GRAPH ALGEBRAS AND ORBIT EQUIVALENCE

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Abstract. We introduce the notion of orbit equivalence of directed graphs, following Matsumoto's notion of continuous orbit equivalence for topological Markov shifts. We show that two graphs in which every cycle has an exit are orbit equivalent if and only if there is a diagonal-preserving isomorphism between their C^*-algebras. We show that it is necessary to assume that every cycle has an exit for the forward implication, but that the reverse implication holds for arbitrary graphs. As part of our analysis of arbitrary graphs E we construct a groupoid $\mathcal{G}_{(C^*(E),\mathcal{D}(E))}$ from the graph algebra $C^*(E)$ and its diagonal subalgebra $\mathcal{D}(E)$ which generalises Renault's Weyl groupoid construction applied to $(C^*(E),\mathcal{D}(E))$. We show that $\mathcal{G}_{(C^*(E),\mathcal{D}(E))}$ recovers the graph groupoid $\mathcal{G}_E$ without the assumption that every cycle in E has an exit, which is required to apply Renault's results to $(C^*(E),\mathcal{D}(E))$. We finish with applications of our results to out-splittings of graphs and to amplified graphs.

1. Introduction

The relationship between orbit equivalence and isomorphism of C^*-algebras has been studied extensively in the last 20 years. The first result of this type was the celebrated theorem of Giordano, Putnam and Skau [5, Theorem 2.4], in which they showed that orbit equivalence for minimal dynamical systems on the Cantor set is equivalent to isomorphism of their corresponding crossed product C^*-algebras. The importance of Giordano, Putnam and Skau's result cannot be overstated. In general there is no direct method of checking whether two Cantor minimal systems are orbit equivalent. However, because the crossed product C^*-algebras are classifiable, Giordano, Putnam and Skau's result means that orbit equivalence can be determined using K-theory. The work in [5] has been generalised in many directions, including Tomiyama's results on topologically free dynamical systems on compact Hausdorff spaces [24], and Giordano, Matui, Putnam and Skau's extension of [5, Theorem 2.4] to minimal $\mathbb{Z}^d$-actions on the Cantor set [6].

More recently, Matsumoto and Matui have shown in [15] that two irreducible one-sided topological Markov shifts (X_A,σ_A) and (X_B,σ_B) are continuously orbit equivalent if and only if the corresponding Cuntz-Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_B$ are isomorphic and $\det(I-A)=\det(I-B)$. The proof of Matsumoto and Matui's theorem relies on two key results. The first of these is [12, Theorem 1.1], in which Matsumoto proves that the following statements are equivalent:

(1) (X_A,σ_A) and (X_B,σ_B) are continuously orbit equivalent,
(2) there exists a $*$-isomorphism $\phi:\mathcal{O}_A\to\mathcal{O}_B$ which maps the maximal abelian subalgebra $\mathcal{D}_A$ onto $\mathcal{D}_B$, and

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(3) the topological full group of (X_A, σ_A) and the topological full group of (X_B, σ_B) are spatially isomorphic. (In [14, Theorem 1.1], Matsumoto showed that this is equivalent to the topological full groups being abstractly isomorphic.)

The second key result is [22, Proposition 4.13], which, as noticed by Matui (see [16, Theorem 5.1]), implies that there exists a $*$-isomorphism $\phi : \mathcal{O}_A \to \mathcal{O}_B$ that maps the maximal abelian subalgebra, or diagonal, $\mathcal{D}_A$ onto $\mathcal{D}_B$ if and only if the corresponding groupoids $\mathcal{G}_A$ and $\mathcal{G}_B$ are isomorphic.

In this paper we initiate the study of orbit equivalence of directed graphs, and we prove the analogous result to [22, Proposition 4.13] for graph algebras. In particular, as part of our main result we prove that if E and F are two graphs in which every cycle has an exit, then the following are equivalent:

(1) There is an isomorphism from $C^*(E)$ to $C^*(F)$ which maps the diagonal subalgebra $\mathcal{D}(E)$ onto $\mathcal{D}(F)$.
(2) The graph groupoids $\mathcal{G}_E$ and $\mathcal{G}_E$ are isomorphic as topological groupoids.
(3) The pseudogroups of E and F are isomorphic.
(4) The graphs E and F are orbit equivalent.

It is natural to ask whether every cycle having an exit is necessary for our results. In our main result we in fact prove that (1) $\iff$ (2), (3) $\iff$ (4) and (2) $\implies$ (3) all hold for arbitrary directed graphs. It is only the implication (3) $\implies$ (2) that requires that every cycle has an exit (and we provide examples that show that (3) $\implies$ (2) does not hold in general without the assumption that every cycle has an exit). Our analysis of these implications for arbitrary graphs provides our most technical innovation, which is the introduction of a groupoid $\mathcal{G}_{(C^*(E),\mathcal{D}(E))}$ associated to $(C^*(E), \mathcal{D}(E))$ that we call the extended Weyl groupoid. Our construction generalises Renault's Weyl groupoid construction from [22, Definition 4.11] applied to $(C^*(E), \mathcal{D}(E))$. We show that $\mathcal{G}_{(C^*(E),\mathcal{D}(E))}$ and $\mathcal{G}_E$ are isomorphic as topological groupoids for an arbitrary graph E, which can be deduced from Renault's results in [22] only when every cycle in E has an exit.

We conclude our paper with two applications of our main theorem. Our first application shows that if two general graphs E and F are conjugate then there is an isomorphism from $C^*(E)$ to $C^*(F)$ which maps $\mathcal{D}(E)$ onto $\mathcal{D}(F)$. As a corollary, we strengthen a result of Bates and Pask [3, Theorem 3.2] on out-splitting of graphs. Our second application adds three additional equivalences to Eilers, Ruiz, and Sørensen's complete invariant for amplified graphs [4, Theorem 1.1]. In addition, we expect that our results will have applications in the study of Leavitt path algebras and further applications to graph algebras.

The paper is organized as follows. Section 2 provides background on graphs, their groupoids and their C^*-algebras. In Section 3 we define orbit equivalence of graphs and associate with each graph a pseudogroup which is the analogue of the topological full group Matsumoto has associated with each irreducible one-sided topological Markov shift, and we show that two graphs are orbit equivalent if and only if their pseudogroups are isomorphic. In Section 4 we construct the extended Weyl groupoid $\mathcal{G}_{(C^*(E),\mathcal{D}(E))}$ from $(C^*(E), \mathcal{D}(E))$, and we show that $\mathcal{G}_{(C^*(E),\mathcal{D}(E))}$ and $\mathcal{G}_E$ are isomorphic as topological groupoids. We use this result to show that if there is a diagonal-preserving isomorphism from $C^*(E)$ to $C^*(F)$, then $\mathcal{G}_E$ and $\mathcal{G}_F$ are isomorphic as topological groupoids. In

Section 5 we finish the proof of our main theorem and provide examples. Finally, in Section 6 we give the two applications of our main theorem.

Remark 1.1. We have learned that Xin Li has also considered orbit equivalence for directed graphs, and has independently proved that two graphs in which every cycle has an exit are orbit equivalent if and only if there is a diagonal-preserving isomorphism between their C*-algebras.

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2. Background on the groupoids and C^*-algebras of directed graphs

We begin with some background on graphs and their C^*-algebras. In this section we recall the definitions of the boundary path space of a directed graph, graph C^*-algebras and graph groupoids.

2.1. **Graphs and their C^*-algebras.** We refer the reader to [19] for a more detailed treatment on graphs and their C^*-algebras. However, we note that the directions of arrows defining a graph are reversed in this paper. We used this convention so that our results can easily be compared with the work of Matsumoto and Matui's work on shift spaces.

A *directed graph* (also called a *quiver*) $E = (E^0, E^1, r, s)$ consists of countable sets E^0 and E^1, and range and source maps $r, s : E^1 \to E^0$. The elements of E^0 are called vertices, and the elements of E^1 are called edges.

A *path* μ of length n in E is a sequence of edges $\mu = \mu_1 \dots \mu_n$ such that $r(\mu_i) = s(\mu_{i+1})$ for all $1 \le i \le n-1$. The set of paths of length n is denoted E^n. We denote by $|\mu|$ the length of μ. The range and source maps extend naturally to paths: $s(\mu) := s(\mu_1)$ and $r(\mu) := r(\mu_n)$. We regard the elements of E^0 as path of length 0, and for $v \in E^0$ we set $s(v) := r(v) := v$. For $v \in E^0$ and $n \in \mathbb{N}$ we denote by vE^n the set of paths of length n with source v, and by $E^n v$ the paths of length n with range v. We define $E^* := \bigcup_{n\in\mathbb{N}} E^n$ to be the collection of all paths with finite length. For $v, w \in E^0$ let $vE^*w := \{\mu \in E^* : s(\mu) = v \text{ and } r(\mu) = w\}$. We define $E^0_{\text{reg}} := \{v \in E^0 : vE^1 \text{ is finite and nonempty}\}$ and $E^0_{\text{sing}} := E^0 \setminus E^0_{\text{reg}}$. If $\mu = \mu_1\mu_2 \cdots \mu_m, \nu = \nu_1\nu_2 \cdots \nu_n \in E^*$ and $r(\mu) = s(\nu)$, then we let $\mu\nu$ denote the path $\mu_1\mu_2 \cdots \mu_m\nu_1\nu_2 \cdots \nu_n$.

A *cycle* in E is a path $\mu \in E^*$ such that $|\mu| \ge 1$ and $s(\mu) = r(\mu)$. If μ is a cycle and k is a positive integer, then μ^k denotes the cycle $\mu\mu \cdots \mu$ where μ is repeated k-times. We say that the cycle μ is *simple* if μ is not equal to ν^k for any cycle ν and any integer $k \ge 2$. Notice than any cycle μ is equal to ν^k for some simple cycle ν and some positive integer k. An edge e is an *exit* to the cycle μ if there exists i such that $s(e) = s(\mu_i)$ and $e \ne \mu_i$. A graph is said to satisfy *condition (L)* if every cycle has an exit.

A *Cuntz-Krieger E-family* $\{P, S\}$ consists of a set of mutually orthogonal projections $\{P_v : v \in E^0\}$ and partial isometries $\{S_e : e \in E^1\}$ satisfying

(CK1) $S_e^* S_e = P_{r(e)}$ for all $e \in E^1$;

(CK2) $S_e S_e^* \le P_{s(e)}$ for all $e \in E^1$;

(CK3) $P_v = \displaystyle\sum_{e \in vE^1} S_e S_e^*$ for all $v \in E^0_{\text{reg}}$.

The *graph C^*-algebra* $C^*(E)$ is the universal C^*-algebra generated by a Cuntz-Krieger E-family. We denote by $\{p, s\}$ the Cuntz-Krieger E-family generating $C^*(E)$. There is a strongly continuous action $\gamma : C^*(E) \to \mathbb{T}$, called the *gauge action*, satisfying $\gamma_z(p_v) = p_v$ and $\gamma_z(s_e) = zs_e$, for all $z \in \mathbb{T}$, $v \in E^0$, $e \in E^1$. If $\{Q, T\}$ is a Cuntz-Krieger E-family in a C^*-algebra B, then we denote by $\pi_{Q,T}$ the homomorphism $C^*(E) \to B$ such that $\pi_{Q,T}(p_v) = Q_v$ for all $v \in E^0$, and $\pi_{Q,T}(s_e) = T_e$ for all $e \in E^1$. an Huef and Raeburn's *gauge invariant uniqueness theorem* [7] says that $\pi_{Q,T}$ is injective if and only if there is an action β of $\mathbb{T}$ on the C^*-algebra generated by $\{Q, T\}$ satisfying $\beta_z(Q_v) = Q_v$ and $\beta_z(T_e) = zT_e$, for all $z \in \mathbb{T}$, $v \in E^0$, $e \in E^1$, and $Q_v \neq 0$ for all $v \in E^0$.

If $\mu = \mu_1 \cdots \mu_n \in E^n$ and $n \geq 2$, then we let $s_\mu := s_{\mu_1} \cdots s_{\mu_n}$. Likewise, we let $s_v := p_v$ if $v \in E^0$. Then $C^*(E) = \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in E^*,\ r(\mu) = r(\nu)\}$. The C^*-subalgebra $\mathcal{D}(E) := \overline{\operatorname{span}}\{s_\mu s_\mu^* : \mu \in E^*\}$ of $C^*(E)$ is a maximal abelian subalgebra if and only if every cycle in E has an exit (see [17, Example 3.3]).

2.2. **The boundary path space of a graph.** An *infinite path* in E is an infinite sequence $x_1 x_2 \ldots$ of edges in E such that $r(e_i) = s(e_{i+1})$ for all i. We let E^∞ be the set of all infinite paths in E. The source map extends to E^∞ in the obvious way. We let $|x| = \infty$ for $x \in E^\infty$. The *boundary path space* of E is the space

$$\partial E := E^\infty \cup \{\mu \in E^* : r(\mu) \in E^0_{\mathrm{sing}}\}.$$

If $\mu = \mu_1 \mu_2 \cdots \mu_m \in E^*$, $x = x_1 x_2 \cdots \in E^\infty$ and $r(\mu) = s(x)$, then we let μx denote the infinite path $\mu_1 \mu_2 \cdots \mu_m x_1 x_2 \cdots \in E^\infty$.

For $\mu \in E^*$, the *cylinder set* of μ is the set

$$Z(\mu) := \{\mu x \in \partial E : x \in r(\mu)\partial E\},$$

where $r(\mu)\partial E := \{x \in \partial E : r(\mu) = s(x)\}$. Given $\mu \in E^*$ and a finite subset $F \subseteq r(\mu)E^1$ we define

$$Z(\mu \setminus F) := Z(\mu) \setminus \left(\bigcup_{e \in F} Z(\mu e) \right).$$

The boundary path space ∂E is a locally compact Hausdorff space with the topology given by the basis $\{Z(\mu \setminus F) : \mu \in E^*,\ F \text{ is a finite subset of } r(\mu)E^1\}$, and each such $Z(\mu \setminus F)$ is compact and open (see [25, Theorem 2.1 and Theorem 2.2]). Moreover, [25, Theorem 3.7] shows that there is a unique homeomorphism h_E from ∂E to the spectrum of $\mathcal{D}(E)$ given by

$$h_E(x)(s_\mu s_\mu^*) = \begin{cases} 1 & \text{if } x \in Z(\mu), \\ 0 & \text{if } x \notin Z(\mu). \end{cases} \tag{2.1}$$

Our next lemma gives a description of the topology on the boundary path space, which we will need in the proof of Proposition 3.3.

Lemma 2.1. *Every nonempty open subset of ∂E is the disjoint union of sets that are both compact and open.*

Proof. Let U be a nonempty open subset of ∂E. For each $x \in U$ let

$$B_x := \{(\mu, F) : \mu \in E^*,\ F \text{ is a finite subset of } r(\mu)E^1,\ x \in Z(\mu \setminus F) \subseteq U\}.$$

If $(\mu, F) \in B_x$, then $x \in Z(\mu)$ and $x \notin Z(\mu e)$ for each $e \in F$. Let μ_x be the shortest $\mu \in E^*$ such that $(\mu, F) \in B_x$ for some finite subset F of $r(\mu)E^1$, and let $F_x := \cap\{F : (\mu_x, F) \in B_x\}$. Then $(\mu_x, F_x) \in B_x$ and $Z(\mu \setminus F) \subseteq Z(\mu_x \setminus F_x)$ for all $(\mu, F) \in B_x$. It follows that if $x, y \in U$, then either $Z(\mu_x \setminus F_x) = Z(\mu_y \setminus F_y)$ or $Z(\mu_x \setminus F_x) \cap Z(\mu_y \setminus F_y) = \emptyset$. Since $U = \cup_{x \in U} Z(\mu_x \setminus F_x)$ and each $Z(\mu_x \setminus F_x)$ is open and compact, this shows that U is the disjoint union of sets that are both compact and open. □

For $n \in \mathbb{N}$, let $\partial E^{\geq n} := \{x \in \partial E : |x| \geq n\}$. Then $\partial E^{\geq n} = \cup_{\mu \in E^n} Z(\mu)$ is an open subset of ∂E. We define the *shift map* on E to be the map $\sigma_E : \partial E^{\geq 1} \to \partial E$ given by $\sigma_E(x_1 x_2 x_3 \cdots) = x_2 x_3 \cdots$ for $x_1 x_2 x_3 \cdots \in \partial E^{\geq 2}$ and $\sigma_E(e) = r(e)$ for $e \in \partial E \cap E^1$. For $n \geq 1$, we let σ_E^n be the n-fold composition of σ_E with itself. We let σ_E^0 denote the identity map on ∂E. Then σ_E^n is a local homeomorphism for all $n \in \mathbb{N}$. When we write $\sigma_E^n(x)$, we implicitly assume that $x \in \partial E^{\geq n}$.

