mu_k(x)$. Hence, $x' \succ_N x$, a contradiction.

2. Let $k \geqq m'$. Observe that the first $m'-1$ elements of $\mu(x')$ and $\mu(y')$ coincide (this is so because they only differ in the i-th coordinate and $\mu_{m'}(x') = x'_i$). From before, we know that the first $m-1$ elements of $\mu(y)$ and $\mu(y')$ coincide, which implies that their first m' elements coincide as well. Hence, the first $m'-1$ elements of $\mu(x')$ coincide with the first $m'-1$ elements of $\mu(y)$. Since x and y only differ in the i-th coordinate and $x_i = \mu_k(x)$, the first m' elements of $\mu(x)$ coincide with the first m' elements of $\mu(y)$. Hence, the first $m'-1$ elements of $\mu(x)$ and $\mu(x')$ coincide. If $\mu_{m'}(x') > \mu_{m'}(x)$, then $x' \succ_N x$, and we obtain a contradiction. Hence, $\mu_{m'}(x') \leqq \mu_{m'}(x)$. But then $x'_i = \mu_{m'}(x') \leqq \mu_{m'}(x) \leqq \mu_k(x) = x_i$, as desired. ■

A.2 The Auxiliary Problems of Proposition 3.4

This section presents a worked out example of the auxiliary problems used in the proof of Proposition 3.4, for some weights vector w. Suppose that the problem and the solution outcome of a problem S are as in Figure 12. Then $L^w(S)$ is reached in two iterations, i.e., $L^w(S) = x^2$.

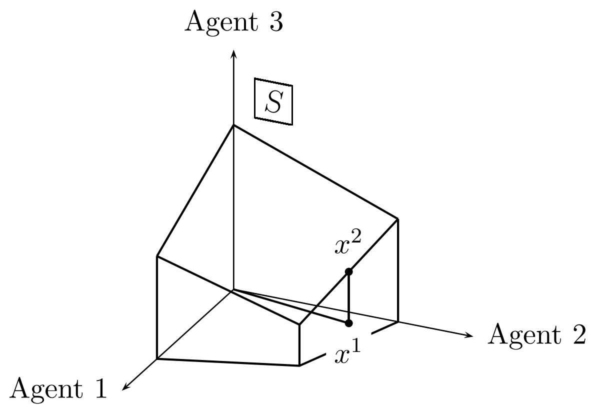
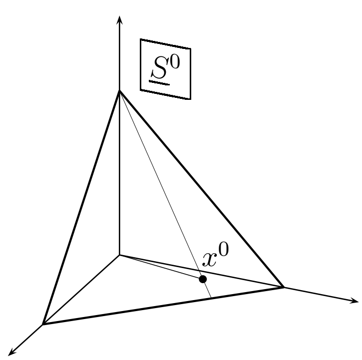
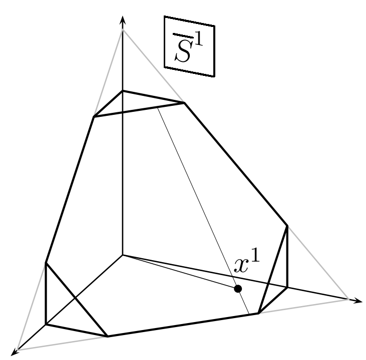


Figure 12: The problem S and the solution outcome $L^w(S) = x^2$.

Recall that $\underline{S}^0 = H(e(N), 1)$. This problem is depicted in Figure 13(a). From $\underline{S}^0$ one can construct the problem $\overline{S}^1$: it is given by $H(e(N), \alpha^1) \cap (e(N) - \mathbb{R}^N_+)$. Thus, the half-space that determines $\underline{S}^0$ slides upwards, and is intersected by a set that limits utilities to 1.



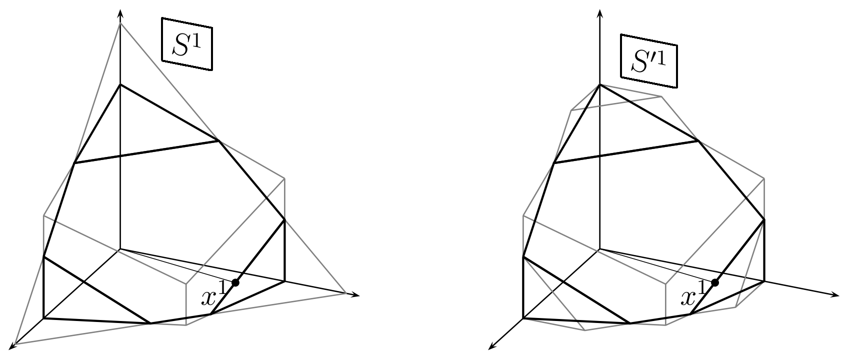
(a) The problem $\underline{S}^0$



(b) The problem $\overline{S}^1$

Figure 13: The problems $\underline{S}^0$ and $\overline{S}^1$.

