ZAPPA-SZÉP PRODUCT GROUPOIDS AND C^*-BLENDS

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Abstract. We study the external and internal Zappa-Szép product of topological groupoids. We show that under natural continuity assumptions the Zappa-Szép product groupoid is étale if and only if the individual groupoids are étale. In our main result we show that the C^*-algebra of a locally compact Hausdorff étale Zappa-Szép product groupoid is a C^*-blend, in the sense of Exel, of the individual groupoid C^*-algebras. We finish with some examples, including groupoids built from $*$-commuting endomorphisms, and skew product groupoids.

1. Introduction

Group theory has for many years provided fertile ground for mathematicians working in C^*-algebras. Indeed, the notion of a group C^*-algebra is as old as the field itself, and possesses many interesting and natural properties. For instance, it is well known that the C^*-algebra of the direct product of groups G and H is the tensor product of the individual group C^*-algebras $C^*(G)$ and $C^*(H)$ (see, for instance, [3, Examples II.10.3.15]). It is also well known that the C^*-algebra of the semidirect product induced by an action $H \curvearrowright G$ is the crossed product C^*-algebra induced by the action $H \curvearrowright C^*(G)$ (see also [3, Examples II.10.3.15]). In this article we are interested in a third, and more general, notion of a product of groups called the Zappa-Szép product. As a consequence of our main result we are able to answer the natural questions: what is the group C^*-algebra of the Zappa-Szép product of two groups G and H, and what does it have to do with $C^*(G)$ and $C^*(H)$?

A Zappa-Szép product of groups G and H is a generalisation of a semidirect product, in the sense that neither group is necessarily normal in the product. Like the semidirect product, there is an internal and external Zappa-Szép product. A group K is the internal Zappa-Szép product of G and H if G, H are subgroups of K such that $K = GH$ as a set, and $G \cap H = \{e\}$. It then follows that K is in bijection with $G \times H$ as a set. If G, H are both normal in K, then K is isomorphic to the direct product of groups $G \times H$ with pointwise multiplication. If only G is normal, then H acts on G by conjugation $(h, g) \mapsto hgh^{-1}$, and K is isomorphic to the semidirect product $G \rtimes H$. In general neither G nor H need be normal in K, so that direct and semidriect products are special cases of the Zappa-Szép product. In any case, since $K = GH$, given any $h \in H$ and $g \in G$ there are elements $h \cdot g \in G$ and $h|_g \in H$ such that $hg = (h \cdot g)h|_g$, and the condition $G \cap H = \{e\}$ forces $h \cdot g$ and $h|_g$ to be uniquely determined. The *action map* $\cdot : H \times G \to G$ is given by $(h, g) \mapsto h \cdot g$, and *the restriction map* $| : H \times G \to H$ is given by $(h, g) \mapsto h|_g$. These maps can be used to define an associative multiplication and inversion on $G \times H$ by

$$(g, h)(g', h') = (g(h \cdot g'), h|_{g'}h') \quad \text{and} \quad (g, h)^{-1} = (h^{-1} \cdot g^{-1}, h^{-1}|_{g^{-1}}),$$

respectively. The resulting group $G \bowtie H$ is called the *Zappa-Szép product* of G and H and there is an isomorphism $G \bowtie H \cong K$ given by $(g, h) \mapsto gh$ (see [26, 20, 21, 22]). Given an arbitrary pair of groups G and H with a left action of H on G and a right action of G on H

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we construct the external Zappa-Szép product $G \bowtie H$ as the set $G \times H$ with a product defined analogously.

In this article we extend the definitions given above to the more general setting of groupoids. Groupoids have featured prominently in the study of C^*-algebras since the seminal work of Renault [17]. The diversity of examples of groupoid C^*-algebras has been a feature of their success, especially in the study of C^*-algebras associated to dynamical systems. Building on the recent work on Zapp-Szép product semigroups [5] (which was influenced by the work on self-similar actions in [13, 11, 10, 23]), we introduce the Zappa-Szép product of topological groupoids. Our notion of the Zappa-Szép product of groupoids is not new on an algebraic level. In [1] Aguiar and Andruskiewitsch introduced the notion of a matched pair of groupoids; a pair of groupoids $(\mathcal{G}, \mathcal{H})$ is a matched pair if and only if there is a well-defined Zappa-Szép product $\mathcal{G} \bowtie \mathcal{H}$. (Note that in [1] $\triangleleft$ corresponds to our restriction, and $\triangleright$ corresponds to our action.) More generally, a Zappa-Szép product of categories was introduced in the work of Brin [4]. However, in the specific case of groupoids, in which every morphism is an isomorphism, Brin's axioms (P1)–(P8) (see [4, p.406]) can be simplified.

After formalising the Zappa-Szép product $\mathcal{G} \bowtie \mathcal{H}$ of topological groupoids, we switch focus to C^*-algebras, and consider the relationship between the groupoid C^*-algebras of $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{G} \bowtie \mathcal{H}$. In our main theorem we prove that $C^*(\mathcal{G} \bowtie \mathcal{H})$ is a C^*-blend of $C^*(\mathcal{G})$ and $C^*(\mathcal{H})$. C^*-blends were recently introduced by Exel in [7] during his examination of the possible algebraic and C^*-algebraic structures one can put on the tensor product $A \otimes B$ of two algebras A and B. While several examples are examined in [7], any new theory in C^*-algebras always benefits from additional examples. Our main theorem allows us to work at the level of groupoids to describe concrete examples of C^*-blends, rather than with their C^*-algebraic completions. We are thus able to describe several new examples, including those involving Deaconu-Renault groupoids, and skew product groupoids.

While we were at first surprised to learn that $C^*(\mathcal{G} \bowtie \mathcal{H})$ is a C^*-blend of the individual groupoid C^*-algebras, once you scratch beneath the surface, the answer is a natural one. As discussed in [7], C^*-blends generalise crossed products of C^*-algebras by groups, in the sense that $A \rtimes G$ is a C^*-blend of A and $C^*(G)$. The theory of C^*-blends then feels like a natural home for the C^*-algebras of Zappa-Szép products of groups, given that they generalise semidirect products, whose C^*-algebras are crossed products. There is also an interesting similarity between the conditions under which the product of two groups is a Zappa-Szép product, and when the product of two C^*-algebras is a C^*-blend. If G and H are subgroups of a group K, then $GH = \{gh : g \in G, h \in H\}$ is a group if and only if $GH = HG$, and is isomorphic to $G \bowtie H$ if and only if $G \cap H = \{e\}$. If A, B are C^*-subalgebras of a C^*-algebra C, it is not hard to show that the statements $AB = \overline{\text{span}}\{ab : a \in A, b \in B\}$ a C^*-algebra, $AB = BA$, and AB a C^*-blend of A and B are equivalent. (See Remark 4.5.)

This article is organised as follows. Section 2 provides preliminaries on topological groupoids and gives three specific examples that will be used heavily throughout the paper: transformation groupoids, Deaconu-Renault groupoids, and skew product groupoids. In Section 3 we discuss the Zappa-Szép product $\mathcal{G} \bowtie \mathcal{H}$ of two groupoids $\mathcal{G}$ and $\mathcal{H}$, and show how the so called "arrow space" of each groupoid along with certain identifications describes the "arrow space" of $\mathcal{G} \bowtie \mathcal{H}$. We then provide an internal characterisation of a Zappa-Szép groupoid and show that if $\mathcal{G}$ and $\mathcal{H}$ are étale groupoids, then so is $\mathcal{G} \bowtie \mathcal{H}$. In Section 4 we prove our main theorem, which says that $C^*(\mathcal{G} \bowtie \mathcal{H})$ is a C^*-blend of $C^*(\mathcal{G})$ and $C^*(\mathcal{H})$. We finish in Section 5 by examining several examples of Zappa-Szép product groupoids and their C^*-algebras.

2. Preliminaries

Let $\mathcal{G}$ be a set and suppose $\mathcal{G}^{(2)} \subset \mathcal{G} \times \mathcal{G}$. We say $\mathcal{G}$ is a *groupoid* if there is a multiplication $(g, h) \mapsto gh$ from $\mathcal{G}^{(2)}$ to $\mathcal{G}$ and an inverse map $g \mapsto g^{-1}$ from $\mathcal{G}$ to $\mathcal{G}$ satisfying the following.

(1) If $(g, h), (h, k) \in \mathcal{G}^{(2)}$ then $(gh, k), (g, hk) \in \mathcal{G}^{(2)}$ and $g(hk) = (gh)k$.
(2) We have $(g^{-1})^{-1} = g$ for all $g \in \mathcal{G}$.