We say that $x \in \partial E$ is *eventually periodic* if there are $m, n \in \mathbb{N}$, $m \neq n$ such that $\sigma_E^m(x) = \sigma_E^n(x)$. Notice that $x \in \partial E$ is eventually periodic if and only if $x = \mu\nu\nu\nu \cdots$ for some path $\mu \in E^*$ and some cycle $\nu \in E^*$ with $s(\nu) = r(\mu)$. By replacing ν by a subcycle if necessary, we can assume that ν is a simple cycle.

2.3. **Graph groupoids.** In [11], Kumjian, Pask, Raeburn, and Renault defined groupoid C^*-algebras associated to a locally-finite directed graph with no sources. Their construction has been generalized to compactly aligned topological k-graphs in [26]. We will now explain this construction in the case that E is an arbitrary graph. The resulting groupoid is isomorphic to the one constructed by Paterson in [18]. Let

$$\mathcal{G}_E := \{(x, m-n, y) : x, y \in \partial E,\ m, n \in \mathbb{N},\ \text{and } \sigma^m(x) = \sigma^n(y)\},$$

with product $(x, k, y)(w, l, z) := (x, k+l, z)$ if $y = w$ and undefined otherwise, and inverse given by $(x, k, y)^{-1} := (y, -k, x)$. With these operations $\mathcal{G}_E$ is a groupoid (cf. [11, Lemma 2.4]). The unit space $\mathcal{G}_E^0$ of $\mathcal{G}_E$ is $\{(x, 0, x) : x \in \partial E\}$ which we will freely identify with ∂E via the map $(x, 0, x) \mapsto x$ throughout the paper. We then have that the range and source maps $r, s : \mathcal{G}_E \to \partial E$ are given by $r(x, k, y) = x$ and $s(x, k, y) = y$.

We now define a topology on $\mathcal{G}_E$. Suppose $m, n \in \mathbb{N}$ and U is an open subset of $\partial E^{\geq m}$ such that the restriction of σ_E^m to U is injective, V is an open subset of $\partial E^{\geq n}$ such that the restriction of σ_E^n to V is injective, and that $\sigma_E^m(U) = \sigma_E^n(V)$, then we define

$$Z(U, m, n, V) := \{(x, k, y) \in \mathcal{G}_E : x \in U,\ k = m - n,\ y \in V,\ \sigma_E^m(x) = \sigma_E^n(y)\}. \tag{2.2}$$

Then $\mathcal{G}_E$ is a locally compact, Hausdorff, étale topological groupoid with the topology generated by the basis consisting of sets $Z(U, m, n, V)$ described in (2.2), see [11, Proposition 2.6] for an analogous situation. For $\mu, \nu \in E^*$ with $r(\mu) = r(\nu)$, let $Z(\mu, \nu) := Z(Z(\mu), |\mu|, |\nu|, Z(\nu))$. It follows that each $Z(\mu, \nu)$ is compact and open, and that the topology ∂E inherits when we consider it as a subset of $\mathcal{G}_E$ by identifying it with $\{(x, 0, x) : x \in \partial E\}$ agrees with the topology described in the previous section. Notice that for all $\mu, \nu \in E^*$, U a compact open subset of $Z(\mu)$, and V a compact open subset of $Z(\nu)$, the collection $\{Z(U, |\mu|, |\nu|, V) : \sigma_E^{|\mu|}(U) = \sigma_E^{|\nu|}(V)\}$ is a basis for the topology of $\mathcal{G}_E$. According to [26, Proposition 6.2], $\mathcal{G}_E$ is topologically amenable in the sense of [1, Definition 2.2.8]. It follows from [1, Proposition 3.3.5] and [1, Proposition 6.1.8] that the reduced and universal C^*-algebras of $\mathcal{G}_E$ are equal, and we denote this C^*-algebra by $C^*(\mathcal{G}_E)$.

Proposition 2.2 (Cf. [11, Proposition 4.1])**.** *Suppose E is a graph. Then there is a unique isomorphism $\pi : C^*(E) \to C^*(\mathcal{G}_E)$ such that $\pi(p_v) = 1_{Z(v,v)}$ for all $v \in E^0$ and $\pi(s_e) = 1_{Z(e,r(e))}$ for all $e \in E^1$, and such that $\pi(\mathcal{D}(E)) = C_0(\mathcal{G}_E^0)$.*

Proof. Using calculations along the lines of those used in the proof of [11, Proposition 4.1], it is straight forward to check that

$$\{Q, T\} := \{Q_v := 1_{Z(v,v)} \text{ and } T_e := 1_{Z(e,r(e))} : v \in E^0, e \in E^1\}$$

is a Cuntz-Krieger E-family. The universal property of $\{p, s\}$ implies that there is a $*$-homomorphism $\pi := \pi_{Q,T} : C^*(E) \to C^*(\mathcal{G}_E)$ satisfying $\pi(p_v) = Q_v$ and $\pi(s_e) = T_e$. An argument similar to the one used in the proof of [11, Proposition 4.1] shows that $C^*(\mathcal{G}_E)$ is generated by $\{Q, T\}$, so π is surjective. The cocycle $(x, k, y) \mapsto k$ induces an action β of $\mathbb{T}$ on $C^*(\mathcal{G}_E)$ satisfying $\beta_z(Q_v) = Q_v$ and $\beta_z(T_e) = zT_e$, for all $z \in \mathbb{T}$, $v \in E^0$, $e \in E^1$ (see [21, Proposition II.5.1]), and since $Q_v = 1_{Z(v,v)} \neq 0$ for all $v \in E^0$, the gauge invariant uniqueness theorem of $C^*(\mathcal{G}_E)$ ([2, Theorem 2.1]) implies that π is injective. Since $\mathcal{D}(E)$ is generated by $\{s_\mu s_\mu^* : \mu \in E^*\}$ and $\pi(s_\mu s_\mu^*) = 1_{Z(\mu,\mu)}$, we have that π maps $\mathcal{D}(E)$ into $C_0(\mathcal{G}_E^0)$. An application of the Stone-Weierstrass theorem implies that $C_0(\mathcal{G}_E^0)$ is generated by $\{1_{Z(\mu,\mu)} : \mu \in E^*\}$. Hence $\pi(\mathcal{D}(E)) = C_0(\mathcal{G}_E^0)$. □

Suppose $\mathcal{G}$ is a groupoid, the isotropy group of $x \in \mathcal{G}^0$ is the group $\text{Iso}(x) := \{\gamma \in \mathcal{G} : s(\gamma) = r(\gamma) = x\}$. In [22], an étale groupoid is said to be *topologically principal* if the set of points of $\mathcal{G}^0$ with trivial isotropy group is dense. We will now characterize when $\mathcal{G}_E$ is topologically principal.

Proposition 2.3. *Let E be a graph. Then the graph groupoid $\mathcal{G}_E$ is topologically principal if and only if every cycle in E has an exit.*

Proof. Let $x \in \partial E$. We claim that $(x, 0, x)$ has nontrivial isotropy group if and only if x is eventually periodic. Indeed, suppose $(x, m - n, x) \in \text{Iso}(x)$ with $m \neq n$, then $\sigma^m(x) = \sigma^n(x)$ and x is eventually periodic. On the other hand, suppose $x = \mu\lambda^\infty$, then $(x, (|\mu| + |\lambda|) - |\mu|, x) \in \text{Iso}(x)$, proving the claim. Now observe that if v is a vertex such that there are two different simple cycles α and β with $s(\alpha) = s(\beta) = v$, then any cylinder set $Z(\delta)$ for which $r(\delta)E^*v \neq \emptyset$ contains a y such that $(y, 0, y)$ has trivial isotropy. To see this, pick $\lambda \in r(\delta)E^*v$, then $y = \delta\lambda\alpha\beta\alpha^2\beta\alpha^3\beta\cdots$ has trivial isotropy since it is not eventually periodic.

Assume that every cycle in E has an exit and suppose for contradiction that U is an nonempty open subset of ∂E such that $(x, 0, x)$ has nontrivial isotropy group for every $x \in U$. Note that $U \subseteq E^\infty$ since $y \in \partial E$ with $|y| < \infty$ implies that the isotropy group of $(y, 0, y)$ is trivial. Let $x \in U$. Since x has nontrivial isotropy group, there exist $\zeta_1 \in E^*$ and a cycle η such that $\zeta_1\eta^\infty \in Z(\zeta_1\eta^k) \subseteq U$ for some $k \in \mathbb{N}$. Since η has an exit and $(x, 0, x)$ has nontrivial isotropy group for every $x \in U$, it follows that there is a $\zeta_2 \in r(\zeta_1)E^*$ such that $Z(\zeta_1\zeta_2) \subseteq U$ and such that $r(\zeta_2)E^*r(\zeta_1) = \emptyset$, for otherwise there would be two distinct simple cycles based at $r(\zeta_1)$. By repeating this argument we get a sequences of paths $\zeta_1, \zeta_2, \zeta_3, \ldots$ such that $s(\zeta_{n+1}) = r(\zeta_n)$, $r(\zeta_{n+1})E^*r(\zeta_n) = \emptyset$ and $Z(\zeta_1\zeta_2 \ldots \zeta_n) \subseteq U$ for all n. The element $y = \zeta_1\zeta_2\zeta_3 \ldots$ then belongs to U, but since it only visits each vertex a finite number of times, $(y, 0, y)$ must have trivial isotropy, which contradicts the assumption that $(x, 0, x)$ has nontrivial isotropy group for every $x \in U$. Thus, $\mathcal{G}_E$ is topologically principal if every cycle in E has an exit.

Conversely, if μ is a cycle without exit and $x = \mu\mu\mu\ldots$, then $(x, 0, x)$ is an isolated point in $\mathcal{G}_E^0$ with nontrivial isotropy group. Thus, $\mathcal{G}_E$ is not topologically principal if there is a cycle in E without an exit. □

Since the reduced and universal C^*-algebras of $\mathcal{G}_E$ are equal, it follows from [21, Proposition II.4.2(i)] that we can regard $C^*(\mathcal{G}_E)$ as a subset of $C_0(\mathcal{G}_E)$. For $f \in C^*(\mathcal{G}_E)$ and $j \in \mathbb{Z}$, we let $\Phi_j(f)$ denote the restriction of f to $\{(x, k, y) \in \mathcal{G}_E : k = j\}$, and for $m \in \mathbb{N}$ we let $\Sigma_m(f) := \sum_{j=-m}^{m}(1 - \frac{|j|}{m+1})\Phi_j(f)$.

Proposition 2.4. *Let E be a graph and let $f \in C^*(\mathcal{G}_E)$. Then each $\Phi_k(f)$ and each $\Sigma_m(f)$ belong to $C^*(\mathcal{G}_E)$, and $(\Sigma_m(f))_{m\in\mathbb{N}}$ converges to f in $C^*(\mathcal{G}_E)$.*

Proof. Let $j \in \mathbb{Z}$. The map $(x, k, y) \mapsto k$ is a continuous cocycle from $\mathcal{G}_E$ to $\mathbb{Z}$. For each $z \in \mathbb{T}$ there is a unique automorphism γ_z on $C^*(\mathcal{G}_E)$ such that $\gamma_z(g)(x, k, y) = z^k g(x, k, y)$ for $g \in C^*(\mathcal{G}_E)$ and $(x, k, y) \in \mathcal{G}_E$, and that the map $z \mapsto \gamma_z$ is a strongly continuous action of $\mathbb{T}$ on $C^*(\mathcal{G}_E)$ (see [21, Proposition II.5.1]). It follows that the integral $\int_{\mathbb{T}} \gamma_z(f)z^{-j}\, dz$, where dz denotes the normalized Haar measure on $\mathbb{T}$, is well-defined and belongs to $C^*(\mathcal{G}_E)$ (see for example [20, Section C.2]). Let $(x, k, y) \in \mathcal{G}_E$. If $k \neq j$, then

$$\int_{\mathbb{T}} \gamma_z(f)z^{-j}\, dz(x, k, y) = \int_{\mathbb{T}} z^{k-j}\, dz\ f(x, k, y) = 0,$$

and if $k = j$, then

$$\int_{\mathbb{T}} \gamma_z(f)z^{-j}\, dz(x, k, y) = \int_{\mathbb{T}} z^{k-j}\, dz\ f(x, k, y) = f(x, k, y).$$

Thus, $\Phi_j(f) = \int_{\mathbb{T}} \gamma_z(f)z^{-j}\, dz$ from which it follows that $\Phi_j(f) \in C^*(\mathcal{G}_E)$.

Since each $\Sigma_m(f)$ is a linear combination of functions of the form $\Phi_j(f)$, each $\Sigma_m(f)$ belongs to $C^*(\mathcal{G}_E)$.

For $m \in \mathbb{N}$, let $\sigma_m : \mathbb{T} \to \mathbb{R}$ be the Fejér's kernel defined by

$$\sigma_m(z) = \sum_{j=-m}^{m} (1 - \frac{|j|}{m+1})z^{-j}.$$

Then $\sigma_m(z) \geq 0$ for all $z \in \mathbb{T}$, $\int_{\mathbb{T}} \sigma_m(z)\, dz = 1$, and

$$\begin{aligned}\Sigma_m(f) &= \sum_{j=-m}^{m} \left(1 - \frac{|j|}{m+1}\right) \Phi_j(f)\\ &= \sum_{j=-m}^{m} \left(1 - \frac{|j|}{m+1}\right) \int_{\mathbb{T}} \gamma_z(f)z^{-j}\, dz = \int_{\mathbb{T}} \gamma_z(f)\sigma_m(z)\, dz.\end{aligned}$$

Thus

$$\|\Sigma_m(f)\| \leq \int_{\mathbb{T}} \|\gamma_z(f)\|\sigma_m(z)dz = \|f\|.$$

If $g \in C_c(\mathcal{G}_E)$, then there is an $m_0 \in \mathbb{N}$ such that $\Phi_j(g) = 0$ for $|j| > m_0$. For m sufficiently large, it follows that

$$\begin{aligned} \|g - \Sigma_m(g)\| &= \Big\| g - \sum_{j=-m}^{m} \Big(1 - \frac{|j|}{m+1}\Big) \Phi_j(g) \Big\| \\ &\leq \Big\| g - \sum_{j=-m}^{m} \Phi_j(g) \Big\| + \Big\| \sum_{j=-m}^{m} \Big(\frac{|j|}{m+1}\Big) \Phi_j(g) \Big\| \\ &\leq \Big\| \sum_{j=-m}^{m} \Big(\frac{|j|}{m+1}\Big) \Phi_j(g) \Big\| \\ &\leq \Big\| \sum_{j=-m_0}^{m_0} \Big(\frac{|j|}{m+1}\Big) \Phi_j(g) \Big\| \quad \text{for } m \geq m_0 \\ &\leq \sum_{j=-m_0}^{m_0} \frac{|j|}{m+1} \|\Phi_j(g)\| \to 0 \quad \text{as } m \to \infty. \end{aligned}$$

Thus, for any $\epsilon > 0$ there exists $g \in C_c(\mathcal{G}_E)$ and an $M \in \mathbb{N}$ such that $||f - g|| < \epsilon/3$ and $||g - \Sigma_m(g)|| < \epsilon/3$ for any $m \geq M$, and then

$$||f - \Sigma_m(f)|| \leq ||f - g|| + ||g - \Sigma_m(g)|| + ||\Sigma_m(g - f)|| < \epsilon$$

for any $m \geq M$. This shows that $(\Sigma_m(f))_{m \in \mathbb{N}}$ converges to f in $C^*(\mathcal{G}_E)$. □

3. Orbit equivalence and pseudogroups

In this section we introduce the notion of orbit equivalence of two graphs, which is a natural generalisation of Matsumoto's continuous orbit equivalence for topological Markov shifts from [12]. We also define the pseudogroup of a graph using Renault's pseudogroups associated to groupoids [22], and then show that two graphs are orbit equivalent if and only if their pseudogroups are isomorphic.

