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Factorisations of distributive laws

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ABSTRACT

Recently, Böhm and Ştefan constructed duplicial (paracyclic) objects from distributive laws between (co)monads. Here we define the category of factorisations of a distributive law, show that it acts on this construction, and give some explicit examples.

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1. Introduction

Distributive laws between monads were originally defined by Beck in [1] and correspond to monad structures on the composite of the two monads. They have found many applications in mathematics as well as computer science; see e.g. [8,10,18,25,26].

Recently, distributive laws have been used by Böhm and Ştefan [4,6] to construct new examples of duplicial objects [13], and hence cyclic homology theories. The paradigmatic example of such a theory is the cyclic homology $HC(A)$ of an associative algebra A [11,24]. It was observed by Kustermans, Murphy, and Tuset [17] that the functor HC can be twisted by automorphisms of A. The aim of the present paper is to extend this procedure to any duplicial object defined by a distributive law.

Given a distributive law χ we define in Section 3.1 the category $\mathcal{F}(\chi)$ of its *factorisations*. The main technical results are the definition of a monoidal structure on $\mathcal{F}(\chi)$ (Lemma 3.2 and Proposition 3.3), a characterisation of the comonoids in $\mathcal{F}(\chi)$ (Proposition 3.5), and the definition of actions of $\mathcal{F}(\chi)$ on the category of admissible data (septuples in [4]) which turns the latter into an $\mathcal{F}(\chi)$-bimodule category (Theorem 3.8 and Corollary 3.9).

The remainder of the paper is devoted to examples. We begin by considering factorisations of distributive laws on Eilenberg–Moore categories, interpreting these as flat connections (Section 4.1). In particular, we present the twisting of cyclic homology in this framework (Section 4.2). We then describe examples arising

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from Hopf algebras (Section 4.3). The final examples are concerned with BD-laws, braidings (Section 4.4), and quantum doubles of Hopf algebras (Section 4.5).

Throughout this paper, $\mathcal{A}, \mathcal{B}, \mathcal{C} \ldots$ are categories, $A, B, C, \ldots$ are functors, and Greek letters are used to denote natural transformations. We use $\circ$ to denote composition of morphisms and vertical composition of natural transformations. The composition of functors and the horizontal composition of natural transformations will be denoted simply by concatenation. The identity morphism, functor and natural transformation is denoted by id. However, we denote the horizontal composition $\alpha\, \mathsf{id}_A \beta$ by $\alpha A \beta$.

2. Preliminaries

In this section, we recall basic definitions and results that are needed later.

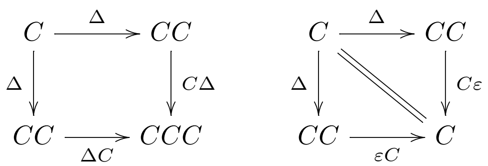
2.1. (Co)monads

Let $\mathcal{A}$ be a category.

Definition 2.1. A *comonad on* $\mathcal{A}$ is a triple $\mathbb{C} = (C, \Delta, \varepsilon)$ where C is an endofunctor on $\mathcal{A}$, and $\Delta: C \longrightarrow CC$ and $\varepsilon: C \longrightarrow \mathsf{id}_{\mathcal{A}}$ are natural transformations such that

$$C\Delta \circ \Delta = \Delta C \circ \Delta, \quad \varepsilon C \circ \Delta = \mathsf{id}_C = C\varepsilon \circ \Delta,$$

that is, the two diagrams



commute.

In other words, a comonad is a comonoid (or coalgebra) in the monoidal category $[\mathcal{A}, \mathcal{A}]$ of endofunctors on $\mathcal{A}$ (with composition as tensor product). Dually, a *monad* on a category $\mathcal{C}$ is a monoid (algebra) in $[\mathcal{C}, \mathcal{C}]$.

2.2. Module categories

Next, we recall the notion of a module category (also known as an $\mathcal{M}$-category) over a monoidal category $(\mathcal{M}, \otimes, \mathbf{1})$, see e.g. [2]. For the purpose of this paper, all monoidal categories and their module categories are strict, and by abuse of notation we will write $\mathcal{M}$ to refer to the whole triple $(\mathcal{M}, \otimes, \mathbf{1})$.

Definition 2.2. A *left module category for* $\mathcal{M}$ is a pair $(\mathcal{C}, \triangleright)$ where $\mathcal{C}$ is a category and $\triangleright: \mathcal{M} \times \mathcal{C} \longrightarrow \mathcal{C}$ is a functor such that we have functorial identities

$$\mathbf{1} \triangleright P = P \quad \text{and} \quad X \triangleright (Y \triangleright P) = (X \otimes Y) \triangleright P$$

for all objects X, Y in $\mathcal{M}$ and P in $\mathcal{C}$. We call $\triangleright$ the *left action of* $\mathcal{M}$ *on* $\mathcal{C}$.

Dually, one defines a *right module category* $(\mathcal{D}, \triangleleft)$. A *bimodule category* is a triple $(\mathcal{C}, \triangleright, \triangleleft)$ where $(\mathcal{C}, \triangleright)$ and $(\mathcal{C}, \triangleleft)$ are right respectively left module categories and the actions commute, i.e. for all objects X, Y in $\mathcal{M}$ and P in $\mathcal{C}$ we have

$$X \triangleright (P \triangleleft Y) = (X \triangleright P) \triangleleft Y,$$

again functorially in X, Y and P. We immediately have the following.

Lemma 2.3. *Let $(\mathcal{C}, \triangleright)$ and $(\mathcal{D}, \triangleleft)$ be left respectively right module categories. Then $\mathcal{C} \times \mathcal{D}$ is a bimodule category with actions given by*

$$X \triangleright (P, Q) \triangleleft Y = (X \triangleright P, Q \triangleleft Y)$$

for all objects X, Y in $\mathcal{M}$, P in $\mathcal{C}$ and Q in $\mathcal{D}$.

2.3. Eilenberg–Moore categories

The comonads we are mostly interested in arise as restrictions of monads to their Eilenberg–Moore categories in the sense of [21]:

Definition 2.4. Let $(\mathcal{C}, \triangleright)$ be a left module category for a monoidal category $\mathcal{M}$, and let $\mathbb{B} = (B, \mu, \eta)$ be a monoid in $\mathcal{M}$. The *Eilenberg–Moore category of* $\mathbb{B}$, denoted by $\mathcal{C}^{\mathbb{B}}$, is the category whose objects are pairs (X, α), where X is an object of $\mathcal{C}$ and $\alpha: B \triangleright X \longrightarrow X$ is a morphism in $\mathcal{C}$ such that the diagrams

$$\begin{array}{ccccc} (B \otimes B) \triangleright X & = & B \triangleright (B \triangleright X) & \xrightarrow{\mathsf{id}_B \triangleright \alpha} & B \triangleright X \\ & \searrow^{\mu \triangleright \mathsf{id}_X} & & & \downarrow \alpha \\ & & B \triangleright X & \xrightarrow[\alpha]{} & X \end{array} \qquad \begin{array}{ccc} \mathbf{1} \triangleright X & \xrightarrow{\eta \triangleright \mathsf{id}_X} & B \triangleright X \\ & \searrow^{=} & \downarrow \alpha \\ & & X \end{array}$$

commute. The morphisms $f: (X, \alpha) \longrightarrow (X', \alpha')$ are morphisms $f: X \longrightarrow X'$ in $\mathcal{C}$ such that the diagram

$$\begin{array}{ccc} B \triangleright X & \xrightarrow{\mathsf{id}_B \triangleright f} & B \triangleright X' \\ \alpha \downarrow & & \downarrow \alpha' \\ X & \xrightarrow[f]{} & X' \end{array}$$

commutes.

