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$\mathcal{E}_\infty$ ring spectra and elements of Hopf invariant 1

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Abstract The 2-primary Hopf invariant 1 elements in the stable homotopy groups of spheres form the most accessible family of elements. In this paper, we explore some properties of the $\mathcal{E}_\infty$ ring spectra obtained from certain iterated mapping cones by applying the free algebra functor. In fact, these are equivalent to Thom spectra over infinite loop spaces related to the classifying spaces $B\mathrm{SO}$, $B\mathrm{Spin}$, $B\mathrm{String}$. We show that the homology of these Thom spectra are all extended comodule algebras of the form $\mathcal{A}_*\square_{\mathcal{A}(r)_*}P_*$ over the dual Steenrod algebra $\mathcal{A}_*$ with $\mathcal{A}_*\square_{\mathcal{A}(r)_*}\mathbb{F}_2$ as an algebra retract. This suggests that these spectra might be wedges of module spectra over the ring spectra $H\mathbb{Z}$, $k\mathrm{O}$ or tmf; however, apart from the first case, we have no concrete results on this.

Keywords Stable homotopy theory · $\mathcal{E}_\infty$ ring spectrum · Power operations · Comodule algebras

Mathematics Subject Classification Primary 55P43; Secondary 55P42 · 57T05

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This paper is dedicated to the memory of **Sam Gitler**.

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Introduction

The 2-primary Hopf invariant 1 elements in the stable homotopy groups of spheres form the most accessible family of elements. In this paper, we explore some properties of the $\mathcal{E}_\infty$ ring spectra obtained from certain iterated mapping cones by applying the free algebra functor. In fact, these are equivalent to Thom spectra over infinite loop spaces related to the classifying spaces BSO, BSpin and BString.

We show that the homology of these Thom spectra are all extended comodule algebras of the form $\mathcal{A}_*\square_{\mathcal{A}(r)_*}P_*$ over the dual Steenrod algebra $\mathcal{A}_*$ with $\mathcal{A}_*\square_{\mathcal{A}(r)_*}\mathbb{F}_2$ as a comodule algebra retract. This suggests that these spectra might be wedges of module spectra over the ring spectra $H\mathbb{Z}$, kO or tmf; however, apart from the first case, we have no concrete results on this.

Our results and methods of proof owe much to the work of Liulevicius [24,25] and Pengelley [30–32], and are also related to the work of Bahri and Mahowald [4] (indeed, there are analogues of our results for $\mathcal{E}_2$ Thom spectra of the kind they discuss). However, we use some additional ingredients: in particular, we make use of formulae for the interaction between the $\mathcal{A}_*$-coaction and the Dyer–Lashof operations in the homology of an $\mathcal{E}_\infty$ ring spectrum described in [9]. We also take a slightly different approach to identifying when the homology of a ring spectrum is a cotensor product of the dual Steenrod algebra $\mathcal{A}_*$ over a finite quotient Hopf algebra $\mathcal{A}(n)_*$, making use of the fact that the dual Steenrod algebra is an extended $\mathcal{A}(n)_*$-comodule; in turn, this is a consequence of the P-algebra property of the Steenrod algebra $\mathcal{A}^*$.

We remark that the finite complexes of Sect. 1 also appear in the recent preprint by Behrens et al. [13]: each is the first of a sequence of generalised integral Brown–Gitler spectra associated with $H\mathbb{Z}$, kO and tmf, see [13, section 2.1] and [5,15,18]. We understand that Bob Bruner and John Rognes have also considered such spectra.

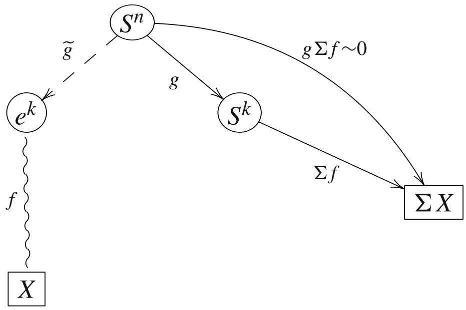
Conventions We will work 2-locally throughout this paper; thus, all simply connected spaces and spectra will be assumed to be localised at the prime 2, and $\mathscr{M}_S$ will denote the category of S-modules where S is the 2-local sphere spectrum as considered in [17]. We will write S^0 for a chosen cofibrant replacement for the S-module S and

$S^n = \Sigma^n S^0$. When discussing CW skeleta of a space X, we will always assume that we have chosen minimal CW models in the sense of [12], so that cells correspond to a basis of $H_*(X) = H_*(X; \mathbb{F}_2)$.

Notation When working with cell complexes (of spaces or spectra), we will often indicate the mapping cone of a coextension $\widetilde{g}$ of a map $g\colon S^n \to S^k$ by writing $X \cup_f e^k \cup_g e^{n+1}$.

Of course, this notation is ambiguous, but nevertheless suggestive. When working stably with spectra, we will often write $h\colon S^{n+r} \to S^{k+r}$ for the suspension $\Sigma^r h$ of a map $h\colon S^n \to S^k$. We will also often identify stable homotopy classes with representing elements.

$$\begin{tikzcd}
 & & S^n \arrow[dl, dashed, "\widetilde{g}"'] \arrow[d, "g"] \\
X \arrow[r] & X \cup_f e^k \arrow[r] & S^k
\end{tikzcd}$$



1 Iterated mapping cones built with elements of Hopf invariant 1

The results of this section can be proved by homotopy theory calculations using basic facts about the elements of Hopf invariant 1 in the homotopy groups of the sphere spectrum S^0,

$$2 \in \pi_0(S^0), \quad \eta \in \pi_1(S^0), \quad \nu \in \pi_3(S^0), \quad \sigma \in \pi_7(S^0).$$

In particular, the following identities are well known; for example, see [33, figure A3.1a]:

$$2\eta = \eta\nu = \nu\sigma = 0. \tag{1.1}$$

Although the next result is probably well known, we outline some details of the constructions of such spectra, and in particular describe their homology as $\mathcal{A}_*$-comodules. Later, we will produce naturally occurring examples of such spectra, but we feel it worthwhile discussing there construction from a homotopy theoretic point of view first. We do not address the question of uniqueness, but it seems possible that they are unique up to equivalence.

Proposition 1.1 *The following CW spectra exist:*

$$S^0 \cup_\eta e^2 \cup_2 e^3, \quad S^0 \cup_\nu e^4 \cup_\eta e^6 \cup_2 e^7, \quad S^0 \cup_\sigma e^8 \cup_\nu e^{12} \cup_\eta e^{14} \cup_2 e^{15}.$$

Sketch of proof In each of the iterated mapping cones below, we will denote the homology generator corresponding to the unique cell in dimension n by x_n.

The case of $S^0 \cup_\eta \cup_2 e^3$ is obvious.

Consider the mapping cone of ν, $C_\nu = S^0 \cup_\nu e^4$. As $\nu\eta = 0$, there is a factorisation of η on the 4-sphere through C_ν.

$$\begin{tikzcd} & & & S^5 \arrow[d, "2"] & \\ & & & S^5 \arrow[dl, dotted, "\widetilde{\eta x_4}"'] \arrow[d, "\eta"] \arrow[dr, "\nu\eta=0"] & \\ S^3 \arrow[r, "\nu"'] & S^0 \arrow[r] & C_\nu \arrow[r] & S^4 \arrow[r, "\nu"'] & S^1. \end{tikzcd}$$

Also, $2\eta = 0$ and $\pi_5(S^0) = 0$, and hence $2\widetilde{\eta x_4} = 0$. A cobar representative for $\widetilde{\eta x_4}$ in the classical Adams E_2-term is

$$[\zeta_1^2 \otimes x_4 + \zeta_2^2 \otimes x_0] \in \mathrm{Ext}^{1,6}_{\mathcal{A}_*}(\mathbb{F}_2, H_*(C_\nu)).$$

We can form the mapping cone $C_{\widetilde{\eta x_4}} = C_\nu \cup_{\widetilde{\eta x_4}} e^6$ and, since $2\widetilde{\eta x_4} = 0$, there is a factorisation of 2 on the 6-sphere through $C_{\widetilde{\eta x_4}}$.

$$\begin{tikzcd} & & & S^6 \arrow[dl, dotted, "\widetilde{2x_6}"'] \arrow[d, "2"] \arrow[dr, "2\widetilde{\eta x_4}=0"] & \\ S^5 \arrow[r, "\phi_1"'] & C_\nu \arrow[r] & C_{\widetilde{\eta x_4}} \arrow[r] & S^6 \arrow[r, "\widetilde{\eta x_4}"'] & \Sigma C_\nu. \end{tikzcd}$$

A cobar representative of $\widetilde{2x_6}$ is

$$[\zeta_1 \otimes x_6 + \zeta_2 \otimes x_4 + \zeta_3 \otimes x_0] \in \mathrm{Ext}^{1,7}_{\mathcal{A}_*}(\mathbb{F}_2, H_*(C_{\widetilde{\eta x_4}})).$$

Consider the mapping cone of σ, $C_\sigma = S^0 \cup_\sigma e^8$. As $\sigma\nu = 0$, there is a factorisation of ν on the 8-cell through C_σ.

$$\begin{tikzcd} & & & S^{12} \arrow[d, "\eta"] & \\ & & & S^{11} \arrow[dl, dotted, "\widetilde{\nu x_8}"'] \arrow[d, "\nu"] \arrow[dr, "\sigma\nu=0"] & \\ S^7 \arrow[r, "\sigma"'] & S^0 \arrow[r] & C_\sigma \arrow[r] & S^8 \arrow[r, "\sigma"'] & S^1. \end{tikzcd}$$

Also, $\nu\eta = 0$ and $\pi_{12}(S^0) = 0 = \pi_{13}(S^0)$, and hence $\eta(\widetilde{\nu x_8}) = 0$.

As $\mathrm{Ext}^{1,12}_{\mathcal{A}_*}(\mathbb{F}_2, H_*(S^0)) = 0$, the element

$$[\zeta_1^4 \otimes x_8 + \zeta_2^4 \otimes x_0] \in \mathrm{Ext}^{1,12}_{\mathcal{A}_*}(\mathbb{F}_2, H_*(C_\sigma))$$

is a cobar representative for $\widetilde{\nu x_8}$.

We can form the mapping cone $C_{\widetilde{\nu x_8}} = C_\sigma \cup_{\widetilde{\nu x_8}} e^{12}$ and, since $\eta\widetilde{\nu x_8} = 0$, there is a factorisation of η on the 12-sphere through $C_{\widetilde{\nu x_8}}$.

$$\begin{tikzcd}
 & & & S^{13} \arrow[d, "2"] & \\
 & & & S^{13} \arrow[dl, dotted, "\widetilde{\eta x_{12}}"'] \arrow[d, "\eta"] \arrow[dr, "\eta(\widetilde{\nu x_8})=0"] & \\
S^{11} \arrow[r, "\widetilde{\nu x_8}"'] & C_\sigma \arrow[r] & C_{\widetilde{\nu x_8}} \arrow[r] & S^{12} \arrow[r, "\widetilde{\nu x_8}"'] & \Sigma C_\sigma .
\end{tikzcd}$$

As part of the long exact sequence for the homotopy of the mapping cone, we have the exact sequence

$$\pi_{13}(S^7) \xrightarrow{\sigma} \pi_{13}(S^0) \longrightarrow \pi_{13}(C_\sigma) \longrightarrow \pi_{13}(S^8),$$

and we have $\pi_{13}(S^0) = 0 = \pi_{13}(S^8)$, so $\pi_{13}(C_\sigma) = 0$. Therefore, $2(\widetilde{\eta x_{12}}) = 0$ and we can factorise 2 on the 14-sphere through the mapping cone of $\widetilde{\eta x_{12}}$, $C_{\widetilde{\eta x_{12}}}$.

$$\begin{tikzcd}
 & & & S^{14} \arrow[dl, dotted, "\widetilde{2x_{14}}"'] \arrow[d, "2"] & \\
S^{13} \arrow[r, "\widetilde{\eta x_{12}}"'] & C_{\widetilde{\nu x_8}} \arrow[r] & C_{\widetilde{\eta x_{12}}} \arrow[r] & S^{14} \arrow[r, "\widetilde{\eta x_{12}}"'] & \Sigma C_{\widetilde{\nu x_8}} .
\end{tikzcd}$$

A cobar representative of $\widetilde{2x_{14}}$ is

$$[\zeta_1 \otimes x_{14} + \zeta_2 \otimes x_{12} + \zeta_3 \otimes x_8 + \zeta_4 \otimes x_0] \in \mathrm{Ext}^{1,15}_{\mathcal{A}_*}(\mathbb{F}_2, H_*(C_{\widetilde{\eta x_{12}}})).$$

The homology of the mapping cone $C_{\widetilde{2x_{14}}}$ has a basis $x_0, x_8, x_{12}, x_{14}, x_{15}$, with coaction given by

$$\psi x_8 = \zeta_1^8 \otimes 1 + 1 \otimes x_8, \tag{1.2a}$$

$$\psi x_{12} = \zeta_2^4 \otimes 1 + \zeta_1^4 \otimes x_8 + 1 \otimes x_{12}, \tag{1.2b}$$

$$\psi x_{14} = \zeta_3^2 \otimes 1 + \zeta_2^2 \otimes x_8 + \zeta_1^2 \otimes x_{12} + 1 \otimes x_{14}, \tag{1.2c}$$

$$\psi x_{15} = \zeta_4 \otimes 1 + \zeta_3 \otimes x_8 + \zeta_2 \otimes x_{12} + 1 \otimes x_{15}. \tag{1.2d}$$

These calculations show that the CW spectra of the stated forms do indeed exist. □

Remark 1.2 The spectra of Proposition 1.1 are all minimal atomic in the sense of [12]; this follows from the fact that in each case the mod 2 cohomology is a cyclic $\mathcal{A}^*$-module.

2 Some $\mathcal{E}_\infty$ Thom spectra

Consider the three infinite loop spaces $B\mathrm{SO} = B\mathrm{O}\langle 2\rangle$, $B\mathrm{Spin} = B\mathrm{O}\langle 4\rangle$ and $B\mathrm{String} = B\mathrm{O}\langle 8\rangle$. The 3-skeleton of $B\mathrm{SO}$ is

$$B\mathrm{SO}^{[3]} = B\mathrm{O}\langle 2\rangle^{[3]} = S^2 \cup_2 e^3,$$

since $\mathrm{Sq}^1 w_2 = w_3$. Similarly, the 7-skeleton of $B\mathrm{Spin}$ is

$$B\mathrm{Spin}^{[7]} = B\mathrm{O}\langle 4\rangle^{[7]} = S^4 \cup_\eta e^6 \cup_2 e^7,$$

since $\mathrm{Sq}^2 w_4 = w_6$ and $\mathrm{Sq}^1 w_6 = w_7$. Finally, the 15-skeleton of $B\mathrm{String}$ is

$$B\mathrm{String}^{[15]} = B\mathrm{O}\langle 8\rangle^{[15]} = S^8 \cup_\nu e^{12} \cup_\eta e^{14} \cup_2 e^{15},$$

since $\mathrm{Sq}^4 w_8 = w_{12}$, $\mathrm{Sq}^2 w_{12} = w_{14}$ and $\mathrm{Sq}^1 w_{14} = w_{15}$.

