A direct approach to the ultradiscrete KdV equation with negative and non-integer site values.

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Abstract

A generalisation of the ultra-discrete KdV equation is investigated using a direct approach. We show that evolution through one time step serves to reveal the entire solitonic content of the system.

1 Introduction

The ultradiscrete KdV equation has been in the literature since the beginning of the 1990s [1, 2, 3]. The equation is a discrete version of the KdV equation where not only are the independent variables x and t discrete but the dependent variable is also discrete and takes two possible values, 0 or 1. Many interesting results have been deduced about the solutions to this system [4, 5, 6, 7]. The most well known is the solitonic behaviour of the solutions. If you take some arbitrary initial profile, in this context this means a row of 0's and 1's and evolve in time, this profile will split into a set of solitons, the larger ones moving faster than the smaller ones. Evolving backwards in time again the profile breaks down into the same selection of solitons. Some very elegant ways of working out the soliton content and deducing conservation laws for this system in both the periodic [8] and non periodic cases have been developed. More recently, it was observed that, if the values of the dependent variable u at the different lattice points were generalised to integers, [9, 10] rather than just the binary 0 and 1, the solutions still exhibited solitonic behaviour. In addition to solitons found in the original case, the generalised version also had solutions that contained 'background material', this is material that moves at a speed one.

In this paper we wish to investigate, using direct methods, solutions to the equation

$$u_n^{t+1} = \min\left(\alpha - u_n^t,\ \sum_{r=-\infty}^{n-1} u_r^t - \sum_{r=-\infty}^{n-1} u_r^{t+1}\right), \tag{1.1}$$

where the subscript n labels the lattice point and the superscript t is the discrete time variable. Notice here to obtain the original ultradiscrete KdV we need to set $\alpha = 1$. In the language of box and ball systems, α represents the capacity of the boxes at each lattice site and consequently will always have a positive value. It is indeed possible to make this α a function of the lattice site but here, for the sake of simplicity, we shall make it constant. In the work of Willox et al and Kanki et al [10, 11, 12, 13] values at the lattice points other than 0 and 1 have been used, in particular in ref [11] real numbers are used. In this paper we will similarly assume the u_i^t take real values.

Before embarking on our direct approach to this problem we want to briefly discuss a simple result due to Tokihiro on transforming the variables of the generalised ultradiscrete KdV which enables us to always have a box and ball interpretation. Let us take the system (1.1), for an initial profile u_n^0, $n \in \mathbb{Z}$ we can always find $u_{\text{low}}, u_{\text{high}}$ chosen so that $u_{\text{low}} \le u_n^0 \le u_{\text{high}}$. Define a new variable $v_n^t = u_n^t + p$, where $p = \max(u_{\text{high}} - \alpha, -u_{\text{low}})$ this new variable v_n^t will always be greater than equal to 0. We obtain

$$v_n^{t+1} = \min\left(\alpha' - v_n^t,\ \sum_{r=-\infty}^{n-1} v_r^t - \sum_{r=-\infty}^{n-1} v_r^{t+1} + p\right) \tag{1.2}$$

where $\alpha' = \alpha + 2p$. This corresponds to a box and ball system with boxes of capacity of α', now no longer with zero boundary conditions, but with boundary condition $v_i^t \to p$ as $i \to \pm\infty$.

In this paper we are going to take a direct approach to calculating the properties of this system, we will do this by considering the effect of evolving the solution through one time step. Evolving through one time step is enough to enable us to fully understand this evolution.

2 Understanding the equation and the initial condition

Without the change of variables above we can still give a box and contents style interpretation of equation (1.1) (we are no longer calling it 'box and ball' as we are using real values rather than integers). The α in the equation represents the size of the boxes at each lattice point and u_i^t represents the amount of material in the box i at time t, so the $\alpha - u_i^t$ represents the space or 'spare capacity' in the box i at that time step. We will define the carrier c_n^t by

$$c_n^t = \sum_{r=-\infty}^{n-1} u_r^t - \sum_{r=-\infty}^{n-1} u_r^{t+1} \tag{2.1}$$

this is the amount of material coming from the left which is looking for empty spaces to go into. So the original equation (1.1) can be rewritten as

$$u_n^{t+1} = \min\left(\alpha - u_n^t, c_n^t\right). \tag{2.2}$$

Let us consider some arbitrary initial profile

$$\cdots u_{-1}, u_0, u_1, \cdots, u_r, u_{r+1}, \cdots. \tag{2.3}$$

We shall assume that there are only a finite number of non zero u's, so sufficiently far to the left and the right all the $u_i = 0$. This assumption is necessary to make sense of the sums to infinity in equation (1.1).