Now observe that the monoid $\mathbb{B}$ defines a comonad $\tilde{\mathbb{B}} = (\tilde{B}, \tilde{\Delta}, \tilde{\varepsilon})$ on $\mathcal{A} = \mathcal{C}^{\mathbb{B}}$ where $\tilde{B}$ is defined on objects and morphisms by

$$\tilde{B}(X, \alpha) = (B \triangleright X, \mu \triangleright \mathsf{id}_X), \qquad \tilde{B}(f) = \mathsf{id}_B \triangleright f,$$

and $\tilde{\Delta}$, $\tilde{\varepsilon}$ are defined on objects (X, α) by

$$B \triangleright X = B \triangleright (\mathbf{1} \triangleright X) \xrightarrow{\mathsf{id}_B \triangleright (\eta \triangleright \mathsf{id}_X)} B \triangleright (B \triangleright X) \qquad B \triangleright X \xrightarrow{\alpha} X$$

respectively.

In particular, every category $\mathcal{C}$ is in an obvious way a module category over $[\mathcal{C}, \mathcal{C}]$. In this case, our definition of Eilenberg–Moore category of a monad $\mathbb{B}$ on $\mathcal{C}$ is the same as the usual definition [19, p. 139].

2.4. Distributive laws

Next we define distributive laws. Note that we consider them between (co)monads and arbitrary endofunctors as is common in the computer science literature, see e.g. [25].

Definition 2.5. Let $\mathbb{T} = (T, \Delta, \varepsilon)$ be a comonad on $\mathcal{A}$ and let C be an endofunctor on $\mathcal{A}$. A *distributive law between the comonad* $\mathbb{T}$ *and the endofunctor* C is a transformation $\chi: TC \longrightarrow CT$ such that the two diagrams

$$\begin{array}{ccccc} TC & \xrightarrow{\chi} & CT & \xrightarrow{C\Delta} & CTT \\ {\scriptstyle \Delta C}\downarrow & & & & \uparrow{\scriptstyle \chi T} \\ TTC & & \xrightarrow[T\chi]{} & & TCT \end{array} \qquad \begin{array}{ccc} TC & \xrightarrow{\chi} & CT \\ & {\scriptstyle \varepsilon C}\searrow & \downarrow{\scriptstyle C\varepsilon} \\ & & C \end{array}$$

commute. We denote this by $\chi: \mathbb{T} \longrightarrow C$. Analogously, we define a distributive law $\chi: T \longrightarrow \mathbb{C}$ between an endofunctor T and a comonad $\mathbb{C}$. A *comonad distributive law* $\chi: \mathbb{T} \longrightarrow \mathbb{C}$ is a transformation χ which is a distributive law between endofunctors and comonads in both ways.

Dually, we can define distributive laws involving monads; distributive laws from a monad to a comonad are usually called mixed distributive laws.

One application of distributive laws is to lift endofunctors to Eilenberg–Moore categories: let $\mathbb{B}$ be a monad on a category $\mathcal{C}$ and $\theta: \mathbb{B} \longrightarrow D$ be a distributive law. We define a functor $\tilde{D}: \mathcal{C}^{\mathbb{B}} \longrightarrow \mathcal{C}^{\mathbb{B}}$ as follows. On objects we define

$$\tilde{D}(X, \alpha) = (DX, D\alpha \circ \theta_X)$$

and we define $\tilde{D}f = Df$ on morphisms. The distributive law θ lifts to give one $\theta: \tilde{\mathbb{B}} \longrightarrow \tilde{D}$ where $\tilde{\mathbb{B}}$ is the comonad described in Section 2.3. If D is part of a comonad $\mathbb{D} = (D, \Delta, \varepsilon)$, and θ is a mixed distributive law $\mathbb{B} \longrightarrow \mathbb{D}$, then $\tilde{D}$ is part of a comonad

$$\tilde{\mathbb{D}} = (\tilde{D}, \Delta, \varepsilon)$$

and θ lifts to a comonad distributive law $\theta: \tilde{\mathbb{B}} \longrightarrow \tilde{\mathbb{D}}$.

See [1,7] for more details on distributive laws.

2.5. The categories of χ-coalgebras

Let $\mathbb{T} = \left(T, \Delta^T, \varepsilon^T\right)$ and $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ be comonads on $\mathcal{A}$, and let $\chi: \mathbb{T} \longrightarrow \mathbb{C}$ be a distributive law.

Definition 2.6. A *right* χ*-coalgebra* is a triple $(M, \mathcal{X}, \rho)$ where $\mathcal{X}$ is a category, $M: \mathcal{X} \longrightarrow \mathcal{A}$ is a functor and $\rho: TM \longrightarrow CM$ is a natural transformation such that the diagrams

$$\begin{array}{ccccc} TM & \xrightarrow{\Delta^T M} & TTM & \xrightarrow{T\rho} & TCM \\ {\scriptstyle \rho}\downarrow & & & & \downarrow{\scriptstyle \chi_M} \\ CM & \xrightarrow[\Delta^C M]{} & CCM & \xleftarrow[C\rho]{} & CTM \end{array} \qquad \begin{array}{ccc} & & TM \\ & {\scriptstyle \varepsilon^T M}\swarrow & \downarrow{\scriptstyle \rho} \\ M & \xleftarrow[\varepsilon^C M]{} & CM \end{array}$$

commute. A *morphism of right χ-coalgebras* between $(M, \mathcal{X}, \rho)$ and $(M', \mathcal{X}', \rho')$ is a pair (φ, F), where $F\colon \mathcal{X} \longrightarrow \mathcal{X}'$ is a functor and $\varphi\colon M \longrightarrow M'F$ is a natural transformation such that the diagram

$$\begin{array}{ccc} TM & \xrightarrow{T\varphi} & TM'F \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\rho' F} \\ CM & \xrightarrow[C\varphi]{} & CM'F \end{array}$$

commutes. We define composition of morphisms by

$$(\varphi', F') \circ (\varphi, F) = (\varphi' F \circ \varphi, F'F)$$

and we define identity morphisms by $\mathsf{id}_{(M,\mathcal{X},\rho)} = (\mathsf{id}_M, \mathsf{id}_{\mathcal{X}})$. We denote the category of right χ-coalgebras by $\mathcal{R}(\chi)$.

Dually, we define the category $\mathcal{L}(\chi)$ of *left χ-coalgebras* $(N, \mathcal{Y}, \lambda)$.

2.6. The construction of Böhm and Ştefan

Finally, we recall the construction of duplicial functors from a comonad distributive law $\chi\colon \mathbb{T} \longrightarrow \mathbb{C}$ on a category $\mathcal{A}$ due to Böhm and Ştefan.

Definition 2.7. The category of *admissible data over* χ is the product category

$$\mathcal{S}(\chi) := \mathcal{R}(\chi) \times \mathcal{L}(\chi).$$

Admissible data are called *admissible septuples* in [4].