The skeletal inclusion maps induce (virtual) bundles whose Thom spectra are themselves skeleta of the universal Thom spectra $M\mathrm{SO}$, $M\mathrm{Spin}$ and $M\mathrm{String}$. Routine calculations with Steenrod operations and the Wu formulae show that

$$\begin{aligned}
M\mathrm{SO}^{[3]} &= M\mathrm{O}\langle 2\rangle^{[3]} = S^0 \cup_\eta e^2 \cup_2 e^3,\\
M\mathrm{Spin}^{[7]} &= M\mathrm{O}\langle 4\rangle^{[7]} = S^0 \cup_\nu e^4 \cup_\eta e^6 \cup_2 e^7,\\
M\mathrm{String}^{[15]} &= M\mathrm{O}\langle 8\rangle^{[15]} = S^0 \cup_\sigma e^8 \cup_\nu e^{12} \cup_\eta e^{14} \cup_2 e^{15}.
\end{aligned}$$

Thus, these Thom spectra are examples of 'iterated Thom complexes' similar in spirit to those discussed in [10].

Each skeletal inclusion factors uniquely through an infinite loop map j_r,

$$\begin{tikzcd}
B\mathrm{O}\langle 2^r\rangle^{[2^{r+1}-1]} \arrow[rr] \arrow[dr] & & B\mathrm{O}\langle 2^r\rangle \\
& \mathrm{Q}B\mathrm{O}\langle 2^r\rangle^{[2^{r+1}-1]} \arrow[ur, "j_1"'] &
\end{tikzcd}$$

where $\mathrm{Q} = \Omega^\infty\Sigma^\infty$ is the free infinite loop space functor. We can also form the associated Thom spectrum Mj_r which is an $\mathcal{E}_\infty$ ring spectrum admitting an $\mathcal{E}_\infty$ morphism $Mj_r \to M\mathrm{O}\langle 2^r\rangle$ factoring the corresponding skeletal inclusion.

Using the algebra of "Appendix 1", it is easy to see that the skeletal inclusions induce monomorphisms in homology whose images contain the lowest degree generators:

$$1, a_{1,0}^{(1)}, a_{3,0} \in H_*(M\mathrm{SO}),$$

$$1, a_{1,0}^{(2)}, a_{3,0}^{(1)}, a_{7,0} \in H_*(M\mathrm{Spin}),$$
$$1, a_{1,0}^{(3)}, a_{3,0}^{(2)}, a_{7,0}^{(1)}, a_{15,0} \in H_*(M\mathrm{String}).$$

Each of the natural orientations $M\mathrm{O}\langle n\rangle \to H\mathbb{F}_2$ above induces an algebra homomorphism $H_*(M\mathrm{O}\langle n\rangle) \to \mathcal{A}_*$ for which

$$a_{1,0}^{(r)} \mapsto \zeta_1^{2^r}, \quad a_{3,0}^{(r)} \mapsto \zeta_2^{2^r}, \quad a_{7,0}^{(r)} \mapsto \zeta_3^{2^r}, \quad a_{15,0}^{(r)} \mapsto \zeta_4^{2^r}.$$

We also note that the skeleta can be identified with skeleta of $H\mathbb{Z}$, $k\mathrm{O}$ and tmf; namely, there are orientations inducing weak equivalences

$$M\mathrm{O}\langle 2\rangle^{[3]} \xrightarrow{\simeq} H\mathbb{Z}^{[3]}, \quad M\mathrm{O}\langle 4\rangle^{[7]} \xrightarrow{\simeq} k\mathrm{O}^{[7]}, \quad M\mathrm{O}\langle 8\rangle^{[15]} \xrightarrow{\simeq} \mathrm{tmf}^{[15]}. \tag{2.1}$$

The first two are induced from well-known orientations, while the third relies on unpublished work of Ando et al. [3]. Actually, such morphisms can be produced using the *reduced free commutative S-algebra* functor $\widetilde{\mathbb{P}}$ of [7], which has a universal property analogous to that of the usual free functor $\mathbb{P}$ of [17].

Proposition 2.1 *For $r = 1, 2, 3$, the natural map $M\mathrm{O}\langle 2^r\rangle^{[2^{r+1}-1]} \to Mj_r$ has unique extensions to a weak equivalence of $\mathcal{E}_\infty$ ring spectra*

$$\widetilde{\mathbb{P}}M\mathrm{O}\langle 2^r\rangle^{[2^{r+1}-1]} \xrightarrow{\sim} Mj_r.$$

The orientations of (2.1) *induce morphisms of $\mathcal{E}_\infty$ ring spectra*

$$\widetilde{\mathbb{P}}M\mathrm{O}\langle 2\rangle^{[3]} \to H\mathbb{Z}, \quad \widetilde{\mathbb{P}}M\mathrm{O}\langle 4\rangle^{[7]} \to k\mathrm{O}, \quad \widetilde{\mathbb{P}}M\mathrm{O}\langle 8\rangle^{[15]} \to \mathrm{tmf}.$$

Proof The existence of such morphisms depends on the universal property of $\widetilde{\mathbb{P}}$. The proof that those of the first kind are equivalences depends on a comparison of the homology rings using Theorem 2.3 below. □

Remark 2.2 In fact, the weak equivalences of (2.1) extend to weak equivalences

$$Mj_1 \sim H\mathbb{Z}^{[4]}, \quad Mj_2 \sim k\mathrm{O}^{[8]}, \quad Mj_3 \sim \mathrm{tmf}^{[16]}. \tag{2.2}$$

The homology of Mj_r can be determined from that of the underlying infinite loop space using the Thom isomorphism, while that for the others it depends on a general description of the homology of $H_*(\widetilde{\mathbb{P}}X)$ which can be found in [7].

Theorem 2.3 *The homology rings of the Thom spectra Mj_r are given by*

$$\begin{aligned}
H_*(Mj_1) &= \mathbb{F}_2[\mathrm{Q}^I x_2, \mathrm{Q}^J x_3 : I, J \textit{ admissible}, \mathrm{exc}(I) > 2, \mathrm{exc}(J) > 3],\\
H_*(Mj_2) &= \mathbb{F}_2[\mathrm{Q}^I x_4, \mathrm{Q}^J x_6, \mathrm{Q}^K x_7 : I, J, K \textit{ admissible}, \mathrm{exc}(I) > 4, \mathrm{exc}(J) > 6, \mathrm{exc}(K) > 7],\\
H_*(Mj_3) &= \mathbb{F}_2[\mathrm{Q}^I x_8, \mathrm{Q}^J x_{12}, \mathrm{Q}^K x_{14}, \mathrm{Q}^L x_{15} : I, J, K, L \textit{ admissible},\\
&\qquad \mathrm{exc}(I) > 8, \mathrm{exc}(J) > 12, \mathrm{exc}(K) > 14, \mathrm{exc}(L) > 15].
\end{aligned}$$

The $\mathcal{E}_\infty$ orientations $Mj_r \to H\mathbb{F}_2$ induce algebra homomorphisms $H_*(Mj_r) \to \mathcal{A}_*$ which have images

$$\begin{aligned} \mathbb{F}_2[\zeta_1^2, \zeta_2, \zeta_3, \ldots] &\cong H_*(H\mathbb{Z}), \\ \mathbb{F}_2[\zeta_1^4, \zeta_2^2, \zeta_3, \zeta_4, \ldots] &\cong H_*(k\mathrm{O}), \\ \mathbb{F}_2[\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \zeta_5, \ldots] &\cong H_*(\mathrm{tmf}). \end{aligned}$$

Recalling Remark 1.2, we note the following where the minimal atomic $\mathcal{E}_\infty$ ring spectrum is used in the sense of Hu, Kriz and May, and was subsequently developed further in [11].

Proposition 2.4 *Each of the $\mathcal{E}_\infty$ ring spectra Mj_r $(r = 1, 2, 3)$ is minimal atomic.*

Proof In [7], we showed that for $X \in S^0/\mathscr{M}_S$ in the slice category of S-modules under a cofibrant replacement of S,

$$\Omega_S(\widetilde{\mathbb{P}}X) \sim \widetilde{\mathbb{P}}X \wedge X/S^0;$$

hence,

$$\mathrm{TAQ}_*(\widetilde{\mathbb{P}}X, S; H) \cong H_*(X/S^0).$$

For $Mj_r \sim \widetilde{\mathbb{P}}M\mathrm{O}\langle 2^r\rangle^{[2^{r+1}-1]}$, this gives

$$\mathrm{TAQ}_*(Mj_r, S; H) \cong H_*(M\mathrm{O}\langle 2^r\rangle^{[2^{r+1}-1]}/S^0).$$

The $(2^{r+1} - 1)$-skeleton for a minimal cell structure on the spectrum Mj_r agrees with $M\mathrm{O}\langle 2^r\rangle^{[2^{r+1}-1]}$, and this is a minimal atomic S-module as noted in Remark 1.2. It follows that the mod 2 Hurewicz homomorphism $\pi_*(Mj_r) \to H_*(Mj_r)$ is trivial in the range $0 < * < 2^{r+1}$. Hence, the TAQ Hurewicz homomorphism

$$\pi_*(Mj_r) \to \mathrm{TAQ}_*(Mj_r, S; H) \xrightarrow{\cong} H_*(Mj_r/S^0)$$

is trivial. Now by [11, theorem 3.3], Mj_r is minimal atomic as claimed. □

3 Some coalgebra

In this section, we review some useful results on comodules over Hopf algebras. Although most of this material is standard, we state some results in a precise form suitable for our requirements. Since writing early versions of this paper, we became aware of work by Hill [20] which uses similar results.

First, we recall a standard algebraic result, for example see [31, lemma 3.1]. We work vector spaces over a field $\Bbbk$ and will set $\otimes = \otimes_\Bbbk$. There are slight modifications required for the graded case which we leave the reader to formulate; however as we work exclusively in characteristic 2, these have no significant effect in this paper.

We refer to the classic paper of Milnor and Moore [28] for background material on coalgebra.

Let A be a commutative Hopf algebra over a field $\Bbbk$, and let B be a quotient Hopf algebra of A. We denote the product and antipode on A by ϕ_A and χ, and the coaction on a left comodule D by ψ_D. We will identify the cotensor product $A\square_B\Bbbk \subseteq A \otimes \Bbbk$ with a subalgebra of A under the canonical isomorphism $A \otimes \Bbbk \xrightarrow{\cong} A$.

Lemma 3.1 *Let D be a commutative A-comodule algebra. Then there is an isomorphism of A-comodule algebras*

$$(\phi_A \otimes \mathrm{Id}_D) \circ (\mathrm{Id}_A \otimes \psi_D)\colon (A\square_B\Bbbk)\otimes D \xrightarrow{\cong} A\square_B D; \quad a\otimes x \longleftrightarrow \sum_i aa_i \otimes x_i, \tag{3.1}$$

where $\psi_D x = \sum_i a_i \otimes x_i$ denotes the coaction on $x \in D$.

Here, the codomain has the diagonal A-comodule structure, while the domain has the left A-comodule structure.

Here is an easily proved generalisation of this result.

Lemma 3.2 *Let C be a commutative B-comodule algebra and let D be a commutative A-comodule algebra, then there is an isomorphism of A-comodule algebras*

$$(A\square_B C) \otimes D \xrightarrow{\cong} A\square_B (C \otimes D), \tag{3.2}$$

where the domain has the diagonal left A-coaction and $C \otimes D$ has the diagonal left B-coaction.

Explicitly, on an element

$$\sum_r u_r \otimes v_r \otimes x \in (A\square_B C) \otimes D \subseteq A \otimes C \otimes D,$$

the isomorphism has the effect

$$\sum_r u_r \otimes v_r \otimes w \longmapsto \sum_r \sum_i u_r a_i \otimes v_r \otimes w_i,$$

where $\psi_D w = \sum_i a_i \otimes w_i$ as above. Similarly, the inverse is given by

$$\sum_r b_r \otimes y_r \otimes w_r \longmapsto \sum_r \sum_i b_r \chi(a_{r,i}) \otimes v_r \otimes w_{r,i}.$$

Now, suppose that H is a finite-dimensional Hopf algebra. If K is a sub-Hopf algebra of H, it is well known that H is a free left or right K-module, i.e. $H \cong K \otimes U$ or $H \cong U \otimes K$ for a vector space U (see [29, theorems 31.1.5 and 3.3.1]). This dualises as follows: If L is a quotient Hopf algebra of H, then H is an extended left or right L-comodule, i.e. $H \cong L \otimes V$ or $H \cong V \otimes L$ for a vector space V; in fact,

$V = H\Box_L\Bbbk$. More generally, according to Margolis [26, pp. 193 and 240], if H is a P-*algebra*, then a result of the first kind holds for any finite-dimensional sub-Hopf algebra K.

We need to make use of the *finite dual* of a Hopf algebra H, namely

$$H^{\mathrm{o}} = \{f \in \mathrm{Hom}_{\Bbbk}(H, \Bbbk) \colon \exists\, I \lhd H \text{ such that } \operatorname{codim} I < \infty \text{ and } I \subseteq \ker f\}.$$

Then, H^{o} becomes an Hopf algebra with product and coproduct obtained from the adjoints of the coproduct and product of H. We will say that H is a P-*coalgebra* if H^{o} is a P-algebra.

Lemma 3.3 *Suppose that A is a commutative Hopf algebra which is a P-coalgebra. If B is a finite dimensional quotient Hopf algebra of A, then A is an extended right (or left) B-comodule, i.e. $A \cong W \otimes B$ (or $A \cong B \otimes W$) for some vector space W, and in fact $W \cong A\Box_B\Bbbk$ (or $W \cong \Bbbk\Box_B A$).*

Corollary 3.4 *For any right B-comodule L or left B-comodule M, as vector spaces,*

$$A\Box_B M \cong (A\Box_B\Bbbk) \otimes M, \quad L\Box_B A \cong L \otimes (\Bbbk\Box_B A).$$

These are isomorphisms of left or right A-comodules for suitable comodule structures on the right hand sides.

To understand the relevant A-comodule structure on $(A\Box_B\Bbbk) \otimes M$, note that there is an isomorphism of left A-comodules

$$\begin{tikzcd}
(A\Box_B\Bbbk) \otimes M \arrow[r, "\mathrm{Id}\otimes\psi_M"'] \arrow[rr, bend left, "\cong"] & (A\Box_B\Bbbk) \otimes B \otimes M \arrow[r, "\cong"'] & A \otimes M ,
\end{tikzcd}$$

where the right hand factor is the isomorphism of Lemma 3.3.