From the initial condition we can calculate

$$a_i = u_i + u_{i+1} - \alpha, \tag{2.4}$$

this a_i is a measure of the spare capacity in moving contents from one box to the next box to the right. If a_i is less than zero then we have space to move all the contents from the box labeled i into the box labeled $i + 1$, this region can be considered as being in an *'under capacity'* region. The effect of this is that if there is no extra material coming from the left at this time step, the contents of the box i will just shift along by one box, we think of this as *'small soliton' or 'non-soliton like'* behaviour. If a_i is greater than zero then when we attempt to evolve the system, the contents from box i won't all fit into box $i + 1$ so some will have to move further forward to find space, this will make up part of the carrier. Boxes in this kind of region are said to be in an *'over capacity'* region as they are sufficiently full that some of the material coming from the left won't fit in and will have to move further along. If $a_i = 0$ then there is exactly the space for the contents of box i to move to $i+1$ but no spare capacity. Note here that in [11] a similar condition is used which divides the material into 3 catagories: bound states with speed greater than 1, bound states with speed equal to one and no bound states (also with speed 1). For the purpose of our analysis we we only need to distinguish between speed 1 material, which is related to the under capacity regions and speed > 1 material which is related to the over capacity regions.

3 Breaking the initial profile into under capacity and over capacity regions

Starting from the left hand end of the initial condition we are going to divide our array into two different kinds of regions:

1. Over capacity
2. Under capacity

We assume that initially if we start far enough to the left that we are in an *'under capacity'* region. Here all the a_i obey

$$a_i = u_i + u_{i+1} - \alpha \leq 0.$$

Traveling from left to right when a_j first satisfies

$$a_j = u_j + u_{j+1} - \alpha > 0$$

we enter an *over capacity* region. The first entry in this region (ie: the j^{th}) is both the head (right hand end) of the *under capacity* region and the tail (left hand end) of the *over capacity* region. Continuing from left to right we remain in the *over capacity* region until $a_k = u_k + u_{k+1} - \alpha < 0$, for some k, this k^{th} position marks the end of the *over capacity* region. So the k^{th} position is the head (right hand end) of the *over capacity* region and also the tail (left hand end) of the next *under capacity* region. We can continue like this traveling to the right. Everytime we find an $a_i > 0$ we move back into an over capacity region and every time we find an $a_i < 0$ we are moving back into an under capacity region. Eventually when we have traveled far enough to the right we will have passed through all the *over capacity* regions and be left finally in an *under capacity* region. These two different types of region have a physical interpretation. The *'under capacity'* region is a region where there is spare capacity between the consecutive boxes, ie if you moved the material out of the i-th box and put it into the $(i+1)$-th box it would all fit. In the *'over capacity'* regions, there is either, not enough space or exactly the right amount of space to fit the material in, but no spare space.

It is possible that for some i, $a_i = u_i + u_{i+1} - \alpha = 0$, Here we have exactly the right amount of material to fill up the next box to the right. From our definition, this situation could occur in either an over or under capacity region.

We shall consider a simple example to show how to break up an initial profile. If we look at a case with $\alpha = 1$

$u_i \rightarrow$	0	0	−1	−2	4	−2	4	−5	−2	1	0	1	0	2	2	1	0	0	0
$a_i \rightarrow$	−1	−2	−4	1	1	1	−2	−8	−2	0	0	0	1	3	2	0	−1	−1	

We can distinguish the over capacity and under capacity parts here

$$u_i \rightarrow \quad 0 \;\; 0 \; -1 \;\; -2 \;\;\; 4 \; -2 \;\;\; 4 \;\; -5 \; -2 \;\;\; 1 \;\;\; 0 \;\;\; 1 \;\;\; 0 \;\;\; 2 \;\;\; 2 \;\;\; 1 \;\;\; 0 \;\;\; 0 \;\;\; 0$$

Brackets over the top are for *over capacity* parts and brackets under are for *under capacity* parts. Some entries have both over and under brackets, these represent the ends of the under capacity and over capacity regions. So the end on an over capacity region is simultaneously the other end of an under capacity region. In general if we take any initial profile, we can always divide the whole profile up, into over capacity and under capacity parts. For the purposes of the next section we are going to sub-divide our under capacity parts in to two different kinds of region, giving us in total 3 distinct types of region:

- Under capacity with low carrier value
- Under capacity with high carrier value
- Over capacity

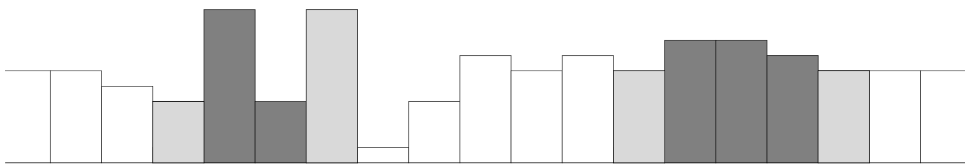

Figure 1: Profile (plotting $y = u_i + 6$) from the above example with two over capacity regions (shown in dark gray). The under capacity parts are coloured white and the light gray represents the ends/beginnings of the under/over capacity regions.

4 Evolution through one time step

The key to understanding the whole evolution is to observe what happens in each of the basic regions and what happens as one crosses from one region to the next. Consequently we shall consider what happens as we evolve an initial configuration starting from the far left and working across to the far right. In sections 4.1, 4.2, 4.3 and 4.4 we will work our way across the lattice from left to right passing through the different types of region.