To every admissible datum $(M, \mathcal{X}, \rho, N, \mathcal{Y}, \lambda)$ there is an associated duplicial functor $\mathcal{X} \longrightarrow \mathcal{Y}$ defined by

$$D_\bullet(M, \mathcal{X}, \rho, N, \mathcal{Y}, \lambda) = NT^{\bullet+1}M$$

which is given objectwise by taking the bar resolution of M with respect to the comonad $\mathbb{T}$, and then applying the functor N. If $\mathcal{Y}$ is an abelian category, we can apply the duplicial functor to an object X in $\mathcal{X}$ resulting in a duplicial object in $\mathcal{Y}$ of which we can take the cyclic homology.

This construction, which unifies and generalises the definition of the cyclic homology of associative algebras and Hopf algebras, is detailed in [4,6].

3. Theory

3.1. The category of factorisations $\mathcal{F}(\chi)$

Throughout this section, let $\mathbb{T} = \left(T, \Delta^T, \varepsilon^T\right)$ and $\mathbb{C} = \left(C, \Delta^C, \varepsilon^C\right)$ be comonads on a category $\mathcal{A}$, and let $\chi\colon \mathbb{T} \longrightarrow \mathbb{C}$ be a distributive law. The main definition of the present paper is the following:

Definition 3.1. A *factorisation of* χ is a triple (Σ, σ, γ) where Σ is an endofunctor on $\mathcal{A}$, and $\sigma\colon \mathbb{T} \longrightarrow \Sigma$ and $\gamma\colon \Sigma \longrightarrow \mathbb{C}$ are distributive laws satisfying the *Yang–Baxter* condition; that is, the hexagon

$$
\begin{tikzcd}
 & \Sigma T C \arrow[r, "\Sigma\chi"] & \Sigma C T \arrow[dr, "\gamma T"] & \\
T\Sigma C \arrow[ur, "\sigma C"] \arrow[dr, "T\gamma"'] & & & C\Sigma T \\
 & T C \Sigma \arrow[r, "\chi\Sigma"'] & C T \Sigma \arrow[ur, "C\sigma"'] &
\end{tikzcd}
$$

commutes. A *morphism* $\alpha\colon (\Sigma, \sigma, \gamma) \longrightarrow (\Sigma', \sigma', \gamma')$ of factorisations is a natural transformation $\alpha\colon \Sigma \longrightarrow \Sigma'$ which is compatible with T and C in the sense that the diagrams

$$
\begin{tikzcd}
T\Sigma \arrow[r, "T\alpha"] \arrow[d, "\sigma"'] & T\Sigma' \arrow[d, "\sigma'"] \\
\Sigma T \arrow[r, "\alpha T"'] & \Sigma' T
\end{tikzcd}
\qquad
\begin{tikzcd}
\Sigma C \arrow[r, "\alpha C"] \arrow[d, "\gamma"'] & \Sigma' C \arrow[d, "\gamma'"] \\
C\Sigma \arrow[r, "C\alpha"'] & C\Sigma'
\end{tikzcd}
$$

commute. There are identity morphisms $\mathsf{id}_{(\Sigma,\sigma,\gamma)} = \mathsf{id}_\Sigma$, and composition of morphisms is given by the vertical composite. This defines the *category of factorisations* which we denote by $\mathcal{F}(\chi)$.

Similarly, we define factorisations of a monad or mixed distributive law.

3.2. The monoidal structure

We define a functor

$$\otimes\colon \mathcal{F}(\chi) \times \mathcal{F}(\chi) \longrightarrow \mathcal{F}(\chi)$$

as follows. On objects we define

$$(\Sigma, \sigma, \gamma) \otimes (\Sigma', \sigma', \gamma') = (\Sigma\Sigma', \Sigma\sigma' \circ \sigma\Sigma', \gamma\Sigma' \circ \Sigma\gamma')$$

and for two morphisms α, β we define $\alpha \otimes \beta$ to be $\alpha\beta$, the horizontal composite of the natural transformations.

Lemma 3.2. *The assignment $\otimes$ is a well-defined functor.*

Proof. Firstly, $\otimes$ is well-defined on objects if $\Sigma\sigma' \circ \sigma\Sigma'$ and $\gamma\Sigma' \circ \Sigma\gamma'$ satisfy the Yang–Baxter condition. Consider the following diagram

$$
\begin{tikzcd}
 & & \Sigma\Sigma' T C \arrow[r, "\Sigma\Sigma'\chi"] & \Sigma\Sigma' C T \arrow[dr, "\Sigma\gamma' T"] & & \\
 & \Sigma T \Sigma' C \arrow[ur, "\Sigma\sigma' C"] \arrow[dr, "\Sigma T\gamma'"'] & & & \Sigma C \Sigma' T \arrow[dr, "\gamma\Sigma' T"] & \\
T\Sigma\Sigma' C \arrow[ur, "\sigma\Sigma' C"] \arrow[dr, "T\Sigma\gamma'"'] & & \Sigma T C \Sigma' \arrow[r, "\Sigma\chi\Sigma'"'] & \Sigma C T \Sigma' \arrow[ur, "\Sigma C\sigma'"'] \arrow[dr, "\gamma T\Sigma'"'] & & C\Sigma\Sigma' T \\
 & T\Sigma C \Sigma' \arrow[ur, "\sigma C\Sigma'"'] \arrow[dr, "T\gamma\Sigma'"'] & & & C\Sigma T \Sigma' \arrow[ur, "C\Sigma\sigma'"'] & \\
 & & T C \Sigma\Sigma' \arrow[r, "\chi\Sigma\Sigma'"'] & C T \Sigma\Sigma' \arrow[ur, "C\sigma\Sigma'"'] & &
\end{tikzcd}
$$

The left square commutes by naturality of σ and the right square commutes by naturality of γ. The inner hexagons commute by the Yang–Baxter conditions. Therefore, the outer hexagon commutes, so the required condition is satisfied.

Secondly, let

$$\alpha\colon(\Sigma,\sigma,\gamma)\longrightarrow(\Gamma,\kappa,\nu)\quad\text{and}\quad\beta\colon(\Sigma',\sigma',\gamma')\longrightarrow(\Gamma',\kappa',\nu')$$

be morphisms in $\mathcal{F}(\chi)$. Consider the diagram

$$\begin{array}{ccccc}
T\Sigma\Sigma' & \xrightarrow{T\alpha\Sigma'} & T\Gamma'\Sigma' & \xrightarrow{T\Gamma\beta} & T\Gamma\Gamma' \\
{\scriptstyle\sigma\Sigma'}\downarrow & & \downarrow{\scriptstyle\kappa\Sigma'} & & \downarrow{\scriptstyle\kappa\Gamma'} \\
\Sigma T\Sigma' & \xrightarrow[\alpha T\Sigma']{} & \Gamma T\Sigma' & \xrightarrow[\Gamma T\beta]{} & \Gamma T\Gamma' \\
{\scriptstyle\Sigma\sigma'}\downarrow & & \downarrow{\scriptstyle\Gamma\sigma'} & & \downarrow{\scriptstyle\Gamma\kappa'} \\
\Sigma\Sigma' T & \xrightarrow[\alpha\Sigma' T]{} & \Gamma\Sigma' T & \xrightarrow[\Gamma\beta T]{} & \Gamma\Gamma' T
\end{array}$$

The bottom-left square commutes by naturality of α, the top-right square commutes by naturality of κ, and the two remaining inner squares commute since α and β are compatible with T. Therefore, the outer square commutes and $\alpha\otimes\beta$ is compatible with T. A similar argument shows that $\alpha\otimes\beta$ is compatible with C. It is clear that $\otimes$ respects composition of morphisms and identity morphisms. Therefore, $\otimes$ is well-defined on morphisms. □

Let $\mathbf{1}$ denote the trivial factorisation $(\mathsf{id}_{\mathcal{A}}, \mathsf{id}_T, \mathsf{id}_C)$.