(3) We have $(g, g^{-1}) \in \mathcal{G}^{(2)}$ for all $g \in \mathcal{G}$ and if $(g, h) \in \mathcal{G}^{(2)}$, then $g^{-1}(gh) = h$ and $(gh)h^{-1} = g$.

We call $\mathcal{G}^{(2)}$ the set of *composable pairs* and $\mathcal{G}^{(0)} := \{gg^{-1} : g \in \mathcal{G}\}$ the set of *units*. The *range* map $\mathcal{G} \to \mathcal{G}^{(0)}$ is given by $g \mapsto gg^{-1}$ and the *source* map $\mathcal{G} \to \mathcal{G}^{(0)}$ is given by $g \mapsto g^{-1}g$.

A useful interpretation of this definition is that an element g of a groupoid $\mathcal{G}$ is an arrow pointing from the source of g to the range of g.

$$gg^{-1} \xleftarrow[g]{} g^{-1}g$$

We think of inversion as reversing the direction of the arrow, and a pair $(g, h) \in \mathcal{G}^{(2)}$ whenever the source of g agrees with the range of h in $\mathcal{G}^{(0)}$; the product gh is then the composition of arrows.

We call $\mathcal{G}$ a *topological groupoid* if $\mathcal{G}$ is a topological space and the multiplication and inversion maps are continuous, where $\mathcal{G}^{(2)}$ has the relative product topology. We call a topological groupoid *étale* if the range (and hence also the source) map is a local homeomorphism.

Example 2.1 (Tranformation groupoids [14, p.8]). Let G be a topological group acting continuously on a topological space X. There is a groupoid $G \ltimes X$ given by

$$\begin{aligned}
&G \ltimes X = G \times X\\
&(G \ltimes X)^{(2)} = \{((g, y), (h, x)) : g, h \in G, x, y \in X, \text{ and } y = h \cdot x\}\\
&(g, h \cdot x)(h, x) = (gh, x)\\
&(g, x)^{-1} = (g^{-1}, g \cdot x)
\end{aligned}$$

We call $G \ltimes X$ a *transformation groupoid*. The unit space is given by $\{e\} \times X \cong X$ and an element (g, x) has range $g \cdot x$ and source x. When given the product topology $G \ltimes X$ is a topological groupoid, and is étale if and only if G has the discrete topology.

Example 2.2 (Deaconu-Renault groupoids [19, Section 3]). The study of topological groupoids associated with endomorphisms began with the seminal paper of Deaconu [6], and has been extended in various directions. In order to associate a groupoid C^*-algebra to a k-graph, Kumjian and Pask [9] began the extension of Deaconu-Renault groupoids to transformations of $\mathbb{N}^k$. More recently, a completely general description of groupoids associated with transformations of $\mathbb{N}^k$ and their topology is given in [19, Section 3].

Let X be a topological space and suppose $k \geq 1$. Following the literature [2] we say that $\sigma : X \to X$ is an endomorphism if σ is a surjective local homeomorphism. Suppose $\theta : \mathbb{N}^k \to \mathrm{End}(X)$ is an action of $\mathbb{N}^k$ on X by continuous endomorphisms. Define the *Deaconu-Renault groupoid* $X \rtimes_\theta \mathbb{N}^k$ by

$$\begin{aligned}
&X \rtimes_\theta \mathbb{N}^k = \{(x, m - n, y) : x, y \in X, m, n \in \mathbb{N}^k, \theta_m(x) = \theta_n(y)\}\\
&(X \rtimes_\theta \mathbb{N}^k)^{(2)} = \{((x, k - l, y), (w, m - n, z)) : y = w\}\\
&(x, k - l, y)(y, m - n, z) = (x, (k + m) - (l + n), z)\\
&(x, m - n, y)^{-1} = (y, n - m, x)
\end{aligned}$$

The unit space is $\{(x, 0, x) : x \in X\} \cong X$ and an element $(x, m - n, y)$ has range x and source y. With the topology described in [19, Section 3], the groupoid $X \rtimes_\theta \mathbb{N}^k$ becomes a topological groupoid.

Example 2.3 (Skew product groupoids [8, Section 4]). Fix an étale groupoid $\mathcal{G}$, a discrete group A and a continuous homomorphism $c : \mathcal{G} \to A$. The *skew-product groupoid* $\mathcal{G}(c)$ is is given by

$$\begin{aligned}
&\mathcal{G}(c) = \mathcal{G} \times A\\
&\mathcal{G}(c)^{(2)} = \{(g,\alpha)(h,\beta) : (g,h) \in \mathcal{G}^{(2)}, \beta = \alpha c(g)\}\\
&(g,\alpha)(h,\alpha c(g)) = (gh,\alpha)\\
&(g,\alpha)^{-1} = (g^{-1},\alpha c(g)).
\end{aligned}$$

We will denote[1] the range and source maps by $b, t : \mathcal{G}(c) \to \mathcal{G}(c)^{(0)}$ respectively. We have

$$\begin{aligned}
b(g,\alpha) &= (g,\alpha)(g^{-1},\alpha c(g)) = (gg^{-1},\alpha)\\
t(g,\alpha) &= (g^{-1},\alpha c(g))(g,\alpha) = (g^{-1}g,\alpha c(g)).
\end{aligned}$$

The unit space is therefore $\mathcal{G}(c)^{(0)} = \mathcal{G}^{(0)} \times A$.

3. Zappa-Szép products of groupoids

In this section we will describe the Zappa-Szép product of two groupoids with bijective unit spaces. To do this we need to recall a fibre product of two sets. Suppose G, H, and X are sets such that $\gamma : G \to X$ and $\eta : H \to X$ are maps. The *fibre product* (or pull-back) of G and H over X is the set

$$G \,{}_\gamma\!\times_\eta H := \{(g,h) \in G \times H : \gamma(g) = \eta(h)\}.$$

Let $\mathcal{G}$ and $\mathcal{H}$ be groupoids and suppose there is a bijection $\mathcal{G}^{(0)} \to \mathcal{H}^{(0)}$. For simplicity, we will consider them the same set and write $\mathcal{G}^{(0)} = \mathcal{U} = \mathcal{H}^{(0)}$. As in [1] we will think of elements of $\mathcal{G}$ as vertical arrows and elements of $\mathcal{H}$ as horizontal arrows as shown below:

$$\begin{array}{ccc}
t(g) = g^{-1}g & & \\
\Big| g & & \\
\downarrow & & \\
b(g) = gg^{-1} & & l(h) = hh^{-1} \xleftarrow{\;h\;} r(h) = h^{-1}h
\end{array}$$

We have denoted the range and source maps of $\mathcal{G}$ by b for bottom and t for top, respectively, and the range and source maps of $\mathcal{H}$ by l for left and r for right, respectively. Suppose that there are maps

$$\cdot : \mathcal{H} \,{}_r\!\times_b \mathcal{G} \to \mathcal{G} \text{ given by } (h,g) \mapsto h \cdot g \quad \text{and} \quad |\, : \mathcal{H} \,{}_r\!\times_b \mathcal{G} \to \mathcal{H} \text{ given by } (h,g) \mapsto h|_g$$

satisfying

(ZS1) $(h_1h_2)\cdot g = h_1 \cdot (h_2 \cdot g)$
(ZS2) $h \cdot (g_1g_2) = (h \cdot g_1)(h|_{g_1} \cdot g_2)$
(ZS3) $h|_{g_1g_2} = (h|_{g_1})|_{g_2}$
(ZS4) $(h_1h_2)|_g = (h_1|_{h_2 \cdot g})(h_2|_g)$
(ZS5) $b(h \cdot g) = l(h)$
(ZS6) $r(h|_g) = t(g)$
(ZS7) $t(h \cdot g) = l(h|_g)$
(ZS8) $b(g) \cdot g = g$
(ZS9) $h|_{r(h)} = h$,

whenever these formulae make sense. We call $\cdot$ the *action* map and $|$ the *restriction* map. These axioms appear in [1], where the action is denoted $\triangleright$, and the restriction is denoted $\triangleleft$.

Before we can construct the Zappa-Szép product groupoid, we need the following lemma.

Lemma 3.1 (cf. [1, Lemma 1.2])**.** *For any $(h,g) \in \mathcal{H} \,{}_r\!\times_b \mathcal{G}$ we have*

(1) $h \cdot r(h) = l(h)$,
(2) $b(g)|_g = t(g)$,
(3) $(h \cdot g)^{-1} = h|_g \cdot g^{-1}$, *and*
(4) $(h|_g)^{-1} = h^{-1}|_{h \cdot g}$.