4.1 Under capacity with low carrier

Let the *carriers* be given by

$$c_n^t = \sum_{r=-\infty}^{n-1} u_r^t - \sum_{r=-\infty}^{n-1} u_r^{t+1}.$$

Starting from the far left we assume we are in an under capacity region (far enough to the left the u_i are all zero), here, because of the spare capacity in the boxes the carriers take low values as there is space for all the material from one box to be put in the adjacent box to the right. Indeed the carrier is given by $c_i^t = u_{i-1}^t$. Thus in this region

$$u_i^{t+1} = \min(\alpha - u_i^t, c_i^t) = \min(\alpha - u_i^t, u_{i-1}^t)$$

but we know that $u_{i-1}^t + u_i^t \leq \alpha$, hence

$$u_i^{t+1} = u_{i-1}^t$$

and the next carrier is

$$c_{i+1}^t = c_i^t + u_i^t - u_i^{t+1} = u_i^t.$$

The diagram shows the effect of evolving one time step in this under capacity region.

$t=0$	$\rightarrow$	$\cdots$		u_{i-1}		u_i		$\cdots$
carrier	$\rightarrow$		$c_{i-1} = u_{i-2}$		$c_i = u_{i-1}$		u_i	
$t=1$	$\rightarrow$	$\cdots$		u_{i-2}		u_{i-1}		$\cdots$

We can see from this, that the evolution in this region is simple and involves each entry shifting one box to the right. This evolution will continue in this manor until we reach the end of the under capacity region.

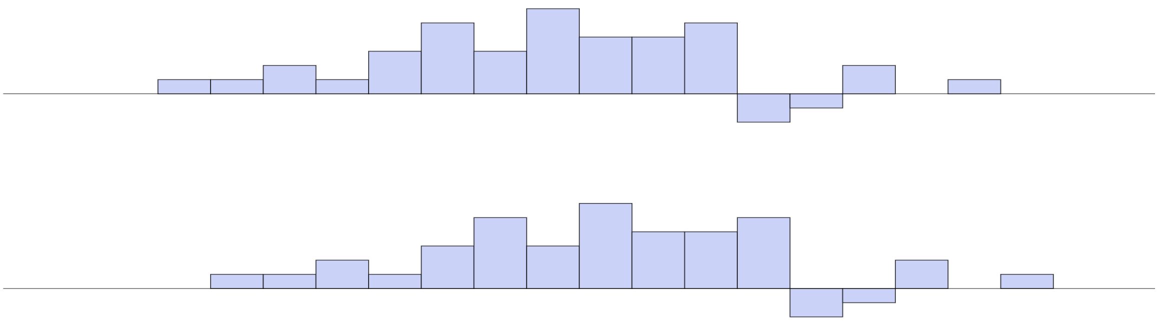

Figure 2: Time steps t=0 and 1 in a region where $u_{i-1}^t + u_i^t \leq \alpha$, with a low carrier value. In one time step the entries all simply shift one place to the right.

4.2 End of under capacity region beginning of over capacity region

If position r is the right hand end of the under capacity region and beginning of the first over capacity region (working from left to right), we have already seen that

$$u_r^{t+1} = u_{r-1}^t$$

with the next carrier being

$$c_{r+1}^t = u_r^t.$$

But now

$$u_{r+1}^{t+1} = \min(\alpha - u_{r+1}^t, u_r^t) = \alpha - u_{r+1}^t + \min(0, u_{r+1}^t + u_r^t - \alpha) = \alpha - u_{r+1}^t$$

and hence the subsequent carrier is

$$c_{r+2}^t = u_r^t + 2u_{r+1}^t - \alpha > u_{r+1}^t.$$

4.3 Crossing the first over capacity region

Following from §4.2 we are now fully in an over capacity region, recall here that $a_i^t = u_i^t + u_{i+1}^t - \alpha \geq 0$. At each step the updated u will be given by

$$u_{r+i}^{t+1} = \alpha - u_{r+i}^t,$$

this can be understood by considering the values of the carriers:

$$c_{r+1}^t = u_r^t = a_r^t + (\alpha - u_{r+1}^t) \geq (\alpha - u_{r+1}^t) \tag{4.1}$$

$$c_{r+2}^t = u_r^t + (2u_{r+1}^t - \alpha) = a_r^t + a_{r+1}^t + (\alpha - u_{r+2}^t) \geq (\alpha - u_{r+2}^t) \tag{4.2}$$

$$c_{r+3}^t = u_r^t + (2u_{r+1}^t - \alpha) + (2u_{r+2}^t - \alpha) = \sum_{i=0}^{i=2} a_{r+i}^t + (\alpha - u_{r+3}^t) \geq (\alpha - u_{r+3}^t) \tag{4.3}$$

$$\vdots \qquad \qquad \vdots$$

$$c_{r+n}^t = u_r^t + \sum_{i=1}^{n-1}(2u_{r+i}^t - \alpha) = \sum_{i=0}^{n-1} a_{r+i}^t + (\alpha - u_{r+n}^t) \geq (\alpha - u_{r+n}^t). \tag{4.4}$$

Each of these conditions is saying that the carrier is greater than or equal to α minus the particular entry. Thus the system evolves according to the diagram below:

$t = 0$	$\rightarrow$	$\cdots$	u_{r-1}		u_r		u_{r+1}		u_{r+2}	$\cdots$			u_{r+n}
carrier	$\rightarrow$	u_{r-2}		u_{r-1}		u_r		c_{r+2}		c_{r+3}	$\cdots$	$\cdots$	
$t = 1$	$\rightarrow$	$\cdots$	u_{r-2}		u_{r-1}		$\alpha - u_{r+1}$		$\alpha - u_{r+2}$	$\cdots$			$\alpha - u_{r+n}$

Where u_r is the left end of the over capacity region and u_{r+n} is the right hand end of the over capacity region (and simultaneously, the left hand end of the next under capacity region).