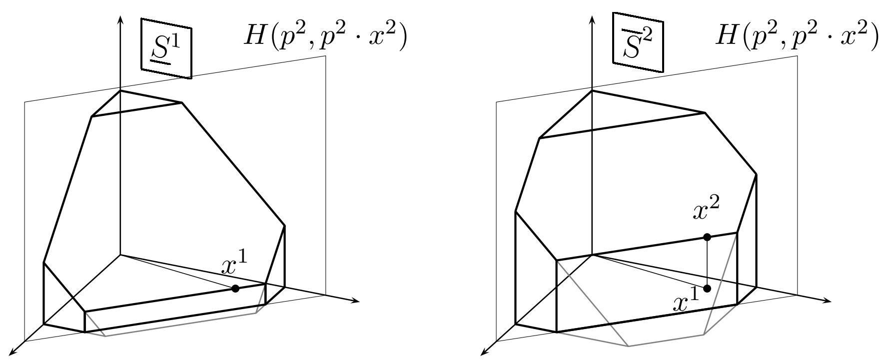
The problem S^1 is the intersection of S with the half-space $H(e(N), \alpha^1)$. Similarly, S'^1 is the intersection of the problem S and the problem $\overline{S}^1$, depicted in Figure 13(b). Note that these two problems coincide. For the first iteration this is always the case.



(a) The problem S^1 (b) The problem S'^1

Figure 14: The problems S^1 and S'^1.

To determine the auxiliary problems for the second iteration, one must determine $\underline{S}^1$. It is equal to $\overline{S}^1$, intersected by $H(p^2, p^2 \cdot x^2)$, a half-space that leaves the utilities of agent 3 free, but restricts those of agents 1 and 2 in a $\{1, 2\}$-symmetric fashion. The problem $\overline{S}^2$ is given by $H(e(N), \alpha^2)$, intersected by that same half-space $H(p^2, p^2 \cdot x^2)$, and the set $(e(N) - \mathbb{R}^N_+)$ that limits the utilities of all agents to 1.

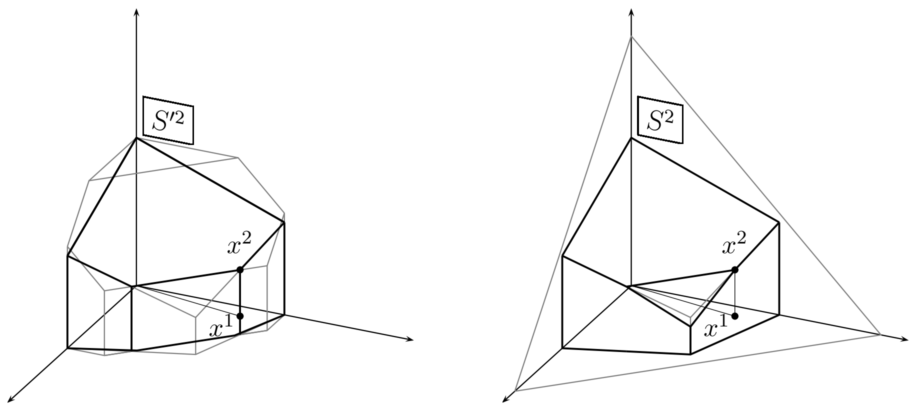


(a) The problem $\underline{S}^1$ (b) The problem $\overline{S}^2$

Figure 15: The problems $\underline{S}^1$ and $\overline{S}^2$.

The problem S'^2 is the intersection of $\overline{S}^2$, as depicted in Figure 15(b), and the original problem S. The end result is depicted in Figure 16(a). The problem

S^2 is given by the intersection of the original problem S and the half-space $H(e(N), \alpha^2)$.



(a) The problem S'^2 (b) The problem S^2

Figure 16: The problems S'^2 and S^2.

A.3 Proof of Claim 1

In order to show that $u(\overline{S}^j) = u(\underline{S}^j) = u(S^j) = u(S'^j) = e(N)$ for all $j = 1, \ldots, k$, it is sufficient to show that e^i is in $\overline{S}^j, \underline{S}^j, S^j$ and S'^j for each j and i. This follows from four observations.