[1]This unusual choice of labelling the range and source maps by b and t is explained in Section 3.

Proof. For (1), using (ZS2), (ZS9), and that $b(g) = r(h)$ we compute

$$h \cdot g = h \cdot (b(g)g) = (h \cdot b(g))(h|_{b(g)} \cdot g) = (h \cdot r(h))(h \cdot g),$$

which implies $h \cdot r(h) = b(h \cdot g) = l(h)$ by (ZS5).

For (2), using (ZS4) and then (ZS8) we compute

$$h|_g = (hr(h))|_g = h|_{(r(h)\cdot g)} r(h)|_g = h|_{b(g)\cdot g} b(g)|g = h|_g b(g)|_g,$$

which implies that $b(g)|_g = r(h|_g) = t(g)$ by (ZS6).

For (3) and (4), using (ZS5), part (1), and (ZS2) we have

$$(h \cdot g)(h \cdot g)^{-1} = b(h \cdot g) = l(h) = h \cdot r(h) = h \cdot b(g) = h \cdot (gg^{-1}) = (h \cdot g)(h|_g \cdot g^{-1})$$

and (3) follows by cancelling $(h \cdot g)$ on the left. Similarly, using (ZS6), part (2), and (ZS4) we have

$$(h|_g)^{-1}(h|_g) = r(h|_g) = t(g) = b(g)|_g = r(h)|_g = (h^{-1}h)|_g = (h^{-1}|_h \cdot g)(h|_g)$$

and (4) follows by cancelling $(h|_g)$ on the right. □

We define the *Zappa-Szép product* as the set

$$\mathcal{G} \bowtie \mathcal{H} = \mathcal{G} \,{}_t\!\times_l \mathcal{H},$$

with the range of $(g, h) \in \mathcal{G} \bowtie \mathcal{H}$ given by $(b(g), b(g)) \in \mathcal{U} \times \mathcal{U}$, and the source of (g, h) given by $(r(h), r(h)) \in \mathcal{U} \times \mathcal{U}$. We have

$$(\mathcal{G} \bowtie \mathcal{H})^{(2)} = \{((g_1, h_1), (g_2, h_2)) : r(h_1) = b(g_2)\}.$$

We define multiplication by

$$(g_1, h_1)(g_2, h_2) = (g_1(h_1 \cdot g_2), h_1|_{g_2} h_2)$$

and inversion by

$$(g, h)^{-1} = (h^{-1} \cdot g^{-1}, h^{-1}|_{g^{-1}}).$$

Proposition 3.2. *With the above structure, $\mathcal{G} \bowtie \mathcal{H}$ is a groupoid with unit space $\mathcal{G}^{(0)} \,{}_t\!\times_l \mathcal{H}^{(0)} \cong \mathcal{U}$.*

Proof. Conditions (ZS5) and (ZS6) imply that the multiplication is well-defined and (ZS7) shows $\mathcal{G} \bowtie \mathcal{H}$ is closed under multiplication. Suppose $((g_1, h_1), (g_2, h_2))$ and $((g_2, h_2), (g_3, h_3)) \in (\mathcal{G} \bowtie \mathcal{H})^{(2)}$. It is easy to see that $(g_1(h_1 \cdot g_2), h_1|_{g_2} h_2), (g_3, h_3)) \in (\mathcal{G} \bowtie \mathcal{H})^{(2)}$ and $((g_1, h_1), (g_2(h_2 \cdot g_3), h_2|_{g_3} h_3)) \in (\mathcal{G} \bowtie \mathcal{H})^{(2)}$. Some parenthetical gymnastics using (ZS1-4) shows that the following associativity holds:

$$((g_1(h_1 \cdot g_2), h_1|_{g_2} h_2))(g_3, h_3) = (g_1, h_1)(g_2(h_2 \cdot g_3), h_2|_{g_3} h_3).$$

For $(g, h) \in \mathcal{G} \bowtie \mathcal{H}$ we have

$$\begin{aligned} ((g, h)^{-1})^{-1} &= (h^{-1} \cdot g^{-1}, h^{-1}|_{g^{-1}})^{-1} \\ &= ((h^{-1}|_{g^{-1}})^{-1} \cdot (h^{-1} \cdot g^{-1})^{-1}, (h^{-1}|_{g^{-1}})^{-1}|_{(h^{-1}\cdot g^{-1})^{-1}}). \end{aligned} \tag{3.1}$$

By Lemma 3.1 (3) the first term in (3.1) becomes

$$(h^{-1}|_{g^{-1}})^{-1} \cdot (h^{-1} \cdot g^{-1})^{-1} = (h^{-1}|_{g^{-1}})^{-1} \cdot (h^{-1}|_{g^{-1}} \cdot g) = g$$

and by Lemma 3.1 (4) the second term in (3.1) becomes

$$(h^{-1}|_{g^{-1}})^{-1}|_{(h^{-1}\cdot g^{-1})^{-1}} = (h|_{(h^{-1}\cdot g^{-1})})|_{(h^{-1}\cdot g^{-1})^{-1}} = h.$$

So $((g,h)^{-1})^{-1} = (g,h)$ as required. From (ZS5) we have $((g,h),(h^{-1}\cdot g^{-1}, h^{-1}|_{g^{-1}})) \in \mathcal{G} \bowtie \mathcal{H}^{(2)}$. Using (ZS1), (ZS3), (ZS8) and Lemma 3.1 (2) we have

$$\begin{aligned}(g,h)(h^{-1}\cdot g^{-1}, h^{-1}|_{g^{-1}}) &= (g(h\cdot(h^{-1}\cdot g^{-1})), h|_{h^{-1}\cdot g^{-1}}h^{-1}|_{g^{-1}})\\ &= (g(l(h)\cdot g^{-1}), l(h)|_{g^{-1}})\\ &= (g(b(g^{-1})\cdot g^{-1}), b(g^{-1})|_{g^{-1}})\\ &= (b(g), b(g)),\end{aligned}$$

so if $((g_1,h_1),(g_2,h_2)) \in (\mathcal{G}\bowtie\mathcal{H})^{(2)}$, then Lemma 3.1 (1) and (ZS9) imply

$$(g_1,h_1)(g_2,h_2)(g_2,h_2)^{-1} = (g_1,h_1)(b(g_2),b(g_2)) = (g_1(h_1\cdot r(h_1)), (h_1|_{r(h_1)})r(h_1)) = (g_1,h_1).$$

A similar argument shows

$$(g,h)^{-1}(g,h) = (r(h),r(h)) \quad \text{and} \quad (g_1,h_1)^{-1}(g_1,h_1)(g_2,h_2) = (g_2,h_2),$$

as required. □

Remark 3.3. In the proof of Proposition 3.2, note that we used all of the axioms (ZS1–9). This shows the necessity of the rather large number of axioms.

From now on we will freely identify $(\mathcal{G}\bowtie\mathcal{H})^{(0)}$ with $\mathcal{U}$. We can consider $\mathcal{G}$ and $\mathcal{H}$ as subgroupids of $\mathcal{G}\bowtie\mathcal{H}$ via

$$\mathcal{G} \cong \{(g,t(g)) : g\in\mathcal{G}\} \quad \text{and} \quad \mathcal{H} \cong \{(l(h),h) : h\in\mathcal{H}\}.$$

Since $\mathcal{G}\bowtie\mathcal{H} = \mathcal{G}\,{}_t\!\times_l\mathcal{H}$ as a set, we may represent elements $(g',h')\in\mathcal{G}\bowtie\mathcal{H}$ and $(h,g)\in\mathcal{H}\,{}_r\!\times_b\mathcal{G}$ as pairs of arrows between elements of $\mathcal{U}$ as follows:

$$\begin{array}{ccc} t(g') = l(h') & \xleftarrow[h']{} & r(h') \\ \Big\downarrow{\scriptstyle g'} & & \\ b(g') & & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} & & t(g) \\ & & \Big\downarrow{\scriptstyle g} \\ l(h) & \xleftarrow[h]{} & r(h) = b(g), \end{array}$$