Definition 3.1. Two graphs E and F are *orbit equivalent* if there exist a homeomorphism $h : \partial E \to \partial F$ and continuous functions $k_1, l_1 : \partial E^{\geq 1} \to \mathbb{N}$ and $k_1', l_1' : \partial F^{\geq 1} \to \mathbb{N}$ such that

$$\sigma_F^{k_1(x)}(h(\sigma_E(x))) = \sigma_F^{l_1(x)}(h(x)) \quad \text{and} \quad \sigma_E^{k_1'(y)}(h^{-1}(\sigma_F(y))) = \sigma_E^{l_1'(y)}(h^{-1}(y)), \tag{3.1}$$

for all $x \in \partial E^{\geq 1}, y \in \partial F^{\geq 1}$.

Example 3.2. Consider the graphs

Then $\partial E = \{e_1e_2e_2\ldots, e_2e_2\ldots\}$ and $\partial F = \{f_1f_2f_1f_2\ldots, f_2f_1f_2f_1\ldots\}$. The map $h : \partial E \to \partial F$ given by

$$h(e_1e_2e_2\ldots) = f_1f_2f_1f_2\ldots \quad \text{and} \quad h(e_2e_2\ldots) = f_2f_1f_2f_1\ldots$$

is a homeomorphism. Consider $k_1, l_1 : \partial E^{\geq 1} \to \mathbb{N}$ given by $k_1(e_1e_2e_2\dots) = 1$ and $k_1(e_2e_2\dots) = 0$, and $l_1(e_1e_2e_2\dots) = 0 = l_1(e_2e_2\dots)$. Then k_1 and l_1 are continuous, and we have $\sigma_F^{k_1(x)}(h(\sigma_E(x))) = \sigma_F^{l_1(x)}(h(x))$ for all $x \in \partial E^{\geq 1}$. Similarly the functions $k_1', l_1' : \partial F^{\geq 1} \to \mathbb{N}$ given by $k_1'(f_1f_2f_1f_2\dots) = 0$ and $k_1'(f_2f_1f_2f_1\dots) = 1$, and $l_1'(f_1f_2f_1f_2\dots) = 1$ and $l_1'(f_2f_1f_2f_1\dots) = 0$, are continuous and satisfy

$$\sigma_E^{k_1'(y)}(h^{-1}(\sigma_F(y))) = \sigma_E^{l_1'(y)}(h^{-1}(y)) \quad \text{for all } y \in \partial F^{\geq 1}.$$

Hence E and F are orbit equivalent.

Sections 5 and 6 contain further examples of orbit equivalent graphs.

In Section 3 of [22], Renault constructs for each étale groupoid $\mathcal{G}$ a pseudogroup in the following way: Define a *bisection* to be a subset A of $\mathcal{G}$ such that the restriction of the source map of $\mathcal{G}$ to A and the restriction of the range map of $\mathcal{G}$ to A both are injective. The set of all open bisections of $\mathcal{G}$ forms an inverse semigroup $\mathcal{S}$ with product defined by $AB = \{\gamma\gamma' : (\gamma, \gamma') \in (A \times B) \cap \mathcal{G}^{(2)}\}$ (where $\mathcal{G}^{(2)}$ denote the set of composable pairs of $\mathcal{G}$), and the inverse of A is defined to be the image of A under the inverse map of $\mathcal{G}$. Each $A \in \mathcal{S}$ defines a unique homeomorphism $\alpha_A : s(A) \to r(A)$ such that $\alpha(s(\gamma)) = r(\gamma)$ for $\gamma \in A$. The set $\{\alpha_A : A \in \mathcal{S}\}$ of partial homeomorphisms on $\mathcal{G}^0$ is the pseudogroup of $\mathcal{G}$.

When E is a graph, then we call the pseudogroup of the étale groupoid $\mathcal{G}_E$ the *pseudogroup of E* and denote it by $\mathcal{P}_E$.

We will now give two alternative characterizations of the partial homeomorphisms of ∂E that belong to $\mathcal{P}_E$.

Proposition 3.3. *Let E be a graph, let U and V be open subsets of ∂E, and let $\alpha : V \to U$ be a homeomorphism. Then the following are equivalent:*

(1) $\alpha \in \mathcal{P}_E$.
(2) *For all $x \in V$, there exist $m, n \in \mathbb{N}$ and an open subset V' such that $x \in V' \subseteq V$, and such that $\sigma_E^m(x') = \sigma_E^n(\alpha(x'))$ for all $x' \in V'$.*
(3) *There exist continuous functions $m, n : V \to \mathbb{N}$ such that $\sigma_E^{m(x)}(x) = \sigma_E^{n(x)}(\alpha(x))$ for all $x \in V$.*

Proof. (1) $\Rightarrow$ (2): Suppose $\alpha \in \mathcal{P}_E$. Let $A \in \mathcal{S}$ be such that $\alpha = \alpha_A$. Let $x \in V$. Then there is a unique $\gamma \in A$ such that $s(\gamma) = x$, and then $r(\gamma) = \alpha(x)$. Since A is an open subset of $\mathcal{G}_E$, there are $m, n \in \mathbb{N}$, an open subset U' of $\partial E^{\geq m}$ such that the restriction of σ_E^m to U' is injective, and an open subset V' of $\partial E^{\geq n}$ such that the restriction of σ_E^n to V' is injective and $\sigma_E^m(U') = \sigma_E^n(V')$, and such that $\gamma \in Z(U', m, n, V') \subseteq A$. Then $x \in V' \subseteq V$ and $\sigma_E^m(x') = \sigma_E^n(\alpha(x'))$ for all $x' \in V'$.

(2) $\implies$ (3): Assume that for all $x \in V$, there exist $m, n \in \mathbb{N}$ and an open subset V' such that $x \in V' \subseteq V$, and such that $\sigma_E^m(x') = \sigma_E^n(\alpha(x'))$ for all $x' \in V'$. According to Lemma 2.1, V is the disjoint union of sets that are both compact and open. Since ∂E is locally compact, it follows that there exists a family $\{V_i : i \in I\}$ of mutually disjoint compact and open sets and a family $\{(m_i, n_i) : i \in I\}$ of pairs of nonnegative integers such that $V = \bigcup_{i \in I} V_i$ and $\sigma_E^{m_i}(x) = \sigma_E^{n_i}(\alpha(x))$ for $x \in V_i$. Define $m, n : V \to \mathbb{N}$ by setting $m(x) = m_i$ and $n(x) = n_i$ for $x \in V_i$. Then m and n are continuous and $\sigma_E^{m(x)}(x) = \sigma_E^{n(x)}(\alpha(x))$ for all $x \in V$.

(3) $\implies$ (1): Assume that $m, n : V \to \mathbb{N}$ are continuous functions such that $\sigma_E^{m(x)}(x) = \sigma_E^{n(x)}(\alpha(x))$ for all $x \in V$. Then there exist for each $x \in V$ a compact and open subset V_x such that $x \in V_x \subseteq V$, $m(x') = m(x)$ and $n(x') = n(x)$ for all $x' \in V_x$, the restriction of $\sigma_E^{n(x)}$ to V_x is injective, and the restriction of $\sigma_E^{m(x)}$ of $\alpha(V_x)$ is injective. According to Lemma 2.1, V is the disjoint union of sets that are both compact and open. It follows that there exists a family $\{V_i : i \in I\}$ of mutually disjoint compact and open sets and a family $\{(m_i, n_i) : i \in I\}$ of pairs of nonnegative integers such that $V = \bigcup_{i \in I} V_i$, $m(x) = m_i$ and $n(x) = n_i$ for all $x \in V_i$, the restriction of $\sigma_E^{n_i}$ to V_i is injective, and the restriction of $\sigma_E^{m_i}$ of $\alpha(V_i)$ is injective. Let $A := \bigcup_{i \in I} Z(\alpha(V_i), m_i, n_i V_i)$. Then $A \in \mathcal{S}$ and $\alpha = \alpha_A$, so $\alpha \in \mathcal{P}_E$. □

Suppose that E and F are two graphs and that there exists a homeomorphism $h : \partial E \to \partial F$. Let U and V be open subsets of ∂E and let $\alpha : V \to U$ be a homeomorphism. We denote by $h \circ \mathcal{P}_E \circ h^{-1} := \{h \circ \alpha \circ h^{-1} : \alpha \in \mathcal{P}_E\}$. We say that the pseudogroups of E and F are isomorphic if there is a homeomorphism $h : \partial E \to \partial F$ such that $h \circ \mathcal{P}_E \circ h^{-1} = \mathcal{P}_F$. We can now state the main result of this section.

Proposition 3.4. *Let E and F be two graphs. Then E and F are orbit equivalent if and only if the pseudogroups of E and F are isomorphic.*

To prove this proposition we will use the following result.

Lemma 3.5. *Suppose two graphs E and F are orbit equivalent, $h : \partial E \to \partial F$ is a homeomorphism and $k_1, l_1 : \partial E^{\geq 1} \to \mathbb{N}$ and $k_1', l_1' : \partial F^{\geq 1} \to \mathbb{N}$ are continuous functions satisfying* (3.1). *Let $n \in \mathbb{N}$. Then there exist continuous functions $k_n, l_n : \partial E^{\geq n} \to \mathbb{N}$ and $k_n', l_n' : \partial F^{\geq n} \to \mathbb{N}$ such that*

$$\sigma_F^{k_n(x)}(h(\sigma_E^n(x))) = \sigma_F^{l_n(x)}(h(x)) \quad \text{and} \quad \sigma_E^{k_n'(y)}(h^{-1}(\sigma_F^n(y))) = \sigma_E^{l_n'(y)}(h^{-1}(y)), \tag{3.2}$$

for all $x \in \partial E^{\geq n}, y \in \partial F^{\geq n}$.

Proof. There is nothing to prove for $n = 0$ and $n = 1$. We will prove the general case by induction. Let $m \geq 1$ and suppose that the lemma holds for $n = m$. Let $x \in \partial E^{\geq m+1}$. Then

$$\sigma_F^{k_1(\sigma_E^m(x))}(h(\sigma_E^{m+1}(x))) = \sigma_F^{l_1(\sigma_E^m(x))}(h(\sigma_E^m(x)))$$

and

$$\sigma_E^{k_m(x)}(h(\sigma_E^m(x))) = \sigma^{l_m(x)}(h(x)).$$

Let

$$k_{m+1}(x) := k_1(\sigma_E^m(x)) + \max\{l_1(\sigma_E^m(x)), k_m(x)\} - l_1(\sigma_E^m(x)) \tag{3.3}$$

$$l_{m+1}(x) := l_m(x) + \max\{l_1(\sigma_E^m(x)), k_m(x)\} - k_m(x). \tag{3.4}$$

Then

$$\sigma_F^{k_{m+1}(x)}(h(\sigma_E^{m+1}(x))) = \sigma_F^{l_{m+1}(x)}(h(x)).$$

Since k_1, l_1, k_m, and l_m are continuous, it follows that $k_{m+1}, l_{m+1} : \partial E^{\geq m+1} \to \mathbb{N}$ defined by (3.3) and (3.4) are also continuous.

Similarly, if we define $k_{m+1}', l_{m+1}' : \partial F^{\geq m+1} \to \mathbb{N}$ by letting

$$k_{m+1}'(y) := k_1'(\sigma_F^m(y)) + \max\{l_1'(\sigma_F^m(y)), k_m'(y)\} - l_1'(\sigma_F^m(y))$$

$$l_{m+1}'(y) := l_m(y) + \max\{l_1'(\sigma_F^m(y)), k_m'(y)\} - k_m'(y)$$

for $y \in \partial F^{\geq m+1}$, then k'_{m+1} and l'_{m+1} are continuous, and

$$\sigma_E^{k'_{m+1}(y)}(h^{-1}(\sigma_F^{m+1}(y))) = \sigma_E^{l'_{m+1}(y)}(h^{-1}(y))$$

for all $y \in \partial E^{\geq m+1}$. Thus, the lemma also holds for $n = m+1$, and the general result holds by induction. □

Proof of Proposition 3.4. Suppose E and F are orbit equivalent. Then there exists a homeomorphism $h : \partial E \to \partial F$ and, for each $n \in \mathbb{N}$, there exists continuous functions $k_n, l_n : \partial E^{\geq n} \to \mathbb{N}$ satisfying the first equation of (3.2). Let $(\alpha : V \to U) \in \mathcal{P}_E$, and let $m, n : V \to \mathbb{N}$ be continuous functions such that $\sigma_E^{m(x)}(x) = \sigma_E^{n(x)}(\alpha(x))$ for all $x \in V$. Let $y \in h(V)$. Then

$$\begin{aligned}\sigma_F^{l_{n(h^{-1}(y))}(\alpha(h^{-1}(y)))}(h(\alpha(h^{-1}(y)))) &= \sigma_F^{k_{n(h^{-1}(y))}(\alpha(h^{-1}(y)))}(h(\sigma_E^{n(h^{-1}(y))}(\alpha(h^{-1}(y))))) \\ &= \sigma_F^{k_{n(h^{-1}(y))}(\alpha(h^{-1}(y)))}(h(\sigma_E^{m(h^{-1}(y))}(h^{-1}(y)))),\end{aligned}$$

and

$$\sigma_F^{k_{m(h^{-1}(y))}(h^{-1}(y))}(h(\sigma_E^{m(h^{-1}(y))}(h^{-1}(y)))) = \sigma_F^{l_{m(h^{-1}(y))}(h^{-1}(y))}(y).$$

So if we let

$$\begin{aligned}m'(y) := l_{m(h^{-1}(y))}(h^{-1}(y)) + \max\{k_{n(h^{-1}(y))}(\alpha(h^{-1}(y))), k_{m(h^{-1}(y))}(h^{-1}(y))\} \\ - k_{m(h^{-1}(y))}(h^{-1}(y))\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}n'(y) := l_{n(h^{-1}(y))}(\alpha(h^{-1}(y))) + \max\{k_{n(h^{-1}(y))}(\alpha(h^{-1}(y))), k_{m(h^{-1}(y))}(h^{-1}(y))\} \\ - k_{n(h^{-1}(y))}(\alpha(h^{-1}(y))),\end{aligned} \tag{3.6}$$

then $\sigma_F^{m'(y)}(y) = \sigma^{n'(y)}(h(\alpha(h^{-1}(y))))$. Since h^{-1}, m, n, and α are continuous, it follows that $m', n' : h(V) \to \mathbb{N}$ defined by (3.5) and (3.6) are also continuous. Thus, it follows from Proposition 3.3 that $h \circ \alpha \circ h^{-1} \in \mathcal{P}_F$. A similar argument proves that if $\alpha' \in \mathcal{P}_F$, then $h^{-1} \circ \alpha' \circ h \in \mathcal{P}_E$. Thus $h \circ \mathcal{P}_E \circ h^{-1} = \mathcal{P}_F$ and the pseudogroups of E and F are isomorphic.