Proposition 3.3. *The triple* $(\mathcal{F}(\chi),\otimes,\mathbf{1})$ *is a strict monoidal category.*

Proof. It is clear that $\mathcal{T}\otimes\mathbf{1}=\mathbf{1}\otimes\mathcal{T}=\mathcal{T}$ for all factorisations $\mathcal{T}$. Consider the products of factorisations

$$\begin{aligned}
&((\Sigma,\sigma,\gamma)\otimes(\Sigma',\sigma',\gamma'))\otimes(\Sigma'',\sigma'',\gamma'')\\
&\quad=(\Sigma\Sigma',\Sigma\sigma'\circ\sigma\Sigma',\gamma\Sigma'\circ\Sigma\gamma')\otimes(\Sigma'',\sigma'',\gamma'')\\
&\quad=(\Sigma\Sigma'\Sigma'',\Sigma\Sigma'\sigma''\circ\Sigma\sigma'\Sigma''\circ\sigma\Sigma'\Sigma'',\gamma\Sigma'\Sigma''\circ\Sigma\gamma'\Sigma''\circ\Sigma\Sigma'\gamma'')
\end{aligned}$$

and

$$\begin{aligned}
&(\Sigma,\sigma,\gamma)\otimes((\Sigma',\sigma',\gamma')\otimes(\Sigma'',\sigma'',\gamma''))\\
&\quad=(\Sigma,\sigma,\gamma)\otimes(\Sigma'\Sigma'',\Sigma'\sigma''\circ\sigma'\Sigma'',\gamma'\Sigma''\circ\Sigma'\gamma'')\\
&\quad=(\Sigma\Sigma'\Sigma'',\Sigma\Sigma'\sigma''\circ\Sigma\sigma'\Sigma''\circ\sigma\Sigma'\Sigma'',\gamma\Sigma'\Sigma''\circ\Sigma\gamma'\Sigma''\circ\Sigma\Sigma'\gamma'').
\end{aligned}$$

These are equal so $\otimes$ is an associative tensor product (observe that all equalities are functorial). □

Remark 3.4. If we ignore set theoretic issues, we can define a 2-category

$$\underline{\mathsf{dist}}:=\mathbf{Cmd}(\mathbf{Cmd}(\underline{\mathsf{CAT}}^{\mathrm{op}})^{\mathrm{op}})$$

where $\underline{\mathsf{CAT}}$ is the 2-category of categories, functors and natural transformations, **Cmd** denotes taking the 2-category of comonads, and op denotes reversal of 1-cells. The 0-cells of this 2-category are comonad distributive laws χ and we have

$$\mathcal{F}(\chi)=\underline{\mathsf{dist}}(\chi,\chi)$$

which is a strict monoidal category. This gives another proof of Proposition 3.3. See [22,3] for the definition of **Cmd**.

3.3. (Co)monads as (co)monoids in $\mathcal{F}(\chi)$

By definition, a pair of morphisms

$$\Delta\colon (\Sigma,\sigma,\gamma) \longrightarrow (\Sigma,\sigma,\gamma)\otimes(\Sigma,\sigma,\gamma), \quad \varepsilon\colon (\Sigma,\sigma,\gamma)\longrightarrow \mathbf{1}$$

is a pair of natural transformations $\Delta\colon \Sigma \longrightarrow \Sigma\Sigma$ and $\varepsilon\colon \Sigma \longrightarrow \mathbf{1}$ that are compatible with the distributive laws σ and γ. This gives us the following characterisation of comonoids in $\mathcal{F}(\chi)$.

Proposition 3.5. *A factorisation (Σ,σ,γ) is a comonoid in $\mathcal{F}(\chi)$ if and only if Σ is part of a comonad and σ, γ are distributive laws of comonads.*

Dually, a factorisation (Σ,σ,γ) is a monoid in $\mathcal{F}(\chi)$ if and only if Σ is part of a monad and σ, γ are mixed distributive laws between monads and comonads.

Corollary 3.6. *Let $\chi\colon \mathsf{id}_{\mathcal{A}} \longrightarrow \mathsf{id}_{\mathcal{A}}$ be the trivial distributive law given by the identity. Then (T,Δ,ε) is a comonad on $\mathcal{A}$ if and only if $(T,\mathsf{id}_T,\mathsf{id}_T)$ is a comonoid in $\mathcal{F}(\chi)$, and (B,μ,η) is a monad on $\mathcal{A}$ if and only if $(B,\mathsf{id}_B,\mathsf{id}_B)$ is a monoid in $\mathcal{F}(\chi)$.*

3.4. Module categories for $\mathcal{F}(\chi)$

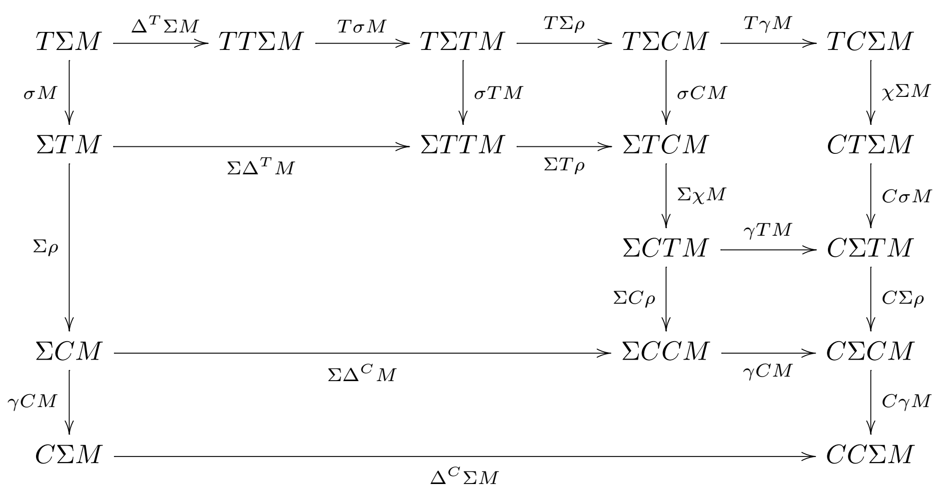
We define a functor $\triangleright\colon \mathcal{F}(\chi)\times\mathcal{R}(\chi)\longrightarrow\mathcal{R}(\chi)$ as follows. On objects we define

$$(\Sigma,\sigma,\gamma)\triangleright(M,\mathcal{X},\rho) = (\Sigma M,\mathcal{X},\gamma M\circ\Sigma\rho\circ\sigma M)$$

and on morphisms we define $\alpha\triangleright(\varphi,F)$ to be the pair $(\alpha\varphi,F)$.