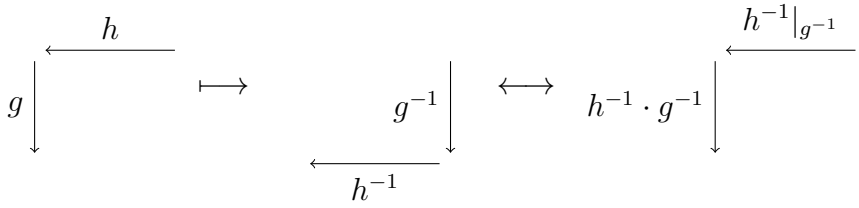
respectively. In the Zappa-Szép product $\mathcal{G}\bowtie\mathcal{H}$, the element $(h,g)\in\mathcal{H}\,{}_r\!\times_b\mathcal{G}$ is identified with $(h\cdot g, h|_g)\in\mathcal{G}\bowtie\mathcal{H}$. Visually, this amounts to the following two diagrams being identified:

$$\begin{array}{ccc} & & t(g) \\ & & \Big\downarrow{\scriptstyle g} \\ l(h) & \xleftarrow[h]{} & r(h) = b(g) \end{array} \qquad \longleftrightarrow \qquad \begin{array}{ccc} t(h\cdot g) = l(h|_g) & \xleftarrow[h|_g]{} & r(h|_g) \\ \Big\downarrow{\scriptstyle h\cdot g} & & \\ b(h\cdot g) & & \end{array}$$

This geometric identification corresponds to the algebraic axioms (ZS5–7). Now suppose $((g_1,h_1),(g_2,h_2)) \in (\mathcal{G}\bowtie\mathcal{H})^{(2)}$, then the product $(g_1,h_1)(g_2,h_2) = (g_1(h_1\cdot g_2), h_1|_{g_2}h_2)$ is represented by

$$\begin{array}{ccccc} & & \xleftarrow{h_2} & & \\ & & \Big\downarrow{\scriptstyle g_2} & & \\ & \xleftarrow{h_1} & & & \\ \Big\downarrow{\scriptstyle g_1} & & & & \end{array} \qquad \longleftrightarrow \qquad \begin{array}{ccccc} & \xleftarrow{h_1|_{g_2}} & & \xleftarrow{h_2} & \\ & \Big\downarrow{\scriptstyle h_1\cdot g_2} & & & \\ & & & & \\ \Big\downarrow{\scriptstyle g_1} & & & & \end{array}$$

Finally, if $(g,h) \in \mathcal{G} \bowtie \mathcal{H}$, then the inverse $(g,h)^{-1} = (h^{-1} \cdot g^{-1}, h^{-1}|_{g^{-1}})$ is represented by



The following proposition determines when a given groupoid decomposes as a Zappa-Szép product.

Proposition 3.4 (Internal Zappa-Szép products)**.** *Let $\mathcal{K}$ be a groupoid and let $\mathcal{G}$ and $\mathcal{H}$ be subgroupoids. Suppose that for any $k \in \mathcal{K}$ there is a unique pair $(g,h) \in (\mathcal{G} \times \mathcal{H}) \cap \mathcal{K}^{(2)}$ such that $k = gh$. Then $\mathcal{K} \cong \mathcal{G} \bowtie \mathcal{H}$.*

Proof. We must first show that $\mathcal{G}$ and $\mathcal{H}$ admit the structure required to take a Zappa-Szép product. First, fix $u \in \mathcal{K}^{(0)}$. Then there is a unique $g \in \mathcal{G}$ and $h \in \mathcal{H}$ with $u = gh$. But then $u = gg^{-1} \in \mathcal{G}$ and $u = h^{-1}h \in \mathcal{H}$. Since $u = uu$, uniqueness forces $h = u = g$. Hence $\mathcal{K}^{(0)} = \mathcal{G}^{(0)} = \mathcal{H}^{(0)}$. We define action and restriction maps using the unique decomposition; given a pair $(g,h) \in (\mathcal{G} \times \mathcal{H}) \cap \mathcal{K}^{(2)}$ let $(g \cdot h, g|_h) \in (\mathcal{G} \times \mathcal{H}) \cap \mathcal{K}^{(2)}$ be the unique pair such that

$$gh = (g \cdot h)(g|_h).$$

Routine calculations show these maps satisfy (ZS1–9), and $k \mapsto (g,h)$, where $k = gh$, is an isomorphism $\mathcal{K} \cong \mathcal{G} \bowtie \mathcal{H}$. □

Using Proposition 3.4 and Lemma 3.1 we can show that taking groupoid Zappa-Szép products is symmetric.

Corollary 3.5. *Any groupoid Zappa-Szép product $\mathcal{G} \bowtie \mathcal{H}$ is isomorphic to a Zappa-Szép product $\mathcal{H} \bowtie \mathcal{G}$.*

Proof. In light of Proposition 3.4 it suffices to notice that any $(g,h) \in \mathcal{G} \bowtie \mathcal{H}$ can be rewritten uniquely as

$$(g,h) = (b(g), h|_{h^{-1} \cdot g^{-1}})(h^{-1}|_{g^{-1}} \cdot g, r(h)). \tag{3.2}$$

That (3.2) holds is a straightforward computation involving several applications of Lemma 3.1 (3) and (4). For the uniqueness, suppose $(g,h) = (l(h'), h')(g', t(g'))$ for some $g' \in \mathcal{G}$ and $h' \in \mathcal{H}$. Then $g = h' \cdot g'$ and $h = h'|_{g'}$. Substituting these equations into (3.2) and applying Lemma 3.1 shows that $h' = h|_{h^{-1} \cdot g^{-1}}$ and $g' = h^{-1}|_{g^{-1}} \cdot g$. Thus the decomposition in (3.2) is unique. □

If $\mathcal{G}$ and $\mathcal{H}$ are topological groupoids with homeomorphic unit spaces, then after endowing $\mathcal{G} \,{}_t\!\times_l \mathcal{H}$ with the relative product topology of $\mathcal{G} \times \mathcal{H}$ it is natural to ask whether $\mathcal{G} \bowtie \mathcal{H}$ is a topological groupoid. It is easy to check that this is true if and only if the action and restriction maps are continuous, where $\mathcal{H} \,{}_r\!\times_b \mathcal{G}$ has the relative product topology.

A similar question may be asked when $\mathcal{G}$ and $\mathcal{H}$ are étale.

Proposition 3.6. *A Zappa-Szép product groupoid $\mathcal{G} \bowtie \mathcal{H}$ endowed with the relative product topology of $\mathcal{G} \times \mathcal{H}$ is étale if and only if both $\mathcal{G}$ and $\mathcal{H}$ are étale and the action and restriction maps are continuous.*

Proof. Since $\mathcal{G}$ and $\mathcal{H}$ are both isomorphic to subgroupoids of $\mathcal{G} \bowtie \mathcal{H}$, assuming $\mathcal{G} \bowtie \mathcal{H}$ is étale immediately implies $\mathcal{G}$ and $\mathcal{H}$ are étale.

For the reverse implication, suppose $\mathcal{G}$ and $\mathcal{H}$ are étale. We must show that $(g,h) \mapsto b(g)$ is a local homeomorphism. To this end, fix $(g,h) \in \mathcal{G} \bowtie \mathcal{H}$. Using that $\mathcal{G}$ and $\mathcal{H}$ are étale we can find open subsets

$$U, V \subset \mathcal{G},\ W \subset \mathcal{H}$$

with $g \in U \cap V$ and $h \in W$ such that $b|_U, t|_V$ and $l|_W$ are all homeomorphisms. Define

$$X := ((U \cap V) \times W) \cap (\mathcal{G}\ {}_t\!\times_l \mathcal{H}) \subset \mathcal{G} \bowtie \mathcal{H}.$$

Then X is open, $(g,h) \in X$ and for any $(g',h'),(g'',h'') \in X$ we have

$$\begin{aligned} b(g') = b(g'') &\iff g' = g'' \text{ since } g', g'' \in U \\ &\iff t(g') = t(g'') \text{ since } g', g'' \in V \\ &\iff l(h') = l(h'') \text{ since } (g',h'),(g'',h'') \in \mathcal{G}\ {}_t\!\times_l \mathcal{H} \\ &\iff h' = h'' \text{ since } h', h'' \in W. \end{aligned}$$

We see that the range map $(g,h) \mapsto b(g)$ is a homeomorphism on X as required. □

Corollary 3.7. *Suppose $\mathcal{G}, \mathcal{H}$ and $\mathcal{K}$ are étale groupoids such that $\mathcal{G}$ and $\mathcal{H}$ are subgroupoids of $\mathcal{K}$ and $\mathcal{K} \cong \mathcal{G} \bowtie \mathcal{H}$, as in Proposition 3.4. Then $\mathcal{K} \cong \mathcal{G} \bowtie \mathcal{H}$ is an isomorphism of topological groupoids and $\mathcal{K}$ is étale.*

Proof. We know from Proposition 3.4 that there is a bijective homomorphism $\mathcal{G} \bowtie \mathcal{H} \to \mathcal{K}$ satisfying $(g,h) \mapsto gh$. Since multiplication is continuous, to see that this map is a homeomorphism it suffices to show it is open. This follows immediately from the fact that $\mathcal{G} \bowtie \mathcal{H}$ is étale. □

Remark 3.8. A subset U of an étale groupoid $\mathcal{G}$ is called a *bisection* if both the range and source maps restricted to U are injective. The collection $\mathrm{B}(\mathcal{G})$ of all open bisections in $\mathcal{G}$ is an inverse semigroup under composition $UV = \{gh : (g,h) \in U \times V \cap \mathcal{G}^{(2)}\}$. If we identify a bisection U with the homeomorphism $r(U) \to s(U)$ on $\mathcal{G}^{(0)}$ satisfying $r(g) \mapsto s(g)$, then it is easily checked that $\mathrm{B}(\mathcal{G})$ is a pseudogroup of homeomorphisms of the topological space $\mathcal{G}^{(0)}$, in the sense of [18, Section 3].