Now suppose that $h : \partial E \to \partial F$ is a homeomorphism such that $h \circ \mathcal{P}_E \circ h^{-1} = \mathcal{P}_F$. Fix $e \in E^1$ and let $\alpha_e := \sigma_E|_{Z(e)}$. Then α_e is a homeomorphism from $Z(e)$ to $\alpha_e(Z(e))$ and since $\alpha_e(x) = \sigma_E(x)$ for all $x \in Z(e)$, it follows from Proposition 3.3 that $\alpha_e \in \mathcal{P}_E$. Thus $h \circ \alpha_e \circ h^{-1} \in \mathcal{P}_F$ by assumption. It follows from Proposition 3.3 that there are continuous functions $m'_e, n'_e : h(Z(e)) \to \mathbb{N}$ such that

$$\sigma_F^{n'_e(y)}(h(\alpha_e(h^{-1}(y)))) = \sigma_F^{m'_e(y)}(y) \quad \text{for } y \in h(Z(e)).$$

Define functions $k_1, l_1 : \partial E^{\geq 1} \to \mathbb{N}$ by $k_1(x) = n'_{x_1}(h(x))$ and $l_1(x) = m'_{x_1}(h(x))$, which are continuous because the $Z(e)$ are pairwise-disjoint compact open sets covering $\partial E^{\geq 1}$. Then for each $x = x_1 x_2 \cdots \in \partial E$ we have

$$\sigma_F^{l_1(x)}(h(x)) = \sigma_F^{m'_{x_1}(h(x))}(h(x)) = \sigma_F^{n'_{x_1}(h(x))}(h(\alpha_{x_1}(x))) = \sigma_F^{k_1(x)}(h(\sigma_E(x))).$$

Hence k_1 and l_1 satisfy the first equation from (3.1). A similar argument gets the second equation from (3.1). Thus E and F are orbit equivalent. □

4. The extended Weyl groupoid of $(C^*(E), \mathcal{D}(E))$

Proposition 2.2 says that the pair $(C^*(E), \mathcal{D}(E))$ is an invariant of $\mathcal{G}_E$, in the sense that if E and F are two graphs such that $\mathcal{G}_E$ and $\mathcal{G}_F$ are isomorphic as topological groupoids, then there is an isomorphism from $C^*(E)$ to $C^*(F)$ which maps $\mathcal{D}(E)$ onto $\mathcal{D}(F)$. In this section we show that $\mathcal{G}_E$ is an invariant of $(C^*(E), \mathcal{D}(E))$, in the sense that if there is an isomorphism from $C^*(E)$ to $C^*(F)$ which maps $\mathcal{D}(E)$ onto $\mathcal{D}(F)$, then $\mathcal{G}_E$ and $\mathcal{G}_F$ are isomorphic as topological groupoids.

To prove this result we build a groupoid from $(C^*(E), \mathcal{D}(E))$ that we call the extended Weyl groupoid, which generalises Renault's Weyl groupoid construction from [22] applied to $(C^*(E), \mathcal{D}(E))$. Recall from [22] that Weyl groupoids are associated to pairs (A, B) consisting of a C^*-algebra A and an abelian C^*-subalgebra B which contains an approximate unit of A. The Weyl groupoid construction has the property that if $\mathcal{G}$ is a topologically principal étale Hausdorff locally compact second countable groupoid and $A = C^*_{\text{red}}(\mathcal{G})$ and $B = C_0(\mathcal{G}^0)$, then the associated Weyl groupoid is isomorphic to $\mathcal{G}$ as a topological groupoid. We will modify Renault's construction for pairs $(C^*(E), \mathcal{D}(E))$ to obtain a groupoid $\mathcal{G}_{(C^*(E),\mathcal{D}(E))}$ such that $\mathcal{G}_{(C^*(E),\mathcal{D}(E))}$ and $\mathcal{G}_E$ are isomorphic as topological groupoids, even when $\mathcal{G}_E$ is not topologically principal. We will then show that if E and F are two graphs such that there is an isomorphism from $C^*(E)$ to $C^*(F)$ which maps $\mathcal{D}(E)$ onto $\mathcal{D}(F)$, then $\mathcal{G}_{(C^*(E),\mathcal{D}(E))}$ and $\mathcal{G}_{(C^*(F),\mathcal{D}(F))}$, and thus $\mathcal{G}_E$ and $\mathcal{G}_F$ are isomorphic as topological groupoids.

As in [22] (and originally defined in [9]), we define the *normaliser* of $\mathcal{D}(E)$ to be the set

$$N(\mathcal{D}(E)) := \{n \in C^*(E) : ndn^*, n^*dn \in \mathcal{D}(E) \text{ for all } d \in \mathcal{D}(E)\}.$$

According to [22, Lemma 4.6], $nn^*, n^*n \in \mathcal{D}(E)$ for $n \in N(\mathcal{D}(E))$. Recalling the definition of h_E given in (2.1), for $n \in N(\mathcal{D}(E))$, we let $\operatorname{dom}(n) := \{x \in \partial E : h_E(x)(n^*n) > 0\}$ and $\operatorname{ran}(n) := \{x \in \partial E : h_E(x)(nn^*) > 0\}$. It follows from [22, Proposition 4.7] that, for $n \in N(\mathcal{D}(E))$, there is a unique homeomorphism $\alpha_n : \operatorname{dom}(n) \to \operatorname{ran}(n)$ such that, for all $d \in \mathcal{D}(E)$,

$$h_E(x)(n^*dn) = h_E(\alpha_n(x))(d)h_E(x)(n^*n). \tag{4.1}$$

From [22, Lemma 4.10] we also know that $\alpha_{n^*} = \alpha_n^{-1}$ and $\alpha_{mn} = \alpha_m \circ \alpha_n$ for each $m, n \in N(\mathcal{D}(E))$.

The following lemma gives an insight into how the homeomorphisms α_n work. We collect further properties of these homeomorphisms in Lemma 4.2.

Lemma 4.1. *Let E be a graph. For each $\mu, \nu \in E^*$ with $r(\mu) = r(\nu)$ we have $s_\mu s_\nu^* \in N(\mathcal{D}(E))$ with*

$$\operatorname{dom}(s_\mu s_\nu^*) = Z(\nu),\ \operatorname{ran}(s_\mu s_\nu^*) = Z(\mu) \text{ and } \alpha_{s_\mu s_\nu^*}(\nu z) = \mu z \text{ for all } z \in r(\nu)\partial E.$$

Proof. Let $\mu, \nu \in E^*$ with $r(\mu) = r(\nu)$. For each $\lambda \in E^*$ we have

$$(s_\mu s_\nu^*)^* s_\lambda s_\lambda^* (s_\mu s_\nu^*) = \begin{cases} s_\nu s_\nu^* & \text{if } \mu = \lambda\mu' \\ s_{\nu\lambda'} s_{\nu\lambda'}^* & \text{if } \lambda = \mu\lambda' \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

So $(s_\mu s_\nu^*)^* s_\lambda s_\lambda^* (s_\mu s_\nu^*) \in \mathcal{D}(E)$, and it follows that $(s_\mu s_\nu^*)^* d (s_\mu s_\nu^*) \in \mathcal{D}(E)$ for all $d \in \mathcal{D}(E)$. A similar argument shows that $(s_\mu s_\nu^*) d (s_\mu s_\nu^*)^* \in \mathcal{D}(E)$ for all $d \in \mathcal{D}(E)$, and hence $s_\mu s_\nu^* \in N(D(E))$. We have

$$h_E(x)((s_\mu s_\nu^*)^* s_\mu s_\nu^*) = h_E(x)(s_\nu s_\nu^*) = \begin{cases} 1 & \text{if } x \in Z(\nu) \\ 0 & \text{if } x \notin Z(\nu), \end{cases}$$

and hence $\operatorname{dom}(s_\mu s_\nu^*) = Z(\nu)$. A similar calculation gives $\operatorname{ran}(s_\mu s_\nu^*) = Z(\mu)$.

Now suppose $z \in r(\nu)\partial E$. We use (4.1) and (4.2) to get

$$\begin{aligned} h_E(\alpha_{s_\mu s_\nu^*}(\nu z))(s_\lambda s_\lambda^*) &= h_E(\nu z)\big((s_\mu s_\nu^*)^* s_\lambda s_\lambda^* (s_\mu s_\nu^*)\big) h_E(\nu z)\big((s_\mu s_\nu^*)^* s_\mu s_\nu^*\big) \\ &= \begin{cases} h_E(\nu z)(s_\nu s_\nu^*) & \text{if } \mu = \lambda\mu' \\ h_E(\nu z)(s_{\nu\lambda'} s_{\nu\lambda'}^*) & \text{if } \lambda = \mu\lambda' \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } \mu z \in Z(\lambda) \\ 0 & \text{otherwise} \end{cases} \\ &= h_E(\mu z)(s_\lambda s_\lambda^*). \end{aligned}$$

It follows that $h_E(\alpha_{s_\mu s_\nu^*}(\nu z)) = h_E(\mu z)$, and hence $\alpha_{s_\mu s_\nu^*}(\nu z) = \mu z$. □

Denote by $\partial E_{\mathrm{iso}}$ the set of isolated points in ∂E. Notice that $x \in \partial E$ belongs to $\partial E_{\mathrm{iso}}$ if and only if the characteristic function $1_{\{x\}}$ belongs to $C_0(\partial E)$. For $x \in \partial E_{\mathrm{iso}}$, we let p_x denote the unique element of $\mathcal{D}(E)$ satisfying that $h_E(y)(p_x)$ is 1 if $y = x$ and zero otherwise.

Lemma 4.2. *Let E be a graph, $n \in N(\mathcal{D}(E))$ and $x \in \partial E_{\mathrm{iso}} \cap \operatorname{dom}(n)$. Then*

(a) $np_x n^* = h_E(x)(n^*n)p_{\alpha_n(x)}$,
(b) $n^* p_{\alpha_n(x)} n = h_E(x)(n^*n)p_x$, *and*
(c) $np_x = p_{\alpha_n(x)} n$.

Proof. We use Equation (4.1) with $d = nn^*$ to get

$$h_E(x)(n^*n)^2 = h_E(x)(n^*nn^*n) = h_E(\alpha_n(x))(nn^*)h_E(x)(n^*n),$$

which implies that

$$h_E(\alpha_n(x))(nn^*) = h_E(x)(n^*n). \tag{4.3}$$

Note that this is a positive number because $x \in \operatorname{dom}(n)$. For (a) we again use (4.1) to get

$$\begin{aligned} h_E(y)\Big(\big(h_E(\alpha_n(x))(nn^*)\big)^{-1} np_x n^*\Big) &= \big(h_E(\alpha_n(x))(nn^*)\big)^{-1} h_E(y)(np_x n^*) \\ &= \big(h_E(\alpha_n(x))(nn^*)\big)^{-1} h_E(\alpha_{n^*}(y))(p_x) h_E(y)(nn^*) \\ &= \begin{cases} 1 & \text{if } y = \alpha_n(x) \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

By the defining property of $p_{\alpha_n(x)}$ we now have $p_{\alpha_n(x)} = \big(h_E(\alpha_n(x))(nn^*)\big)^{-1} np_x n^*$. Using (4.3) gives $np_x n^* = h_E(x)(n^*n)p_{\alpha_n(x)}$, which is (a). Identity (b) follows from (a) by replacing n with n^* and x with $\alpha_n(x)$ and then use (4.3).

To prove (c) we first notice that

$$h_E(y)\big((h_E(x)(n^*n))^{-1}n^*np_x\big) = (h_E(x)(n^*n))^{-1}h_E(y)(n^*n)h_E(y)(p_x) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases}$$

Hence by the defining property of p_x we have

$$n^*np_x = h_E(x)(n^*n)p_x. \tag{4.4}$$

We now use (4.4) and (a) to get (c):

$$np_x = n((h_E(x)(n^*n))^{-1}n^*np_x) = (h_E(x)(n^*n))^{-1}np_xn^*n = p_{\alpha_n(x)}n. \qquad \square$$

Lemma 4.3. *Suppose $x \in \partial E_{\text{iso}}$. If x is not eventually periodic, then $p_xC^*(E)p_x = p_x\mathcal{D}(E)p_x = \mathbb{C}p_x$. If $x = \mu\eta\eta\eta\cdots$ for some $\mu \in E^*$ and some simple cycle $\eta \in E^*$ with $s(\eta) = r(\mu)$, then $p_xC^*(E)p_x$ is isomorphic to $C(\mathbb{T})$ by the isomorphism mapping $p_xs_\mu s_\eta s_\mu^*p_x$ to the identity function on $\mathbb{T}$, and $p_x\mathcal{D}(E)p_x = \mathbb{C}p_x$.*

Proof. Let $(\mathcal{G}_E)_x^x$ denote the isotropy group $\{\gamma \in \mathcal{G}_E : s(\gamma) = r(\gamma) = x\}$ of $(x, 0, x)$. Assume that x is not eventually periodic. Then $(\mathcal{G}_E)_x^x = \{(x, 0, x)\}$. Proposition 2.2 implies that there is an isomorphism from $p_xC^*(E)p_x$ to $C^*((\mathcal{G}_E)_x^x)$, and consequently $p_xC^*(E)p_x = p_x\mathcal{D}(E)p_x = \mathbb{C}p_x$, completing the first assertion in the lemma.

Assume then that $x = \mu\eta\eta\eta\cdots$ for some $\mu \in E^*$ and some simple cycle $\eta \in E^*$ with $s(\eta) = r(\mu)$. We then have that $(\mathcal{G}_E)_x^x = \{(x, k|\eta|, x) : k \in \mathbb{Z}\}$. Now Proposition 2.2 implies that there is an isomorphism from $p_xC^*(E)p_x$ to $C(\mathbb{T})$ which maps $p_xs_\mu s_\eta s_\mu^*p_x$ to the identity function on $\mathbb{T}$, and that $p_x\mathcal{D}(E)p_x = \mathbb{C}p_x$. $\square$

The extended Weyl groupoid associated to $(C^*(E), \mathcal{D}(E))$ is built using an equivalence relation defined on pairs of normalisers and boundary paths. For isolated boundary paths x the equivalence relation is defined using a unitary in the corner of $C^*(E)$ determined by p_x.

Lemma 4.4. *Let E be a graph. For $x \in \partial E_{\text{iso}}$, $n_1, n_2 \in N(\mathcal{D}(E))$, $x \in \text{dom}(n_1) \cap \text{dom}(n_2)$, and $\alpha_{n_1}(x) = \alpha_{n_2}(x)$, we denote*

$$U_{(x,n_1,n_2)} := (h_E(x)(n_1^*n_1n_2^*n_2))^{-1/2}p_xn_1^*n_2p_x.$$

Then

(1) $U_{(x,n_1,n_2)}U_{(x,n_1,n_2)}^* = U_{(x,n_1,n_2)}^*U_{(x,n_1,n_2)} = p_x$, *and*
(2) $U_{(x,n_1,n_2)}^* = U_{(x,n_2,n_1)}$.

Moreover, if $n_3 \in N(\mathcal{D}(E))$, $x \in \text{dom}(n_3)$, and $\alpha_{n_3}(x) = \alpha_{n_1}(x) = \alpha_{n_2}(x)$, then

(3) $U_{(x,n_1,n_2)}U_{(x,n_2,n_3)} = U_{(x,n_1,n_3)}$.

Proof. Suppose that $x \in \partial E_{\text{iso}}$, $n_1, n_2 \in N(\mathcal{D}(E))$, $x \in \text{dom}(n_1)\cap\text{dom}(n_2)$, and $\alpha_{n_1}(x) = \alpha_{n_2}(x)$. First note that since $x \in \text{dom}(n_1) \cap \text{dom}(n_2)$, we have $h_E(n_1^*n_1), h_E(n_2^*n_2) > 0$, and the formula for $U_{(x,n_1,n_2)}$ makes sense. We now claim that

$$p_xn_1^*n_2p_xn_2^*n_1p_x = h_E(x)(n_1^*n_1n_2^*n_2)p_x. \tag{4.5}$$

To see this, we apply identities (a) and (b) of Lemma 4.2 to get

$$\begin{aligned} p_x n_1^* n_2 p_x n_2^* n_1 p_x &= p_x n_1^* (n_2 p_x n_2^*) n_1 p_x \\ &= p_x n_1^* \big(h_E(x)(n_2^* n_2) p_{\alpha_{n_2}(x)} \big) n_1 p_x \\ &= h_E(x)(n_2^* n_2) p_x n_1^* p_{\alpha_{n_1}(x)} n_1 p_x \\ &= h_E(x)(n_1^* n_1) h_E(x)(n_2^* n_2) p_x \\ &= h_E(x)(n_1^* n_1 n_2^* n_2) p_x. \end{aligned}$$

We now use (4.5) to get

$$U_{(x,n_1,n_2)} U^*_{(x,n_1,n_2)} = (h_E(n_1^* n_1 n_2^* n_2))^{-1} p_x n_1^* n_2 p_x n_2^* n_1 p_x = p_x$$

Similarly, and using that $p_x C^*(E) p_x$ is commutative, we have

$$\begin{aligned} U^*_{(x,n_1,n_2)} U_{(x,n_1,n_2)} &= (h_E(n_1^* n_1 n_2^* n_2))^{-1} p_x n_2^* n_1 p_x n_1^* n_2 p_x \\ &= (h_E(n_1^* n_1 n_2^* n_2))^{-1} p_x n_1^* n_2 p_x n_2^* n_1 p_x \\ &= p_x. \end{aligned}$$

So (1) holds.