Crucially for our purposes, for a prime p, the Steenrod algebra $\mathcal{A}^*$ is a P-algebra in the sense of Margolis [26], i.e. it is a union of finite sub-Hopf algebras. When $p = 2$,

$$\mathcal{A}^* = \bigcup_{n\geqslant 0} \mathcal{A}(n)^*,$$

and it follows from the preceding results that if $n \geqslant 0$, $\mathcal{A}^*$ is free as a right or left $\mathcal{A}(n)^*$-module; see [26, pp. 193 and 240]. Dually, $(\mathcal{A}_*)^{\mathrm{o}} = \mathcal{A}^*$ and $\mathcal{A}_*$ is an extended $\mathcal{A}(n)_*$-comodule:

$$\mathcal{A}_* \cong (\mathcal{A}_*\Box_{\mathcal{A}(n)_*}\mathbb{F}_2) \otimes \mathcal{A}(n)_*, \tag{3.3}$$

$$\mathcal{A}_* \cong \mathcal{A}(n)_* \otimes (\mathbb{F}_2\Box_{\mathcal{A}(n)_*}\mathcal{A}_*). \tag{3.4}$$

Given this, we see that for any left $\mathcal{A}(n)_*$-comodule M_*, as vector spaces

$$\mathcal{A}_*\Box_{\mathcal{A}(n)_*} M_* \cong (\mathcal{A}_*\Box_{\mathcal{A}(n)_*}\mathbb{F}_2) \otimes M_*. \tag{3.5}$$

In fact, this is also an isomorphism left $\mathcal{A}_*$-comodules.

Here is an explicit description of isomorphisms of the type given by Lemma 3.3. For $n \geqslant 0$, we will use the function

$$\mathrm{e}_n \colon \mathbb{N} \to \mathbb{N}; \quad \mathrm{e}_n(i) = \begin{cases} 2^{n+2-i} & \text{if } 1 \leqslant i \leqslant n+2, \\ 1 & \text{if } i \geqslant n+3. \end{cases}$$

For any natural number r, write

$$r = r'(n,i)\mathrm{e}_n(i) + r''(n,i),$$

where $0 \leqslant r''(n,i) < \mathrm{e}_n(i)$. We note that

$$\mathcal{A}_* \square_{\mathcal{A}(n)_*} \mathbb{F}_2 = \mathbb{F}_2[\zeta_1^{\mathrm{e}_n(1)}, \zeta_2^{\mathrm{e}_n(2)}, \zeta_3^{\mathrm{e}_n(3)}, \ldots] \subseteq \mathcal{A}_*,$$

and

$$\mathcal{A}(n)_* = \mathcal{A}_* /\!/ (\mathcal{A}_* \square_{\mathcal{A}(n)_*} \mathbb{F}_2) = \mathcal{A}_*/(\zeta_1^{\mathrm{e}_n(1)}, \zeta_2^{\mathrm{e}_n(2)}, \zeta_3^{\mathrm{e}_n(3)}, \ldots).$$

We will indicate elements of $\mathcal{A}(n)_*$ by writing $\|z\|$ for the coset of z which is always chosen to be a sum of monomials $\zeta_1^{s_1}\zeta_2^{s_2}\ldots\zeta_\ell^{s_\ell}$ with exponents satisfying $0 \leqslant s_i < \mathrm{e}_n(i)$.

Proposition 3.5 *For $n \geqslant 0$, there is an isomorphism of right $\mathcal{A}(n)_*$-comodules*

$$\mathcal{A}_* \xrightarrow{\cong} (\mathcal{A}_* \square_{\mathcal{A}(n)_*} \mathbb{F}_2) \otimes \mathcal{A}(n)_*$$

given on basic tensors by

$$\zeta_1^{r_1}\zeta_2^{r_2}\ldots\zeta_\ell^{r_\ell} \longleftrightarrow \zeta_1^{r_1'(n,1)\mathrm{e}_n(1)} \ldots \zeta_\ell^{r_\ell'(n,\ell)\mathrm{e}_n(\ell)} \otimes \left\| \zeta_1^{r_1''(n,1)} \ldots \zeta_\ell^{r_\ell''(n,\ell)} \right\|.$$

We will also use the following result to construct algebraic maps in lieu of geometric ones. The proof is a straightforward generalisation of a standard one for the case where $B = \Bbbk$.

Lemma 3.6 *Suppose that M is a left A-comodule and N is a left B-comodule. Then there is a natural isomorphism*

$$\mathrm{Comod}_B(M, N) \xrightarrow{\cong} \mathrm{Comod}_A(M, A\square_B N); \quad f \mapsto \widetilde{f},$$

where $\widetilde{f}$ *is the unique factorisation of* $(\mathrm{Id} \otimes f)\psi_M$ *through* $A\square_B N$.

$$
\begin{tikzcd}
M \arrow[r, "\widetilde{f}"] \arrow[d, "\psi_M"'] \arrow[dr, "(\mathrm{Id}\otimes f)\psi_M"'] & A\square_B N \arrow[d, hook] \\
A \otimes M \arrow[r, "\mathrm{Id}\otimes f"'] & A \otimes N.
\end{tikzcd}
$$

Furthermore, if M is an A-comodule algebra and N is a B-comodule algebra, then if f is an algebra homomorphism, so is $\widetilde{f}$.

As an example of the multiplicative version of this result, suppose that M is an A-comodule algebra which is augmented. Then there is a composite homomorphism of B-comodule algebras $\alpha\colon M \to \Bbbk \to N$ giving rise to homomorphism of A-comodule algebras

$$\widetilde{\alpha}\colon M \to A\square_B N; \quad \widetilde{\alpha}(x) = a \otimes 1,$$

where $\psi_M(x) = a \otimes 1 + \cdots + 1 \otimes x$.

4 The homology of Mj_r for $r = 1, 2, 3$

Now we analyse the specific cases for $H_*(Mj_r)$ for $r = 1, 2, 3$. Since some of the details differ in each case, we treat these separately. In each case, there is a commutative diagram of commutative $\mathcal{A}_*$-comodule algebras

$$
\begin{tikzcd}
H_*(Mj_r) \arrow[rr, dashed, bend left, "(\cong)"] \arrow[dr] \arrow[ddr, "\psi"'] & (\mathcal{A}_*\square_{\mathcal{A}(r-1)_*}\mathbb{F}_2) \otimes H_*(Mj_r) \arrow[r, "\pi"] & (\mathcal{A}_*\square_{\mathcal{A}(r-1)_*}\mathbb{F}_2) \otimes H_*(Mj_r)/I_r \\
 & \mathcal{A}_*\square_{\mathcal{A}(r-1)_*} H_*(Mj_r) \arrow[r, "\pi"] \arrow[u, "\cong"] \arrow[d, hook] & \mathcal{A}_*\square_{\mathcal{A}(r-1)_*} H_*(Mj_r)/I_r \arrow[u, "\cong"'] \arrow[d, hook] \\
 & \mathcal{A}_* \otimes H_*(Mj_r) \arrow[r, "\pi"] & \mathcal{A}_* \otimes H_*(Mj_r)/I_r
\end{tikzcd}
\tag{4.1}
$$

in which $I_r \lhd H_*(Mj_r)$ is a certain $\mathcal{A}(r-1)_*$-comodule ideal. In each case, the proof involves showing that the dashed arrow is an isomorphism.

4.1 The homology of Mj_1

By Theorem 2.3,

$$H_*(Mj_1) = \mathbb{F}_2[\mathrm{Q}^I x_2, \mathrm{Q}^J x_3 \colon I, J \text{ admissible}, \operatorname{exc}(I) > 2, \operatorname{exc}(J) > 3], \tag{4.2}$$

where the left $\mathcal{A}_*$-coaction is determined by

$$\psi x_2 = 1 \otimes x_2 + \zeta_1^2 \otimes 1, \quad \psi x_3 = 1 \otimes x_3 + \zeta_1 \otimes x_2 + \zeta_2 \otimes 1.$$

To calculate the coaction on the other generators $\mathrm{Q}^I x_2$ and $\mathrm{Q}^J x_3$, we follow [9] and use the right coaction

$$\widetilde{\psi} \colon H_*(Mj_1) \to H_*(Mj_1) \otimes \mathcal{A}_*; \quad \widetilde{\psi}(z) = \sum_i z_i \otimes \chi(\alpha_i),$$

where $\psi(z) = \sum_i \alpha_i \otimes z_i$ and χ is the antipode of $\mathcal{A}_*$. So,

$$\widetilde{\psi} x_2 = x_2 \otimes 1 + 1 \otimes \zeta_1^2, \quad \widetilde{\psi} x_3 = x_3 \otimes 1 + x_2 \otimes \zeta_1 + 1 \otimes \xi_2.$$

In general, if z has degree m, then

$$\widetilde{\psi}\mathrm{Q}^r z = \sum_{m \leqslant k \leqslant r} \mathrm{Q}^k(\widetilde{\psi} z)[\zeta(t)^k]_{t^r} = \sum_{m \leqslant k \leqslant r} \mathrm{Q}^k(\widetilde{\psi} z)\left[\left(\frac{\zeta(t)}{t}\right)^k\right]_{t^{r-k}}. \tag{4.3}$$

By (4.3),

$$\begin{aligned}
\widetilde{\psi}\mathrm{Q}^4 x_3 &= \mathrm{Q}^3(x_3 \otimes 1 + x_2 \otimes \zeta_1 + 1 \otimes \xi_2)\left[\left(\frac{\zeta(t)}{t}\right)^3\right]_t \\
&\quad + \mathrm{Q}^4(x_3 \otimes 1 + x_2 \otimes \zeta_1 + 1 \otimes \xi_2) \\
&= x_3^2 \otimes \zeta_1 + x_2^2 \otimes \zeta_1^3 + 1 \otimes \zeta_1\xi_2^2 \\
&\quad + \mathrm{Q}^4 x_3 \otimes 1 + (\mathrm{Q}^3 x_2 \otimes \zeta_1^2 + x_2^2 \otimes \mathrm{Q}^2\zeta_1) + 1 \otimes \mathrm{Q}^4\xi_2 \\
&= x_3^2 \otimes \zeta_1 + x_2^2 \otimes \zeta_1^3 + 1 \otimes \zeta_1\xi_2^2 + \mathrm{Q}^4 x_3 \otimes 1 \\
&\quad + \mathrm{Q}^3 x_2 \otimes \zeta_1^2 + x_2^2 \otimes \zeta_2 + 1 \otimes (\xi_3 + \zeta_1\xi_2^2) \\
&= (\mathrm{Q}^4 x_3 \otimes 1 + x_3^2 \otimes \zeta_1 + x_2^2 \otimes \xi_2 + 1 \otimes \xi_3) + \mathrm{Q}^3 x_2 \otimes \zeta_1^2.
\end{aligned}$$

We also have

$$\widetilde{\psi}\mathrm{Q}^3 x_2 = \mathrm{Q}^3 x_2 \otimes 1, \quad \widetilde{\psi}\mathrm{Q}^5 x_2 = \mathrm{Q}^5 x_2 \otimes + \mathrm{Q}^3 x_2 \otimes \zeta_1^2.$$

Combining these, we obtain

$$\widetilde{\psi}(\mathrm{Q}^4 x_3 + \mathrm{Q}^5 x_2) = (\mathrm{Q}^4 x_3 + \mathrm{Q}^5 x_2) \otimes 1 + x_3^2 \otimes \zeta_1 + x_2^2 \otimes \xi_2 + 1 \otimes \xi_3, \tag{4.4}$$

or equivalently,

$$\psi(\mathrm{Q}^4x_3+\mathrm{Q}^5x_2)=1\otimes(\mathrm{Q}^4x_3+\mathrm{Q}^5x_2)+\zeta_1\otimes x_3^2+\zeta_2\otimes x_2^2+\zeta_3\otimes 1. \qquad (4.5)$$

We will consider the sequence of elements $X_{1,1}$ and $X_{1,s}\in H_{2^s-1}(Mj_1)$ ($s\geqslant 2$) defined by

$$X_{1,s}=\begin{cases} x_2 & \text{if } s=1,\\ x_3 & \text{if } s=2,\\ \mathrm{Q}^4x_3+\mathrm{Q}^5x_2 & \text{if } s=3,\\ \mathrm{Q}^{(2^{s-1},\dots,2^4,2^3)}(\mathrm{Q}^4x_3+\mathrm{Q}^5x_2)=\mathrm{Q}^{2^{s-1}}X_{1,s-1} & \text{if } s\geqslant 4,\end{cases}$$

where $\mathrm{Q}^{(i_1,i_2,\dots,i_\ell)}=\mathrm{Q}^{i_1}\mathrm{Q}^{i_2}\dots\mathrm{Q}^{i_\ell}$. We claim that $X_{1,s}$ have the following right and left coactions:

$$\begin{aligned}\widetilde{\psi}X_{1,s}&=X_{1,s}\otimes 1+X_{1,s-1}^2\otimes\zeta_1+\cdots+X_{1,3}^{2^{s-3}}\otimes\xi_{s-3} \qquad (4.6)\\ &\quad+X_{1,2}^{2^{s-2}}\otimes\xi_{s-2}+X_{1,1}^{2^{s-2}}\otimes\xi_{s-1}+1\otimes\xi_s,\\ \psi X_{1,s}&=1\otimes X_{1,s}+\zeta_1\otimes X_{1,s-1}^2+\cdots+\zeta_{s-3}\otimes X_{1,3}^{2^{s-3}}\\ &\quad+\zeta_{s-2}\otimes X_{1,2}^{2^{s-2}}+\zeta_{s-1}\otimes X_{1,1}^{2^{s-2}}+\zeta_s\otimes 1. \qquad (4.7)\end{aligned}$$