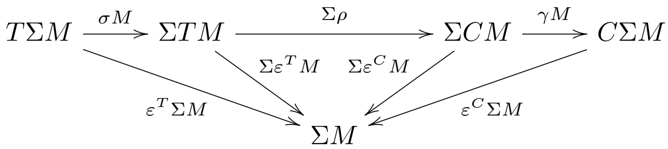
Proposition 3.7. *The assignment $\triangleright$ is a well-defined functor.*

Proof. Consider the diagram



The top-left and bottom rectangles commute by the distributive law axioms, the middle-left rectangle commutes because $(M, \mathcal{X}, \rho)$ is a right χ-coalgebra, the top-right diagram commutes by the Yang–Baxter condition, and the remaining squares commute by naturality of σ, γ. Therefore, the outer rectangle commutes.

Consider the triangle



The middle triangle commutes because $(M, \mathcal{X}, \rho)$ is a right χ-coalgebra, and the other two inner triangles commute by the distributive law axioms. Therefore, the outer triangle commutes. This shows that $\triangleright$ is well-defined on objects.

Let $(\varphi, F)\colon (M, \mathcal{X}, \rho) \longrightarrow (M', \mathcal{X}', \rho')$ and $\alpha\colon (\Sigma, \sigma, \gamma) \longrightarrow (\Sigma', \sigma', \gamma')$ be morphisms of right χ-coalgebras and factorisations, respectively. Consider the diagram

$$\begin{array}{ccccc}
T\Sigma M & \xrightarrow{T\alpha M} & T\Sigma' M & \xrightarrow{T\Sigma'\varphi} & T\Sigma' M' F \\
{\scriptstyle \sigma M}\downarrow & & \downarrow{\scriptstyle \sigma M'} & & \downarrow{\scriptstyle \sigma' M' F} \\
\Sigma T M & \xrightarrow{\alpha T M} & \Sigma' T M & \xrightarrow{\Sigma' T\varphi} & \Sigma' T M' F \\
{\scriptstyle \Sigma\rho}\downarrow & & \downarrow{\scriptstyle \Sigma'\rho} & & \downarrow{\scriptstyle \Sigma'\rho' F} \\
\Sigma C M & \xrightarrow[\alpha C M]{} & \Sigma' C M & \xrightarrow[\Sigma' C\varphi]{} & \Sigma' C M' F
\end{array}$$

The top-left square commutes since α is compatible with T, the top-right square commutes by naturality of σ, the bottom-left square commutes by naturality of α, and the bottom-right square commutes since (φ, F) is a right χ-coalgebra morphism. Thus the outer square commutes, which shows that $\alpha \triangleright (\varphi, F)$ is a right χ-coalgebra morphism.

It is clear that $\triangleright$ respects identities and composition of morphisms (because the vertical and horizontal compositions of natural transformations are compatible with each other), so $\triangleright$ is well-defined on morphisms. □

Dually, we also define a functor

$$\triangleleft\colon \mathcal{L}(\chi) \times \mathcal{F}(\chi) \longrightarrow \mathcal{L}(\chi).$$

Theorem 3.8. *The category $\mathcal{R}(\chi)$ is a strict left module category for $\mathcal{F}(\chi)$, with left action given by the functor $\triangleright$. Furthermore, the category $\mathcal{L}(\chi)$ is a strict right module category for $\mathcal{F}(\chi)$, with right action given by the functor $\triangleleft$.*

Proof. We will prove only the first statement, as the second follows by a similar argument. It is clear that **1** acts as the identity. Let $(\Sigma, \sigma, \gamma), (\Sigma', \sigma', \gamma')$ be two factorisations and let $(M, \mathcal{X}, \rho)$ be a right χ-coalgebra. We have

$$\begin{aligned}
&((\Sigma, \sigma, \gamma) \otimes (\Sigma', \sigma', \gamma')) \triangleright (M, \mathcal{X}, \rho) \\
&\quad = (\Sigma\Sigma', \Sigma\sigma' \circ \sigma\Sigma', \gamma\Sigma' \circ \Sigma\gamma') \triangleright (M, \mathcal{X}, \rho) \\
&\quad = (\Sigma\Sigma' M, \mathcal{X}, \gamma\Sigma' M \circ \Sigma\gamma' M \circ \Sigma\Sigma'\rho \circ \Sigma\sigma' M \circ \sigma\Sigma' M)
\end{aligned}$$

and

$$\begin{aligned}(\Sigma,\sigma,\gamma) &\rhd ((\Sigma',\sigma',\gamma') \rhd (M,\mathcal{X},\rho))\\ &= (\Sigma,\sigma,\gamma) \rhd (\Sigma M,\mathcal{X},\gamma M \circ \Sigma\rho \circ \sigma M)\\ &= (\Sigma\Sigma' M,\mathcal{X},\gamma\Sigma' M \circ \Sigma\gamma' M \circ \Sigma\Sigma'\rho \circ \Sigma\sigma' M \circ \sigma\Sigma' M)\end{aligned}$$

These are functorially equal, so $\rhd$ is a left action of $\mathcal{F}(\chi)$. □

Corollary 3.9. *The category $\mathcal{S}(\chi)$ is a strict bimodule category for $\mathcal{F}(\chi)$.*

Proof. This follows immediately by applying Lemma 2.3 to Theorem 3.8. □

4. Examples

4.1. Flat connections

Let $\mathbb{B} = (B,\mu,\eta)$ be a monad on a category $\mathcal{C}$. The forgetful functor $U: \mathcal{C}^{\mathbb{B}} \longrightarrow \mathcal{C}$ has a left adjoint F defined by

$$F(X,\alpha) = (BX,\mu_X), \qquad F(f) = Bf.$$

The unit of this adjunction is given by η and the counit is $\tilde{\varepsilon}_{(X,\alpha)} = \alpha$. Let $\tilde{B}$ denote the functor FU and let $\tilde{\Delta}$ denote the natural transformation $F\eta U$. The adjunction gives rise to a comonad $\tilde{\mathbb{B}} = (\tilde{B},\tilde{\Delta},\tilde{\varepsilon})$, which is the same as the comonad discussed in Section 2.3.

Let $\Sigma: \mathcal{C}^{\mathbb{B}} \longrightarrow \mathcal{C}^{\mathbb{B}}$ be an endofunctor. For every object (X,α) in $\mathcal{C}^{\mathbb{B}}$ there are natural isomorphisms

$$\mathcal{C}^{\mathbb{B}}(\tilde{B}\Sigma(X,\alpha),\Sigma\tilde{B}(X,\alpha)) \cong \mathcal{C}(U\Sigma(X,\alpha),U\Sigma\tilde{B}(X,\alpha))$$

given by the adjunction, so there is a one-to-one correspondence between natural transformations $\sigma: \tilde{B}\Sigma \longrightarrow \Sigma\tilde{B}$ and natural transformations $\nabla: U\Sigma \longrightarrow U\Sigma\tilde{B}$. In fact, σ is a distributive law if and only if the diagrams

$$\begin{array}{ccc} U\Sigma & \xrightarrow{\nabla} & U\Sigma\tilde{B} \\ {\scriptstyle\nabla}\downarrow & & \downarrow{\scriptstyle\nabla\tilde{B}} \\ U\Sigma\tilde{B} & \xrightarrow[U\Sigma\tilde{\Delta}]{} & U\Sigma\tilde{B}\tilde{B} \end{array} \qquad\qquad \begin{array}{ccc} U\Sigma & \xrightarrow{\nabla} & U\Sigma\tilde{B} \\ & \searrow\!\!\!\!= & \downarrow{\scriptstyle U\Sigma\tilde{\varepsilon}} \\ & & U\Sigma \end{array}$$

commute.