Identity (2) holds because $n_1^* n_1, n_2^* n_2 \in \mathcal{D}(E)$, and hence

$$\begin{aligned} U^*_{(x,n_1,n_2)} := (h_E(x)(n_1^* n_1 n_2^* n_2))^{-1/2} p_x n_2^* n_1 p_x &= (h_E(x)(n_2^* n_2 n_1^* n_1))^{-1/2} p_x n_2^* n_1 p_x \\ &= U_{(x,n_2,n_1)}. \end{aligned}$$

We use identities (a) and (c) of Lemma 4.2 to get

$$\begin{aligned} U_{(x,n_1,n_2)} U_{(x,n_2,n_3)} &= (h_E(x)(n_1^* n_1 n_2^* n_2) h_E(x)(n_2^* n_2 n_3^* n_3))^{-1/2} p_x n_1^* n_2 p_x n_2^* n_3 p_x \\ &= (h_E(x)(n_2^* n_2))^{-1} (h_E(x)(n_1^* n_1 n_3^* n_3))^{-1/2} p_x n_1^* \big(h_E(x)(n_2^* n_2) p_{\alpha_{n_2}(x)} \big) n_3 p_x \\ &= (h_E(n_1^* n_1 n_3^* n_3))^{-1/2} p_x n_1^* p_{\alpha_{n_3}(x)} n_3 p_x \\ &= (h_E(n_1^* n_1 n_3^* n_3))^{-1/2} p_x n_1^* n_3 p_x \\ &= U_{(x,n_1,n_3)}. \end{aligned}$$

So (3) holds. □

Notation 4.5. For x, n_1 and n_2 as in Lemma 4.4 we denote

$$\lambda_{(x,n_1,n_2)} := h_E(x)(n_1^* n_1 n_2^* n_2).$$

So $U_{(x,n_1,n_2)} = \lambda^{-1/2}_{(x,n_1,n_2)} p_x n_1^* n_2 p_x$. It follows from identity (1) of Lemma 4.4 that $U_{(x,n_1,n_2)}$ is a unitary element of $p_x C^*(E) p_x$. We denote by $[U_{(x,n_1,n_2)}]_1$ the class of $U_{(x,n_1,n_2)}$ in $K_1(p_x C^*(E) p_x)$. Notice that since $p_x C^*(E) p_x$ is commutative (see Lemma 4.3), $[U_{(x,n_1,n_2)}]_1 = 0$ if and only if $U_{(x,n_1,n_2)}$ is homotopic to p_x.

Proposition 4.6. *Let E be a graph. For each $x_1, x_2 \in \partial E$ and $n_1, n_2 \in N(\mathcal{D}(E))$ such that $x_1 \in \operatorname{dom}(n_1)$ and $x_2 \in \operatorname{dom}(n_2)$ we write $(n_1, x_1) \sim (n_2, x_2)$ if either*

(a) *$x_1 = x_2 \in \partial E_{\text{iso}}$, $\alpha_{n_1}(x_1) = \alpha_{n_2}(x_2)$, and $[U_{(x_1,n_1,n_2)}]_1 = 0$; or*
(b) *$x_1 = x_2 \notin \partial E_{\text{iso}}$ and there is an open set V such that $x_1 \in V \subseteq \operatorname{dom}(n_1) \cap \operatorname{dom}(n_2)$ and $\alpha_{n_1}(y) = \alpha_{n_2}(y)$ for all $y \in V$.*

Then $\sim$ is an equivalence relation on $\{(n, x) : n \in N(\mathcal{D}(E)),\ x \in \operatorname{dom}(n)\}$.

Proof. The only nontrivial parts to prove are that $\sim$ is symmetric and transitive when the boundary paths are isolated points. Suppose $(n_1, x_1) \sim (n_2, x_2)$ with $x := x_1 = x_2 \in \partial E_{\text{iso}}$. We know from Lemma 4.4(2) that $U_{(x,n_2,n_1)} = U^*_{(x,n_1,n_2)}$. So

$$[U_{(x,n_1,n_2)}]_1 = 0 \Longrightarrow [U_{(x,n_2,n_1)}]_1 = [U^*_{(x,n_1,n_2)}]_1 = 0,$$

and hence $(n_2, x_2) \sim (n_1, x_1)$.

For transitivity, suppose $(n_1, x_1) \sim (n_2, x_2)$ and $(n_2, x_2) \sim (n_3, x_3)$ with $x := x_1 = x_2 = x_3 \in \partial E_{\text{iso}}$. We know from Lemma 4.4(2) that $U_{(x,n_1,n_2)}U_{(x,n_2,n_3)} = U_{(x,n_1,n_3)}$. So

$$[U_{(x,n_1,n_2)}]_1 = 0 = [U_{(x,n_2,n_3)}]_1 \Longrightarrow [U_{(x,n_1,n_3)}]_1 = [U_{(x,n_1,n_2)}]_1[U_{(x,n_2,n_3)}]_1 = 0,$$

and hence $(n_1, x_1) \sim (n_3, x_3)$. □

Proposition 4.7. *Let E be a graph, and $\sim$ the equivalence relation on $\{(n,x) : n \in N(\mathcal{D}(E)),\ x \in \operatorname{dom}(n)\}$ from Proposition 4.6. Denote the collection of equivalence classes by $\mathcal{G}_{(C^*(E),\mathcal{D}(E))}$. Define a partially-defined product on $\mathcal{G}_{(C^*(E),\mathcal{D}(E))}$ by*

$$[(n_1, x_1)][(n_2, x_2)] := [(n_1 n_2, x_2)] \quad \text{if } \alpha_{n_2}(x_2) = x_1,$$

and undefined otherwise. Define an inverse map by $[(n,x)]^{-1} := [(n^*, \alpha_n(x))]$. Then these operations make $\mathcal{G}_{(C^*(E),\mathcal{D}(E))}$ into a groupoid.

Proof. We only check that composition and inversion are well-defined. That composition is associative and every element is composable with its inverse (in either direction) is left to the reader. To see that composition is well-defined, suppose $[(n_1, x_1)] = [(n_1', x_1')]$ and $[(n_2, x_2)] = [(n_2', x_2')]$ with $[(n_1, x_1)]$ and $[(n_2, x_2)]$ composable. We need to show that $[(n_1', x_1')]$ and $[(n_2', x_2')]$ are also composable with

$$[(n_1 n_2, x_2)] = [(n_1' n_2', x_2')]. \tag{4.6}$$

We immediately know that $x_1 = x_1'$, $x_2 = x_2'$, $x_2 = \alpha_{n_2}^{-1}(x_1)$, $\alpha_{n_1}(x_1) = \alpha_{n_1'}(x_1')$, and $\alpha_{n_2}(x_2) = \alpha_{n_2'}(x_2')$. This gives

$$\alpha_{n_2'}^{-1}(x_1') = \alpha_{n_2'}^{-1}(x_1) = \alpha_{n_2'}^{-1}(\alpha_{n_2}(x_2)) = \alpha_{n_2'}^{-1}(\alpha_{n_2'}(x_2')) = x_2'.$$

So $\alpha_{n_2'}(x_2') = x_1'$, and hence $[(n_1', x_1')]$ and $[(n_2', x_2')]$ are composable.

To see that (4.6) holds we have two cases:

Case 1: Suppose $x_1 \notin \partial E_{\text{iso}}$. Then $x_1' = x_1 \notin \partial E_{\text{iso}}$, $x_2 = \alpha_{n_2}^{-1}(x_1) \notin \partial E_{\text{iso}}$, and $x_2' = \alpha_{n_2'}^{-1}(x_1') \notin \partial E_{\text{iso}}$. We also know there exists an open set V_1 such that $x_1 \in V_1 \subseteq \operatorname{dom}(n_1) \cap \operatorname{dom}(n_1')$ with $\alpha_{n_1}|_{V_1} = \alpha_{n_1'}|_{V_1}$, and an open set V_2 such that $x_2 \in V_2 \subseteq \operatorname{dom}(n_2) \cap \operatorname{dom}(n_2')$ with $\alpha_{n_2}|_{V_2} = \alpha_{n_2'}|_{V_2}$. Let $V := V_2 \cap \alpha_{n_2}^{-1}(V_1)$, which is an open set containing x_2. We claim that

$$V \subseteq \operatorname{dom}(n_1 n_2) \cap \operatorname{dom}(n_1' n_2').$$

To see this, let $x \in V$. Then using (4.1) we have

$$h_E(x)((n_1 n_2)^* n_1 n_2) = h_E(\alpha_{n_2}(x))(n_1^* n_1) h_E(x)(n_2^* n_2),$$

which is positive because $\alpha_{n_2}(x) \in \operatorname{dom}(n_1)$ and $x \in \operatorname{dom}(n_2)$. So $V \subseteq \operatorname{dom}(n_1 n_2)$. A similar argument gives $V \subseteq \operatorname{dom}(n_1' n_2')$, and so the claim holds. For each $x \in V$ we have $\alpha_{n_2}(x) = \alpha_{n_2'}(x) \in V_1$, which means

$$\alpha_{n_1 n_2}(x) = \alpha_{n_1}(\alpha_{n_2}(x)) = \alpha_{n_1'}(\alpha_{n_2'}(x)) = \alpha_{n_1' n_2'}(x).$$

So $\alpha_{n_1n_2}|_V = \alpha_{n_1'n_2'}|_V$. Hence $(n_1n_2, x_2) \sim (n_1'n_2', x_2')$, and (4.6) holds in this case.

Case 2: Suppose $x_1 \in \partial E_{\text{iso}}$. Then $x_1' = x_1 \in \partial E_{\text{iso}}$, $x_2 = \alpha_{n_2}^{-1}(x_1) \in \partial E_{\text{iso}}$, and $x_2' = \alpha_{n_2'}^{-1}(x_1') \in \partial E_{\text{iso}}$. We also have $\alpha_{n_1}(x_1) = \alpha_{n_1'}(x_1')$, $\alpha_{n_2}(x_2) = \alpha_{n_2'}(x_2')$, and hence

$$\alpha_{n_1n_2}(x_2) = \alpha_{n_1}(\alpha_{n_2'}(x_2')) = \alpha_{n_1}(x_1') = \alpha_{n_1'}(x_1') = \alpha_{n_1'}(\alpha_{n_2'}(x_2')) = \alpha_{n_1'n_2'}(x_2').$$

To get $(n_1n_2, x_2) \sim (n_1'n_2', x_2')$ in this case it now suffices to show that $U_{(x_2,n_1n_2,n_1'n_2')}$ is homotopic to p_{x_2}. We use that $\alpha_{n_2}(x_2) = x_1$ and $\alpha_{n_2'}(x_2') = x_1' = x_1$ and apply Lemma 4.2(c) twice to get

$$p_{x_2}n_2^*n_1^*n_1'n_2'p_{x_2} = (n_2p_{x_2})^*n_1^*n_1'(n_2'p_{x_2}) = (p_{\alpha_{n_2}(x_2)}n_2)^*n_1^*n_1'p_{\alpha_{n_2'}(x_2)}n_2' = n_2^*p_{x_1}n_1^*n_1'p_{x_1}n_2'.$$

Now we can write

$$\begin{aligned} U_{(x_2,n_1n_2,n_1'n_2')} &= \lambda^{-1/2}_{(x_2,n_1n_2,n_1'n_2')}p_{x_2}n_2^*n_1^*n_1'n_2'p_{x_2} \\ &= \lambda^{-1/2}_{(x_2,n_1n_2,n_1'n_2')}n_2^*p_{x_1}n_1^*n_1'p_{x_1}n_2' \\ &= \lambda^{-1/2}_{(x_2,n_1n_2,n_1'n_2')}\lambda^{1/2}_{(x_1,n_1,n_1')}n_2^*\big(\lambda^{-1/2}_{(x_1,n_1,n_1')}p_{x_1}n_1^*n_1'p_{x_1}\big)n_2' \\ &= \lambda^{-1/2}_{(x_2,n_1n_2,n_1'n_2')}\lambda^{1/2}_{(x_1,n_1,n_1')}n_2^*U_{(x_1,n_1,n_1')}n_2' \end{aligned}$$

Since $(n_1, x_1) \sim (n_1', x_1')$ implies that $U_{(x_1,n_1,n_1')}$ is homotopic to p_{x_1}, we see that $U_{(x_2,n_1n_2,n_1'n_2')}$ is homotopic to

$$\lambda^{-1/2}_{(x_2,n_1n_2,n_1'n_2')}\lambda^{1/2}_{(x_1,n_1,n_1')}n_2^*p_{x_1}n_2'.$$

We use Lemma 4.2(c) to get

$$n_2^*p_{x_1}n_2'n_2^*p_{\alpha_{n_2}(x_2)}p_{\alpha_{n_2'}(x_2')}n_2' = p_{x_2}n_2^*n_2'p_{x_2}.$$

Hence $U_{(x_2,n_1n_2,n_1'n_2')}$ is homotopic to

$$\begin{aligned} \lambda^{-1/2}_{(x_2,n_1n_2,n_1'n_2')}\lambda^{1/2}_{(x_1,n_1,n_1')}n_2^*p_{x_1}n_2' &= \lambda^{-1/2}_{(x_2,n_1n_2,n_1'n_2')}\lambda^{1/2}_{(x_1,n_1,n_1')}p_{x_2}n_2^*n_2'p_{x_2} \\ &= \lambda^{-1/2}_{(x_2,n_1n_2,n_1'n_2')}\lambda^{1/2}_{(x_1,n_1,n_1')}\lambda^{1/2}_{(x_2,n_2,n_2')}U_{(x_2,n_2,n_2')}. \end{aligned}$$

But $(n_2, x_2) \sim (n_2', x_2')$ implies that $U_{(x_2,n_2,n_2')}$ is homotopic to p_{x_2}, and hence $U_{(x_2,n_1n_2,n_1'n_2')}$ is homotopic to p_{x_2}.

This complete the proof that composition is well-defined. To see that inversion is well-defined, suppose $[(n_1, x_1)] = [(n_1', x_1')]$. We need to show that $[(n_1^*, \alpha_{n_1}(x_1))] = [((n_1')^*, \alpha_{n_1'}(x_1')]$. We again have two cases.

Case 1: Suppose that $x_1 = x_1' \notin \partial E_{\text{iso}}$. We know that there is open V such that $x_1 \in V \subseteq \text{dom}(n_1) \cap \text{dom}(n_1')$ and $\alpha_{n_1}|_V = \alpha_{n_1'}|_V$. A straightforward argument shows that the open set $V' := \alpha_{n_1}(V)$ satisfies $\alpha_{n_1}(x_1) \in V' \subseteq \text{dom}(n_1^*) \cap \text{dom}((n_1')^*)$ and $\alpha_{n_1^*}|_{V'} = \alpha_{(n_1')^*}|_{V'}$. So $[(n_1^*, \alpha_{n_1}(x_1))] = [((n_1')^*, \alpha_{n_1'}(x_1')]$ in this case.