To prove these, we use induction on s, where the early cases $s=1,2,3$ are known already. For the inductive step, assume that (4.6) holds for some $s\geqslant 3$. Then,

$$\begin{aligned}\widetilde{\psi}X_{1,s+1}&=\widetilde{\psi}\mathrm{Q}^{2^s}X_{1,s}=(\widetilde{\psi}X_{1,s})^2\zeta_1+\mathrm{Q}^{2^s}(\widetilde{\psi}X_{1,s})\\ &=X_{1,s}^2\otimes\zeta_1+X_{1,s-1}^{2^2}\otimes\zeta_1^3+\cdots+X_{1,3}^{2^{s-2}}\otimes\xi_{s-3}^2\zeta_1\\ &\quad+X_{1,2}^{2^{s-1}}\otimes\xi_{s-2}^2\zeta_1+X_{1,1}^{2^{s-1}}\otimes\xi_{s-1}^2\zeta_1+1\otimes\xi_s^2\zeta_1\\ &\quad+\mathrm{Q}^{2^s}(X_{1,s}\otimes 1+X_{1,s-1}^2\otimes\zeta_1+\cdots+X_{1,3}^{2^{s-3}}\otimes\xi_{s-3}\\ &\quad+X_{1,2}^{2^{s-2}}\otimes\xi_{s-2}+X_{1,1}^{2^{s-2}}\otimes\xi_{s-1}+1\otimes\xi_s)\\ &=X_{1,s}^2\otimes\zeta_1+X_{1,s-1}^{2^2}\otimes\zeta_1^3+\cdots+X_{1,3}^{2^{s-2}}\otimes\xi_{s-3}^2\zeta_1+X_{1,2}^{2^{s-1}}\otimes\xi_{s-2}^2\zeta_1\\ &\quad+X_{1,1}^{2^{s-1}}\otimes\xi_{s-1}^2\zeta_1+1\otimes\xi_s^2\zeta_1\\ &\quad+\mathrm{Q}^{2^s}X_{1,s}\otimes 1+X_{1,s-1}^{2^2}\otimes\mathrm{Q}^2\zeta_1+\cdots+X_{1,3}^{2^{s-2}}\otimes\mathrm{Q}^{2^{s-3}}\xi_{s-3}\\ &\quad+X_{1,2}^{2^{s-1}}\otimes\mathrm{Q}^{2^{s-2}}\xi_{s-2}+X_{1,1}^{2^{s-1}}\otimes\mathrm{Q}^{2^{s-1}}\xi_{s-1}+1\otimes\mathrm{Q}^{2^s}\xi_s\\ &=X_{1,s}^2\otimes\zeta_1+X_{1,s-1}^{2^2}\otimes\zeta_1^3+\cdots+X_{1,3}^{2^{s-2}}\otimes\xi_{s-3}^2\zeta_1+X_{1,2}^{2^{s-1}}\otimes\xi_{s-2}^2\zeta_1\\ &\quad+X_{1,1}^{2^{s-1}}\otimes\xi_{s-1}^2\zeta_1+1\otimes\xi_s^2\zeta_1\\ &\quad+X_{1,s+1}\otimes 1+X_{1,s-1}^{2^2}\otimes(\xi_2+\zeta_1^3)+\cdots+X_{1,3}^{2^{s-2}}\otimes(\xi_{s-2}+\xi_{s-3}^2\zeta_1)\\ &\quad+X_{1,2}^{2^{s-1}}\otimes(\xi_{s-1}+\xi_{s-2}^2\zeta_1)+X_{1,1}^{2^{s-1}}\otimes(\xi_s+\xi_{s-1}^2\zeta_1)+1\otimes(\xi_{s+1}+\xi_s^2\zeta_1)\end{aligned}$$

$$
\begin{aligned}
&= X_{1,s+1} \otimes 1 + X_{1,s}^2 \otimes \zeta_1 + X_{1,s-1}^{2^2} \otimes \xi_2 + \cdots + X_{1,3}^{2^{s-2}} \otimes \xi_{s-2} \\
&\quad + X_{1,2}^{2^{s-1}} \otimes \xi_{s-1} + X_{1,1}^{2^{s-1}} \otimes \xi_s + 1 \otimes \xi_{s+1},
\end{aligned}
$$

giving the result for $s+1$. Here for terms of form $\mathrm{Q}^{|u|+|v|+1}(u \otimes v)$, we have

$$
\begin{aligned}
\mathrm{Q}^{|u|+|v|+1}(u \otimes v) &= \mathrm{Q}^{|u|+1} u \otimes \mathrm{Q}^{|v|} v + \mathrm{Q}^{|u|+1} u \otimes \mathrm{Q}^{|v|+1} v \\
&= \mathrm{Q}^{|u|+1} u \otimes v^2 + u^2 \otimes \mathrm{Q}^{|v|+1} v
\end{aligned}
$$

by the Cartan formula and unstable conditions.

Under the homomorphism $\rho\colon H_*(Mj_1) \to \mathcal{A}_*$ induced by the orientation $Mj_1 \to H\mathbb{F}_2$, we have

$$
\rho(x_2) = \zeta_1^2, \quad \rho(x_3) = \zeta_2, \quad \rho(X_{1,s}) = \zeta_s \quad (s \geqslant 3).
$$

Also,

$$
\rho(\mathrm{Q}^3 x_2) = \mathrm{Q}^3(\rho x_2) = \mathrm{Q}^3(\zeta_1^2) = 0,
$$

and for each admissible monomial I, $\rho(\mathrm{Q}^I x_2) \in \mathcal{A}_*$ is a square.

This shows that the restriction of ρ to the subalgebra generated by the $X_{1,s}$ is an isomorphism of $\mathcal{A}_*$-comodule algebras

$$
\mathbb{F}_2[X_{1,s} : s \geqslant 1] \xrightarrow{\cong} \mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2 \subseteq \mathcal{A}_*,
$$

where

$$
\mathcal{A}(0)_* = \mathcal{A}_* /\!/ \mathbb{F}_2[\zeta_1^2, \zeta_2, \zeta_3, \ldots], \quad \mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2 = \mathbb{F}_2[\zeta_1^2, \zeta_2, \zeta_3, \ldots] \subseteq \mathcal{A}_*.
$$

In the algebra $H_*(Mj_1)$, the regular sequence $X_{1,s}$ $(s \geqslant 1)$ generates an ideal

$$
I_1 = (X_{1,s} \colon s \geqslant 1) \lhd H_*(Mj_1).
$$

This is not an $\mathcal{A}_*$-subcomodule since, for example,

$$
\psi X_{1,3} = \psi(\mathrm{Q}^4 x_3 + \mathrm{Q}^5 x_2) = (1 \otimes X_{1,3} + \zeta_1 \otimes X_{1,2}^2 + \zeta_2 \otimes X_{1,1}^2) + \zeta_3 \otimes 1.
$$

However, under the induced $\mathcal{A}(0)_*$-coaction

$$
\psi' \colon H_*(Mj_1) \to \mathcal{A}(0)_* \otimes H_*(Mj_1),
$$

the last term becomes trivial; in fact,

$$
\psi' X_{1,3} = 1 \otimes X_{1,3} + \zeta_1 \otimes X_{1,2}^2,
$$

where we identify elements of $\mathcal{A}(0)_*$ with representatives in $\mathcal{A}_*$. More generally, by (4.7), for $s \geqslant 2$,

$$\psi' X_{1,s} = 1 \otimes X_{1,s} + \zeta_1 \otimes X_{1,s-1}^2.$$

It follows that I_1 is an $\mathcal{A}(0)_*$-invariant ideal.

Proposition 4.1 *There is an isomorphism of commutative $\mathcal{A}_*$-comodule algebras*

$$H_*(Mj_1) \xrightarrow{\cong} \mathcal{A}_* \square_{\mathcal{A}(0)_*} H_*(Mj_1)/I_1.$$

Proof Taking $r = 1$, from (4.1), we obtain a commutative diagram of commutative $\mathcal{A}_*$-comodule algebras

$$\begin{tikzcd}
H_*(Mj_1) \arrow[rr, dashed, bend left] \arrow[dr] \arrow[ddr, "\psi"'] & (\mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2) \otimes H_*(Mj_1) \arrow[r, "\pi"] & (\mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2) \otimes H_*(Mj_1)/I_1 \\
 & \mathcal{A}_* \square_{\mathcal{A}(0)_*} H_*(Mj_1) \arrow[r, "\pi"] \arrow[u, "\cong"] \arrow[d, hook] & \mathcal{A}_* \square_{\mathcal{A}(0)_*} H_*(Mj_1)/I_1 \arrow[u, "\cong"] \arrow[d, hook] \\
 & \mathcal{A}_* \otimes H_*(Mj_1) \arrow[r, "\pi"] & \mathcal{A}_* \otimes H_*(Mj_1)/I_1
\end{tikzcd}$$

and furthermore

$$\begin{aligned}
\psi X_{1,1} &= \zeta_1^2 \otimes 1 + 1 \otimes X_{1,1}, \\
\psi X_{1,2} &= \zeta_2 \otimes 1 + \zeta_1 \otimes X_{1,1} + 1 \otimes X_{1,1}, \\
\psi X_{1,s} &= \zeta_{s+1} \otimes 1 + \cdots + 1 \otimes X_{1,s} \quad (s \geqslant 3),
\end{aligned}$$

giving

$$\pi\, \psi X_{1,1} = \zeta_1^2 \otimes 1, \quad \pi\, \psi X_{1,2} = \zeta_2 \otimes 1, \quad \pi\, \psi X_{1,s} = \zeta_{s+1} \otimes 1 + \cdots .$$

The latter form part of a set of polynomial generators for the polynomial ring

$$\mathcal{A}_* \otimes H_*(Mj_1)/I_1 \cong (\mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2) \otimes H_*(Mj_1)/I_1.$$

Now, a straightforward argument shows that the dashed arrow is surjective; but as the Poincaré series of $H_*(Mj_1)$ and $(\mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2) \otimes H_*(Mj_1)/I_1$ are equal, it is actually an isomorphism. Therefore,

$$H_*(Mj_1) \cong \mathcal{A}_* \square_{\mathcal{A}(0)_*} H_*(Mj_1)/I_1.$$

$\square$

Remark 4.2 For the purposes of proving such a result, we might as well have set $X_{1,3} = \mathrm{Q}^4 x_3$ and

$$X_{1,s} = \mathrm{Q}^{2^{s-1}} X_{1,s-1} \quad (s \geqslant 3),$$

since

$$\psi' X_{1,3} = 1 \otimes X_{1,3} + \zeta_1 \otimes x_3^2$$

and so on. However, the cases of Mj_2 and Mj_3 will require modifications similar to the ones we have used above which give an indication of the methods required.

We have the following splitting result.

Proposition 4.3 *There is a splitting of $\mathcal{A}_*$-comodule algebras*

$$\begin{tikzcd}
\mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2 \arrow[rr, "\cong"] \arrow[dr] & & \mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2 \ , \\
& H_*(Mj_1) \arrow[ur] &
\end{tikzcd}$$

where $H_*(Mj_1) \to H_*(H\mathbb{Z}) = \mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2$ is induced by the $\mathcal{E}_\infty$ orientation $Mj_1 \to H\mathbb{Z}$.

Proof This is proved using Lemma 3.6 together with the trivial $\mathcal{A}(0)_*$-comodule algebra homomorphism $\mathcal{A}_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2 \to H_*(Mj_1)/I_1$. □

4.2 The homology of Mj_2

We have

$$H_*(Mj_2) = \mathbb{F}_2[\mathrm{Q}^I x_4, \mathrm{Q}^J x_6, \mathrm{Q}^K x_7 : I, J, K \text{ admissible}, \mathrm{exc}(I) > 4, \mathrm{exc}(J) > 6, \mathrm{exc}(K) > 7],$$

with right coaction satisfying

$$\begin{aligned}
\widetilde{\psi} x_4 &= x_4 \otimes 1 + 1 \otimes \zeta_1^4, \\
\widetilde{\psi} x_6 &= x_6 \otimes 1 + x_4 \otimes \zeta_1^2 + 1 \otimes \xi_2^2, \\
\widetilde{\psi} x_7 &= x_7 \otimes 1 + x_6 \otimes \zeta_1 + x_4 \otimes \xi_2 + 1 \otimes \xi_3.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\widetilde{\psi} \mathrm{Q}^8 x_7 = x_7^2 \otimes \zeta_1 &+ x_6^2 \otimes \zeta_1^3 + x_4^2 \otimes \zeta_1 \xi_2^2 + 1 \otimes \xi_3^2 \zeta_1 \\
&+ \mathrm{Q}^8 (x_7 \otimes 1 + x_6 \otimes \zeta_1 + x_4 \otimes \xi_2 + 1 \otimes \xi_3) \\
&= x_7^2 \otimes \zeta_1 + x_6^2 \otimes \zeta_1^3 + x_4^2 \otimes \zeta_1 \xi_2^2 + 1 \otimes \xi_3^2 \zeta_1 + \mathrm{Q}^8 x_7 + \mathrm{Q}^7 x_6 \otimes \zeta_1^2
\end{aligned}$$

$$\begin{aligned}&+ Q^5 x_4 \otimes \xi_2^2 + 1 \otimes (\xi_4 + \zeta_1 \xi_3^2) + x_6^2 \otimes \zeta_2 + x_4^2 \otimes (\xi_3 + \zeta_1 \xi_2^2)\\ &= (Q^8 x_7 + x_7^2 \otimes \zeta_1 + x_6^2 \otimes \xi_2 + x_4^2 \otimes \xi_3 + 1 \otimes \xi_4) + Q^7 x_6 \otimes \zeta_1^2 + Q^5 x_4 \otimes \xi_2^2,\end{aligned}$$

so the left $\mathcal{A}(1)_*$-coproduct

$$\psi' \colon H_*(Mj_2) \to \mathcal{A}(1)_* \otimes H_*(Mj_2)$$

has

$$\begin{aligned}\psi' Q^8 x_7 &= (Q^8 x_7 + \zeta_1 \otimes x_7^2 + \zeta_2 \otimes x_6^2 + \zeta_3 \otimes x_4^2 + \zeta_4 \otimes 1) + \zeta_1^2 \otimes Q^7 x_6 + \zeta_2^2 \otimes Q^5 x_4\\ &= (Q^8 x_7 + \zeta_1 \otimes x_7^2 + \zeta_2 \otimes x_6^2) + \zeta_1^2 \otimes Q^7 x_6.\end{aligned}$$

We also have

$$\begin{aligned}\psi' Q^9 x_6 &= 1 \otimes Q^9 x_6 + \zeta_1^2 \otimes Q^7 x_6 + \zeta_1^4 \otimes Q^7 x_4 + \zeta_2^2 \otimes Q^5 x_4\\ &= 1 \otimes Q^9 x_6 + \zeta_1^2 \otimes Q^7 x_6,\end{aligned}$$

so

$$\psi'(Q^8 x_7 + Q^9 x_6) = Q^8 x_7 + \zeta_1 \otimes x_7^2 + \zeta_2 \otimes x_6^2 \in \mathcal{A}(1)_* \otimes H_*(Mj_2).$$

Now, we define a sequence of elements $X_{2,s}$ $(s \geqslant 1)$ by

$$X_{2,s} = \begin{cases} x_4 & \text{if } s = 1,\\ x_6 & \text{if } s = 2,\\ x_7 & \text{if } s = 3,\\ Q^8 x_7 + Q^9 x_6 & \text{if } s = 4,\\ Q^{(2^{s-1},\dots,2^5,2^4)}(Q^8 x_7 + Q^9 x_6) = Q^{2^{s-1}} X_{2,s-1} & \text{if } s \geqslant 5.\end{cases}$$

An inductive calculation shows that for $s \geqslant 4$,

$$\psi' X_{2,s} = 1 \otimes X_{2,s} + \zeta_1 \otimes X_{2,s-1}^2 + \zeta_2 \otimes X_{2,s-2}^4 \in \mathcal{A}(1)_* \otimes I_2.$$

So this sequence is regular and generates an $\mathcal{A}(1)_*$-invariant ideal

$$I_2 = (X_{2,s} : s \geqslant 1) \lhd H_*(Mj_2).$$

The next result follows using similar arguments to those in the proof of Proposition 4.1 using the diagram (4.1).

Proposition 4.4 *There is an isomorphism of $\mathcal{A}_*$-comodule algebras*

$$H_*(Mj_2) \xrightarrow{\cong} \mathcal{A}_* \square_{\mathcal{A}(1)_*} H_*(Mj_2)/I_2.$$

The $\mathcal{E}_\infty$ morphism $Mj_2 \to k\mathrm{O}$ induces an algebra homomorphism $H_*(Mj_2) \to H_*(k\mathrm{O}) \subseteq \mathcal{A}_*$ under which

$$X_{2,1} \mapsto \zeta_1^4, \quad X_{2,2} \mapsto \zeta_2^2, \quad X_{2,s} \mapsto \zeta_s \quad (s \geqslant 3).$$

We have the following splitting result analogous to Proposition 4.3.