Definition 4.1. We say that the natural transformation σ is a *connection* if $\tilde{\varepsilon}$ is compatible with σ, i.e. the second diagram above commutes for the corresponding natural transformation ∇. We say that a connection σ is *flat* if $\tilde{\Delta}$ is compatible with σ, i.e. σ is a distributive law, or equivalently, both diagrams above commute.

The terminology is motivated by the special case discussed in detail in the following section.

4.2. (A,A)-bimodules

Let k be a commutative ring and let A be a unital associative algebra over k. Let $\mathcal{C} = A\text{-}\mathsf{Mod}$ be the category of left A-modules. The functor $B = - \otimes_k A: \mathcal{C} \longrightarrow \mathcal{C}$, together with the natural transformations

$$\begin{aligned}\mu_M\colon M\otimes_k A\otimes_k A &\longrightarrow M\otimes_k A & \eta_M\colon M &\longrightarrow M\otimes_k A\\ m\otimes a\otimes b&\longmapsto m\otimes ab & m&\longmapsto m\otimes 1\end{aligned}$$

defines a monad $\mathbb{B}$ on $\mathcal{C}$ which lifts to a comonad $\tilde{\mathbb{B}}$ on $\mathcal{C}^{\mathbb{B}}$. The latter is isomorphic to the category of (A,A)-bimodules (with symmetric action of k).

The functor $D = A\otimes_k -\colon \mathcal{C}\longrightarrow\mathcal{C}$, together with the natural transformations

$$\begin{aligned}\Delta_M\colon A\otimes_k M &\longrightarrow A\otimes_k A\otimes_k M & \varepsilon_M\colon A\otimes_k M &\longrightarrow M\\ a\otimes m&\longmapsto a\otimes 1\otimes m & a\otimes m&\longmapsto am\end{aligned}$$

defines a comonad $\mathbb{D}$ on $\mathcal{C}$. There is a mixed distributive law $\theta\colon\mathbb{B}\longrightarrow\mathbb{D}$ given by rebracketing on components

$$\theta_M\colon (A\otimes_k M)\otimes_k A\longrightarrow A\otimes_k (M\otimes_k A)$$

so this lifts to a comonad distributive law $\theta\colon\tilde{\mathbb{B}}\longrightarrow\tilde{\mathbb{D}}$.

Let N be an (A,A)-bimodule and $\Sigma\colon\mathcal{C}^{\mathbb{B}}\longrightarrow\mathcal{C}^{\mathbb{B}}$ be the functor defined by $\Sigma(M) = M\otimes_A N$. We have that $\Sigma\tilde{D} = \tilde{D}\Sigma$ so the identity $\mathsf{id}_{\Sigma\tilde{D}}\colon\Sigma\longrightarrow\mathbb{D}$ is a distributive law.

In this case, the components of a natural transformation $\nabla\colon U\Sigma\longrightarrow U\Sigma\tilde{B}$ are given by a left A-linear map

$$\nabla_M\colon M\otimes_A N\longrightarrow (M\otimes_k A)\otimes_A N\cong M\otimes_k N$$

The corresponding natural transformation $\sigma\colon\tilde{\mathbb{B}}\longrightarrow\Sigma$ is given by

$$\begin{aligned}\sigma_M\colon (M\otimes_A N)\otimes_k A&\longrightarrow (M\otimes_k A)\otimes_A N\cong M\otimes_k N\\ (m\otimes_A n)\otimes b&\longmapsto \nabla_M(m\otimes_A n)b.\end{aligned}$$

The natural transformation ∇ defines a connection if and only if each ∇_M splits the quotient map $M\otimes_k N\longrightarrow M\otimes_A N$. Taking $M = A$ yields an A-linear splitting of the action $A\otimes_k N\longrightarrow N$, so N is k-relative projective. Conversely, given a splitting $n\mapsto n_{(-1)}\otimes n_{(0)}$ of the action, we obtain ∇_M as $\nabla_M(m\otimes_A n) = mn_{(-1)}\otimes n_{(0)}$.

Thus we have:

Proposition 4.2. *The functor Σ admits a connection σ if and only if N is k-relative projective as a left A-module.*

Composing ∇_M with the noncommutative De Rham differential

$$\mathrm{d}\colon A\longrightarrow\Omega^1_{A,k},\quad a\longmapsto 1\otimes a - a\otimes 1$$

gives the notion of connection in noncommutative geometry [12, III.3.5].

If N is not just k-relative projective but k-relative free, i.e. $N\cong A\otimes_k V$ as left A-modules, for some k-module V, then the assignment $\nabla_M(m\otimes_A (a\otimes v)) = ma\otimes(1\otimes v)$ defines a flat connection. Thus we have:

Proposition 4.3. *The triple $(\Sigma,\sigma,\mathsf{id}_{\Sigma\tilde{D}})$ is a factorisation of θ.*

In particular, let $\sigma\colon A\longrightarrow A$ be an algebra map and $N = A_\sigma$, the (A,A)-bimodule which is A as a left A-module with right action of $a\in A$ given by right multiplication by $\sigma(a)$. Then we have $\Sigma(M) =$

$M \otimes_A A_\sigma \cong M_\sigma$. Since A_σ is free as a left A-module we get a factorisation $(\Sigma, \sigma, \mathsf{id}_{\Sigma\tilde{D}})$ by Proposition 4.3, where $\sigma: \tilde{\mathbb{B}} \longrightarrow \Sigma$ is the flat connection defined on components by

$$\sigma_M: M_\sigma \otimes_k A \longrightarrow (M \otimes_k A)_\sigma$$
$$m \otimes a \longmapsto m \otimes \sigma(a).$$

Note that we use σ to denote both the algebra map and the flat connection.

From the general theory developed in Section 3 we obtain therefore an action of the group of endomorphisms of A on the category of admissible data for θ. In particular, we can act on the standard cyclic object associated to A [11,24], which corresponds to the following admissible datum.

Consider A as a functor $A: \{*\} \longrightarrow \mathcal{C}^{\mathbb{B}}$ from the one-morphism category to the category of (A, A)-bimodules. Since $\tilde{B}A = \tilde{D}A = A \otimes_k A$ we have a natural transformation $\rho = \mathsf{id}_{A\otimes_k A}: \tilde{B}A \longrightarrow \tilde{D}A$. The triple $(A, \{*\}, \rho)$ is a right θ-coalgebra.

Considering (A, A)-bimodules as either left or right $A^e := A \otimes_k A^{\mathrm{op}}$-modules, we view the zeroth Hochschild homology as a functor $H = - \otimes_{A^e} A: \mathcal{C}^{\mathbb{B}} \longrightarrow k\text{-}\mathsf{Mod}$. We define a natural transformation $\lambda: H\tilde{D} \longrightarrow H\tilde{B}$ by

$$\lambda_M: (A \otimes_k M) \otimes_{A^e} A \longrightarrow (M \otimes_k A) \otimes_{A^e} A \cong M$$
$$(a \otimes m) \otimes_{A^e} b \longmapsto mba$$

The pair $(H, k\text{-}\mathsf{Mod}, \lambda)$ is a left θ-coalgebra, and the duplicial k-module associated to the admissible datum $(A, \{*\}, \rho, H, k\text{-}\mathsf{Mod}, \lambda)$ is indeed the cyclic object defining the cyclic homology $HC(A)$.