The Zappa-Szép product of inverse semigroups (and semigroups more broadly) are studied in [24]. It is natural to examine the existence of a Zappa-Szép product of $B(\mathcal{G})$ and $B(\mathcal{H})$, and whether there is a connection to the collection of bisections of $\mathcal{G} \bowtie \mathcal{H}$. We considered these problems, and there seems to be no obvious definitions for the action and restriction maps. In particular, there is no reason that the sets

$$V \cdot U := \{h \cdot g : h \in V, g \in U, r(h) = b(g)\} \quad \text{and} \quad V|_U = \{h|_g : h \in V, g \in U, r(h) = b(g)\},$$

for $U \in B(\mathcal{G}), V \in B(\mathcal{H})$, are open bisections in $B(G)$ and $B(\mathcal{H})$, respectively.

4. The C^*-algebra of a Zappa-Szép groupoid

In this section we prove the main result of this paper, which says that the groupoid C^*-algebra of the Zappa-Szép product of two groupoids $\mathcal{G}$ and $\mathcal{H}$ is a C^*-blend of the two groupoid C^*-algebras $C^*(\mathcal{G})$ and $C^*(\mathcal{H})$. Before we state this result, we briefly recall the construction of groupoid C^*-algebras, and the formal definition of a C^*-blend from [7].

For $\mathcal{G}$ a locally compact Hausdorff étale groupoid with range and source maps r and s, define a multiplication and involution on $C_c(\mathcal{G})$ by

$$\xi * \eta(g) = \sum_{g_1 g_2 = g} \xi(g_1)\eta(g_2) \quad \text{and} \quad \xi^*(g) = \overline{\xi(g^{-1})}.$$

With these operations, pointwise scalar multiplication and addition, and $*$-algebra norm given by

$$\|\xi\|_I = \sup_{u \in \mathcal{G}^{(0)}} \max\left\{ \sum_{r(g)=u} |\xi(g)|, \sum_{s(g)=u} |\xi(g)| \right\},$$

$C_c(\mathcal{G})$ becomes a normed $*$-algebra. This norm, called the I-norm, is typically not a C^*-norm. However, there is a C^*-norm on $C^*(\mathcal{G})$ given by

$$\|\xi\| = \sup\{\|\pi(\xi)\| : \pi \text{ is a } \|\cdot\|_I\text{-bounded } *\text{-representation of } C_c(\mathcal{G})\},$$

and the completion of $C_c(\mathcal{G})$ under $\|\cdot\|$ is called the *full groupoid C^*-algebra of $\mathcal{G}$*.

There is also a reduced groupoid C^*-algebra. For each $u \in \mathcal{G}^{(0)}$ there is an I-norm-bounded representation Ind_u of $C_c(\mathcal{G})$ on $\ell^2(s^{-1}(u))$ given by $\mathrm{Ind}_u(f)\delta_g = \sum_{r(h)=r(g)} f(h^{-1}g)\delta_h$. The reduced norm is given by $\|f\|_r = \sup_{u\in\mathcal{G}^{(0)}} \|\mathrm{Ind}_u(f)\|$. The completion of $C_c(\mathcal{G})$ under $\|\cdot\|_r$ is called the *reduced groupoid C^*-algebra of $\mathcal{G}$*. For more details of these constructions we refer the reader to Renault's original treatment [17].

We now recall Exel's notion of a C^*-blend from [7].

Definition 4.1. For C^*-algebras A and B we denote by $A \otimes_{\mathbb{C}} B$ the algebraic tensor product. Given a C^*-algebra X and $*$-homomorphisms

$$i : A \to M(X) \quad \text{and} \quad j : B \to M(X),$$

the bilinear maps $(a, b) \mapsto i(a)j(b)$ and $(b, a) \mapsto j(b)i(a)$ induce linear maps

$$i \circledast j : A \otimes_{\mathbb{C}} B \to M(X) \quad \text{satisfying} \quad a \otimes b \mapsto i(a)j(b)$$

and

$$j \circledast i : B \otimes_{\mathbb{C}} A \to M(X) \quad \text{satisfying} \quad b \otimes a \mapsto j(b)i(a).$$

A *C^*-blend* is a quintuple (A, B, i, j, X), consisting of: C^*-algebras A, B, and X; and $*$-homomorphisms i and j as above, with the property that the range of $i \circledast j$ is contained and dense in X (or, equivalently, $\mathrm{range}(j \circledast i) = (\mathrm{range}(i \circledast j))^*$ is contained and dense in X).

We can now state our main theorem.

Theorem 4.2. *Suppose $\mathcal{G} \bowtie \mathcal{H}$ is a locally compact Hausdorff étale Zappa-Szép product groupoid. The maps $i : C_c(\mathcal{G}) \to C^*(\mathcal{G} \bowtie \mathcal{H})$ and $j : C_c(\mathcal{H}) \to C^*(\mathcal{G} \bowtie \mathcal{H})$, given by*

$$i(\xi)(g, h) = \delta_{h,t(g)}\xi(g) \quad \text{and} \quad j(\eta)(g, h) = \delta_{g,l(h)}\eta(h),$$

extend to $*$-homomorphisms $i : C^*(\mathcal{G}) \to C^*(\mathcal{G} \bowtie \mathcal{H})$ and $j : C^*(\mathcal{H}) \to C^*(\mathcal{G} \bowtie \mathcal{H})$, and the quintuple $(C^*(\mathcal{G}), C^*(\mathcal{H}), i, j, C^*(\mathcal{G} \bowtie \mathcal{H}))$ is a C^*-blend.

Remark 4.3. Notice that the range of i and j are in $C^*(\mathcal{G} \bowtie \mathcal{H})$, rather than the multiplier algebra, since $\mathcal{G} \bowtie \mathcal{H}$ is étale by Proposition 3.6.

To prove this result we need a lemma about *slices*, which are precompact open subsets of a groupoid on which the range and source maps are bijective. We think this lemma is well known, but we could not find a proof, so we include one here. Note that $\|\cdot\|_\infty$ denotes the usual supremum norm on functions.

Lemma 4.4. *Let $\mathcal{G}$ be a locally compact Hausdorff étale groupoid. Let $\xi \in C_c(\mathcal{G}) \subset C^*(\mathcal{G})$ such that $\mathrm{supp}(\xi)$ is a slice. Then*

$$\|\xi\| = \|\xi\|_r = \|\xi\|_I = \|\xi\|_\infty.$$

Proof. We first show that $\|\xi\|_I = \|\xi\|_\infty$. For any unit $u \in \mathcal{G}^{(0)}$ we have

$$|r^{-1}(u) \cap \mathrm{supp}(\xi)|, |s^{-1}(u) \cap \mathrm{supp}(\xi)| \leq 1$$

since r, s are local homeomorphisms on $\mathrm{supp}(\xi)$. Therefore

$$\|\xi\|_I = \sup_{u\in\mathcal{G}^{(0)}} \max\left\{ \sum_{r(g)=u} |\xi(g)|, \sum_{s(g)=u} |\xi(g)| \right\} = \sup_{g\in\mathcal{G}} |\xi(g)| = \|\xi\|_\infty.$$

To see that $\|\xi\|_r = \|\xi\|_\infty$, first fix $g \in \mathcal{G}$. We have

$$\xi^*\xi(g) = \sum_{hk=g} \xi^*(h)\xi(k) = \sum_{hk=g} \overline{\xi(h^{-1})}\xi(k).$$