Proposition 4.5 *There is a splitting of $\mathcal{A}_*$-comodule algebras*

$$\begin{tikzcd}
\mathcal{A}_*\square_{\mathcal{A}(1)_*}\mathbb{F}_2 \arrow[rr, "\cong"] \arrow[dr] & & \mathcal{A}_*\square_{\mathcal{A}(1)_*}\mathbb{F}_2\ , \\
& H_*(Mj_2) \arrow[ur] &
\end{tikzcd}$$

where $H_*(Mj_2) \to H_*(k\mathrm{O}) = \mathcal{A}_*\square_{\mathcal{A}(1)_*}\mathbb{F}_2$ is induced by the $\mathcal{E}_\infty$ orientation $Mj_2 \to k\mathrm{O}$.

4.3 The homology of Mj_3

In $H_*(Mj_3)$, consider the regular sequence

$$X_{3,s} = \begin{cases} x_8 & \text{if } s = 1, \\ x_{12} & \text{if } s = 2, \\ x_{14} & \text{if } s = 3, \\ x_{15} & \text{if } s = 4, \\ \mathrm{Q}^{16}x_{15} + \mathrm{Q}^{17}x_{14} + \mathrm{Q}^{19}x_{12} & \text{if } s = 5, \\ \mathrm{Q}^{(2^{s-1},\ldots,2^6,2^5)}(\mathrm{Q}^{16}x_{15} + \mathrm{Q}^{17}x_{14} + \mathrm{Q}^{19}x_{12}) = \mathrm{Q}^{2^{s-1}}X_{3,s-1} & \text{if } s \geqslant 6. \end{cases}$$

We leave the reader to verify that the ideal

$$I_3 = (X_{3,s} \colon s \geqslant 1) \lhd H_*(Mj_3)$$

is $\mathcal{A}(2)_*$-invariant. The proof of the following result is similar to those of Propositions 4.1 and 4.4 using the diagram (4.1).

Proposition 4.6 *There is an isomorphism of $\mathcal{A}_*$-comodule algebras*

$$H_*(Mj_3) \xrightarrow{\cong} \mathcal{A}_*\square_{\mathcal{A}(2)_*}H_*(Mj_3)/I_3.$$

The $\mathcal{E}_\infty$ morphism $Mj_3 \to \mathrm{tmf}$ induces an algebra homomorphism $H_*(Mj_3) \to H_*(\mathrm{tmf}) \subseteq \mathcal{A}_*$ under which

$$X_{3,1} \mapsto \zeta_1^8, \quad X_{3,2} \mapsto \zeta_2^4, \quad X_{3,3} \mapsto \zeta_3^2, \quad X_{3,s} \mapsto \zeta_s \quad (s \geqslant 3).$$

We have the following splitting result analogous to Propositions 4.3 and 4.5.

Proposition 4.7 *There is a splitting of $\mathcal{A}_*$-comodule algebras*

$$\begin{tikzcd} \mathcal{A}_*\Box_{\mathcal{A}(2)_*}\mathbb{F}_2 \arrow[rr, "\cong"] \arrow[dr] & & \mathcal{A}_*\Box_{\mathcal{A}(2)_*}\mathbb{F}_2\ , \\ & H_*(Mj_3) \arrow[ur] & \end{tikzcd}$$

where $H_*(Mj_3) \to H_*(\mathrm{tmf}) = \mathcal{A}_*\Box_{\mathcal{A}(2)_*}\mathbb{F}_2$ is induced by the $\mathcal{E}_\infty$ orientation $Mj_3 \to$ tmf.

We end this discussion by recording the following result which was in part motivated by a result of Lawson and Naumann [23].

Theorem 4.8 *There is a morphism of $\mathcal{E}_\infty$ ring spectra $Mj_3 \to k\mathrm{O}$ which induces an epimorphism*

$$H_*(Mj_3) \twoheadrightarrow \mathbb{F}_2[\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \zeta_5, \ldots] \subseteq \mathbb{F}_2[\zeta_1^4, \zeta_2^2, \zeta_3, \zeta_4, \zeta_5, \ldots] \cong H_*(k\mathrm{O})$$

on $H_*(-)$ and an epimorphism $\pi_k(Mj_3) \to \pi_k(k\mathrm{O})$ for $k \neq 4$.

Proof We will use the fact that $Mj_3 \sim \widetilde{\mathbb{P}}\mathrm{tmf}^{[15]}$ and show the existence of a suitable $\mathcal{E}_\infty$ morphism $\widetilde{\mathbb{P}}\mathrm{tmf}^{[15]} \to k\mathrm{O}$.

We first require a map $\mathrm{tmf}^{[15]} \to k\mathrm{O}$ extending the unit map $S^0 \to k\mathrm{O}$. The existence of maps can be shown using classical obstruction theory, since the successive obstructions lie in the groups $H^8(\mathrm{tmf}^{[15]}; \pi_7(k\mathrm{O}))$, $H^{12}(\mathrm{tmf}^{[15]}; \pi_{11}(k\mathrm{O}))$, $H^{14}(\mathrm{tmf}^{[15]}; \pi_{13}(k\mathrm{O}))$ and $H^{15}(\mathrm{tmf}^{[15]}; \pi_{14}(k\mathrm{O}))$, all of which are trivial. For definiteness, choose such a map as $\theta\colon \mathrm{tmf}^{[15]} \to k\mathrm{O}$.

Let us examine the induced $\mathcal{A}_*$-comodule homomorphism $\theta_*\colon H_*(\mathrm{tmf}^{[15]}) \to H_*(k\mathrm{O}) \subseteq \mathcal{A}_*$. By Lemma 3.6, we have

$$\begin{aligned} \mathrm{Comod}_{\mathcal{A}_*}(H_*(\mathrm{tmf}^{[15]}), H_*(k\mathrm{O})) &\cong \mathrm{Comod}_{\mathcal{A}_*}(H_*(\mathrm{tmf}^{[15]}), \mathcal{A}_*\Box_{\mathcal{A}(1)_*}\mathbb{F}_2) \\ &\cong \mathrm{Comod}_{\mathcal{A}(1)_*}(H_*(\mathrm{tmf}^{[15]}), \mathbb{F}_2) \cong \mathbb{F}_2, \end{aligned}$$

so θ_* is a uniquely determined. Recall the formulae for the coaction on $H_*(\mathrm{tmf}^{[15]})$ given in (1.2a); we find that

$$\theta_*(x_8) = \zeta_1^8, \quad \theta_*(x_{12}) = \zeta_2^4, \quad \theta_*(x_{14}) = \zeta_3^2, \quad \theta_*(x_{15}) = \zeta_4.$$

There is a unique extension of θ to a morphism of $\mathcal{E}_\infty$ ring spectra $\widetilde{\theta}\colon \widetilde{\mathbb{P}}\mathrm{tmf}^{[15]} \to k\mathrm{O}$. The homology of $\widetilde{\mathbb{P}}\mathrm{tmf}^{[15]}$ is given in Theorem 2.3, and for $s \geqslant 5$

$$\widetilde{\theta}_*(X_{3,s}) = \mathrm{Q}^{(2^{s-1},\ldots,2^6,2^5)}(\theta_*(x_{15})) = \mathrm{Q}^{(2^{s-1},\ldots,2^6,2^5)}(\zeta_4) = \zeta_s.$$

It follows that

$$\operatorname{im}\widetilde{\theta}_* = \mathbb{F}_2[\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \zeta_5, \ldots] \cong H_*(\mathrm{tmf}).$$

To prove the result about homotopy groups, we show first that $\theta_*\colon \pi_k(\mathrm{tmf}^{[15]}) \to \pi_k(k\mathrm{O})$ is surjective when $k = 8, 9, 10, 12$. We will use some arguments about some Toda brackets in $\pi_*(\mathrm{tmf}^{[15]})$ and $\pi_*(k\mathrm{O})$; similar results were used in [12, section 7]. Given an S-module X, we can define Toda brackets of the form $\langle \alpha, \beta, \gamma\rangle \subseteq \pi_{a+b+c+1}(X)$, where $\alpha \in \pi_a(S)$, $\beta \in \pi_b(S)$ and $\gamma \in \pi_c(X)$ satisfy $\alpha\beta = 0$ in $\pi_{a+b}(S)$ and $\beta\gamma = 0$ in $\pi_{b+c}(X)$. The indeterminacy here is as usual

$$\mathrm{indet}\langle \alpha, \beta, \gamma\rangle = \alpha\pi_{b+c+1}(X) + \pi_{a+b+1}(S)\gamma \subseteq \pi_{a+b+c+1}(X).$$

The case $k = 8$ follows from the well-known facts that the Toda brackets $\langle 16, \sigma, 1\rangle \subseteq \pi_8(\mathrm{tmf})$ and $\langle 16, \sigma, 1\rangle \subseteq \pi_8(k\mathrm{O})$ contain generators $c_4' \in \pi_8(\mathrm{tmf}) \cong \pi_8(\mathrm{tmf}^{[15]})$ and $w \in \pi_8(k\mathrm{O})$, respectively. Naturality shows that $\theta_*\colon \pi_8(\mathrm{tmf}^{[15]}) \to \pi_8(k\mathrm{O})$ is surjective.

For the cases $k = 9, 10$, we can use multiplication by η and η^2 in $\pi_*(\mathrm{tmf})^{[15]}$ and $\pi_*(k\mathrm{O})$ to see that $\theta_*\colon \pi_k(\mathrm{tmf})^{[15]} \to \pi_k(k\mathrm{O})$ is surjective in these cases.

For $k = 12$, we need to know the classical result $\nu w = 0$ as well as $\nu c_4' = 0$; the latter can be read off of the Adams spectral sequence diagrams in [16, chapter 13]. Given these facts, it follows that the Toda brackets $\langle 8, \nu, c_4'\rangle \subseteq \pi_{12}(\mathrm{tmf}) \cong \pi_{12}(\mathrm{tmf}^{[15]})$ and $\langle 8, \nu, w\rangle \subseteq \pi_{12}(k\mathrm{O})$ contain generators and naturality shows that $\theta_*\colon \pi_{12}(\mathrm{tmf}^{[15]}) \to \pi_{12}(k\mathrm{O})$ is surjective.

To finish our argument, we know that when $k = 8, 9, 10, 12$, the composition

$$\begin{tikzcd} \pi_k(\mathrm{tmf}^{[15]}) \arrow[r] \arrow[rr, bend left, "\theta_*"] & \pi_k(\widetilde{\mathbb{P}}\mathrm{tmf}^{[15]}) \arrow[r, "\widetilde{\theta}_*"'] & \pi_k(k\mathrm{O}) \end{tikzcd}$$

is surjective. Using multiplication by the image of c_4' in $\pi_*(\widetilde{\mathbb{P}}\mathrm{tmf}^{[15]})$, it is straightforward to show that $\theta_*\colon \pi_k(\mathrm{tmf}^{[15]}) \to \pi_k(k\mathrm{O})$ is surjective for all $k > 4$. $\square$

In [23], Lawson and Naumann have shown the existence of an $\mathcal{E}_\infty$ map $\mathrm{tmf} \to k\mathrm{O}$ whose restriction to $\mathrm{tmf}^{[15]}$ could be used in the proof above. However, our argument does not assume the prior existence of such a map and seems more elementary. Indeed, our result suggests the possibility of a more direct approach to building an $\mathcal{E}_\infty$ morphism $\mathrm{tmf} \to k\mathrm{O}$ in comparison with the approach of Lawson and Naumann: it would suffice to show that the map $\mathcal{I} \to k\mathrm{O}$ from the homotopy fibre $\mathcal{I}$ of the $\mathcal{E}_\infty$ morphism $\widetilde{\mathbb{P}}\mathrm{tmf}^{[15]} \to k\mathrm{O}$ was null homotopic, so there is an $\mathcal{E}_\infty$ morphism $\mathrm{tmf} \to k\mathrm{O}$ making the following diagram homotopy commutative.

$$\begin{tikzcd} \mathcal{I} \arrow[d] \arrow[dr] & \\ \widetilde{\mathbb{P}}\mathrm{tmf}^{[15]} \arrow[r, "\widetilde{\theta}"'] \arrow[d] & k\mathrm{O}. \\ \mathrm{tmf} \arrow[ur, dotted, bend right] & \end{tikzcd}$$

To date, we have been unable to make this approach work.

5 Some other examples

Our approach to proving algebraic splittings of the homology of $\mathcal{E}_\infty$ Thom spectra can be used to rederive many known results for classical examples such as $M\mathrm{O}$, $M\mathrm{SO}$, $M\mathrm{SO}$, $M\mathrm{Spin}$, $M\mathrm{String} = M\mathrm{O}\langle 8\rangle$ and $M\mathrm{U}$. We can also obtain some other new examples with these methods.

5.1 An example related to $k\mathrm{U}$

Our first example is based on similar ideas to those used to construct the spectra Mj_r, but using $\mathrm{Spin}^{\mathrm{c}}$. The low-dimensional homology of $B\mathrm{Spin}^{\mathrm{c}}$ can be read off from Theorem 7.2 and Remark 7.3. Passing to the Thom spectrum over the 7-skeleton $(B\mathrm{Spin}^{\mathrm{c}})^{[7]}$, we have for its homology

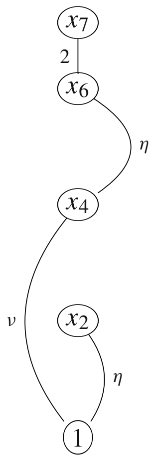
$$H_*((M\mathrm{Spin}^{\mathrm{c}})^{[7]}) = \mathbb{F}_2\{1, a_{1,0}^{(1)}, a_{1,1}^{(1)}, (a_{1,0}^{(1)})^2, a_{3,0}^{(1)}, a_{7,0}\}.$$

For our purposes, the fact that there are two 4-cells is problematic, so we instead restrict to a smaller complex. The map $B\mathrm{Spin}^{[7]} \to B\mathrm{Spin}^{c}$ induces an epimorphism in cohomology, and the resulting map $S^2 \vee B\mathrm{Spin}^{[7]} \to B\mathrm{Spin}^{c}$ induces a monomorphism in homology with image

$$\mathbb{F}_2\{1, a_{1,0}^{(1)}, a_{1,1}^{(1)}, a_{3,0}^{(1)}, a_{7,0}\}.$$

The Thom spectrum over this space has a cell structure of the form

$$(S^0 \cup_\eta e^2) \cup_\nu e^4 \cup_\eta e^6 \cup_2 e^7.$$



The skeletal inclusion factors through an infinite loop map

$$\begin{tikzcd} S^2 \vee B\mathrm{Spin}^{[7]} \arrow[rr] \arrow[dr] & & B\mathrm{Spin}^{c} \\ & \mathrm{Q}(S^2 \vee B\mathrm{Spin}^{[7]}) \arrow[ur, "j^{\mathrm{c}}"'] & \end{tikzcd}$$

and we obtain an $\mathcal{E}_\infty$ Thom spectrum Mj^{c} over $\mathrm{Q}(S^2 \vee B\mathrm{Spin}^{[7]})$ whose homology is

$$H_*(Mj^{\mathrm{c}}) = \mathbb{F}_2[\mathrm{Q}^{I_2}x_2, \mathrm{Q}^{I_4}x_4, \mathrm{Q}^{I_6}x_6, \mathrm{Q}^{I_7}x_7 \colon I_r \text{ admissible}, \operatorname{exc}(I_r) > r].$$