The cyclic homology of the duplicial object associated to the admissible datum

$$(\Sigma, \sigma, \mathsf{id}_{\Sigma\tilde{D}}) \triangleright (A, \{*\}, \rho, H, k\text{-}\mathsf{Mod}, \lambda) = (A_\sigma, \{*\}, \rho \circ \sigma_A, H, k\text{-}\mathsf{Mod}, \lambda)$$

is $HC^\sigma(A)$, the σ-twisted cyclic homology of A. This was first considered in [17] and is discussed in Section 5.2 of [14] in the context of Hopf algebroids. Thus the action of the category of factorisations generalises this twisting procedure.

4.3. Mixed factorisations

Let $\mathbb{B} = (B, \mu, \eta)$ be a monad on a category $\mathcal{C}$ and let $\Sigma: \mathcal{C}^{\mathbb{B}} \longrightarrow \mathcal{C}^{\mathbb{B}}$ be a functor. In this section, we consider a special case of Section 4.1: when the functor Σ is a lift of a functor $S: \mathcal{C} \longrightarrow \mathcal{C}$, i.e. there is a commutative diagram

$$\begin{array}{ccc} \mathcal{C}^{\mathbb{B}} & \xrightarrow{\Sigma} & \mathcal{C}^{\mathbb{B}} \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\ \mathcal{C} & \xrightarrow[S]{} & \mathcal{C} \end{array}$$

Let $\mathbb{D}$ be a comonad on $\mathcal{C}$ and let $\theta: \mathbb{B} \longrightarrow \mathbb{D}$ be a distributive law. Distributive laws $\gamma: S \longrightarrow \mathbb{D}$ lift to give distributive laws $\gamma: \Sigma \longrightarrow \tilde{\mathbb{D}}$, and if γ is part of a factorisation (S, σ, γ) of $\theta: \mathbb{B} \longrightarrow \mathbb{D}$ then we get a factorisation (Σ, σ, γ) of $\theta: \tilde{\mathbb{B}} \longrightarrow \tilde{\mathbb{D}}$.

We consider three special cases of this construction. The distributive laws used therein are instances of one defined on the category of right U-modules, where U is a left Hopf algebroid, which is defined and discussed in [15].

Example 4.4. Suppose that $\sigma: \mathbb{B} \longrightarrow \mathbb{B}$ is a monad morphism which is compatible with θ; that is $\sigma: B \longrightarrow B$ is a natural transformation such that the three diagrams

$$
\begin{array}{ccc} BB & \xrightarrow{\sigma\sigma} & BB \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\mu} \\ B & \xrightarrow[\sigma]{} & B \end{array}
\qquad
\begin{array}{ccc} \mathsf{id}_{\mathcal{C}} & \xrightarrow{\eta} & B \\ & {\scriptstyle\eta}\searrow & \downarrow{\scriptstyle\sigma} \\ & & B \end{array}
\qquad
\begin{array}{ccc} BD & \xrightarrow{\sigma D} & BD \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\theta} \\ DB & \xrightarrow[D\sigma]{} & DB \end{array}
$$

commute. The first two diagrams say that $\sigma: \mathbb{B} \longrightarrow \mathsf{id}_{\mathcal{C}}$ is a distributive law. The triple $(\mathsf{id}_{\mathcal{C}}, \sigma, \mathsf{id}_{SD})$ is a factorisation of $\theta: \mathbb{B} \longrightarrow \mathbb{D}$, so we get a factorisation $(\Sigma, \sigma, \mathsf{id}_{\Sigma\tilde{D}})$ of $\theta: \tilde{\mathbb{B}} \longrightarrow \tilde{\mathbb{D}}$. Explicitly, $\Sigma: \mathcal{C}^{\mathbb{B}} \longrightarrow \mathcal{C}^{\mathbb{B}}$ is given by

$$\Sigma(X, \alpha) = (X, \alpha \circ \sigma_X), \qquad \Sigma(f) = f.$$

Observe that the composition of monad morphisms corresponds under this assignment to the monoidal structure in $\mathcal{F}(\theta)$, so when viewing the monad morphisms as a monoidal category with composition as tensor product and the identity id_B as unit object, we have:

Proposition 4.5. *The assignment* $\sigma \longmapsto (\Sigma, \sigma, \mathsf{id}_{\Sigma\tilde{D}})$ *is a monoidal functor.*

The factorisation given in Proposition 4.3 arises in this way.

Example 4.6. Let k be a commutative ring and let U be a Hopf algebra over k. We use Sweedler notation to denote the coproduct

$$\Delta(u) = u_{(1)} \otimes u_{(2)}.$$

See [23,20] for more information about Hopf algebras.

Consider the category $\mathcal{C} = k\text{-}\mathsf{Mod}$. The functor $B = - \otimes_k U: \mathcal{C} \longrightarrow \mathcal{C}$ is part of a monad $\mathbb{B}$ where the multiplication is given by the multiplication of the algebra U and the unit is given by the unit of the algebra U. Dually, the functor $D = U \otimes_k -: \mathcal{C} \longrightarrow \mathcal{C}$ is part of a comonad, whose structure is given by the comultiplication and counit of the coalgebra U. There is a mixed distributive law $\theta: \mathbb{B} \longrightarrow \mathbb{D}$ given by

$$\theta_X: U \otimes_k X \otimes_k U \longrightarrow U \otimes_k X \otimes_k U$$
$$u \otimes x \otimes v \longmapsto S(v_{(2)})u \otimes x \otimes v_{(1)}.$$

Let P be any right U-module. This defines a functor $P \otimes_k -: \mathcal{C} \longrightarrow \mathcal{C}$. The maps

$$\sigma_X: P \otimes_k X \otimes_k U \longrightarrow P \otimes_k X \otimes_k U$$
$$p \otimes x \otimes u \longmapsto pu_{(1)} \otimes x \otimes u_{(2)}$$

define a distributive law $\sigma: \mathbb{B} \longrightarrow P \otimes_k -$ and the maps

$$\gamma_X: P \otimes_k U \otimes_k X \longrightarrow U \otimes_k P \otimes_k X$$
$$p \otimes u \otimes x \longmapsto u \otimes p \otimes x$$

define a distributive law $\gamma: P \otimes_k - \longrightarrow \mathbb{D}$. The triple $(P \otimes_k -, \sigma, \gamma)$ is a factorisation of $\theta: \mathbb{B} \longrightarrow \mathbb{D}$, and so this gives a factorisation of $\theta: \tilde{\mathbb{B}} \longrightarrow \tilde{\mathbb{D}}$ in the category $\mathcal{C}^{\mathbb{B}} \cong \mathrm{Mod}\text{-}U$.