If $\xi^*\xi(g) \neq 0$, there exists $h, k \in \mathcal{G}$ with $hk = g$ and $\overline{\xi(h^{-1})}\xi(k) \neq 0$. Then $h^{-1}, k \in \mathrm{supp}(\xi)$, and since $r|_{\mathrm{supp}(\xi)}$ is injective, we have

$$g = hk \Longrightarrow r(h^{-1}) = r(k) \Longrightarrow h^{-1} = k \Longrightarrow g \in \mathcal{G}^{(0)}.$$

Hence $\xi^*\xi \in C_0(\mathcal{G}^{(0)})$. Now, fix $u \in \mathcal{G}^{(0)}$ and $\delta_g \in \ell^2(\mathcal{G}_u)$, where $\mathcal{G}_u := \{g \in \mathcal{G} : g^{-1}g = u\}$. Since $\operatorname{supp}(\xi^*\xi) \subset \mathcal{G}^{(0)}$ we have

$$\operatorname{Ind}_u(\xi^*\xi)\delta_g = \sum_{r(h)=r(g)} \xi^*\xi(h^{-1}g)\delta_h = \xi^*\xi(g^{-1}g)\delta_g = \xi^*\xi(u)\delta_g,$$

and so $\|\operatorname{Ind}_u(\xi^*\xi)\| = |\xi^*\xi(u)|$. Hence

$$\|\xi\|_r^2 = \|\xi^*\xi\|_r = \sup_{u\in\mathcal{G}^{(0)}} \|\operatorname{Ind}_u(\xi^*\xi)\| = \sup_{u\in\mathcal{G}^{(0)}} |\xi^*\xi(u)| = \|\xi^*\xi\|_\infty = \|\xi\|_\infty^2.$$

Finally, $\|\xi\|_r \le \|\xi\| \le \|\xi\|_I = \|\xi\|_r$, so we have shown all the required equalities. □

Proof of Theorem 4.2. Fix $\xi \in C_c(\mathcal{G})$. We claim that $i(\xi) \in C_c(\mathcal{G} \bowtie \mathcal{H})$. To see that $i(\xi)$ is continuous, fix an open subset $V \subseteq \mathbb{C}$. If $0 \notin V$, then

$$i(\xi)^{-1}(V) = \{(g, t(g)) : \xi(g) \in V\} = (\xi^{-1}(V) \times \mathcal{H}^{(0)}) \cap (\mathcal{G}\ {}_t\!\times_l \mathcal{H}).$$

If $0 \in V$, then

$$\begin{aligned} i(\xi)^{-1}(V) &= \{(g, t(g)) : \xi(g) \in V\} \cup \{(g, h) \in \mathcal{G}\ {}_t\!\times_l \mathcal{H} : h \in \mathcal{H} \setminus \mathcal{H}^{(0)}\} \\ &= ((\xi^{-1}(V) \times \mathcal{H}^{(0)}) \cup (\mathcal{G} \times \mathcal{H} \setminus \mathcal{H}^{(0)})) \cap (\mathcal{G}\ {}_t\!\times_l \mathcal{H}). \end{aligned}$$

Since ξ is continuous, we have $\xi^{-1}(V)$ open, and since $\mathcal{H}$ is Hausdorff and étale, both $\mathcal{H}^{(0)}$ and $\mathcal{H} \setminus \mathcal{H}^{(0)}$ are open. So in either case, $i(\xi)^{-1}(V)$ is open in the relative product topology, and $i(\xi)$ is continuous. The support of $i(\xi)$ is the set $\operatorname{supp}(\xi)\ {}_t\!\times_l \mathcal{H}^{(0)}$, which is homeomorphic to $\operatorname{supp}(\xi)$ via $(g, t(g)) \mapsto g$. Since $\operatorname{supp}(\xi)$ is compact, we have $i(\xi) \in C_c(\mathcal{G} \bowtie \mathcal{H})$, as claimed. A symmetric argument using that $\mathcal{G}$ is Hausdorff and étale shows that $j(\eta) \in C_c(\mathcal{G} \bowtie \mathcal{H})$ for any $\eta \in C_c(\mathcal{H})$.

We extend i and j to $*$-homomorphisms $C^*(\mathcal{G}) \to C^*(\mathcal{G} \bowtie \mathcal{H})$ and $C^*(\mathcal{H}) \to C^*(\mathcal{G} \bowtie \mathcal{H})$, respectively, and we now claim that $(C^*(\mathcal{G}), C^*(\mathcal{H}), i, j, C^*(\mathcal{G} \bowtie \mathcal{H}))$ is a C^*-blend. Firstly, for each $\xi \in C_c(\mathcal{G})$ and $\eta \in C_c(\mathcal{H})$ we have

$$i \circledast j(\xi \otimes \eta)(g, h) = \xi(g)\eta(h),$$

from which we see that $i \circledast j(\xi \otimes \eta)$ is continuous. We also have

$$\operatorname{supp}(i \circledast j(\xi \otimes \eta)) = (\operatorname{supp}(\xi) \times \operatorname{supp}(\eta)) \cap (\mathcal{G}\ {}_t\!\times_l \mathcal{H}),$$

which is compact. So the image of $i \circledast j$ is contained in $C_c(\mathcal{G} \bowtie \mathcal{H})$. To complete the proof we need to show that this image is dense in $C^*(\mathcal{G} \bowtie \mathcal{H})$.

For an arbitrary function $\theta \in C_c(\mathcal{G} \bowtie \mathcal{H})$ we can cover the support by a finite number of precompact open bisections $\{U_k : 1 \le k \le n\}$. If $\{\pi_k : 1 \le k \le n\}$ is a partition of unity for $\operatorname{supp}(\theta)$ with $\operatorname{supp}(\pi_k) \subset U_k$, then $\theta = \sum_{k=1}^n \theta\pi_k$, where $\operatorname{supp}(\theta\pi_k)$ is a precompact open bisection. Since we know from Lemma 4.4 that $\|\theta\pi_k\|_\infty = \|\theta\pi_k\|_I$, it suffices to show that the image of $i \circledast j$ is *uniformly* dense in $C_c(\mathcal{G} \bowtie \mathcal{H})$. To this end, fix $(g, h) \ne (g', h')$. By the Stone-Weierstrass theorem for locally compact spaces, it is enough to find $\xi \in C_c(\mathcal{G})$ and $\eta \in C_c(\mathcal{H})$ with $i \circledast j(\xi \otimes \eta)(g, h) = 1$ and $i \circledast j(\xi \otimes \eta)(g', h') = 0$. Without loss of generality assume $g \ne g'$. Fix $\xi \in C_c(\mathcal{G})$ with $\xi(g) = 1, \xi(g') = 0$ and $\eta \in C_c(\mathcal{H})$ with $\eta(h) = 1$. Then

$$i \circledast j(\xi \otimes \eta)(g, h) = \xi(g)\eta(h) = 1 \quad \text{and} \quad i \circledast j(\xi \otimes \eta)(g', h') = \xi(g')\eta(h') = 0,$$

as required. □

Remark 4.5. If G and H are subgroups of a group K, then $GH = \{gh : g \in G, h \in H\}$ is a group if and only if $GH = HG$. Moreover, GH is isomorphic to $G \bowtie H$ if and only if $G \cap H = \{e\}$ (see [16, Satz 6]). There is a similar characterisation for the product of subsets of a C^*-algebra to be a C^*-blend. Suppose A and B are C^*-subalgebras of a C^*-algebra C, and denote by AB the set $\overline{\operatorname{span}}\{ab : a \in A, b \in B\}$. Then the following are equivalent:

(1) $AB = BA$,
(2) AB is a C^*-algebra,

(3) there exist C^*-homomorphisms $i : A \to M(AB)$ and $j : B \to M(AB)$ such that (A, B, i, j, AB) is a C^*-blend.

For (2) $\implies$ (3) we use maps $i : A \to M(AB)$ and $j : B \to M(AB)$ given by $i(a)x = ax$ and $j(b)x = bx$. Implications (3) $\implies$ (2) and (1) $\iff$ (2) are straightforward exercises.

5. Examples

In our final section we examine several examples of Zappa-Szép product groupoids and their C^*-algebras.

5.1. $*$-commuting endomorphisms.

In this section we show that every pair of $*$-commuting endomorphisms of a topological space gives rise to a Zappa-Szép product of Deaconu-Renault groupoids (see Example 2.2).