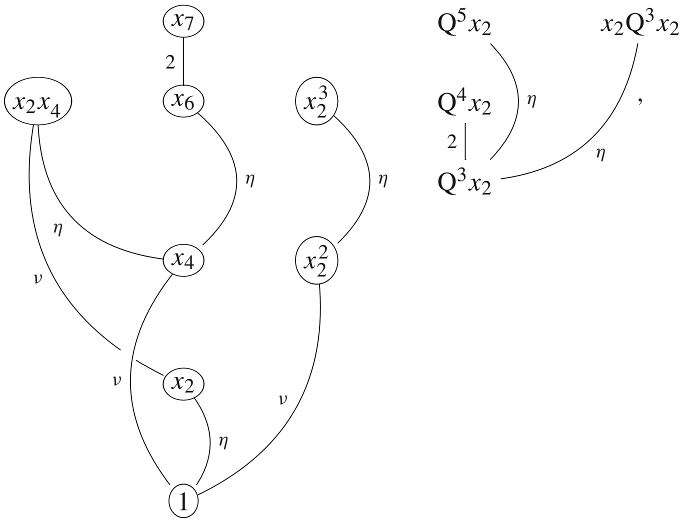
It is easy to see that there is a morphism of $\mathcal{E}_\infty$ ring spectra

$$\widetilde{\mathbb{P}}(S^0 \cup_\nu e^4 \cup_\eta e^6 \cup_2 e^7) \to k\mathrm{U}$$

inducing an epimorphism on $H_*(-)$ under which

$$x_2 \mapsto \zeta_1^2, \quad x_4 \mapsto \zeta_1^4, \quad x_6 \mapsto \zeta_2^2, \quad x_7 \mapsto \zeta_3.$$

The 7-skeleton of Mj^{c} has the form



since $\pi_3(C_\eta) \cong \pi_3(S^0)/\eta\pi_1(S^0) = \pi_3(S^0)/4\pi_3(S^0)$ and the generators are detected by Sq^4. It follows that there is an element $\pi_4(Mj^{\mathrm{c}})$ with Hurewicz image $x_4 + x_2^2$, and if $w\colon S^4 \to Mj^{\mathrm{c}}$ is a representative, we can form the $\mathcal{E}_\infty$ cone $Mj^{\mathrm{c}}//w$ as the pushout in the diagram

$$\begin{tikzcd}
\mathbb{P}S^4 \arrow[r, "w"] \arrow[d, "\widetilde{w}"'] \arrow[dr, phantom, "\ulcorner"] & \mathbb{P}D^5 \arrow[d] \\
Mj^{\mathrm{c}} \arrow[r] & Mj^{\mathrm{c}}//w
\end{tikzcd}$$

taken in the category $\mathscr{C}_S$ of commutative S-algebras. There is a Künneth spectral sequence of the form

$$\mathrm{E}^2_{s,t} = \mathrm{Tor}^{H_*(\mathbb{P}S^4)}_{s,t}(\mathbb{F}_2, H_*(Mj^{\mathrm{c}})) \Longrightarrow H_{s+t}(Mj^{\mathrm{c}}//w)$$

where the $H_*(Mj^{\mathrm{c}})$ is the $H_*(\mathbb{P}S^4)$-module algebra

$$H_*(\mathbb{P}S^4) = \mathbb{F}_2[\mathrm{Q}^I z_4\colon I \text{ admissible}, \operatorname{exc}(I) > 4] \to H_*(Mj^{\mathrm{c}});$$

where

$$\mathrm{Q}^I z_4 \mapsto \mathrm{Q}^I(x_2^2) + \mathrm{Q}^I x_4.$$

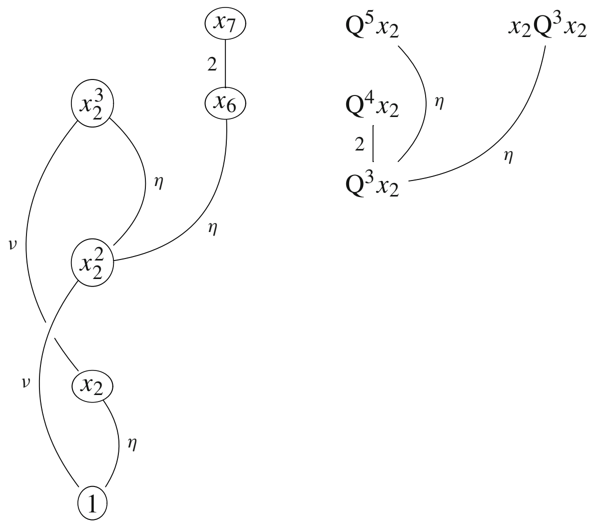
Notice that the term $\mathrm{Q}^I(x_2^2)$ is either trivial (if at least one term in I is odd) or a square (if all terms in I are even), hence can be used as a polynomial generator of $H_*(Mj^{\mathrm{c}})$ in place of $\mathrm{Q}^I x_4$. It follows that $H_*(Mj^{\mathrm{c}})$ is a free $H_*(\mathbb{P}S^4)$-module, so the spectral sequence is trivial with

$$\begin{aligned}
\mathrm{E}^2_{*,*} &= \mathrm{Tor}^{H_*(\mathbb{P}S^4)}_{0,*}(\mathbb{F}_2, H_*(Mj^{\mathrm{c}})) \\
&= H_*(Mj^{\mathrm{c}})/(\mathrm{Q}^I(x_2^2) + \mathrm{Q}^I x_4\colon I \text{ admissible}, \operatorname{exc}(I) > 4),
\end{aligned}$$

therefore we have

$$H_*(Mj^{\mathrm{c}}//w) = \mathbb{F}_2[\mathrm{Q}^{I_2}x_2, \mathrm{Q}^{I_6}x_6, \mathrm{Q}^{I_7}x_7\colon I_r \text{ admissible}, \operatorname{exc}(I_r) > r]. \tag{5.1}$$

Here is the 7-skeleton of $Mj^{\mathrm{c}}//w$.



We define a sequence of elements X_s in $H_*(Mj^{\mathrm{c}}//w)$ by

$$X_s = \begin{cases} x_2 & \text{if } s = 1, \\ x_6 & \text{if } s = 2, \\ x_7 & \text{if } s = 3, \\ \mathrm{Q}^{(2^{s-1},\ldots,2^4,2^3)}x_7 = \mathrm{Q}^{2^{s-1}}X_{s-1} & \text{if } s \geqslant 4. \end{cases}$$

This is a regular sequence and the induced coaction over the quotient Hopf algebra

$$\mathcal{E}(1)_* = \mathcal{A}_*/(\zeta_1^2, \zeta_2^2, \zeta_3, \ldots) = \mathcal{A}_*//\mathbb{F}_2[\zeta_1^2, \zeta_2^2, \zeta_3, \ldots] = \Lambda(\zeta_1, \zeta_2)$$

satisfies

$$\psi' X_s = \begin{cases} 1 \otimes X_1 & \text{if } s = 1, 2, \\ 1 \otimes X_3 + \zeta_1 \otimes X_2 + \zeta_2 \otimes X_1^2 & \text{if } s = 3, \\ 1 \otimes X_s + \zeta_1 \otimes X_{s-1} + \zeta_2 \otimes X_{s-2} & \text{if } s \geqslant 4. \end{cases}$$

therefore the ideal $I^{\mathrm{c}} = (X_s : s \geqslant 1) \lhd H_*(Mj^{\mathrm{c}}//w)$ is an $\mathcal{E}(1)_*$-invariant regular ideal.

Recall that

$$\mathcal{A}_* \square_{\mathcal{E}(1)_*} \mathbb{F}_2 = \mathbb{F}_2[\zeta_1^2, \zeta_2^2, \zeta_3, \ldots] \cong H_*(k\mathrm{U}).$$

We have proved the following analogues of earlier results.

Proposition 5.1 *There is an isomorphism of $\mathcal{A}_*$-comodule algebras*

$$H_*(Mj^{\mathrm{c}}//w) \xrightarrow{\cong} \mathcal{A}_* \square_{\mathcal{E}(1)_*} H_*(Mj^{\mathrm{c}}//w)/I^{\mathrm{c}}.$$

Proposition 5.2 *There is a splitting of $\mathcal{A}_*$-comodule algebras*

$$\begin{tikzcd}
\mathcal{A}_*\square_{\mathcal{E}(1)_*}\mathbb{F}_2 \arrow[rr, "\cong"] \arrow[dr] & & \mathcal{A}_*\square_{\mathcal{E}(1)_*}\mathbb{F}_2, \\
& H_*(Mj^{\mathrm{c}}/\!/w) \arrow[ur] &
\end{tikzcd}$$

where $H_*(Mj^{\mathrm{c}}/\!/w) \to H_*(k\mathrm{U}) = \mathcal{A}_*\square_{\mathcal{E}(1)_*}\mathbb{F}_2$ is induced by a factorisation $Mj^{\mathrm{c}} \to Mj^{\mathrm{c}}/\!/w \to k\mathrm{U}$ of the $\mathcal{E}_\infty$ orientation.

Of course, in principle use of the well-known lightning flash technology of [1,2] should lead to a description of $H_*(Mj^{\mathrm{c}}/\!/w)/I^{\mathrm{c}}$ as an $\mathcal{E}(1)_*$-comodule. For example, there are many infinite lightning flashes such as the following

$$\begin{tikzcd}
& \mathrm{Q}^4x_2 \arrow[dl, "\mathrm{q}^0_*"'] & & \mathrm{Q}^6x_2 \arrow[dlll, two heads, "\mathrm{q}^1_*"'] \arrow[dl] & & \mathrm{Q}^8x_2 \arrow[dlll, two heads] \arrow[dl] & & \cdots \arrow[dlll, dotted, two heads] \\
\mathrm{Q}^3x_2 & & \mathrm{Q}^5x_2 & & \mathrm{Q}^7x_2 & & &
\end{tikzcd},$$

as well as parallelograms such as

$$\begin{tikzcd}
& & \mathrm{Q}^8\mathrm{Q}^4x_2 \arrow[dl, two heads] \arrow[dd] \\
& \mathrm{Q}^6\mathrm{Q}^3x_2 \arrow[ddl, "\mathrm{q}^0_*"'] & \\
& & \mathrm{Q}^7\mathrm{Q}^4x_2, \arrow[dll, two heads, "\mathrm{q}^1_*"'] \\
\mathrm{Q}^5\mathrm{Q}^3x_2 & &
\end{tikzcd}$$

which can be determined by using [9, proposition 7.3].

5.2 An example related to the Brown–Peterson spectrum

From [8, section 4] we recall the 2-local $\mathcal{E}_\infty$ ring spectrum R_∞ for which there is a map of commutative ring spectra $R_\infty \to BP$ inducing a rational equivalence, an epimorphism $\pi_*(R_\infty) \to \pi_*(BP)$, and $H_*(R_\infty)$ contains a regular sequence $z_s \in H_{2^{2+1}-2}(R_\infty)$ mapping to the generators $t_s \in H_{2^{2+1}-2}(BP)$ which in turn map to $\zeta_s^2 \in H_{2^{2+1}-2}(H) = \mathcal{A}_{2^{2+1}-2}$ under the induced ring homomorphisms

$$H_*(R_\infty) \to H_*(BP) \to H_*(H) = \mathcal{A}_*.$$

We note that both of these homomorphisms are compatible with the Dyer–Lashof operations, even though BP is not known to be an $\mathcal{E}_\infty$ ring spectrum. These elements z_s have the following coactions:

$$\psi(z_r) = 1 \otimes z_r + \zeta_1^2 \otimes z_{r-1}^2 + \zeta_2^2 \otimes z_{r-2}^4 + \cdots + \zeta_{r-1}^2 \otimes z_1^{2^{r-1}} + \zeta_r^2 \otimes 1,$$

and generate an ideal $I_\infty \lhd H_*(R_\infty)$.

Let

$$\mathcal{E}_* = \mathcal{A}_*/(\zeta_i^2 : i \geqslant 1),$$

the exterior quotient Hopf algebra. Although it $\mathcal{E}_*$ is not finite dimensional, it is still true that $\mathcal{A}_*$ is an extended right $\mathcal{E}_*$-comodule,

$$\mathcal{A}_* \cong (\mathcal{A}_* \square_{\mathcal{E}_*} \mathbb{F}_2) \otimes \mathcal{E}_*.$$

Under the induced $\mathcal{E}_*$-coaction on $H_*(R_\infty)$, I_∞ is an $\mathcal{E}_*$-comodule ideal, therefore $H_*(R_\infty)/I_\infty$ is an $\mathcal{E}_*$-comodule algebra.

Proposition 5.3 *There is an isomorphism of commutative $\mathcal{A}_*$-comodule algebras*

$$H_*(R_\infty) \xrightarrow{\cong} \mathcal{A}_* \square_{\mathcal{E}_*} H_*(R_\infty)/I_\infty,$$

and a splitting $\mathcal{A}_*$-comodule algebras

$$\begin{tikzcd} \mathcal{A}_* \square_{\mathcal{E}_*} \mathbb{F}_2 \arrow[rr, "\cong"] \arrow[dr] & & \mathcal{A}_* \square_{\mathcal{E}_*} \mathbb{F}_2, \\ & H_*(R_\infty) \arrow[ur] & \end{tikzcd}$$

where $\mathcal{A}_* \square_{\mathcal{E}_*} \mathbb{F}_2 \cong H_*(BP)$ and the right hand homomorphism is induced from the morphism of commutative ring spectra $R_\infty \to BP$.

This result supports the view that R_∞ admits a map $BP \to R_\infty$ extending the unit $S^0 \to R_\infty$ and then the composition

$$BP \to R_\infty \to BP$$

would necessarily be a weak equivalence since BP is minimal atomic in the sense of [12].

6 Speculation and conjectures

Our algebraic splittings of $H_*(Mj_r)$ are consistent with spectrum-level splittings. Indeed, in the case of $r = 1$, a result of Mark Steinberger [14] already shows that

Mj_1 splits as a wedge of suspensions of $H\mathbb{Z}$ and $H\mathbb{Z}/2^s$ for $s \geqslant 1$, all of which are $H\mathbb{Z}$-module spectra. In fact a direct argument is also possible.

Using Lemma 3.2, it is easy to see that if a spectrum X is a module spectrum over one of $H\mathbb{Z}$, $k\mathrm{O}$ or tmf then its homology is a retract of the extended comodule $\mathcal{A}_*\square_{\mathcal{A}(r)_*}H_*(X)$ for the relevant value of r; a similar observation holds for a module spectrum over $k\mathrm{U}$ and $\mathcal{A}_*\square_{\mathcal{E}(1)_*}H_*(X)$. Thus our algebraic results provide evidence for the following conjectural splittings.