Case 2: Suppose that $x_1 = x_1' \in \partial E_{\text{iso}}$. We have to show that $U_{(\alpha_{n_1}(x_1),n_1^*,(n_1')^*)}$ is homotopic to $p_{\alpha_{n_1}(x_1)}$. We use Lemma 4.2(c) to get

$$\begin{aligned} U_{(\alpha_{n_1}(x_1),n_1^*,(n_1')^*)} &= \lambda^{-1/2}_{(\alpha_{n_1}(x_1),n_1^*,(n_1')^*)}p_{\alpha_{n_1}(x_1)}n_1(n_1')^*p_{\alpha_{n_1}(x_1)} \\ &= \lambda^{-1/2}_{(\alpha_{n_1}(x_1),n_1^*,(n_1')^*)}n_1p_{x_1}(n_1')^*. \end{aligned}$$

Since $(n_1, x_1) \sim (n_1', x_1')$, we have that $U_{(x_1,n_1,n_1')}$ is homotopic to p_{x_1}. Hence $U_{(\alpha_{n_1}(x_1),n_1^*,(n_1')^*)}$ is homotopic to

$$\lambda^{-1/2}_{(\alpha_{n_1}(x_1),n_1^*,(n_1')^*)} n_1 U_{(x_1,n_1,n_1')}(n_1')^* = \lambda^{-1/2}_{(\alpha_{n_1}(x_1),n_1^*,(n_1')^*)} \lambda^{-1/2}_{(x_1,n_1,n_1')} n_1 p_{x_1} n_1^* n_1' p_{x_1} (n_1')^*.$$

Now, using (4.3) we have

$$\lambda^{-1/2}_{(\alpha_{n_1}(x_1),n_1^*,(n_1')^*)} \lambda^{-1/2}_{(x_1,n_1,n_1')} = h_E(x_1)(n_1^* n_1)^{-1} h_E(x_1)((n_1')^* n_1')^{-1}.$$

So $U_{(\alpha_{n_1}(x_1),n_1^*,(n_1')^*)}$ is homotopic to

$$\begin{aligned} h_E(x_1)(n_1^*n_1)^{-1} h_E(x_1)((n_1')^*n_1')^{-1} n_1 p_{x_1} n_1^* n_1' p_{x_1} (n_1')^* \\ &= \big(h_E(x_1)(n_1^*n_1)^{-1} n_1 p_{x_1} n_1^*\big)\big(h_E(x_1)((n_1')^*n_1')^{-1} n_1' p_{x_1} (n_1')^*\big) \\ &= p_{\alpha_{n_1}(x_1)}, \end{aligned}$$

where the last equality follows from Lemma 4.2(a). □

We equip $\mathcal{G}_{(C^*(E),\mathcal{D}(E))}$ with the topology generated by $\{\{[(n,x)] : x \in \operatorname{dom}(n)\} : n \in N(\mathcal{D}(E))\}$. It can be proven directly that $\mathcal{G}_{(C^*(E),\mathcal{D}(E))}$ is a topological groupoid with this topology, however, it also follows from our next result.

Proposition 4.8. *Let E be a graph. Then $\mathcal{G}_{(C^*(E),\mathcal{D}(E))}$ is a topological groupoid, and $\mathcal{G}_{(C^*(E),\mathcal{D}(E))}$ and $\mathcal{G}_E$ are isomorphic as topological groupoids.*

Remark 4.9. If $\mathcal{G}_E$ is topologically principal, which we know from Proposition 2.3 is equivalent to E satisfying condition (L), then $\mathcal{G}_{(C^*(E),\mathcal{D}(E))}$ is isomorphic to the Weyl groupoid $\mathcal{G}_{C_0(\mathcal{G}_E^0)}$ of $(C^*(\mathcal{G}_E), C_0(\mathcal{G}_E^0))$ as in [22]. In this case the isomorphism of $\mathcal{G}_E$ and $\mathcal{G}_{(C^*(E),\mathcal{D}(E))}$ proved below follows from [22, Proposition 4.14]. However, for a general graph E, the proof does not follow from [22, Proposition 4.14] and becomes much more technical.

To prove Proposition 4.8 we need the following result. The proof can be deduced from the proof of [22, Proposition 4.8], but we include a proof for completeness. As in [22], we let $\operatorname{supp}'(f) := \{y \in \mathcal{G}_E : f(\gamma) \neq 0\}$ for $f \in C^*(\mathcal{G}_E)$.

Lemma 4.10. *Let E be a graph and $\pi : C^*(E) \to C^*(\mathcal{G}_E)$ the isomorphism from Proposition 2.2. Let $n \in N(\mathcal{D}(E))$, and $f := \pi(n)$. Then $\operatorname{supp}'(f)$ satisfies*

(i) $s(\operatorname{supp}'(f)) = \operatorname{dom}(n)$;
(ii) $(x, k, y) \in \operatorname{supp}'(f) \Longrightarrow \alpha_n(y) = x$; *and*
(iii) $y \in \operatorname{dom}(n) \Longrightarrow (\alpha_n(y), k, y) \in \operatorname{supp}'(f)$ *for some* $k \in \mathbb{Z}$.

Proof. Identity (i) follows because

$$h_E(y)(n^*n) = \pi(n^*n)(y,0,y) = f^*f(y,0,y) = \sum_{\substack{\gamma\in\mathcal{G}_E \\ s(\gamma)=(y)}} |f(\gamma)|^2.$$

For (ii) we first consider the function f^*gf where g is any element of $\pi(\mathcal{D}(E)) = C_0(\mathcal{G}_E^{(0)})$. Using the convolution product we have

$$f^*gf(y,0,y) = \sum_{\substack{\gamma\in\mathcal{G}_E \\ s(\gamma)=(y)}} |f(\gamma)|^2 g(r(\gamma)) \quad \text{for all } y \in \partial E. \tag{4.7}$$

Alternatively, we can also apply (4.1) to, say, $g = \pi(d)$ to get

$$\begin{aligned} f^*gf(y,0,y) = \pi(n^*dn)(y,0,y) = h_E(y)(n^*dn) &= h_E(\alpha_n(y))(d)h_E(y)(n^*n) \\ &= g(\alpha_n(y),0,\alpha_n(y))|f(y,0,y)|^2. \end{aligned} \tag{4.8}$$

Now suppose for contradiction that $(x,k,y) \in \operatorname{supp}'(f)$ but $\alpha_n(y) \neq x$. Choose $g \in C_0(\mathcal{G}_E^{(0)})$ a positive function with $g(x,0,x) = 1$ and $g(\alpha_n(y),0,\alpha_n(y)) = 0$. Then (4.7) gives

$$f^*gf(y,0,y) \geq |f(x,k,y)|^2 g(x,0,x) > 0,$$

whereas (4.8) gives

$$f^*gf(y,0,y) = g(\alpha_n(y),0,\alpha_n(y))|f(y,0,y)|^2 = 0.$$

So (ii) holds.

Implication (iii) follows immediately from (i) and (ii). □

Proof of Proposition 4.8. Let $(x,k,y) \in \mathcal{G}_E$. Then there are $\mu,\nu \in E^*$ and $z \in \partial E$ such that $x = \mu z$, $y = \nu z$, and $k = |\mu| - |\nu|$. We know from Lemma 4.1 that $s_\mu s_\nu^* \in N(\mathcal{D}(E))$, $y \in \operatorname{dom}(s_\mu s_\nu^*)$, and that $\alpha_{s_\mu s_\nu^*}(y) = x$. Define $\phi : \mathcal{G}_E \to \mathcal{G}_{(C^*(E),\mathcal{D}(E))}$ by

$$\phi((x,k,y)) = [(s_\mu s_\nu^*, y)].$$

It is routine to check that ϕ is well-defined, in the sense that if $\mu,\nu,\mu',\nu' \in E^*$, $z,z' \in \partial E$, $\mu z = \mu' z'$, $\nu z = \nu' z'$, and $|\mu| - |\nu| = |\mu'| - |\nu'|$, then $[(s_\mu s_\nu^*, \nu z)] = [(s_{\mu'} s_{\nu'}^*, \nu' z')]$. It is also routine to check that ϕ is a groupoid homomorphism. We now have to show that ϕ is a homeomorphism.

To show that ϕ is injective, assume that $\phi((x,k,y)) = \phi((x',k',y'))$. Then $x = x'$ and $y = y'$. Suppose for contradiction that $k \neq k'$. Then y must be eventually periodic, because otherwise we would have $\alpha_{s_\kappa s_\lambda^*}(y) \neq \alpha_{s_{\kappa'} s_{\lambda'}^*}(y)$ for $|\kappa| - |\lambda| = k$ and $|\kappa'| - |\lambda'| = k'$. Thus $x = \mu\eta\eta\eta\cdots$ and $y = \nu\eta\eta\eta\cdots$ for some $\mu,\nu \in E^*$ and a simple cycle $\eta \in E^*$ such that $s(\eta) = r(\mu) = r(\nu)$. It follows that $\phi((x,k,y)) = [(s_{\mu(\eta)^m} s_{\nu(\eta)^n}^*, y)]$ and $\phi((x,k',y)) = [(s_{\mu(\eta)^{m'}} s_{\nu(\eta)^{n'}}^*, y)]$ where m,n,m',n' are nonnegative integers such that $|\mu(\eta)^m| - |\nu(\eta)^n| = k$ and $|\mu(\eta)^{m'}| - |\nu(\eta)^{n'}| = k'$. Suppose that η has an exit. Then $y \notin \partial E_{\mathrm{iso}}$, and there is a $\zeta \in E^*$ such that $s(\zeta) = s(\eta)$, $|\zeta| \leq |\eta|$, and $\zeta \neq \eta_1\eta_2\cdots\eta_{|\zeta|}$ (where $\eta = \eta_1\eta_2\cdots\eta_{|\eta|}$). Then for any open set U with $y \in U \subseteq \operatorname{dom}(s_{\mu(\eta)^m} s_{\nu(\eta)^n}^*) \cap \operatorname{dom}(s_{\mu(\eta)^{m'}} s_{\nu(\eta)^{n'}}^*)$, there is a positive integer l such that $\emptyset \neq Z(\nu(\eta)^l\zeta) \subseteq U$, and that $\alpha_{s_{\mu(\eta)^m} s_{\nu(\eta)^n}^*}(z) \neq \alpha_{s_{\mu(\eta)^{m'}} s_{\nu(\eta)^{n'}}^*}(z)$ for any $z \in Z(\mu(\eta)^l\zeta)$. This contradicts the assumption that $\phi((x,k,y)) = \phi((x,k',y))$. If η does not have an exit, then $y \in \partial E_{\mathrm{iso}}$. Without loss of generality assume $k > k'$, then we can use Lemma 4.2(c) to compute

$$[p_y(s_{\mu(\eta)^m} s_{\nu(\eta)^n}^*)^* s_{\mu(\eta)^{m'}} s_{\nu(\eta)^{n'}}^* p_y]_1 = [p_y s_\nu s_{\eta^{k-k'}} s_\nu^* p_y]_1 = [(p_y s_\nu s_\eta s_\nu^* p_y)^{k-k'}]_1,$$

and the second assertion in Lemma 4.3 implies that $[(p_y s_\nu s_\eta s_\nu^* p_y)^{k-k'}]_1 \neq 0$. Thus,

$$[U_{(y, s_{\mu(\eta)^m} s_{\nu(\eta)^n}^*, s_{\mu(\eta)^{m'}} s_{\nu(\eta)^{n'}}^*)}]_1 \neq 0$$

and hence $(s_{\mu(\eta)^m} s_{\nu(\eta)^n}^*, y) \not\sim (s_{\mu(\eta)^{m'}} s_{\nu(\eta)^{n'}}^*, y)$. But this means $\phi((x,k,y)) \neq \phi((x,k',y))$, which is a contradiction. So we must have $k = k'$, and hence ϕ is injective.

To show that ϕ is surjective, let $[(n,x)]$ be an arbitrary element of $\mathcal{G}_{(C^*(E),\mathcal{D}(E))}$. Let $f := \pi(n)$, where $\pi : C^*(E) \to C^*(\mathcal{G}_E)$ is the isomorphism from Proposition 2.2. We know from (iii) of Lemma 4.10 that $(\alpha_n(x), k, x) \in \operatorname{supp}'(f)$ for some $k \in \mathbb{Z}$. Suppose first that $x \notin \partial E_{\mathrm{iso}}$. Choose $\mu, \nu \in E^*$, a clopen neighborhood U of $\alpha_n(x)$, and a clopen neighborhood V of x such that $U \subseteq Z(\mu)$, $V \subseteq Z(\nu)$, $\sigma_E^{|\mu|}(U) = \sigma_E^{|\nu|}(V)$, $k = |\mu| - |\nu|$, and $Z(U, |\mu|, |\nu|, V) \subseteq \operatorname{supp}'(f)$. Then $\alpha_{s_\mu s_\nu^*}(y) = \alpha_n(y)$ for all $y \in V$, and hence $\phi(\alpha_n(x), |\mu| - |\nu|, x) = [(s_\mu s_\nu^*, x)] = [(n, x)]$.

Now suppose that $x \in \partial E_{\mathrm{iso}}$ is not eventually periodic. Choose $\mu, \nu \in E^*$ and $z \in \partial E$ such that $x = \nu z$, $\alpha_n(x) = \mu z$, and $k = |\mu| - |\nu|$. It follows from Lemma 4.3 that $[U_{(x,n,s_\mu s_\nu^*)}]_1 = 0$ (because $K_1(\mathbb{C}) = 0$), and thus that $\phi((\alpha_n(x), k, x)) = [(s_\mu s_\nu^*, x)] = [(n, x)]$. Assume then that x is eventually periodic. Then there are $\mu, \nu \in E^*$ and a simple cycle $\eta \in E^*$ such that $s(\eta) = r(\mu) = r(\nu)$, $x = \nu\eta\eta\eta\cdots$, $\alpha_n(x) = \mu\eta\eta\eta\cdots$, and $k = |\mu| - |\nu|$. By Lemma 4.3 there are positive integers l and m such that $[U_{(x,n,s_\mu s_\nu^*)}]_1 = l - m$ such that

$$[p_x s_\mu s_{\eta^l} s_{\eta^m}^* s_\nu^* p_x]_1 = [(p_x s_\mu s_\eta s_\nu^* p_x)^l (p_x s_\mu s_\eta^* s_\nu^* p_x)^m]_1 = l - m,$$

and hence $[U_{(x,n,s_\mu s_\nu^*)}]_1 = [p_x s_\nu s_{\eta^l} s_{\eta^m}^* s_\nu^* p_x]_1$. Since

$$h_E(s_{\nu\eta^m} s_{\nu\eta^m}^*)^{-1/2} U_{(x,n,s_\mu s_\nu^*)} (p_x s_\nu s_{\eta^l} s_{\eta^m}^* s_\nu^* p_x)^* = U_{(x,n,s_{\mu\eta^m} s_{\nu\eta^l}^*)},$$

We have $[U_{(x,n,s_{\mu\eta^m} s_{\nu\eta^l}^*)}]_1 = [U_{(x,n,s_\mu s_\nu^*)}]_1 - [p_x s_\nu s_{\eta^l} s_{\eta^m}^* s_\nu^* p_x]_1 = 0$. Hence

$$\phi((\alpha_n(x), |\mu(\eta)^m| - |\nu(\eta)^l|, x)) = [(s_{\mu(\eta)^m} s_{\nu(\eta)^l}^*, x)] = [(n, x)],$$

which shows that ϕ is surjective.

To see that ϕ is open, let $\mu, \nu \in E^*$ and let U and V be clopen subsets of ∂E such that $U \subseteq Z(\mu)$, $V \subseteq Z(\nu)$, and $\sigma_E^{|\mu|}(U) = \sigma_E^{|\nu|}(V)$. Then there is a $p_V \in \mathcal{D}(E)$ such that $h_E(x)(p_V) = 1$ if $x \in V$, and $h_E(x)(p_V) = 0$ if $x \in \partial E \setminus V$; and then $\phi(Z(U, |\mu|, |\nu|, V)) = \{[s_\mu s_\nu^* p_V, x] : x \in \operatorname{dom}(s_\mu s_\nu^* p_V)\}$. This shows that ϕ is open.