Recall from [2] that a pair of commuting endomorphisms S and T of a topological space X are said to *$*$-commute* if, for every $x, y \in X$ with $Tx = Sy$, there exists a unique $z \in X$ with $Sz = x$ and $Tz = y$. We call such S and T *$*$-commuting endomorphisms*.

Proposition 5.1. *Suppose S and T are $*$-commuting endomorphisms of a topological space X. Then there is an action θ of $\mathbb{N}^2$ on X given by $\theta_{(m_1,m_2)} = S^{m_1}T^{m_2}$, and the Deaconu-Renault groupoid for this action is the internal Zappa-Szép product of the individual Deaconu-Renault groupoids for the actions of $\mathbb{N}$ on X induced by S and T.*

Since S and T commute, $\theta_{(m_1,m_2)} := S^{m_1}T^{m_2}$ gives an action θ of $\mathbb{N}^2$ by continuous endomorphisms of X. Let $X \rtimes_\theta \mathbb{N}^2$ be the corresponding Deaconu-Renault groupoid, and $X \rtimes_S \mathbb{N}$ and $X \rtimes_T \mathbb{N}$ be the Deaconu-Renault groupoids for the actions of $\mathbb{N}$ on X induced by S and T, respectively. Notice that $X \rtimes_S \mathbb{N}$ and $X \rtimes_T \mathbb{N}$ can be viewed as subgroupoids of $X \rtimes_\theta \mathbb{N}^2$ via

$$X \rtimes_S \mathbb{N} \cong \{(x, m - n, y) : m, n \in \mathbb{N} \times \{0\}\}, \text{ and}$$
$$X \rtimes_T \mathbb{N} \cong \{(x, m - n, y) : m, n \in \{0\} \times \mathbb{N}\}.$$

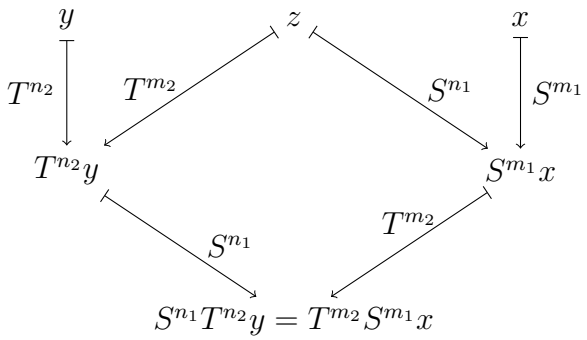
So to prove Proposition 5.1 we need to show that

$$X \rtimes_\theta \mathbb{N}^2 \cong (X \rtimes_S \mathbb{N}) \bowtie (X \rtimes_T \mathbb{N}).$$

Proof of Proposition 5.1. We aim to use Proposition 3.4. Fix $(x, m - n, y) \in X \rtimes_\theta \mathbb{N}^2$. Write $m = (m_1, m_2)$ and $n = (n_1, n_2)$. By definition of $X \rtimes_\theta \mathbb{N}^2$ we have

$$S^{n_1}T^{n_2}y = T^{m_2}S^{m_1}x$$

Since S and T $*$-commute, the maps S^{m_1} and T^{m_2} also $*$-commute. Therefore, there is a unique $z \in X$ such that $S^{n_1}z = S^{m_1}x$ and $T^{m_2}z = T^{n_2}y$. This information is summarised in the following diagram:



Hence, we have elements

$$(x, (m_1, 0) - (n_1, 0), z) \in X \rtimes_S \mathbb{N} \subset X \rtimes_\theta \mathbb{N}^2$$

and

$$(z, (0, m_2) - (0, n_2), y) \in X \rtimes_T \mathbb{N} \subset X \rtimes_\theta \mathbb{N}^2$$

with $(x,(m_1,0)-(n_1,0),z)(z,(0,m_2)-(0,n_2),y)=(x,m-n,y)$. Since z was unique, this decomposition is also unique and so Proposition 3.4 provides us with the desired isomorphism. □

Remark 5.2. Applying Theorem 4.2 in this setting gives a C^*-blend

$$(C^*(X\rtimes_S\mathbb{N}),C^*(X\rtimes_T\mathbb{N}),i,j,C^*(X\rtimes_\theta\mathbb{N}^2)).$$

5.2. **1-coaligned 2-graphs.** We know from [12, Defintion 2.1] (also see [25]) that examples of $*$-commuting maps come from the shift map on certain 2-graphs. We recall the details.

We view the monoid $\mathbb{N}^2$ as a category with one object in the usual way. We write $e_1=(1,0)$ and $e_2=(0,1)$ for the canonical generators. Recall from [9] that a 2-*graph* is a small category Λ equipped with a degree functor $d:\Lambda\to\mathbb{N}^2$ which satisfies the factorisation property, in the sense that whenever $\lambda\in\Lambda$ and $m,n\in\mathbb{N}^2$ satisfy $d(\lambda)=m+n$, there are unique elements $\mu,\nu\in\Lambda$ satisfying $d(\mu)=m$, $d(\nu)=n$ and $\lambda=\mu\nu$. The objects of Λ can be identified with $\Lambda^0:=d^{-1}(0)$. The codomain and domain maps are denoted by r and s, and are called the range and source maps. A 2-graph is called *row-finite and with no sources* if for every $v\in\Lambda^0$ and $n\in\mathbb{N}^2$ the set $\{\lambda\in\Lambda:d(\lambda)=n,\,r(\lambda)=v\}$ is nonempty and finite.

Let Ω_2 be the category with objects $\mathbb{N}^2$, morphisms $\{(m,n):m,n\in\mathbb{N}^2,m\leq n\}$ where $\mathbb{N}^2$ has the usual partial order, and range and source maps $r(m,n)=m,s(m,n)=n$. With the degree functor $d(m,n)=n-m$, Ω_2 is a 2-graph. If Λ is a 2-graph, an *infinite path* in Λ is a functor $x:\Omega_2\to\Lambda$. We write Λ^∞ for the space of infinite paths. If Λ is row-finite and with no sources, then Λ^∞ equipped with the topology generated by cylinder sets $Z(\lambda):=\{x\in\Lambda^\infty:x(0,d(\lambda))=\lambda\}$ is a totally disconnected locally compact Hausdorff space. For each $p\in\mathbb{N}^2$ consider the shift map $\sigma^p:\Lambda^\infty\to\Lambda^\infty$ given by $\sigma^p(x)(m,n)=x(m+p,n+p)$. Each σ^p is a local homeomorphism. If in addition Λ has no sinks, in the sense that for the each $v\in\Lambda^0$ and $n\in\mathbb{N}^2$ the set $\{\lambda\in\Lambda:d(\lambda)=n,\,s(\lambda)=v\}$ is nonempty, then each σ^p is also surjective. So for Λ a 2-graph which is row-finite and with no sinks or sources, the map $k\mapsto\sigma^k$ determines an action of $\mathbb{N}^2$ by endomorphisms Λ^∞. Let $\mathcal{G}_\Lambda=\Lambda^\infty\rtimes\mathbb{N}^2$ be the associated Deaconu-Renault groupoid.

Definition 5.3. [12, Defintion 2.1] A 2-graph Λ is 1-*coaligned* if for every pair $(e^1,e^2)\in\Lambda^{e_1}\,{}_s\!\times_s\Lambda^{e_2}$ there exists a unique pair $(f^1,f^2)\in\Lambda^{e_1}\,{}_r\!\times_r\Lambda^{e_2}$ such that $f^1e^2=f^2e^1$.

A large class of examples of 1-coaligned 2-graphs are provided in [12, Theorem 3.1]. The connection between 1-coaligned 2-graphs and $*$-commuting endomorphisms comes from the following result (which applies to more general k-graphs, but we state only for 2-graphs).

Theorem 5.4. [12, Corollary 2.4] *If Λ is a 1-coaligned row-finite 2-graph with no sinks or sources, then for each $i\neq j$, σ^{e_i} and σ^{e_j} are $*$-commuting surjective local homeomorphisms.*

Using our results we can now decompose both the graph groupoid of a 1-coaligned 2-graph, and its groupoid C^*-algebra, the graph algebra. We direct the reader to [15] for an account of directed graphs, k-graphs, and their C^*-algebras. For a 2-graph Λ we define the *blue graph* B_Λ and the *red graph* R_Λ to be the directed graphs

$$B_\Lambda=(B^0_\Lambda:=\Lambda^0,B^1_\Lambda:=\Lambda^{e_1},r|_{\Lambda^{e_1}},s|_{\Lambda^{e_1}})\quad\text{and}\quad R_\Lambda=(R^0_\Lambda:=\Lambda^0,R^1_\Lambda:=\Lambda^{e_2},r|_{\Lambda^{e_2}},s|_{\Lambda^{e_2}}).$$

Theorem 5.5. *For every 1-coaligned row-finite 2-graph Λ with no sinks or sources we have*

$$\mathcal{G}_\Lambda\cong(\Lambda^\infty\rtimes_{\sigma^{e_1}}\mathbb{N})\bowtie(\Lambda^\infty\rtimes_{\sigma^{e_2}}\mathbb{N}).$$

Moreover, there are $*$-homomorphisms $i:C^*(B_\Lambda)\to C^*(\Lambda)$ and $j:C^*(R_\Lambda)\to C^*(\Lambda)$ which make $(C^*(B_\Lambda),C^*(R_\Lambda),i,j,C^*(\Lambda))$ a C^*-blend.