4.4 Under capacity region with high carrier value

We can now follow the evolution going from left to right, through this next under capacity region. Unlike the left most under capacity region discussed in §4.1, the value of the carrier when we enter the region is high. As we cross the region the carrier value will drop due to the spare capacity in the boxes. One of two things can happen here. (i) Either c_i drops down below $\alpha - u_i$ while we are still in the under capacity region or (ii) we reach the right hand end of the under capacity region with the value of the carrier remaining above the value of $\alpha - u_i$. We shall consider each case seperately.

4.4.1 Case (i)

Firstly in case (i) u_i updates to $\alpha - u_i$ until we reach the point where c_i drops down below $\alpha - u_i$. If this happens at a position p, we have $c_{p-1}^t \geq \alpha - u_{p-1}^t$ and $c_p^t < \alpha - u_p^t$. Again we can examine how the carriers change:

$$
\begin{aligned}
c^t_{r+n+1} &= c^t_{r+n} + (2u^t_{r+n} - \alpha) \\
c^t_{r+n+2} &= c^t_{r+n} + (2u^t_{r+n} - \alpha) + (2u^t_{r+n+1} - \alpha) \qquad (4.5) \\
\vdots \;&=\; \vdots \qquad\qquad \vdots \\
c^t_{r+n+i} &= c^t_{r+n} + \sum_{j=0}^{i-1}(2u^t_{r+n+j} - \alpha) = d + \left(\sum_{j=r+n}^{r+n+i-1} a_j\right) + (\alpha - u^t_{r+n+i}) \qquad (4.6) \\
\vdots \;&=\; \vdots \qquad\qquad \vdots \\
c^t_{p-1} &= c^t_{r+n} + \sum_{j=r+n}^{p-2}(2u^t_j - \alpha) = d + \left(\sum_{j=r+n}^{p-2} a_j\right) + (\alpha - u^t_{p-1}) \qquad (4.7) \\
c^t_{p} &= c^t_{r+n} + \sum_{j=r+n}^{p-1}(2u^t_j - \alpha) = d + \left(\sum_{j=r+n}^{p-1} a_j\right) + (\alpha - u^t_{p}) \qquad (4.8)
\end{aligned}
$$

where $d = c^t_{r+n} + u^t_{r+n} - \alpha$. All the a_i's in this region are less than or equal to zero, so the carriers will decrease as we cross the region. At the pth position the $u^{t+1}_p = c^t_p$ and the next carrier will just be $c^t_{p+1} = u^t_p$ which is smaller than $\alpha - u^t_{p+1}$ thus giving $u^{t+1}_{p+1} = u^t_p$. So now we are back in an under capacity region with low carrier, the evolution in this type of region we have already discussed in §4.1. The evoloution here is just the same as to the left of the first over capacity region. Eventually we will hit the next over capacity region and the evolution will be similar to what has already been described. Below shows a diagram of evolution through one time step in this under capacity region:

$t = 0$	$\rightarrow$	$\cdots$	u_{r+n}		u_{r+n+1}		u_{r+n+2}		u_{r+n+3}	$\cdots$
carrier	$\rightarrow$	c_{r+n}		c_{r+n+1}		c_{r+n+2}		c_{r+n+3}		$\cdots$
$t = 1$	$\rightarrow$	$\cdots$	$\alpha - u_{r+n}$		$\alpha - u_{r+n+1}$		$\alpha - u_{r+n+2}$		$\alpha - u_{r+n+3}$	$\cdots$

$t = 0$	$\rightarrow$		u_{p-1}		u_p		u_{p+1}		u_{p+2}	$\cdots$
carrier	$\rightarrow$	c_{p-1}		c_p		u_p		u_{p+1}		$\cdots$
$t = 1$	$\rightarrow$		$\alpha - u_{p-1}$		c_p		u_p		u_{p+1}	$\cdots$

In figure 3 below we present an example of the case (i). The boxes with purely light gray in them make up the under capacity regions. The boxes with dark gray and light gray (or for the ends of that region purely dark gray) represent a single over capacity region. The light gray here, represents what we shall call the background. In the under capacity region it is exactly the values of the u_i's and moves with speed 1. In the over capacity region it is given by $\alpha - u_i$. At the left and right endmost positions of an over capacity region there is no background present. The dark gray material on its own, represents a speed > 1 soliton.