Conjecture 6.1 *As a spectrum, Mj_2 is a wedge of $k\mathrm{O}$-module spectra, Mj_3 is a wedge of* tmf*-module spectra and Mj^{c} is a wedge of $k\mathrm{U}$-module spectra.*

Here the phrase 'module spectrum' can be interpreted either purely homotopically, or strictly in the sense of [17]. In each case, it is enough to produce any map $E \to Mj$ extending the unit (up to homotopy), for then the $\mathcal{E}_\infty$ structure on Mj gives rise to a homotopy commutative diagram of the following form.

$$\begin{tikzcd}
S^0 \wedge Mj \arrow[r] \arrow[dr] \arrow[rrrr, bend left, "\sim"] & Mj \wedge Mj \arrow[r] & \widetilde{\mathbb{P}}Mj \wedge Mj \arrow[r] & Mj \wedge Mj \arrow[r] & Mj. \\
& E \wedge Mj \arrow[r] \arrow[u] & \widetilde{\mathbb{P}}E \wedge Mj \arrow[u] & &
\end{tikzcd}$$

Related to this conjecture, and indeed implied by it, is the following where we know that analogues hold for the cases Mj_1, Mj_2, Mj^{c}, i.e. the natural homomorphisms

$$\pi_*(Mj_1) \to \pi_*(H\mathbb{Z}), \quad \pi_*(Mj_2) \to \pi_*(k\mathrm{O}), \quad \pi_*(Mj^{\mathrm{c}}) \to \pi_*(k\mathrm{U})$$

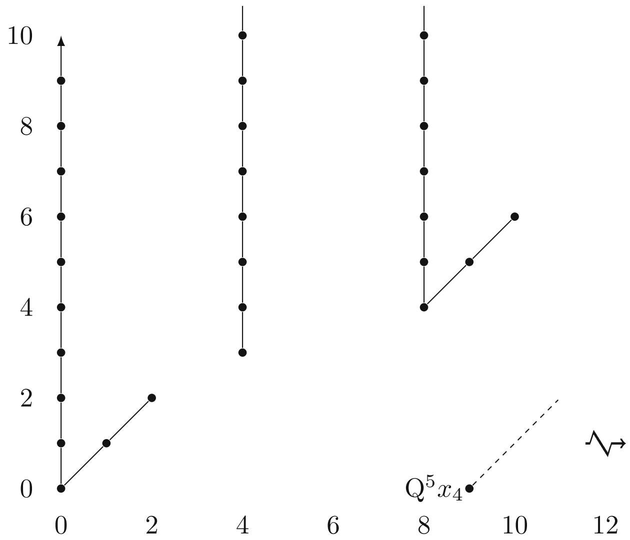
are epimorphisms. One approach to verifying these is by using the Adams spectral sequence: in each of the first two cases, the lowest degree element in the E_2-term not associated with the $\mathcal{A}_*\square_{\mathcal{A}(r-1)_*}\mathbb{F}_2$ summand is one of the elements Q^3x_2 or Q^5x_4, and this is too far along to give elements supporting anomalous differentials on this summand, and the multiplicative structure completes the argument. Here is a small portion of the Adams spectral sequence for Mj_2 to illustrate this, with Q^5x_4 at position $(9, 0)$ and most of the diagram being part of the E_2-term for $k\mathrm{O}$. Since

$$\psi\mathrm{Q}^6x_4 = \zeta_1 \otimes \mathrm{Q}^5x_4 + 1 \otimes \mathrm{Q}^6x_4,$$

this element Q^5x_4 does not produce an h_0 tower; in fact, the $\mathcal{A}(1)_*$-subcomodule

$$\mathbb{F}_2\{\mathrm{Q}^5x_4, \mathrm{Q}^6x_4\} \subseteq H_*(Mj_2)/I_2$$

gives rise to a copy of the Adams E_2-term for $k\mathrm{O} \wedge (S^0 \cup_2 e^1)$ carried on Q^5x_4.



In the third case, the first element not in the kU summand is $Q^3 x_2$ and a similar argument applies.

Conjecture 6.2 *The $\mathcal{E}_\infty$ orientation $Mj_3 \to$ tmf induces a ring epimorphism $\pi_*(Mj_3) \to \pi_*(\mathrm{tmf})$.*

This is easily seen to be true up to degree 16 and also holds rationally. To go further seems to require a detailed examination of the Adams spectral sequences for $\pi_*(Mj_3)$ and $\pi_*(\mathrm{tmf})$, and to date we have checked it up to degree 26. Of course, this conjecture is implied by the above splitting conjecture.

To understand how the splitting question might be resolved, let us examine the settled case of Mj_1. This provides a universal example for the general splitting result of Steinberger [14, theorem III.4.2], and the general case is implied by that of Mj_1. Since

$$H_*(Mj_1) \cong \mathcal{A}_* \square_{\mathcal{A}(0)_*} H_*(Mj_1)/I_1,$$

we have

$$\mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathbb{F}_2, H_*(Mj_1)) \cong \mathrm{Ext}^{*,*}_{\mathcal{A}(0)_*}(\mathbb{F}_2, H_*(Mj_1)/I_1).$$

Following the strategy of Steinberger's proof for the general case, we consider the $\mathcal{A}(0)_*$-comodule structure of $H_*(Mj_1)/I_1$, or equivalently its $\mathcal{A}(0)^*$-module structure. Of course, here there is only one copy of $H\mathbb{Z}$, and the remaining summands are suspensions of $H\mathbb{Z}/2^r$ for various r.

The Bockstein spectral sequence for $H_*(Mj_1; \mathbb{Z}_{(2)})$ can be determined from this using formulae for higher Bocksteins of [27, proposition 6.8], which we learnt about from Rolf Hoyer and Peter May.

Let E be a connective finite type 2-local $\mathcal{E}_\infty$ ring spectrum and let $x \in H_{2m}(E)$ where $m \in \mathbb{Z}$. Writing β_k for the kth higher Bockstein operation, and assuming that $\beta_{k-1}x$ is defined, we have

$$\beta_k(x^2) = \begin{cases} x\beta_1 x + \mathrm{Q}^{2m}(\beta_1 x) & \text{if } k = 2, \\ x\beta_{k-1}x & \text{if } k > 2. \end{cases} \tag{6.1}$$

These formulae determine higher differentials in the Bockstein spectral sequence for $H_*(E;\mathbb{Z}_{(2)})$. The first differential $\beta_1 = \mathrm{Sq}^1_*$ is given on polynomial generators by

$$\beta_1\mathrm{Q}^I x_2 = \begin{cases} \mathrm{Q}^{(i_1-1,i_2,\ldots,i_k)}x_2 & \text{if } k > 0 \text{ and } I = (i_1, i_2, \ldots, i_k) \text{ with } i_1 \text{ even}, \\ 0 & \text{otherwise}, \end{cases} \tag{6.2}$$

$$\beta_1\mathrm{Q}^I x_3 = \begin{cases} x_2 & \text{if } I = () \text{ is the empty sequence}, \\ \mathrm{Q}^{(i_1-1,i_2,\ldots,i_k)}x_3 & \text{if } k > 0 \text{ and } I = (i_1, i_2, \ldots, i_k) \text{ with } i_1 \text{ even}, \\ 0 & \text{otherwise}. \end{cases} \tag{6.3}$$

In each of the cases with i_1 even, $\mathrm{Q}^{(i_1-1,i_2,\ldots,i_k)}x_s$ is a polynomial generator except when $i_1 = i_2 + \cdots + i_k + s + 1$ and then

$$\beta_1\mathrm{Q}^I x_s = (\mathrm{Q}^{(i_2,\ldots,i_k)}x_s)^2.$$

As a dga with respect to β_1, $H_*(Mj_1)$ is a tensor product of acyclic subcomplexes of the form $\mathbb{F}_2[\beta_1\mathrm{Q}^I x_s, \mathrm{Q}^I x_s]$ where $s = 2, 3$ and $I = (i_1, \ldots, i_k) \neq ()$ with i_1 even, together with $\mathbb{F}_2[x_2, x_3]$ and the polynomial ring generated by the squares not already accounted for. In particular, the E^2-term of the Bockstein spectral sequence agrees with the β_1-homology of $H_*(Mj_1)/I_1$. The higher Bocksteins now follow from the above formulae (6.2) and (6.3).

This approach might be generalised to the cases of Mj_2, Mj_3 and Mj^{c} by studying suitable Bockstein spectral sequences for $k\mathrm{O}_*(Mj_2)$, $\mathrm{tmf}_*(Mj_3)$ and $k\mathrm{U}_*(Mj^{\mathrm{c}})$. We remark that the $\mathcal{E}_\infty$ ring spectra $H\mathbb{Z}\wedge Mj_1$, $k\mathrm{O}\wedge Mj_2$ and $\mathrm{tmf}\wedge Mj_3$ can be identified in different guises using the Thom diagonals associated with the $\mathcal{E}_\infty$ orientations $Mj_1 \to H\mathbb{Z}$, $MJ_2 \to k\mathrm{O}$ and $Mj_3 \to \mathrm{tmf}$, giving weak equivalences of $\mathcal{E}_\infty$ ring spectra

$$H\mathbb{Z}\wedge Mj_1 \xrightarrow{\sim} H\mathbb{Z}\wedge \Sigma^\infty_+ \mathrm{Q}(B\mathrm{SO}^{[3]}),$$
$$k\mathrm{O}\wedge Mj_2 \xrightarrow{\sim} k\mathrm{O}\wedge \Sigma^\infty_+ \mathrm{Q}(B\mathrm{Spin}^{[7]}),$$
$$\mathrm{tmf}\wedge Mj_3 \xrightarrow{\sim} \mathrm{tmf}\wedge \Sigma^\infty_+ \mathrm{Q}(B\mathrm{String}^{[15]}),$$

and there are isomorphisms of $\mathcal{A}_*$-comodule algebras

$$H_*(H\mathbb{Z}\wedge Mj_1) \xrightarrow{\cong} \mathcal{A}_* \square_{\mathcal{A}(0)_*} H_*(\mathrm{Q}(B\mathrm{SO}^{[3]})),$$

$$H_*(k\mathrm{O} \wedge Mj_2) \xrightarrow{\cong} \mathcal{A}_* \square_{\mathcal{A}(1)_*} H_*(\mathrm{Q}(B\mathrm{Spin}^{[7]})),$$

$$H_*(\mathrm{tmf} \wedge Mj_3) \xrightarrow{\cong} \mathcal{A}_* \square_{\mathcal{A}(2)_*} H_*(\mathrm{Q}(B\mathrm{String}^{[15]})).$$

The referee has raised the question of whether the approach of Subsection 5.1 can be used to produce an $\mathcal{E}_\infty$ Thom spectrum related to $\mathrm{tmf}_1(3)$ as Mj^{c} is related to $k\mathrm{U}$. We recall from [23] that there is a commutative diagram of 2-local $\mathcal{E}_\infty$ ring spectra

$$\begin{tikzcd}
\mathrm{tmf} \arrow[r] \arrow[d] & k\mathrm{O} \arrow[d] \\
\mathrm{tmf}_1(3) \arrow[r] & k\mathrm{U}.
\end{tikzcd}$$

On applying $H_*(-)$, this induces the following diagram of $\mathcal{A}_*$-comodule subalgebras of $\mathcal{A}_*$.

$$\begin{tikzcd}
\mathcal{A}_* \square_{\mathcal{A}(2)_*} \mathbb{F}_2 \arrow[r, equal] & \mathbb{F}_2[\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \ldots] \arrow[r, hook] \arrow[d, hook] & \mathbb{F}_2[\zeta_1^4, \zeta_2^2, \zeta_3, \ldots] \arrow[r, equal] \arrow[d, hook] & \mathcal{A}_* \square_{\mathcal{A}(1)_*} \mathbb{F}_2 \\
\mathcal{A}_* \square_{\mathcal{E}(2)_*} \mathbb{F}_2 \arrow[r, equal] & \mathbb{F}_2[\zeta_1^2, \zeta_2^2, \zeta_3^2, \zeta_4, \ldots] \arrow[r, hook] & \mathbb{F}_2[\zeta_1^2, \zeta_2^2, \zeta_3, \ldots] \arrow[r, equal] & \mathcal{A}_* \square_{\mathcal{E}(1)_*} \mathbb{F}_2.
\end{tikzcd}$$

We propose using the space

$$S^2 \vee B\mathrm{Spin}^{[6]} \vee B\mathrm{O}\langle 8\rangle^{[15]},$$

which admits a map to $B\mathrm{Spin}^{\mathrm{c}}$ that restricts to a map inducing an epimorphism in cohomology on each wedge summand. Extending this to an infinite loop map

$$j\colon \mathrm{Q}(S^2 \vee B\mathrm{Spin}^{[6]} \vee B\mathrm{O}\langle 8\rangle^{[15]}) \to B\mathrm{Spin}^{\mathrm{c}} \to B\mathrm{SO},$$

we obtain an $\mathcal{E}_\infty$ Thom spectrum Mj.

Conjecture 6.3 *There is an $\mathcal{E}_\infty$ morphism $Mj \to \mathrm{tmf}_1(3)$ which factors through an $\mathcal{E}_\infty$ 3-cell complex $Mj//w_4, w_8, w_{12}$ with $\mathcal{E}_\infty$ cells of dimensions 5, 9 and 13 attached by maps w_4, w_8, w_{12}. Moreover, the morphism $Mj//w_4, w_8, w_{12} \to \mathrm{tmf}_1(3)$ induces an epimorphism on $H_*(-)$ which is an isomorphism up to degree 15.*

We have not yet checked all the details, but it seems plausible that the approach used for Mj^{c} offers a route to doing this. Of course, we might then expect a splitting of $Mj//w_4, w_8, w_{12}$ into $\mathrm{tmf}_1(3)$-module spectra, or at least that the map $Mj//w_4, w_8, w_{12} \to \mathrm{tmf}_1(3)$ induces an epimorphism on $\pi_*(-)$.

Appendix 1: On the homology of connective covers of BO

We review the structure of the homology Hopf algebras $H_*(BO\langle n\rangle;\mathbb{F}_2)$ for $n = 1, 2, 4, 8$. The dual cohomology rings were originally determined by Stong, but later a body of literature by Bahri, Kochman, Pengelley as well as the present author evolved describing these homology rings. We will use the Husemoller–Witt decompositions of [6] to give explicit algebra generators; the actions of Steenrod and Dyer–Lashof operations on these can be determined using the work of Kochman and Lance [21,22].