Example 4.7. Let $\mathcal{C} = k\text{-}\mathsf{Mod}$ where k is a commutative ring, and consider the functor $B = U \otimes_k -: \mathcal{C} \longrightarrow \mathcal{C}$. Similarly to Example 4.6, this is simultaneously part of a monad $\mathbb{B}$ and a comonad $\mathbb{D}$. There is a mixed distributive law $\theta: \mathbb{B} \longrightarrow \mathbb{D}$ given by

$$\theta_X: U \otimes_k U \otimes_k X \longrightarrow U \otimes_k U \otimes_k X$$
$$u \otimes v \otimes x \longmapsto vS(u_{(2)}) \otimes u_{(1)} \otimes x$$

and a distributive law $\tau: \mathbb{B} \longrightarrow B$ given by

$$\tau_X: U \otimes_k U \otimes_k X \longrightarrow U \otimes_k U \otimes_k X$$
$$u \otimes v \otimes x \longmapsto v \otimes u \otimes x.$$

If U is commutative (or even just if the antipode S maps into the centre of U), then (B, τ, θ) is a factorisation of $\theta: \mathbb{B} \longrightarrow \mathbb{D}$ and so $(\tilde{B}, \tau, \theta)$ is a factorisation of $\theta: \tilde{\mathbb{B}} \longrightarrow \tilde{\mathbb{B}}$ in $\mathcal{C}^{\mathbb{B}} \cong U\text{-}\mathsf{Mod}$.

4.4. Braided distributive laws

Let $\chi: \mathbb{T} \longrightarrow \mathbb{C}$ be a comonad distributive law on a category $\mathcal{A}$.

Definition 4.8. A distributive law $\tau: \mathbb{T} \longrightarrow T$ between the comonad $\mathbb{T}$ and the endofunctor T is *braided with respect to* χ if the hexagon

$$\begin{array}{ccccccc} & & TTC & \xrightarrow{T\chi} & TCT & & \\ & \overset{\tau C}{\nearrow} & & & & \overset{\chi T}{\searrow} & \\ TTC & & & & & & CTT \\ & \underset{T\chi}{\searrow} & & & & \underset{C\tau}{\nearrow} & \\ & & TCT & \xrightarrow[\chi T]{} & CTT & & \end{array}$$

commutes. Dually, we say that a distributive law $\varphi: C \longrightarrow \mathbb{C}$ between the endofunctor C and the comonad $\mathbb{C}$ is *braided with respect to* χ if a similar hexagon commutes.

Clearly, τ is braided if and only if (T, τ, χ) is a factorisation of χ, since the above hexagon is just the Yang–Baxter condition in that case. In the dual case, (C, χ, φ) would be a factorisation of χ.

Example 4.9. In Example 4.7, the distributive law τ is braided with respect to θ.

Example 4.10. Let $\tau: \mathbb{T} \longrightarrow \mathbb{T}$ be a BD-law. These are defined in [16] and are exactly those distributive laws which are braided with respect to themselves. Thus (T, τ, τ) is a factorisation of τ.

Example 4.11. For this example we relax the assumption that monoidal categories are strict. Let $\mathcal{A}$ be a braided monoidal category with tensor product $\otimes$, associator morphisms α and braiding morphisms b. Let $\mathbf{U} = (U, \Delta^U, \varepsilon^U)$ and $\mathbf{V} = (V, \Delta^V, \varepsilon^V)$ be comonoids in $\mathcal{A}$. The comonoids $\mathbf{U}$, $\mathbf{V}$ define two comonads $\mathbb{U}$, $\mathbb{V}$ with endofunctors $U \otimes -$, $V \otimes -$ respectively, and three distributive laws $\chi: \mathbb{U} \longrightarrow \mathbb{V}$, $\tau: \mathbb{U} \longrightarrow \mathbb{U}$ and $\varphi: \mathbb{V} \longrightarrow \mathbb{V}$ defined by

$$U \otimes (V \otimes X) \xrightarrow{\alpha^{-1}_{U,V,X}} (U \otimes V) \otimes X \xrightarrow{b_{U,V} \otimes \mathsf{id}} (V \otimes U) \otimes X \xrightarrow{\alpha_{V,U,X}} V \otimes (U \otimes X)$$

$$U \otimes (U \otimes X) \xrightarrow{\alpha^{-1}_{U,U,X}} (U \otimes U) \otimes X \xrightarrow{b_{U,U} \otimes \mathsf{id}} (U \otimes U) \otimes X \xrightarrow{\alpha_{U,U,X}} U \otimes (U \otimes X)$$

$$V \otimes (V \otimes X) \xrightarrow{\alpha^{-1}_{V,V,X}} (V \otimes V) \otimes X \xrightarrow{b_{V,V} \otimes \mathsf{id}} (V \otimes V) \otimes X \xrightarrow{\alpha_{V,V,X}} V \otimes (V \otimes X)$$

respectively. The distributive laws τ and φ are both braided with respect to χ so we get two factorisations $(U \otimes -, \tau, \chi)$ and $(V \otimes -, \chi, \varphi)$ of χ. By Proposition 3.5 these are both comonoids in $\mathcal{F}(\chi)$. This example comes from the dual of Example 1.11 in [5].

4.5. Quantum doubles

In our final example, we consider the distributive laws corresponding to quantum doubles: let B and C be two Hopf algebras over a commutative ring k and $\mathcal{R} \in C \otimes_k B$ be an invertible 2-cycle, meaning that we have

$$(\Delta^C \otimes_k \mathsf{id}_B)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \quad (\mathsf{id}_C \otimes_k \Delta^B)(\mathcal{R}) = \mathcal{R}_{12}\mathcal{R}_{13},$$
$$(\mathsf{id}_C \otimes_k S^B)(\mathcal{R}) = \mathcal{R}^{-1}, \quad (S^C \otimes_k \mathsf{id}_B)(\mathcal{R}) = \mathcal{R}^{-1},$$

where $\mathcal{R}^{-1}$ refers to the multiplicative inverse in the tensor product algebra $C \otimes_k B$ and subscripts denote components in $C \otimes_k C \otimes_k B$ respectively $C \otimes_k B \otimes_k B$. We refer to [9] for more background information.

The coalgebras B and C define comonads $\mathbb{T}$ and $\mathbb{C}$ on $\mathcal{A} = k\text{-}\mathsf{Mod}$ given by $B \otimes_k -$ and $C \otimes_k -$ with structure maps given by the coproducts and the counits. The 2-cycle $\mathcal{R}$ defines a distributive law $\chi \colon \mathbb{T} \longrightarrow \mathbb{C}$ given by

$$\chi_X \colon B \otimes_k C \otimes_k X \longrightarrow C \otimes_k B \otimes_k X$$
$$b \otimes c \otimes x \longmapsto \mathcal{R}(c \otimes b)\mathcal{R}^{-1} \otimes x.$$

In this case, every (B, C^{op})-bimodule M, that is, a k-module M with two commuting left actions of B and C, gives rise to a factorisation of χ: let $\Sigma \colon \mathcal{A} \longrightarrow \mathcal{A}$ be the functor $M \otimes_k -$. We define distributive laws

$$\sigma_X \colon B \otimes_k M \otimes_k X \longrightarrow M \otimes_k B \otimes_k X, \qquad \gamma_X \colon M \otimes_k C \otimes_k X \longrightarrow C \otimes_k M \otimes_k X,$$
$$b \otimes m \otimes x \longmapsto \mathcal{R}_{12}(m \otimes b \otimes x), \qquad m \otimes c \otimes x \longmapsto \mathcal{R}_{12}(c \otimes m \otimes x).$$

Then a straightforward computation shows that (Σ, σ, γ) is a factorisation of χ.

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