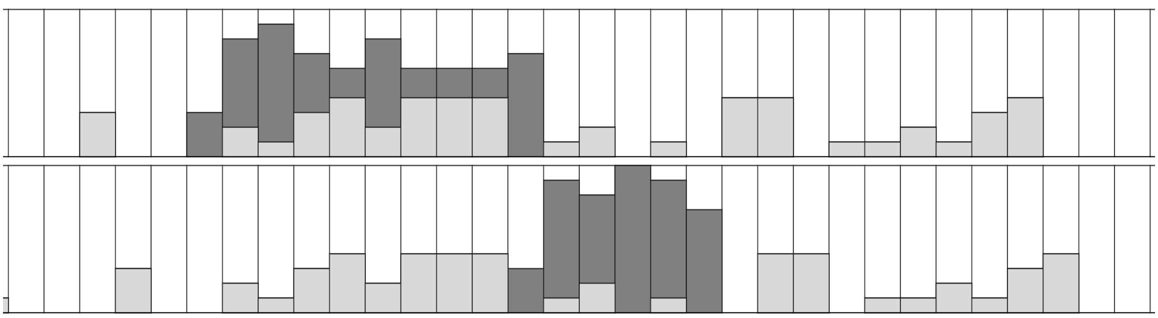

Figure 3: Case (i), time steps t=0 and 1 in a region with a single over capacity region. Notice that the position of the right hand end of the over capacity region at one time step becomes the position of the left hand end at the next time step.

4.4.2 Case (ii)

Here we have two over capacity regions seperated by a sufficiently small under capacity region. When evolving through one time step, when the right hand end of this central under capacity region (let this be the position q) is reached, the value of the carrier is above the value of $\alpha - u_q^t$. Hence we are entering the second over capacity region with the carrier being larger than this $\alpha - u_q^t$, consequently $u_q^{t+1} = \alpha - u_q^t$. The behaviour through this second over capacity region is exactly as discussed in §4.3.

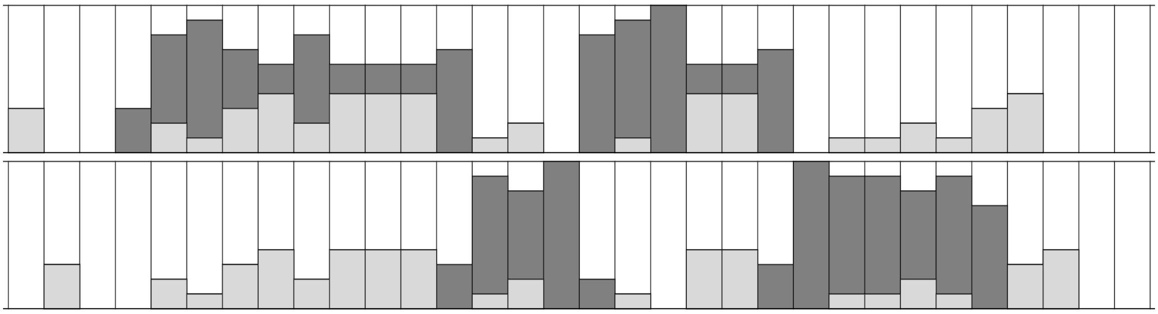

Figure 4: Case (ii), time steps t=0 and 1 in a region with two, over capacity regions. Here the carrier does not go below the value $\alpha - u_q^t$ between the two regions. The background material (the light gray) for this example has been chosen exactly the same as in figure 3. As with case (i) the position of the right hand end of over capacity region at one time step becomes the position of the left hand end at the next time step. Additionally now, the left hand end of the second over capacity region at time t=0, becomes the left hand end of an under capacity region at t=1.

4.5 Summary of evolution through one time step

We have now evolved far enough to see what the whole evolution from time step t to $t+1$ will look like. So the basic picture is that at time step t we can divide the solution into a finite number of over capacity regions. Between each of these over capacity regions, there will be an under capacity region. If at this time step we go far enough to the left or indeed far enough to the right we assume we are in an under capacity region. The last entry in each over/under capacity region is the first entry in each under/over capacity region.

The right hand end of the over capacity regions at time step t become the left hand ends of the over capacity regions at time $t+1$. The right hand ends of the new over capacity regions at time $t+1$ either coincide with the left hand end of an over capacity region region from the previous time step t or lie further to the left.

The parts depicted in dark grey in figures 3 and 4 represent solitons or parts of solitons moving with speed > 1, The parts depicted in light grey in figures 3 and 4 represent anything moving with speed 1.

5 A simple example

The mathematical calculations to show the evolution from one time step t the next are simple but a little opaque, so in this next section we will show a simple explicit example to demonstrate the evolution. We will take our box size as one ie $\alpha = 1$.