We recall that there are polynomial generators $a_{k,s} \in H_{2^s k}(BO)$ (k odd, $s \geqslant 0$) such that

$$\mathrm{B}[k]_* = \mathbb{F}_2[a_{k,s} : s \geqslant 0] \subseteq H_*(BO)$$

is a polynomial sub-Hopf algebra and there is a decomposition of Hopf algebras

$$H_*(BO) = \bigotimes_{k \text{ odd}} \mathrm{B}[k]_*.$$

For each odd k, there is an isomorphism of Hopf algebras

$$\mathrm{B}[k]^*/(a_{k,0}) \cong \mathrm{Hom}(\mathrm{B}^{(1)}[k]_*, \mathbb{F}_2).$$

Here, the dual Hopf algebra $\mathrm{B}[k]^* = \mathrm{Hom}(\mathrm{B}[k]_*, \mathbb{F}_2)$ is isomorphic to $\mathrm{B}[k]_*$, i.e. these are self-dual Hopf algebras. There is also a decomposition of Hopf algebras

$$H^*(BO) = \bigotimes_{k \text{ odd}} \mathrm{B}[k]^*.$$

For each $h \geqslant 1$, there is a monomorphism of Hopf algebras which multiplies degrees by 2^h,

$$\mathrm{B}[k]_* \to \mathrm{B}[k]_*; \quad x \mapsto x^{(h)} = x^{2^h},$$

whose image is denoted by $\mathrm{B}^{(h)}[k]_*$. Notice that the primitives in $\mathrm{B}^{(h)}[k]_*$ are the powers

$$(a_{k,0}^{(h)})^{2^s} = (a_{k,0})^{2^{s+h}} \quad (s \geqslant 0).$$

Dually, there is an epimorphism of Hopf algebras

$$\mathrm{B}[k]^* \to \mathrm{B}[k]^*; \quad a_{k,s} \mapsto \begin{cases} a_{k,s-h} & \text{if } s \geqslant h, \\ 0 & \text{if } s < h, \end{cases}$$

and this induces an isomorphism of Hopf algebras

$$\mathrm{B}[k]^*/(a_{k,0}, a_{k,1}, \dots, a_{k,h-1}) \cong \mathrm{B}[k]^*$$

which divides degrees by 2^h. The dual Hopf algebra of $\mathrm{B}^{(h)}[k]_*$ is

$$\mathrm{B}^{(h)}[k]^* \cong \mathrm{B}[k]^*/(a_{k,0}, a_{k,1}, \ldots, a_{k,h-1}).$$

Let $\alpha = \alpha_2$ denote the dyadic number function which counts the number of non-zero coefficients in the binary expansion of a natural number.

Theorem 7.1 *The natural infinite loop maps $B\mathrm{O}\langle n\rangle \to B\mathrm{O}\langle 1\rangle = B\mathrm{O}$ ($n = 2, 4, 8$) induce monomorphisms of Hopf algebras $H_*(B\mathrm{O}\langle n\rangle) \to H_*(B\mathrm{O})$ whose images are the following sub-Hopf algebras of $H_*(B\mathrm{O})$:*

$$\begin{aligned}
&\mathrm{B}^{(1)}[1]_* \otimes \bigotimes_{\text{odd } k>1} \mathrm{B}[k]_* && \text{if } n = 2,\\
&\mathrm{B}^{(2)}[1]_* \otimes \bigotimes_{\substack{k \text{ odd}\\ \alpha(k)=2}} \mathrm{B}^{(1)}[k]_* \otimes \bigotimes_{\substack{k \text{ odd}\\ \alpha(k)>2}} \mathrm{B}[k]_* && \text{if } n = 4,\\
&\mathrm{B}^{(3)}[1]_* \otimes \bigotimes_{\substack{k \text{ odd}\\ \alpha(k)=2}} \mathrm{B}^{(2)}[k]_* \otimes \bigotimes_{\substack{k \text{ odd}\\ \alpha(k)=3}} \mathrm{B}^{(1)}[k]_* \otimes \bigotimes_{\substack{k \text{ odd}\\ \alpha(k)>3}} \mathrm{B}[k]_* && \text{if } n = 8.
\end{aligned}$$

By dualising and using the above observations, we obtain Hopf algebra decompositions of the cohomology of these spaces. For example,

$$\begin{aligned}
H^*(B\mathrm{SO}) = H^*(B\mathrm{O}\langle 2\rangle) &= \mathrm{B}^{(1)}[1]^* \otimes \bigotimes_{\text{odd } k>1} \mathrm{B}[k]^*\\
&= \mathrm{B}[1]^*/(a_{1,0}) \otimes \bigotimes_{\text{odd } k>1} \mathrm{B}[k]^*.
\end{aligned}$$

We may identify $H_*(M\mathrm{O}\langle n\rangle)$ with $H_*(B\mathrm{O}\langle n\rangle)$ using the Thom isomorphism which is an isomorphism of algebras over the Dyer–Lashof algebra, but not over the Steenrod algebra. To avoid excessive notation, we will often treat the Thom isomorphism as an equality and write $a_{k,s}^{(r)}$ for each of the corresponding elements.

The generators $a_{2^s-1,0}$ are particularly interesting. In $H_*(B\mathrm{O})$, $a_{2^s-1,0}$ is primitive, and in $H_*(M\mathrm{O})$ there is a simple formula for the $\mathcal{A}_*$-coaction:

$$\psi(a_{2^s-1,0}) = 1 \otimes a_{2^s-1,0} + \zeta_1 \otimes a_{2^{s-1}-1,0}^2 + \zeta_2 \otimes a_{2^{s-2}-1,0}^4 + \cdots + \zeta_{s-1} \otimes a_{1,0}^{2^{s-1}} + \zeta_s \otimes 1. \tag{7.1}$$

The natural orientation $M\mathrm{O} \to H\mathbb{F}_2$ induces an algebra homomorphism over both of the Dyer–Lashof and Steenrod algebras under which

$$a_{2^s-1,0} \mapsto \zeta_s. \tag{7.2}$$

For completeness, we also describe the homology of $B\mathrm{Spin}^{\mathrm{c}}$ in similar algebraic form to that of Theorem 7.1, since we are not aware of this being documented anywhere else; note that [35, p. 293] contains an apparently incorrect statement on the mod 2 cohomology, while [19] describes the cohomology of $B\mathrm{Spin}^{\mathrm{c}}(n)$.

Theorem 7.2 *The natural infinite loop map $B\mathrm{Spin}^{\mathrm{c}} \to BO$ induces a monomorphism of Hopf algebras $H_*(B\mathrm{Spin}^{\mathrm{c}}) \to H_*(BO)$ with image*

$$\bigotimes_{\substack{k \text{ odd} \\ \alpha(k)\leqslant 2}} \mathrm{B}^{(1)}[k]_* \otimes \bigotimes_{\substack{k \text{ odd} \\ \alpha(k)>2}} \mathrm{B}[k]_*.$$

Sketch of proof The cohomology ring $H^*(B\mathrm{Spin}^{\mathrm{c}})$ can be calculated using the Serre spectral sequence

$$\mathrm{E}_2^{r,s} = H^r(B\mathrm{SO}; H^s(K(\mathbb{Z},2))) \Longrightarrow H^{r+s}(B\mathrm{Spin}^{\mathrm{c}})$$

for the fibration sequence

$$K(\mathbb{Z},2) \to B\mathrm{Spin}^{\mathrm{c}} \to B\mathrm{SO}.$$

Then,

$$\mathrm{E}_2^{*,*} = \mathbb{F}_2[w_k : k \geqslant 2] \otimes \mathbb{F}_2[x],$$

where $w_k \in H^k(B\mathrm{SO})$ is the image of the k-th Stiefel-Whitney class, while $x \in H^2(K(\mathbb{Z},2))$ and $x^{2^t} \in H^{2^{t+1}}(K(\mathbb{Z},2))$ transgresses to

$$d_{2^{t+1}+1}(x^{2^t}) = w_{2^{t+1}+1} \quad \text{(mod decomposables)}.$$

As $d_{2^{t+1}+1}(x^{2^t})$ has to be a primitive, it must agree with the element $a_{2^{t+1}+1,0}$. It follows that the natural map $B\mathrm{Spin}^{\mathrm{c}} \to BO$ induces an epimorphism $H^*(BO) \to H^*(B\mathrm{Spin}^{\mathrm{c}})$, while dually $H_*(B\mathrm{Spin}^{\mathrm{c}}) \to H_*(BO)$ is a monomorphism. Also, $H^*(B\mathrm{Spin}^{\mathrm{c}})$ is polynomial with one generator in each degree k where either $\alpha(k) > 2$ or k is even with $\alpha(k) \leqslant 2$. Indeed, there is an isomorphism of Hopf algebras

$$H^*(B\mathrm{Spin}^{\mathrm{c}}) \cong \bigotimes_{\substack{k \text{ odd} \\ \alpha(k)\leqslant 2}} \mathrm{B}[k]^*/(a_{k,0}) \otimes \bigotimes_{\substack{k \text{ odd} \\ \alpha(k)>2}} \mathrm{B}[k]^*.$$

The claimed description of the homology $H_*(B\mathrm{Spin}^{\mathrm{c}})$ follows. □

Remark 7.3 The natural map $B\mathrm{Spin} \to B\mathrm{Spin}^{\mathrm{c}}$ induces a homomorphism in homology whose image contains $(a_{1,0}^{(1)})^2$, $a_{3,0}^{(1)}$ and $a_{7,0}$.

Appendix 2: Dyer–Lashof operations and Steenrod coactions

For the convenience of the reader, we summarise some results from [9] which are based on the work of Kochman and Steinberger [14,21].

The mod 2 Steenrod algebra $\mathcal{A}_*$ is the homology of the mod 2 Eilenberg–Mac Lane spectrum $H = H\mathbb{F}_2$ which is an $\mathcal{E}_\infty$ ring spectrum and so $\mathcal{A}_*$ supports an action of the Dyer–Lashof operations. However, when dealing with the left $\mathcal{A}_*$-coaction on the homology of an $\mathcal{E}_\infty$ ring spectrum, it is often convenient to consider a twisted version formed using the antipode χ and given by

$$\widetilde{\mathrm{Q}}^s = \chi \mathrm{Q}^s \chi.$$

Based on Steinberger's determination of the usual action [14], by [9, lemma 4.4] we have the following equivalent formulae for all $s \geqslant 1$:

$$\mathrm{Q}^{2^s} \xi_s = \xi_{s+1} + \xi_1 \xi_s^2, \tag{8.1a}$$

$$\widetilde{\mathrm{Q}}^{2^s} \zeta_s = \zeta_{s+1} + \zeta_1 \zeta_s^2. \tag{8.1b}$$

The spectra $H\mathbb{Z}$, $k\mathrm{O}$ and tmf are all $\mathcal{E}_\infty$ ring spectra and there are $\mathcal{E}_\infty$ morphisms $H\mathbb{Z} \to H\mathbb{F}_2$, $k\mathrm{O} \to H\mathbb{F}_2$ and tmf $\to H\mathbb{F}_2$ inducing monomorphisms on $H_*(-)$ identifying their homology with the subalgebras

$$\mathbb{F}_2[\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \zeta_5, \ldots] \subseteq \mathbb{F}_2[\zeta_1^4, \zeta_2^2, \zeta_3, \zeta_4, \ldots] \subseteq \mathbb{F}_2[\zeta_1^2, \zeta_2, \zeta_3, \ldots] \subseteq \mathcal{A}_*.$$

It follows that each of these subalgebras is closed under the Dyer–Lashof operations. More generally, from the work of Stong [34], each of the $\mathcal{E}_\infty$ morphisms $M\mathrm{O}\langle 2^d \rangle \to H\mathbb{F}_2$ induces a ring homomorphism whose image is $\mathbb{F}_2[\zeta_1^{2^d}, \zeta_2^{2^{d-1}}, \ldots, \zeta_d^2, \zeta_{d+1}, \zeta_{d+2}, \ldots]$ and this must be closed under the Dyer–Lashof operations.

We will give a purely algebraic generalisation of these observations.

For $n \geqslant 0$, let

$$\mathcal{I}(n) = (\zeta_1^{2^{n+1}}, \zeta_2^{2^n}, \zeta_3^{2^{n-1}}, \ldots, \zeta_n^4, \zeta_{n+1}^2, \zeta_{n+2}, \zeta_{n+3}, \ldots) \lhd \mathcal{A}_*.$$

This is a Hopf ideal and $\mathcal{A}(n)_* = \mathcal{A}_*/\mathcal{I}(n)$ is a well-known finite quotient Hopf algebra. We also set

$$\mathcal{I}(n)^{[d]} = \{\alpha^{2^d} : \alpha \in \mathcal{I}(n)\} \lhd \mathcal{A}_*,$$

and observe that

$$\mathcal{I}(n)^{[d+1]} \subseteq \mathcal{I}(n+1)^{[d]} \subseteq \mathcal{I}(n+d). \tag{8.2}$$

Lemma 8.1 *Let $s \geqslant 1$. If $k \in \mathbb{N}$, then $\mathrm{Q}^k \zeta_s \in \mathcal{I}(s-1)$; more generally, for $r \geqslant 0$, $\mathrm{Q}^k(\zeta_s^{2^r}) \in \mathcal{I}(s+r-1)$.*

Proof We make use of the results of [9, section 5].

The proof is by induction on s. When $s = 1$, for $k \geqslant 1$, write $k = 2m$ or $k = 2m+1$. Then,

$$\mathrm{Q}^{2m} \zeta_1 = \mathrm{N}_{2m+1}(\xi) = \xi_1 \mathrm{N}_m(\xi)^2 + \xi_2 \mathrm{N}_{m-1}(\xi)^2 + \xi_3 \mathrm{N}_{m-3}(\xi)^2 + \cdots \in \mathcal{I}(0),$$

and

$$\begin{aligned}
Q^{2m+1}\zeta_1 &= N_{2m+2}(\xi) = N_{m+1}(\xi)^2 \\
&= \xi_1^2 N_m(\xi)^4 + \xi_2^2 N_{m-2}(\xi)^4 + \xi_3^2 N_{m-6}(\xi)^2 + \cdots \in \mathcal{I}(0).
\end{aligned}$$

Now, suppose that the result holds for all $s < n$. Recall that for $k \geqslant 2^n - 1$, $Q^k\zeta_n = 0$ unless $k \equiv 0 mod 2^n$ or $k \equiv 2^n - 1 mod 2^n$ when

$$\begin{aligned}
Q^{2^n m}\zeta_n &= N_{2^n m+2^n-1}(\xi) \\
&= \xi_1 N_{2^{n-1}m+2^{n-1}-1}(\xi)^2 + \xi_2 N_{2^{n-2}m+2^{n-2}-1}(\xi)^4 \\
&\quad + \xi_3 N_{2^{n-3}m+2^{n-3}-1}(\xi)^8 + \cdots \\
&= \xi_1(Q^{2^{n-1}m}\xi_{n-1})^2 + \xi_2(Q^{2^{n-2}m}\xi_{n-2})^4 + \xi_3(Q^{2^{n-3}m}\xi_{n-3})^8 + \cdots \\
&\in \mathcal{I}(n-2)^{[1]} + \mathcal{I}(n-3)^{[2]} + \cdots \quad \subseteq \mathcal{I}(n-1),
\end{aligned}$$

and similarly $Q^{2^n m+2^n-1}\zeta_n \in \mathcal{I}(n-1)$.

For $r \geqslant 0$, $Q^k(\zeta_s^{2^r}) = 0$ unless $2^r | k$, and then by (8.2),

$$Q^{2^r\ell}(\zeta_s^{2^r}) = (Q^\ell\zeta_s)^{2^r} \in \mathcal{I}(n-1)^{[r]} \subseteq \mathcal{I}(n+r-1).$$

$\square$

Corollary 8.2 *For $n \geqslant 0$, the cotensor product $\mathcal{A}_*\square_{\mathcal{A}(n)_*}\mathbb{F}_2 \subseteq \mathcal{A}_*$ is closed under the Dyer–Lashof operations, and the Dyer–Lashof operations commute with the Hopf algebra quotient homomorphism $\mathcal{A}_* \to \mathcal{A}(n)_*$.*

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