(a) $e^i \in H(p^j,\ p^j \cdot x^j)$ for each i and j;

(b) $e^i \in H(e(N), \alpha^j)$ for each i and j;

(c) $e^i \in S$ for all i;

(d) $e^i \in (e(N) - \mathbb{R}^N_+)$ for all i.

Note that (a), (b) and (d) together imply $e^i \in \overline{S}^j$ for each i and j. Then by (c), $e^i \in S'^j$ for each i and j; for $j < k$, it is implied by (a) that each e^i is in $\underline{S}^j$. Finally, (b) and (c) together imply $e^i \in S^j$ for each i and j.

Observation (d) is trivial. Observation (c) follows from comprehensiveness of S and the assumption that $u(S) = e(N)$. We now show (a) and (b). Denote $N \setminus Q^2$ by Q, Q^2 by Q', and for $i \in Q$, denote $Q' \cup \{i\}$ by Q_i. Let $\bar{w} := (\bar{w}_i)_{i \in N}$ where $\bar{w}_i := w_i / \sum_{i' \in Q} w_{i'}$ for all $i \in N$. Note that $\sum_{i \in Q} \bar{w}_i = 1$.

By the supporting hyperplane theorem and the definition of x^1, there is a $p \in \mathbb{R}^N_+$ with $p_i = 0$ for all $i \in Q'$, such that $p \cdot z \leqq p \cdot x^1$ for all $z \in S$. By observation (c), $p\cdot e^i \leqq p\cdot x^1$ for all $i \in Q$. Since $p_i = 0$ for all $i \in Q'$, this implies $p \cdot e(Q_i) = p \cdot e^i \leqq p \cdot x^1$ for all $i \in Q$. It follows that $p \cdot \sum_{i\in Q} \bar{w}_i e(Q_i) \leqq p \cdot x^1$. Note that $\sum_{i\in Q} \bar{w}_i e(Q_i) = \bar{w} + (e(Q') - \bar{w}e(Q'))$. Since $p_i = 0$ for all $i \in Q'$, we obtain $p\cdot\bar{w} \leqq p\cdot x^1$. Note that $x^1 = \alpha^1 w_N = [\alpha^1 \sum_{i\in Q} w_i]\bar{w}$. Then $p\cdot\bar{w} \leqq p\cdot x^1$ is equivalent to $\alpha^1 \sum_{i\in Q} w_i \geqq 1$. Hence, $x^1 = \alpha^1 w_N \geqq \frac{w_N}{\sum_{i\in Q} w_i} = \bar{w}$.

Since $p^1 \cdot e^i = 0$ for all $i \in N$, each e^i is trivially in $H(p^1,\ p^1 \cdot x^1)$. Consider some $j \in \{2, \dots, k\}$. Note that $p^j_i = 0$ for all $i \in Q^j$. Hence,

$$p^j \cdot \bar{w} = \sum_{i\in N\setminus Q^j} \bar{w}_i = \frac{\sum_{i\in N\setminus Q^j} w_i}{\sum_{i\in N\setminus Q^2} w_i} \geqq \frac{\sum_{i\in N\setminus Q^2} w_i}{\sum_{i\in N\setminus Q^2} w_i} = 1.$$

The inequality follows from the observation that $N \setminus Q^2$ is a subset of $N \setminus Q^j$. By the above, $x^j \geqq x^1 \geqq \bar{w}$. Hence, $p^j\cdot x^j \geqq p^j\cdot x^1 \geqq p^j\cdot\bar{w} \geqq 1$. Since $p^j\cdot e^i \leqq 1$ for all $i \in N$, we obtain $p^j\cdot x^j \geqq p^j\cdot e^i$ for all $i \in N$. This establishes observation (a).

Since $x^1 \geqq \bar{w}$ and $\sum_{i\in N} \bar{w}_i \geqq \sum_{i\in Q} \bar{w}_i = 1$, $e(N)\cdot e^i = 1 \leqq e(N)\cdot\bar{w} \leqq e(N)\cdot x^1 = \alpha^1$ for all $i \in N$. Hence, $e^i \in H(e(N), \alpha^1)$ for all $i \in N$. Since $H(e(N), \alpha^1) \subset H(e(N), \alpha^j)$ for each $1 < j \leqq k$, observation (b) follows. ■

A.4 Proof of Proposition 3.4 (Continued)

The proof of Proposition 3.4 is concluded by the following three claims.