5.1 The solitons and background

If we take our initial condition and evolve it sufficiently far forward or backwards in time it will split in to a string of solitons with speeds > 1. The larger ones moving faster than the smaller ones. In addition to these solitons there will be some 'background' material. This background material moves at a speed 1 to the right. To analyse the whole solution we shall break it into two parts, these parts are 1) the 'background' and 2) the solitons with speed greater than one. At a time t, the background can easily be seen to consist of the u_i^t in the under capacity regions and $\alpha - u_i^t$ in the over capacity regions. The ends of the over/under capacity regions don't contribute to the background. Thus for example, with $\alpha = 1$, given an initial condition:

$$\cdots \quad 0 \quad 0 \quad 2 \quad -1 \quad 2 \quad -1 \quad 0 \quad 0 \quad 2 \quad 0 \quad -2 \quad 1 \quad -1 \quad 2 \quad 1 \quad 0 \quad 1 \quad 0 \quad 2 \quad -1 \quad 0 \quad \cdots \tag{5.1}$$

we firstly identify where the over and under capacity regions are, by checking whether $a_i = u_i + u_{i+1} - \alpha$ is greater than or less than or equal to zero. We will put brackets over the top of the over capacity regions.

$$\cdots \; 0 \; \overbrace{0 \;\; 2 \;\; -1 \;\; 2 \;\; -1} \; 0 \; \overbrace{0 \;\; 2 \;\; 0} \; -2 \;\; 1 \;\; -1 \; \overbrace{2 \;\; 1 \;\; 0 \;\; 1 \;\; 0 \;\; 2 \;\; -1} \; 0 \; \cdots \tag{5.2}$$

Sometimes, we will put brackets underneath to identify the under capacity regions:

$$\underbrace{\cdots \;\; 0 \;\; 0} \;\; 2 \;\; -1 \;\; 2 \;\; \underbrace{-1 \;\; 0 \;\; 0} \;\; 2 \;\; \underbrace{0 \;\; -2 \;\; 1 \;\; -1 \;\; 2} \;\; 1 \;\; 0 \;\; 1 \;\; 0 \;\; 2 \;\; \underbrace{-1 \;\; 0 \;\; \cdots} \tag{5.3}$$

We can then identify the background by first removing the ends of each region:

$$\cdots \; 0 \; \overbrace{. \;\; 2 \;\; -1 \;\; 2 \;\; .} \; 0 \; \overbrace{. \;\; 2 \;\; .} \; -2 \;\; 1 \;\; -1 \; \overbrace{. \;\; 1 \;\; 0 \;\; 1 \;\; 0 \;\; 2 \;\; .} \; 0 \; \cdots$$

and then taking u_i^t in the under capacity regions and $\alpha - u_i^t$ in the over capacity regions

$$\cdots \; 0 \;\; . \;\; -1 \;\; 2 \;\; -1 \;\; . \;\; 0 \;\; . \;\; -1 \;\; . \;\; -2 \;\; 1 \;\; -1 \;\; . \;\; 0 \;\; 1 \;\; 0 \;\; 1 \;\; -1 \;\; . \;\; 0 \; \cdots$$

Finally giving us the background material:

$$\cdots \; 0 \;\; -1 \;\; 2 \;\; -1 \;\; 0 \;\; -1 \;\; -2 \;\; 1 \;\; -1 \;\; 0 \;\; 1 \;\; 0 \;\; 1 \;\; -1 \;\; 0 \; \cdots \tag{5.4}$$

This is the material moving at the slowest speed of 1, so asymptotically if we evolve far enough forward in time, this material will be to the left of the rest of the (faster moving) material and if we evolve backwards in time this material will be to the right of the rest of the material.

We also wish to extract information about the higher speed solitons. For this we need to calculate the amount of faster flowing material and the spaces in between this material.

The size of a typical over capacity region can be obtained by considering the carriers in the relevant region. We find the size of this over capacity region can be written in terms of the a_i's in the region, as

$$\alpha + \sum_{i=\text{left}}^{\text{right}-1} a_i \;=\; u_{\text{left}} \;+\; \sum_{\text{internal parts}} (2u_i - \alpha) \;+\; u_{\text{right}},$$

where $u_{\text{left}}, u_{\text{right}}$ respectively are the left and right hand ends of the particular over capacity region. The sum over *internal parts* means the sum over all the lattice points between (but not including) u_{left} and u_{right}. This is really a measure of how much extra material there is in this over capacity region that will need to flow forward to find space (it is this material that will eventually form a soliton or part of a soliton).

So in the above example there are three over capacity regions (marked by the over brackets in (5.2)) of sizes

$$\begin{aligned} e_1 &= 0 + (3 - 3 + 3) - 1 = 2 \\ e_2 &= 0 + 3 + 0 = 3 \\ e_3 &= 2 + (1 - 1 + 1 - 1 + 3) - 1 = 4. \end{aligned}$$

The under capacity regions in this particular example (marked by the under brackets in (5.3)), are to the far left and far right and also the two regions seperating the three over capacity regions. The sizes of these can also be calculated (this is really a measure of the empty space between the adjacent over capacity regions). In general it will be given by

$$(\alpha - u_{\text{left}}) \;+\; \sum_{\text{internal parts}} (\alpha - 2u_i) \;+\; (\alpha - u_{\text{right}}), \tag{5.5}$$

where now $u_{\text{left}}, u_{\text{right}}$ respectively are the left and right hand ends of the particular under capacity region. Again in terms of the a_i's the size is

$$\alpha - \sum_{i=\text{left}}^{\text{right}-1} a_i$$

We can see that using (5.5), the two under capacity regions seperating the over capacity regions, in this example have size:

$$f_1 = (\alpha - (-1)) + (\alpha - 2\times 0) + (\alpha - 0) = 2 + (1) + 1 = 4$$
$$f_2 = (\alpha - 0) + [(\alpha - 2\times(-2)) + (\alpha - 2\times 1) + (\alpha - 2\times(-1))] + (\alpha - 2) = 1 + [5 - 1 + 3] - 1 = 7.$$

This is really a measure of how much spare capacity there is in these regions Thus we can write down a table containing the sizes of both the under and over capacity regions:

f_0	e_1	f_1	e_2	f_2	e_3	f_3
∞	2	4	3	7	4	∞

(5.6)

Although here in this example we have used integers for simplicity, the formulae work just the same with any real numbers. It is worth pointing out here that the time evolution of the (e_i, f_i) is known to satisfy the ultradiscrete Toda molecule equation with boundary condition $f_0 = f_N = \infty$ [14].