To prove that ϕ is continuous we will show that $\phi^{-1}(\{[(n, y)] : y \in \operatorname{dom}(n)\})$ is open for each $n \in N(\mathcal{D}(E))$. Fix $n \in N(\mathcal{D}(E))$ and $z \in \operatorname{dom}(n)$. We claim that there is an open subset $V_{(n,z)}$ in $\mathcal{G}_E$ such that

$$\phi^{-1}([(n, z)]) \in V_{(n,z)} \subseteq \phi^{-1}(\{[(n, y)] : y \in \operatorname{dom}(n)\}).$$

Let $\pi : C^*(E) \to C^*(\mathcal{G}_E)$ be the isomorphism from Proposition 2.2, and $f := \pi(n)$. We know from (iii) of Lemma 4.10 that $(\alpha_n(z), k, z) \in \operatorname{supp}'(f)$ for some $k \in \mathbb{Z}$. Suppose that there are two different integers k_1 and k_2 such that both $(\alpha_n(z), k_1, z)$ and $(\alpha_n(z), k_2, z)$ belong to $\operatorname{supp}'(f)$. Then there are $\mu_1, \nu_1, \mu_2, \nu_2 \in E^*$ such that $(\alpha_n(z), k_1, z) \in Z(\mu_1, \nu_1)$, $(\alpha_n(z), k_2, z) \in Z(\mu_2, \nu_2)$ and $Z(\mu_1, \nu_1), Z(\mu_2, \nu_2) \subseteq \operatorname{supp}'(f)$. Without loss of generality we can assume that $|\mu_1| = |\mu_2|$, and then we have $\mu_1 = \mu_2$. We also have $\nu_1 = \nu_2\xi$ or $\nu_2 = \nu_1\xi$ for some $\xi \in E^* \setminus E^0$; we assume that $\mu_2 = \mu_1\xi$ and denote $\mu := \mu_1 = \mu_2$. We claim that z is an isolated point, and that

$$s(Z(\mu, \nu_1)) \cap s(Z(\mu, \nu_1\xi)) = \{(z)\}.$$

To see this, suppose $(x) \in s(Z(\mu, \nu_1)) \cap s(Z(\mu, \nu_1\xi))$. Then $x = \nu_1 y$ for some y such that $\alpha_n(y) = \mu y$, and $x = \nu_1\xi y'$ for some y' such that $\alpha_n(y) = \mu y'$. It follows that $y' = y = \xi y'$, and hence $y = \nu_1\xi\xi\xi\ldots$. So

$$s(Z(\mu, \nu_1)) \cap s(Z(\mu, \nu_1\xi)) = \{(\nu_1\xi\ldots)\} = \{(z)\},$$

and hence z is an isolated point. Now $\phi^{-1}([(n,z)]) = (\mu\xi\dots, |\mu|-|\nu_1|, \nu_1\xi\dots)$ is isolated because $\{\phi^{-1}([(n,z)])\} = Z(\{\mu\xi\dots\}, |\mu|, |\nu_1|, \{\nu_1\xi\dots\})$ is open. So in this case we take $V_{(n,z)} = \{\phi^{-1}([(n,z)])\}$.

Now assume that there is a unique k such that $(\alpha_n(z), k, z) \in \operatorname{supp}'(f)$. Choose $\mu, \nu \in E^*$ with $r(\mu) = r(\nu)$ and an open subset $V \subseteq r(\mu)\partial E$ such that $(\alpha_n(z), k, z) \in Z(\mu V, |\mu|, |\nu|, \nu V) \subseteq \operatorname{supp}'(f)$. Lemma 4.10 implies that $\alpha_n(x) = \alpha_{s_\mu s_\nu^*}(x)$ for all $x \in \nu V$. We aim to find an open subset $W \subseteq \nu V$ such that $z \in W$ and

$$[U_{(x,n,s_\mu s_\nu^*)}]_1 = 0 \text{ for all } x \in W \cap \partial E_{\text{iso}}; \tag{4.9}$$

for then we have $\phi((\alpha_n(x), |\mu| - |\nu|, x)) = [(s_\mu s_\nu^*, x)] = [(n,x)]$ for all $x \in W$, and the open subset $V_{(n,z)} := Z(\alpha_n(W), |\mu|, |\nu|, W)$ satisfies the desired $\phi^{-1}([(n,z)]) \in V_{(n,z)} \subseteq \phi^{-1}(\{[(n,y)] : y \in \operatorname{dom}(n)\})$.

Let $\delta := |f(\alpha_n(z), k, z)|$. Then

$$h_E(z)(n^*n) = f^*f(z,0,z) = \sum_{\substack{\gamma \in \mathcal{G}_E \\ s(\gamma) = (z)}} |f(\gamma)|^2 = \delta^2.$$

Choose an open subset $V_0 \subseteq \nu V$ such that $z \in V_0$ and $h_E(x)(n^*n) > (\delta/2)^2$ for all $x \in V_0$. Define

$$g := f^* 1_{Z(\mu,\nu)} - \lambda (f^*f)^{1/2},$$

where $\lambda = \overline{f(\alpha_n(z), k, z)}/|f(\alpha_n(z), k, z)| \in \mathbb{T}$. We claim that $g(z,j,z) = 0$ for all $j \in \mathbb{Z}$. When $j = 0$ we have

$$g(z,0,z) = \sum_{\gamma_1\gamma_2 = (z,0,z)} f^*(\gamma_1) 1_{Z(\mu,\nu)}(\gamma_2) - \lambda \sum_{\eta_1\eta_2 = (z,0,z)} (f^*(\eta_1) f(\eta_2))^{1/2}.$$

Implication (ii) of Lemma 4.10 ensures that the only terms in the sums which produce nonzero entries are $\gamma_1, \eta_1 = (z, -k, \alpha_n(z))$ and $\gamma_2, \eta_2 = (\alpha_n(z), k, z)$. Hence

$$g(z,0,z) = \overline{f(\alpha_n(z), k, z)} - \lambda |f(\alpha_n(z), k, z)| = 0.$$

When $j \neq 0$, both terms in the expression for g contain $f(\alpha_n(z), k-j, z)$, which is zero. Hence $g(z,j,z) = 0$.

Use Proposition 2.4 to choose $m \in \mathbb{N}$ such that $\|g - \Sigma_m(g)\| < \delta/2$. Since $g(z,j,z) = 0$ for all $j \in \mathbb{Z}$, there is an open set W such that $z \in W \subseteq V_0$ and

$$\Big|\Big(1 - \frac{|j|}{m+1}\Big) g(x,j,x)\Big| < \frac{\delta}{2(m+1)} \tag{4.10}$$

for all $-m \leq j \leq m$ and $x \in W$. Then $|\Sigma_m(g)(x,j,x)| < \delta/2$ for all $(x,j,x) \in \mathcal{G}_E$ with $x \in W$. It follows from the definition of the norm on $C^*(\mathcal{G}_E)$ that for all $x \in W \cap \partial E_{\text{iso}}$ we have

$$\begin{aligned}
\|\pi(p_x)\Sigma_m(g)\pi(p_x)\| &\leq \Big|\sum_{\gamma \in \mathcal{G}_E} (\pi(p_x)\Sigma_m(g)\pi(p_x))(\gamma)\Big| \\
&= \Big|\sum_{\gamma \in \mathcal{G}_E} \sum_{\gamma_1\gamma_2\gamma_3 = \gamma} 1_{\{(x,0,x)\}}(\gamma_1)\Sigma_m(g)(\gamma_2) 1_{\{(x,0,x)\}}(\gamma_3)\Big| \\
&= \Big|\sum_{j \in \mathbb{Z}} \Sigma_m(g)(x,j,x)\Big| < \frac{\delta}{2}.
\end{aligned}$$

Hence

$$\|\pi(p_x)g\pi(p_x)\| \le \|g - \Sigma_m(g)\| + \|\pi(p_x)\Sigma_m(g)\pi(p_x)\| < \delta. \tag{4.11}$$

We now claim that $\|U_{(x,n,s_\mu s_\nu^*)} - \lambda p_x\| < 2$ for all $x \in W \cap \partial E_{\text{iso}}$. To see this, first note that

$$\pi(p_x) = (f^*f)^{-1/2}(x,0,x)(f^*f)^{1/2}(x,0,x)\pi(p_x) = (f^*f)(x,0,x)^{-1/2}(f^*f)^{1/2}\pi(p_x).$$

Thus

$$\begin{aligned}\pi(U_{(x,n,s_\mu s_\nu^*)} - \lambda p_x) &= \pi\big(h_E(x)(n^*n)^{-1/2}p_x n^* s_\mu s_\nu^* p_x - \lambda p_x\big)\\ &= (f^*f)(x,0,x)^{-1/2}\pi(p_x)f^*1_{Z(\mu,\nu)}\pi(p_x) - \lambda\pi(p_x)\\ &= (f^*f)(x,0,x)^{-1/2}\big(\pi(p_x)f^*1_{Z(\mu,\nu)}\pi(p_x) - \lambda(f^*f)^{1/2}\pi(p_x)\big)\\ &= (f^*f)(x,0,x)^{-1/2}\pi(p_x)g\pi(p_x).\end{aligned}$$

Using (4.11) we now get

$$\begin{aligned}\|U_{(x,n,s_\mu s_\nu^*)} - \lambda p_x\| = \|\pi(U_{(x,n,s_\mu s_\nu^*)} - \lambda p_x)\| &= (f^*f)(x,0,x)^{-1/2}\|\pi(p_x)g\pi(p_x)\|\\ &< (f^*f)(x,0,x)^{-1/2}\delta.\end{aligned}$$

Recall that $x \in W \cap \partial E_{\text{iso}} \subseteq V_0$, and hence $(f^*f)(x,0,x)^{-1/2} = h_E(x)(n^*n)^{-1/2} < 2/\delta$. So

$$\|U_{(x,n,s_\mu s_\nu^*)} - \lambda p_x\| < 2.$$

But this means $[U_{(x,n,s_\mu s_\nu^*)}]_1 = 0$, and so W satisfies the desired (4.9). As mentioned, this means $V_{(n,z)} := Z(\alpha_n(W), |\mu|, |\nu|, W)$ satisfies

$$\phi^{-1}([(n,z)]) \in V_{(n,z)} \subseteq \phi^{-1}(\{[(n,y)] : y \in \text{dom}(n)\}),$$

as required. □

Proposition 4.11. *Let E and F be two graphs. If there is an isomorphism from $C^*(E)$ to $C^*(F)$ which maps $\mathcal{D}(E)$ onto $\mathcal{D}(F)$, then $\mathcal{G}_{(C^*(E),\mathcal{D}(E))}$ and $\mathcal{G}_{(C^*(F),\mathcal{D}(F))}$ are isomorphic as topological groupoids, and consequently $\mathcal{G}_E$ and $\mathcal{G}_F$ are isomorphic as topological groupoids.*

Proof. Suppose ϕ is an isomorphism from $C^*(E)$ to $C^*(F)$ which maps $\mathcal{D}(E)$ onto $\mathcal{D}(F)$. Then there is a homeomorphism $\kappa : \partial E \to \partial F$ such that $h_E(x)(f) = h_F(\kappa(x))\phi(f)$ for all $f \in \mathcal{D}(E)$ and all $x \in \partial E$. It is routine to check that the map $[(n,x)] \mapsto [(\phi(n), \kappa(x))]$ is an isomorphism between the topological groupoids $\mathcal{G}_{(C^*(E),\mathcal{D}(E))}$ and $\mathcal{G}_{(C^*(F),\mathcal{D}(F))}$. Then Proposition 4.8 implies that $\mathcal{G}_E$ and $\mathcal{G}_F$ are isomorphic as topological groupoids. □

5. Main result and examples

Theorem 5.1. *Let E and F be graphs. Consider the following four statements.*

1. *There is an isomorphism from $C^*(E)$ to $C^*(F)$ which maps $\mathcal{D}(E)$ onto $\mathcal{D}(F)$.*
2. *The graph groupoids $\mathcal{G}_E$ and $\mathcal{G}_F$ are isomorphic as topological groupoids.*
3. *The pseudogroups of E and F are isomorphic.*
4. *E and F are orbit equivalent.*

Then (1) $\iff$ (2), (3) $\iff$ (4) and (2) $\implies$ (3). If E and F satisfy condition (L), then (3) $\implies$ (2) and the four statements are equivalent.

Proof. (1) $\implies$ (2) is proved in Proposition 4.11. (2) $\implies$ (1) follows from Proposition 2.2 and [16, Theorem 5.1]. (3) $\iff$ (4) is proved in Proposition 3.4. (2) $\implies$ (3) follows directly from the definition of the pseudogroups $\mathcal{P}_E$ and $\mathcal{P}_F$.

Assume that E and F satisfy condition (L). Then it follows from Proposition 2.3 and [22, Proposition 3.6(i)] that $\mathcal{G}_E$ is isomorphic to the groupoid of germs of the pseudogroup $\mathcal{P}_E$ constructed on page 8 of [22], and that $\mathcal{G}_F$ is isomorphic to the groupoid of germs of the pseudogroup $\mathcal{P}_F$. It follows that if $\mathcal{P}_E$ and $\mathcal{P}_F$ are isomorphic, then $\mathcal{G}_E$ and $\mathcal{G}_F$ are isomorphic. Thus (3) $\implies$ (2), and all 4 statements are equivalent when E and F satisfy condition (L). ☐

Example 5.2. We show that (3) does not imply (2) in general. Consider the single vertex and single loop graphs

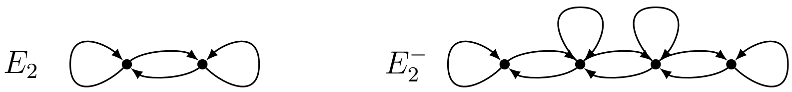
We have $\partial E = \{v\}$ and $\partial F = \{ee\dots\}$. So E and F are orbit equivalent, but $C^*(E) \cong \mathbb{C}$ is not isomorphic to $C^*(F) \cong C(\mathbb{T})$. Obviously F does not satisfy condition (L), so E and F provide a simple counterexample to the equivalence of statements (1) and (4) of Theorem 5.1 without the presence of condition (L).

Example 5.3. The graphs

provide a similar counterexample to the equivalence of statements (1) and (4) of Theorem 5.1 without the presence of condition (L). In this case $\partial E = \mathbb{N} = \partial F$ (and, unlike Example 5.2, the shift map is defined on all of ∂E and ∂F), but $C^*(E) \cong \mathcal{K} \not\cong \mathcal{K} \otimes C(\mathbb{T}) \cong C^*(F)$.

Example 5.4. There exist graphs E and F such that $C^*(E)$ and $C^*(F)$ are isomorphic, and $\mathcal{D}(E)$ and $\mathcal{D}(F)$ isomorphic, but E and F are not orbit equivalent.

Consider for example the graphs E_2 and E_2^- below.



It follows from [19, Remark 2.8] that the C^*-algebra of E_2 is isomorphic to $\mathcal{O}_2$ (see for example [23]) and that the C^*-algebra of E_2^- is isomorphic to $\mathcal{O}_2^-$ (see for example [23]). It is proved in [23, Lemma 6.4] that $\mathcal{O}_2$ and $\mathcal{O}_2^-$ are isomorphic. We also have that $\mathcal{D}(E_2)$ and $\mathcal{D}(E_2^-)$ because both ∂E_2 and ∂E_2^- are Cantor sets. However, E_2 and E_2^- cannot be orbit equivalent because if they were, then it would follow from Theorem 5.1 and [15, Theorem 3.6] that $\det(I - A_2) = \det(I - A_2^-)$ where

$$A_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \text{and} \qquad A_2^- = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

However, $\det(I - A_2) = -1$ and $\det(I - A_2^-) = 1$.

6. Applications

In this section we provide two applications of Theorem 5.1. Our first result shows that conjugacy of general graphs implies that their C^*-algebras are isomorphic and the isomorphism decends to their maximal abelian subalgebras. As a corollary we obtain a strengthening of [3, Theorem 3.2]. Our second application adds three additional equivalences to [4, Theorem 1.1], which provides a complete invariant for amplified graphs.

6.1. **Conjugacy and out-splitting.** Two graphs E and F are said to be *conjugate* if there is a homeomorphism $h : \partial E \to \partial F$ such that $h(\partial E^{\geq 1}) = \partial F^{\geq 1}$ and $h(\sigma_E(x)) = \sigma_F(h(x))$ for all $x \in \partial E^{\geq 1}$. It is routine to verify that if E and F are conjugate, then they are also orbit equivalent. Thus Theorem 5.1 implies that if E and F both satisfy condition (L) and they are conjugate, then there is an isomorphism from $C^*(E)$ to $C^*(F)$ which maps $\mathcal{D}(E)$ onto $\mathcal{D}(F)$. In Theorem 6.1 we will prove that if E and F are conjugate, then $\mathcal{G}_E$ and $\mathcal{G}_F$ are isomorphic, and hence there is an isomorphism from $C^*(E)$ to $C^*(F)$ which maps $\mathcal{D}(E)$ onto $\mathcal{D}(F)$, even if E and F do not satisfy condition (L). As a corollary, we strengthen [3, Theorem 3.2] for out-splittings of graphs.