Claim 7 $\varphi(\overline{S}^1) = \varphi(\underline{S}^1) = \varphi(S^1) = \varphi(S'^1) = x^1$.

Proof. Since $\varphi(\underline{S}^0) = x^0$, $\varphi(\overline{S}^1) = x^1$ by Claim 6. Since $x^1 \in \overline{S}^1$ and $x^1 \in S$, $x^1 \in S'^1$. Then $x^1 = \varphi(\overline{S}^1) \in S'^1 \subseteq \overline{S}^1$, and by Claim 1, $u(S'^1) = u(\overline{S}^1)$. Thus by IIIA, $\varphi(S'^1) = \varphi(\overline{S}^1) = x^1$. For all $i \in N \setminus Q^2$, $x^1_i = x^2_i$, implying $p^2 \cdot x^1 = p^2 \cdot x^2$. Hence, $x^1 \in H(p^2, p^2 \cdot x^2)$. Then $x^1 = \varphi(\overline{S}^1) \in \underline{S}^1 \subseteq \overline{S}^1$; by Claim 1, $u(\underline{S}^1) = u(\overline{S}^1)$. Then by IIIA, $\varphi(\underline{S}^1) = \varphi(\overline{S}^1) = x^1$. To see that $\varphi(S^1) = x^1$, observe first that $S'^1 = S^1 \cap (e(N) - \mathbb{R}^N_+)$. Since $\varphi(S^1) \leqq u(S^1) = e(N)$, $\varphi(S^1) \in (e(N) - \mathbb{R}^N_+)$. Hence, $\varphi(S^1) \in S'^1 \subseteq S^1$, and by Claim 1, $u(S'^1) = u(S^1)$. Thus, $\varphi(S^1) = \varphi(S'^1) = x^1$ by IIIA. ■

Claim 8 $\varphi(\overline{S}^j) = \varphi(\underline{S}^j) = \varphi(S^j) = \varphi(S'^j) = x^j$ *for each* $j = 1, \dots, k$ *(or* $j = 1, \dots, k-1$ *for* $\varphi(\underline{S}^j)$*).*

Proof. Consider $j \in \{2, \dots, k\}$, and assume $\varphi(\overline{S}^{j-1}) = \varphi(\underline{S}^{j-1}) = \varphi(S^{j-1}) = \varphi(S'^{j-1}) = x^{j-1}$. By Claim 6, $\varphi(\overline{S}^j) = x^j$. Then $\varphi(S'^j) = x^j$ follows as in Claim

7. Furthermore, if $j < k$, then also $\varphi(\underline{S}^j) = x^j$ follows as in Claim 7. What is left to show is that $\varphi(S^j) = x^j$. To this end, it is first argued that $\varphi(S^j)$ is an element of $\overline{S}^j$.

1. Since $S^j \subseteq H(e(N), \alpha^j)$, $\varphi(S^j) \in H(e(N), \alpha^j)$.
2. Since $\varphi(S^j) \leqq u(S^j) = e(N)$, $\varphi(S^j) \in (e(N) - \mathbb{R}^N_+)$.
3. By part ii) of Lemma 3.5, $S^{j-1}_{-i} = S^j_{-i}$ for all $i \in N$. Furthermore, $S^{j-1} \subseteq S^j$. Thus by an $|N|$-fold application of IM, $\varphi(S^j) \geqq \varphi(S^{j-1}) = x^{j-1}$. As in Claim 3, this implies $\varphi_i(S^j) = x^j_i$ for all $i \in N \setminus Q^j$. From this it follows that for all $j' \in \{1, \ldots, j\}$, $\varphi(S^j) \in H(p^{j'}, p^{j'} \cdot x^{j'})$. Thus, $\varphi(S^j) \in \bigcap_{j'=1}^{j} H(p^{j'}, p^{j'} \cdot x^{j'})$.

It follows that $\varphi(S^j) \in \overline{S}^j$. Since $S^j \subseteq S$, we further have $\varphi(S^j) \in S$. Thus, $\varphi(S^j) \in S'^j \subseteq S^j$, and by Claim 1, $u(S^j) = u(S'^j)$. Then by IIIA, $\varphi(S^j) = \varphi(S'^j) = x^j$. ■

Claim 9 *$\varphi(S) = x^k$.*

Proof. Since $S^k = S \cap H(e(N), \alpha^k)$, $\varphi(S) \geqq \varphi(S^k)$ by an $|N|$-fold application of IM and part ii) of Lemma 3.5. Then by Claim 8, $\varphi(S) \geqq x^k$. The claim follows from the observation that $x^k \in P(S)$. ■