In the next section we shall examine how to identify the sizes of the solitons asymptotically.

5.2 The solitons

Having looked at what constitutes the speed 1 material in the previous section and calculated the sizes of the over and under capacity regions we now have a choice of several known techniques for analysing the actual soliton content. One of the simplest, which we shall describe here is due to Nagai et al [14, 4].

Firstly, though, we shall write down our initial solution with the background removed. At each lattice point we will have either 0, α (the size of the boxes) or at either end of the over capacity regions, but NOT internal/external to the regions, some values β where each β is such that $0 \le \beta \le \alpha$. So in our particular (rather simple) example above $\alpha = 1$, using table (5.6) we have:

$$\cdots\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ \cdots$$

for the solution with the background removed. Relabeling $f_i = g_{2i}$ and $e_i = g_{2i-1}$ table (5.6) becomes:

g_0	g_1	g_2	g_3	g_4	g_5	g_6
∞	2	4	3	7	4	∞

(5.7)

The actual lengths of the solitons can be calculated following a simple procedure. Let the solitons lengths be $S_1 \le S_2 \le S_3 \cdots \le S_n$, where n is the number of solitons (notice here, unlike continuous solitons it is possible to have 2 identical size solitons). Then

$$\begin{aligned} S_1 &= \min[g_i], \qquad \forall i. \\ S_1 + S_2 &= \min[g_i + g_j], \qquad \text{where } i, j \text{ not adjacent} \\ S_1 + S_2 + S_3 &= \min[g_i + g_j + g_k], \qquad i, j, k \text{ not adjacent} \\ &\vdots \\ S_1 + \cdots + S_n &= \min[g_{i_1} + \cdots + g_{i_n}], \qquad i_1, i_2 \cdots i_n \text{ not adjacent} \end{aligned}$$

Thus for our example

$$\begin{aligned} S_1 &= \min[g_i] = g_1 = 2 \\ S_1 + S_2 &= \min[g_i + g_j] = g_1 + g_3 = 2 + 3 = 5 \\ S_1 + S_2 + S_3 &= \min[g_i + g_j + g_k] = g_1 + g_3 + g_5 = 2 + 3 + 4 = 9. \end{aligned}$$

giving

$$S_1 = 2, \qquad S_2 = 3, \qquad S_3 = 4.$$

We stress here that although in our simple example we have only integer sizes, the calculations work identically for non integer cases.

6 Locations of the speed > 1 solitons

We have seen that it is possible, in a straight forward manner, to obtain the speed > 1 soliton and background contents of a solution by considering the details of one time step. It is also possible to say something about the locations of the solitons.

Consider a soliton of size s_2 overtaking a soliton of size $s_1 < s_2$. The faster soliton will be shifted $2\times s_1/\alpha$ forward. At the same time, the smaller soliton will be retarded by $2\times s_1/\alpha$. We can include speed 1 material in this discussion by considering the speed of the material rather than the size of the individual parts. So if a faster moving soliton passes material of size b say, where b could take any value (including negative values), then the faster moving soliton will be shifted forward by $2b/\alpha$. The material of size b, if $b \leq \alpha$ (the box size) will be phase shifted backwards by a distance $2\times$ (the speed of this material). For example let us evolve a simple profile with $\alpha = 1$:

$$\begin{array}{cccccccccccccccccccccccccccc}
0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & \frac{2}{3} & 1 & 1 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 1 & 1 & 1 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{2}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 1 & 1 & 1 & \frac{2}{3} & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 1 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 1 & 1 & 1 & \frac{2}{3} & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 1 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \cdots
\end{array} \tag{6.1}$$

It is clear to see that here we have a size 4 soliton, a size 2 soliton and a size 1/3 piece of background.

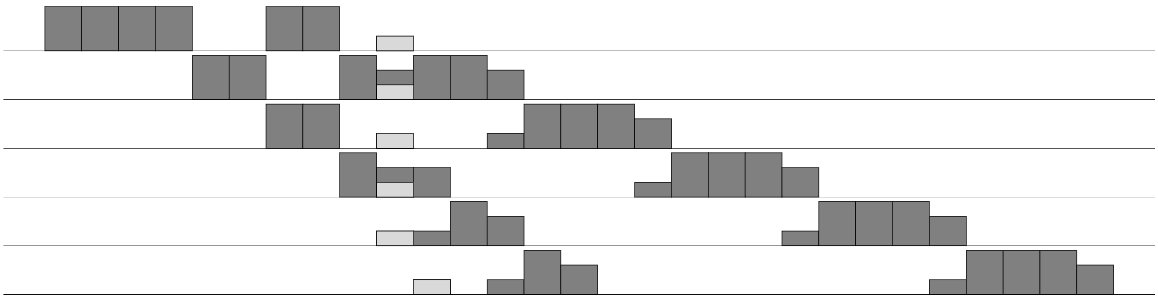

Figure 5: Barchart view of table 6.1.