This result follows from Theorem 4.2 once the isomorphisms $C^*(B_\Lambda)\cong C^*(\Lambda^\infty\rtimes_{\sigma^{e_1}}\mathbb{N})$ and $C^*(R_\Lambda)\cong C^*(\Lambda^\infty\rtimes_{\sigma^{e_2}}\mathbb{N})$ are established; this is an exercise in finding appropriate Cuntz-Krieger families in $C^*(\Lambda^\infty\rtimes_{\sigma^{e_1}}\mathbb{N})$ and $C^*(\Lambda^\infty\rtimes_{\sigma^{e_2}}\mathbb{N})$, and applying the gauge-invariant uniqueness theorem. We leave the details to the reader.

5.3. **Skew product groupoids.** Fix an étale groupoid $\mathcal{G}$, a discrete group A and a continuous homomorphism $c : \mathcal{G} \to A$. Recall from Example 2.3 the construction of the skew product groupoid $\mathcal{G}(c)$; this groupoid is also étale because A is discrete.

The formula $\beta \cdot (g, \alpha) := (g, \alpha\beta^{-1})$ defines a left action of A on the space $\mathcal{G}(c)$. For a composable pair $((g, \alpha), (h, \alpha c(g))) \in \mathcal{G}(c)^{(2)}$, this action satisfies

$$\begin{aligned}\beta \cdot (g, \alpha)(h, \alpha c(g)) = \beta \cdot (gh, \alpha) &= (gh, \alpha\beta^{-1}) \\ &= (g, \alpha\beta^{-1})(h, \alpha\beta^{-1}c(g)) \\ &= \big(\beta \cdot (g, \alpha)\big)(h, \alpha c(g)c(g)^{-1}\beta^{-1}c(g)) \\ &= \big(\beta \cdot (g, \alpha)\big)\big(c(g)^{-1}\beta c(g) \cdot (h, \alpha c(g))\big).\end{aligned}$$

This identity is suggestive of a Zappa-Szép product structure on $\mathcal{G}(c) \times A$ with restriction given by $\beta|_{(g,\alpha)} := c(g)^{-1}\beta c(g)$. The next result says that is indeed the case, although we have to be careful with the unit spaces, which makes the details a little more complicated.

Proposition 5.6. *Fix an étale groupoid $\mathcal{G}$, a discrete group A and a continuous homomorphism $c : \mathcal{G} \to A$. There is a left action of A on the space $\mathcal{G}^{(0)} \times A$ given by $\beta \cdot (u, \alpha) := (u, \alpha\beta^{-1})$. If $\mathcal{H} := A \ltimes (\mathcal{G}^{(0)} \times A)$ denotes the corresponding transformation groupoid, with range and source maps denoted l and r, then the maps $\cdot : \mathcal{H}\ {}_r\!\times_b \mathcal{G}(c) \to \mathcal{G}(c)$ and $| : \mathcal{H}\ {}_r\!\times_b \mathcal{G}(c) \to \mathcal{H}$ given by*

$$(\beta, (gg^{-1}, \alpha)) \cdot (g, \alpha) := (g, \alpha\beta^{-1}) \quad \text{and} \quad (\beta, (gg^{-1}, \alpha))|_{(g,\alpha)} := (c(g)^{-1}\beta c(g), (g^{-1}g, \alpha c(g)))$$

satisfy (ZS1–9), and hence induce a Zappa-Szép product groupoid $\mathcal{G}(c) \bowtie \mathcal{H}$.

The proof of this result is nothing more than a checklist of what it takes for $\beta \cdot (u, \alpha) := (u, \alpha\beta^{-1})$ to give an action, and the axioms (ZS1–9). As each calculation is routine, we leave the details to the reader. Notice that an arbitrary element of $\mathcal{G}(c) \bowtie \mathcal{H}$ has the form $((g, \alpha), (\beta, (g^{-1}g, \alpha c(g)\beta)))$, and is completely determined by the elements $(g, \alpha) \in \mathcal{G}(c)$ and $\beta \in A$. So as a space it is homeomorphic to $\mathcal{G}(c) \times A$ and in some sense should be considered as the Zappa-Szép product of the groupoid $\mathcal{G}(c)$ with the group A.

The C^*-algebras of skew product groupoids are well studied in [17, 8], and we use the notation of [17]. We know from [8] that c induces a coaction $\delta_c : C^*(\mathcal{G}) \to C^*(\mathcal{G}) \otimes C^*(A)$ satisfying $\delta_c(\xi) = \xi \otimes U_b$ whenever $\xi \in C_c(\mathcal{G})$ satisfies $\operatorname{supp}\xi \subset c^{-1}(\{b\})$. In [8, Theorem 4.3] the authors show that $C^*(\mathcal{G}(c))$ is isomorphic to the coaction crossed product $C^*(\mathcal{G}) \rtimes_{\delta_c} A$. The canonical left-action $\beta \cdot (g, \alpha) \mapsto (g, \beta\alpha)$ commutes with right multiplication in $\mathcal{G}(c)$ and hence induces an action $\gamma : A \to \operatorname{Aut} C^*(\mathcal{G}(c))$ characterised by $\gamma_\beta(\xi)(g, \alpha) = \xi(\beta^{-1} \cdot (g, \alpha))$, for $\xi \in C_c(\mathcal{G}(c))$. In [8, Corollary 4.5] it is shown that γ is dual to the coaction δ_c, so that $C^*(\mathcal{G}(c)) \rtimes_\gamma A \cong C^*(\mathcal{G}) \otimes \mathcal{K}(\ell^2(A))$.

The alternative left action $\beta \cdot (g, \alpha) = (g, \alpha\beta^{-1})$ that we used to build the Zappa-Szép product does *not* commute with right multiplication in $\mathcal{G}(c)$, and hence does not induce an action of A on $C^*(\mathcal{G}(c))$. We do not therefore expect a crossed-product description of $C^*(\mathcal{G}(c) \bowtie \mathcal{H})$. Theorem 4.2 does apply and says that $C^*(\mathcal{G}(c) \bowtie \mathcal{H})$ is the blend of $C^*(\mathcal{G}(c))$ and $C^*(\mathcal{H})$.

We can also say a little more about the Zappa-Szép product groupoid $\mathcal{G}(c) \bowtie \mathcal{H}$. There is a right action of $\mathcal{G}$ by automorphisms of A given by $\alpha \cdot g = c(g)^{-1}\alpha c(g)$, from which we can form the semidirect product groupoid $\mathcal{G} \ltimes A$. Pairs $((g, \alpha), (h, \beta))$ are composable in $\mathcal{G} \ltimes A$ if $(g, h) \in \mathcal{G}^{(2)}$, and composition and inversion are given by $(g, \alpha)(h, \beta) = (gh, c(h)^{-1}\alpha c(h)\beta)$ and $(g, \alpha)^{-1} = (c(g)\alpha^{-1}c(g)^{-1}, \alpha^{-1})$. One can check that the map $\widetilde{c} : \mathcal{G} \ltimes A \to A$ satisfying $\widetilde{c}(g, \alpha) = c(g)\alpha$ is a continuous groupoid homomorphism, and that the map $((g, \alpha), \beta) \mapsto ((g, \alpha), (\beta, (g^{-1}g, \alpha c(g)\beta)))$ is a groupoid isomorphism of $(\mathcal{G} \ltimes A)(\widetilde{c})$ onto $\mathcal{G}(c) \bowtie \mathcal{H}$. So despite $C^*(\mathcal{G} \bowtie \mathcal{H})$ not admitting a natural crossed-product description, we can use the results of [8] to describe it as the coaction crossed product $C^*(\mathcal{G} \ltimes A) \rtimes_{\delta_{\widetilde{c}}} A$.

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