Theorem 6.1. *Let E and F be graphs. If E and F are conjugate, then $\mathcal{G}_E$ and $\mathcal{G}_F$ are isomorphic as topological groupoids, and hence there is an isomorphism from $C^*(E)$ to $C^*(F)$ which maps $\mathcal{D}(E)$ onto $\mathcal{D}(F)$.*

Proof. Let $h : \partial E \to \partial F$ be a homeomorphism such that $h(\partial E^{\geq 1}) = \partial F^{\geq 1}$ and $h(\sigma_E(x)) = \sigma_F(h(x))$ for all $x \in \partial E^{\geq 1}$. Define $\phi : \mathcal{G}_E \to \mathcal{G}_F$ by $\phi((x, k, y)) = (h(x), k, h(y))$. Then ϕ is a homeomorphism, and $\mathcal{G}_E$ and $\mathcal{G}_F$ are isomorphic as topological groupoids. Then Theorem 5.1 implies that there is an isomorphism from $C^*(E)$ to $C^*(F)$ which maps $\mathcal{D}(E)$ onto $\mathcal{D}(F)$. □

As a corollary we are able to strengthen [3, Theorem 3.2]. Before we state the corollary we recall the terminology of [3].

Let E be a graph and let $\mathcal{P}$ be a partition of E^1 constructed in the following way. For each $v \in E^0$ with $vE^1 \neq \emptyset$, partition vE^1 into disjoint nonempty subsets $\mathcal{E}_v^1, \ldots, \mathcal{E}_v^{m(v)}$ where $m(v) \geq 1$, and let $m(v) = 0$ when $vE^1 = \emptyset$. The partition $\mathcal{P}$ is *proper* if for each $v \in E^0$ we have that $m(v) < \infty$ and that $\mathcal{E}_v^i$ is infinite for at most one i. The *out-split* of E with respect to $\mathcal{P}$ is the graph $E_s(\mathcal{P})$ where

$$E_s(\mathcal{P})^0 := \{v^i : v \in E^0,\ 1 \leq i \leq m(v)\} \cup \{v : v \in E^0,\ m(v) = 0\},$$
$$E_s(\mathcal{P})^1 := \{e^j : e \in E^1,\ 1 \leq j \leq m(r(e))\} \cup \{e : e \in E^1,\ m(r(e)) = 0\}$$

and $r, s : E_s(\mathcal{P})^1 \to E_s(\mathcal{P})^0$ are given by

$$s(e^j) := s(e)^i \text{ and } r(e^j) := r(e)^j \text{ for } e \in \mathcal{E}_{s(e)}^i \text{ with } m(r(e)) \geq 1, \text{ and}$$
$$s(e) := s(e)^i \text{ and } r(e) := r(e) \text{ for } e \in \mathcal{E}_{s(e)}^i \text{ with } m(r(e)) = 0.$$

Corollary 6.2. *Let $\mathcal{P}$ be a proper partition of E^1 as above. Then E and $E_s(\mathcal{P})$ are conjugate and there is an isomorphism from $C^*(E)$ to $C^*(E_s(\mathcal{P}))$ which maps $\mathcal{D}(E)$ onto $\mathcal{D}(E_s(\mathcal{P}))$.*

Proof. Notice that since $\mathcal{P}$ is proper, we have that $v^i \in E_s(\mathcal{P})^0_{\mathrm{reg}}$ if $v \in E^0_{\mathrm{reg}}$, and that if vE^1 is infinite, then $v^i E_s(\mathcal{P})^1$ is infinite for exactly one i.

For $x = x_1 x_2 \cdots \in \partial E$, let $h(x) = y = y_1 y_2 \cdots \in \partial E_s(\mathcal{P})$ be defined by $h(x)$ having the same length as x and

$$y_n := \begin{cases} x_n & \text{if } m(r(x_n)) = 0, \\ x_n^i & \text{if } r(x_n)^i E_s(\mathcal{P})^1 \text{ is infinite and the length of } x \text{ is } n, \\ x_n^j & \text{if } x_{n+1} \in \mathcal{E}^j_{r(e)}. \end{cases}$$

Then the map $x \mapsto h(x)$ is a homeomorphism from ∂E to $\partial E_s(\mathcal{P})$, $h(\partial E^{\geq 1}) = \partial E_s(\mathcal{P})^{\geq 1}$ and $h(\sigma_E(x)) = \sigma_{E_s(\mathcal{P})}(h(x))$ for all $x \in \partial E^{\geq 1}$. Thus, E and $E_s(\mathcal{P})$ are conjugate and it follows from Theorem 6.1 that there is an isomorphism from $C^*(E)$ to $C^*(E_s(\mathcal{P}))$ which maps $\mathcal{D}(E)$ onto $\mathcal{D}(E_s(\mathcal{P}))$. □

Remark 6.3. To see that orbit equivalence is weaker than conjugacy consider the graphs E and F from Example 3.2. We have already seen that E and F are orbit equivalent. They are not, however, conjugate because $\sigma_E(e_2 e_2 \dots) = e_2 e_2 \dots$ and $\sigma_F(y) \neq y$ for all $y \in \partial F$, and fixed points are a conjugacy invariant.

6.2. **Amplified graphs and orbit equivalence.** In [4], a graph is called *amplified* if whenever there is an edge between two vertices in the graph, there are infinitely many. Theorem 1.1 in [4] characterises when the C^*-algebras of amplified graphs are isomorphic. Using our main result, we improve this result by adding three additional equivalences, see Theorem 6.4. Before we precisely state the result of [4] and our improvement, we will first recall the notation of [4].

If E is a graph, then the *amplification* of E is the graph $\overline{E}$ defined by $\overline{E}^0 := E^0$, $\overline{E}^1 := \{e(v,w)^n : e \in E^1,\ s(e) = v,\ r(e) = w,\ n \in \mathbb{N}\}$, $s(e(v,w)^n) := v$, and $r(e(v,w)^n) := w$. It is routine to see that a graph E is amplified if and only if $E = \overline{E}$.

If E is a graph, then the *transitive closure* of E is the graph $\mathtt{t}E$ defined by $\mathtt{t}E^0 := E^0$, $\mathtt{t}E^1 := E^1 \cup \{e(v,w) : \mu \in E^* \setminus (E^0 \cup E^1),\ s(\mu) = v,\ r(\mu) = w\}$, with source and range maps that extend those of E and satisfy $s(e(v,w)) := v$, and $r(e(v,w)) := w$.

Theorem 1.1 of [4] says that if E and F are graphs with E^0 and F^0 finite, then the following 6 statements are equivalent.

(1) The graphs $\overline{\mathtt{t}E}$ and $\overline{\mathtt{t}F}$ are isomorphic, in the sense that there are bijections $\phi^0 : \overline{\mathtt{t}E}^0 \to \overline{\mathtt{t}F}^0$ and $\phi^1 : \overline{\mathtt{t}E}^1 \to \overline{\mathtt{t}F}^1$ such that $s(\phi^1(e)) = \phi^0(s(e))$ and $r(\phi^1(e)) = \phi^0(r(e))$ for all $e \in \overline{\mathtt{t}E}^1$.
(2) The C^*-algebras $C^*(\overline{\mathtt{t}E})$ and $C^*(\overline{\mathtt{t}F})$ are isomorphic.
(3) The C^*-algebras $C^*(\overline{E})$ and $C^*(\overline{F})$ are isomorphic.
(4) The C^*-algebras $C^*(\overline{E})$ and $C^*(\overline{F})$ are stably isomorphic.
(5) The tempered primitive ideal spaces $\mathrm{Prim}^\tau(C^*(\overline{E}))$ and $\mathrm{Prim}^\tau(C^*(\overline{F}))$ are isomorphic (see [4, Definition 4.8]).
(6) The ordered filtered K-theories $FK(C^*(\overline{E}))$ and $FK(C^*(\overline{F}))$ of $C^*(\overline{E})$ and $C^*(\overline{F})$ are isomorphic (see [4, Definition 4.4]).

The following result improves on [4, Theorem 1.1].

Theorem 6.4. *Let E and F be graphs with E^0 and F^0 finite. Then each of the following 3 statements is equivalent to each of the statements (1)–(6) above.*

(7) *The graphs* $\overline{E}$ *and* $\overline{F}$ *are orbit equivalent.*
(8) *The graph groupoids* $\mathcal{G}_{\overline{E}}$ *and* $\mathcal{G}_{\overline{F}}$ *are isomorphic as topological groupoids.*
(9) *There exists an isomorphism from* $C^*(\overline{E})$ *to* $C^*(\overline{F})$ *which maps* $\mathcal{D}(\overline{E})$ *onto* $\mathcal{D}(\overline{F})$.

Hence the statements (1)–(9) are all equivalent.

To prove Theorem 6.4 we need two results. We start with a modification of [4, Theorem 3.8].

Lemma 6.5. *Let* E *be a graph and* $\mu = \mu_1\mu_2\ldots,\mu_m \in E^*$. *Let* F *be the graph with* $F^0 := E^0$, $F^1 := E^1 \cup \{\mu^n : n \in \mathbb{N}\}$, *and range and source maps that extend those of* E *and satisfy* $s(\mu^n) := s(\mu)$ *and* $r(\mu^n) := r(\mu)$. *If the set* $\{e \in E^1 : s(e) = s(\mu),\ r(e) = r(\mu)\}$ *is infinite, then* E *and* F *are orbit equivalent.*

Proof. Let $A := \{e \in E^1 : s(e) = s(\mu),\ r(e) = r(\mu)\}$ and assume A is infinite. Then there are injective functions $\eta_1 : \mathbb{N} \to A$ and $\eta_2 : A \to A$ such that $\eta_1(\mathbb{N}) \cap \eta_2(A) = \emptyset$ and $\eta_1(\mathbb{N}) \cup \eta_2(A) = A$. For each $x \in \partial F$, let $h(x)$ be the element of ∂E obtained by, for each $n \in \mathbb{N}$, replacing every occurrence of μ^n by the path $\eta_1(n)\mu_2\mu_3\ldots\mu_m$ and, for each $e \in A$, replacing every occurrence of the path $e\mu_2\mu_3\ldots\mu_m$ by the path $\eta_2(e)\mu_2\mu_3\ldots\mu_m$. Then $x \mapsto h(x)$ is a homeomorphism from ∂F to ∂E.

Define $k_1, l_1 : \partial F^{\geq 1} \to \mathbb{N}$ by

$$k_1(x) := 0 \text{ for all } x \in \partial F^{\geq 1}, \qquad l_1(x) := \begin{cases} m & \text{if } x \in \cup_{n\in\mathbb{N}} Z(\mu^n), \\ 1 & \text{if } x \notin \cup_{n\in\mathbb{N}} Z(\mu^n). \end{cases}$$

Then k_1 and l_1 are both continuous, and $\sigma_E^{k_1(x)}(h(\sigma_F(x))) = \sigma_E^{l_1(x)}(h(x))$ for all $x \in \partial F^{\geq 1}$. Similarly, define $k_1', l_1' : \partial E^{\geq 1} \to \mathbb{N}$ by

$$k_1'(y) := \begin{cases} m-1 & \text{if } y \in \cup_{e\in\eta_1(\mathbb{N})} Z(e\mu_2\mu_3\ldots\mu_m), \\ 0 & \text{if } y \in \cup_{e\in\eta_2(A)} Z(e\mu_2\mu_3\ldots\mu_m), \\ 0 & \text{if } y \notin \cup_{e\in A} Z(e\mu_2\mu_3\ldots\mu_m). \end{cases}$$
$$l_1'(y) := 1 \text{ for all } x \in \partial F^{\geq 1}.$$

Then k_1' and l_1' are both continuous, and $\sigma_F^{k_1'(y)}(h^{-1}(\sigma_E(y))) = \sigma_F^{l_1'(y)}(h^{-1}(y))$ for all $y \in \partial E^{\geq 1}$. This shows that E and F are orbit equivalent. □

Proposition 6.6. *Let* E *be a graph with* E^0 *finite. Then* $\overline{E}$ *and* $\overline{\mathrm{t}E}$ *are orbit equivalent.*

Proof. Notice that $\overline{\mathrm{t}E}$ can be obtained from $\overline{E}$ by adding infinitely many edges from v to w whenever there is a path from v to w. Thus, that $\overline{E}$ and $\overline{\mathrm{t}E}$ are orbit equivalent follows from finitely many applications of Lemma 6.5. □

Proof of Theorem 6.4. Since both $\overline{E}$ and $\overline{F}$ satisfy condition (L), it follows from our main theorem that (7)–(9) are equivalent, and it is obvious that (9) implies (3). Proposition 6.6 shows that (1) implies (7). □

References

[1] C. Anantharaman-Delaroche and J. Renault, *Amenable Groupoids*, Monographies de L'Enseignement Mathématique, vol. 36, L'Enseignement Mathématique, Geneva, 2000.
[2] T. Bates, J. Hong, I. Raeburn, and W. Szymanski, *The ideal structure of the* C^**-algebras of infinite graphs*, Illinois J. Math. **46** (2002), 1159–1176.

[3] T. Bates and D. Pask, *Flow equivalence of graph algebras*, Ergodic Th. and Dynam. Sys. **24** (2004), 367–382.
[4] S. Eilers, E. Ruiz, and A. Sørensen, *Amplified graph C^*-algebras*, Münster J. Math. **5** (2012), 121–150.
[5] T. Giordano, I. Putnam, and C. Skau, *Topological orbit equivalence and C^*-crossed products*, J. Reine Angew. Math. **469** (1995), 51–111.
[6] T. Giordano, H. Matui, I. Putnam, and C. Skau, *Orbit equivalence for Cantor minimal $\mathbb{Z}^d$-systems*, Invent. Math. **179** (2010), 119–158.
[7] A. an Huef and I. Raeburn, *The ideal structure of Cuntz-Krieger algebras*, Ergodic Th. and Dynam. Sys. **17** (1997), 611–624.
[8] A. Kumjian, *On localizations and simple C^*-algebras*, Pacific J. Math. **112** (1984), 141–192.
[9] A. Kumjian, *On C^*-diagonals*, Can. J. Math. **4** (1986), 969–1008.
[10] A. Kumjian, D. Pask, and I. Raeburn *Cuntz-Krieger algebras of directed graphs*, Pacific J. Math. **184** (1998), 161–174.
[11] A. Kumjian, D. Pask, I. Raeburn, and J. Renault, *Graphs, groupoids, and Cuntz-Krieger algebras*, J. Func. Anal. **144** (1997), 505–541.
[12] K. Matsumoto, *Orbit equivalence of topological Markov shifts and Cuntz-Krieger algebras*, Pacific J. Math. **246** (2010), 199–225.
[13] K. Matsumoto, *Orbit equivalence of one-sided subshifts and the associated C^*-algebras*, Yokohama Math. J. **56** (2010), 59–85.
[14] K. Matsumoto, *Full groups of one-sided topological Markov shifts*, preprint, `arXiv:1205.1320.`
[15] K. Matsumoto and H. Matui, *Continuous orbit equivalence of topological Markov shifts and Cuntz-Krieger algebras*, to appear in Kyoto J. Math., `arXiv1307.1299.`
[16] H. Matui, *Homology and topological full groups of étale groupoids on totally disconnected spaces*, Proc. Lond. Math. Soc. **104** (2012), 27–56.
[17] G. Nagy and S. Reznikoff, *Pseudo-diagonals and uniqueness theorems*, Proc. Amer. Math. Soc. **142** (2014), 263–275.
[18] A.L.T Paterson, *Graph Inverse Semigroups, Groupoids and their C^*-Algebras*, J. Operator Theory, **48** (2002), 645–662.
[19] I. Raeburn, *Graph Algebras*, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2005, vi+113.
[20] I. Raeburn and D. Williams, *Morita Equivalence and Continuous-Trace C^*-Algebras*, Mathematical Surveys and Monographs Volume 60, the American Mathematical Society, Providence, RI, 1998, xiv+327.
[21] J.N. Renault, *A Groupoid Approach to C^*-Algebras*, Lecture Notes in Math., vol. 793, Springer-Verlag, Berlin 1980.
[22] J.N. Renault, *Cartan subalgebras in C^*-algebras*, Irish Math. Soc. Bull. **61** (2008), 29–63.
[23] M. Rørdam, *Classification of Cuntz-Krieger algebras*, K-Theory **9** (1995), 31–58.
[24] J. Tomiyama, *Topological full groups and structure of normalizers in transformation group C^*-algebras*, Pacific J. Math. **173** (1996), 571–583.
[25] S. Webster, *The path space of a directed graph*, Proc. Amer. Math. Soc. **142** (2014), 213–225.
[26] T. Yeend, *Groupoid models for the C^*-algebras of topological higher-rank graphs*, J. Operator Theory **57** (2007), 95–120.

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