- The size 4-soliton is shifted forwards, in t steps by

$$\phi_4 = 2(\text{size of all the material it overtakes}) + 4t/\alpha = 2(2+\frac{1}{3}) + 4t$$

- The size 2 soliton is shifted by

$$\begin{aligned}\phi_2 &= 2(\text{size of all the material it overtakes}) - (\text{contribution due to being overtaken}) + 2t \\ &= 2\left(\frac{1}{3}\right) - 2(2) + 2t = -\frac{10}{3} + 2t.\end{aligned}$$

  The contribution due to being overtaken, is $2\times$ (the size of the soliton in question) $\times$ (number of solitons overtaking).

- Finally the background is shifted by

$$\begin{aligned}\phi_{bg} &= -2(\text{the number of solitons overtaking it}) + t \\ &= -2(2) + t = -4 + t.\end{aligned}$$

Note here that as the background was of size 1/3, the 4-soliton and the 2-soliton undergo non-integer phase shifts, this can be seen in the last line of table (6.1).

In multiple soliton/background interactions it is possible to keep track of where the individual solitons and background are. Locating the the solitons whilst in the middle of an interaction is possible but not so straight forward, so we shall leave that for a further publication. Instead we will only consider the overall phase shift in going from a large negative time to a large positive time. Consider a system with m solitons of sizes S_i (all greater than one), $i = 1 \cdots m$ and background material which we will label b_j. If we consider the soliton labeled by r with size $S_r > 1$. This soliton will be overtaken by all solitons that are larger than it. It will overtake all solitons that are smaller than it, and it will also overtake any speed 1 material. So the phase shift for such a soliton is:

$$\phi_r = 2 \times (\text{amount of material it overtakes}) - 2 \times (\text{size of the soliton}) \times (\text{no of solitons overtaking it })$$

$$= 2 \left(\sum_{i:S_i<S_r} S_i - \sum_{j:S_j>S_r} S_r + \sum_{\text{all background}} b_i \right).$$

The overall phase shift for the background material and speed 1 solitons, is

$$\begin{aligned} \phi_{bg} &= -2 \times (\text{the speed of the background}) \times (\text{no of solitons overtaking it}) \\ &= -2\ (\text{number of solitons of size} > 1 \text{ in the system}) \\ &= -2m. \end{aligned}$$

We can understand the form of these phase shifts directly from the observation that during an interaction of a bigger soliton with a smaller soliton, after one time step (i.e. in the middle of the interaction) there is a gap where the smaller soliton was. After a second time step the larger soliton will have passed through the smaller soliton and the gap where the smaller soliton was is now filled again with material. Leaving the smaller soliton unmoved after two time steps, so effectively phase shifted back by twice its size. To balance this out, the large soliton is phase shifted forwards by the same amount. Similarly as a faster soliton passes through the speed 1 material, after one time step the $u_i \to (\alpha - u_i)$, then after a second time step the $(\alpha - u_i) \to u_i$, thus in two time steps the speed 1 material hasn't moved thus effectively it has been phase shifted back by 2. These formulae can also be found in Willox et al [11] which the authors obtain by an asymptotic analysis of the dressing formula for the ultra discrete τ function.

7 Conclusions

The aim of this paper was to take a direct approach to soliton evolution in ultra discrete systems. We have shown that in systems where the lattice values are not necessarily integers we can use simple techniques to understand the evolution, the soliton content and the background content of the system. The key observation in this paper is that just by considering the profile at one time step and looking at the evolution to a subsequent time step, the contents of the system can be seen. We found that the solution at a time t can be simply broken into regions of over capacity (where there is not enough space to move all material from the adjacent box on the left into the box in question) and regions of under capacity (where there is space to move all material from the adjacent box on the left into the box in question). These correspond to faster than speed 1 soliton like behaviour and essentially non solitonic behaviour. From this observation it is possible to easily calculate key features such as the soliton content.

Comparing the $u_i = 0, 1$ case of equation (1.1) with, for instance, the KdV equation, we can see that the ultra discrete case solutions appear to be more limited, as an initial profile breaks only into solitons and nothing else. For the continuous KdV, in addition to the solitons there will be a dispersive part. However we have seen that by extending the dependent variables to reals in the ultra discrete case, a new feature is introduced 'the background' material. Although this material doesn't disperse it appears related to the dispersive part from the continuous case [11].

It is is reassuring to find that our results agree with those found previously by Willox et al [10, 11] who used a discrete spectral approach, identifing the bound states and using an undressing procedure on the τ-function to break the solution down in to it constituent parts.

From the ideas presented in this paper there are further interesting avenues to be investigated. In particular, it is possible to use some simple graphical methods to establish the soliton content and the evolution of the system, we shall return to this elsewhere.

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