Direct, stigmatic, imaging with curved surfaces

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We study the possibilities of direct (using one intersection with each light ray) stigmatic imaging with a curved surface that can change ray directions in an arbitrary way. By purely geometric arguments we show that the only possible case of such imaging is the trivial one where the image of any point is identical to the point itself and the surface does not perform any change of the ray direction at all. We also discuss an example of a curved surface which performs indirect stigmatic imaging after twice intersecting each light ray.

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1. Introduction

Euclidean transformation optics is a way of designing optical instruments by considering a volume of (standard) flat space ("electromagnetic space"), imagining a distortion of this volume, and then designing spatially-varying material parameters in the volume which distort any light ray passing through the volume in the same way. (In non-Euclidean transformation optics [1], electromagnetic space can be curved.) The distortion is what actually happens, which is why the distorted space is called "physical space", but to light (and to any observer who looks at the volume) it looks like there is no distortion, that is as if the instrument was not there. The positions of any intersections of light rays within the volume in electromagnetic space are mapped to different intersection positions in physical space, and in this sense the instrument stigmatically (perfectly sharp in terms of geometrical optics) images any point in electromagnetic space to a corresponding position in physical space. After passage through the instrument, light rays travel along the straight-line continuation of their initial trajectory, which implies that transmission through the instrument maps any object position to the identical image position.

We are interested in stigmatic imaging by thin surfaces, by which we mean that any transmitted light ray leaves the surface from the same position where it intersected the surface, but on the other side. Imaging with surfaces is of particular interest to transformation-optics devices when the transformation is piecewise homogeneous, like in the cloak described in Ref. [2], as any interface between piecewise homogeneous media then must stigmatically image the surrounding space. Our study assumes that general light-ray-direction mappings (i.e. completely generalised laws of refraction) can be realised. Note that wave-optical limitations mean that this is not actually the case, but by introducing discontinuities into transmitted phase fronts it is possible to realise very good approximations to wave-optically forbidden light-ray-direction mappings [3]. For us, this topic is interesting as it is at the intersection of two of our areas of research: surfaces that perform generalised refraction, and perfect imaging in general and transformation optics in particular.

Generalised refraction has recently gained prominence with a metamaterial surface that changes the direction of transmitted light rays according to a generalised law of refraction [4]. It is also possible to achieve generalised refraction using transformation optics [5]. Our own interest is in pixellated refraction: the "pixels" are pieces of the transmitted beam that are transformed independently; between the pixels, the beam is riddled with discontinuities that develop into optical vortices [6, 7]. This compromises beam quality, but it does allow us to perform refraction so general that it transforms many incident light-ray fields into apparently (but not actually) wave-optically forbidden light-ray fields [3].

We previously studied stigmatic imaging with planar inhomogeneous surfaces [8] and planar homogeneous surfaces [9]. Here we investigate a natural generalisation of this work: stigmatic imaging with curved surfaces. Our argument is purely geometric, and therefore more fundamental than previous arguments about imaging that use optical path length (e.g. [10, 11]) in that it does not

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assume the existence of a continuous wave field that corresponds to the ray field. Our results therefore apply not only to the ray fields corresponding to continuous light waves, but also to those corresponding to the discontinuous light waves resulting from wave-optically forbidden light-ray-direction mappings.

2. Statement of the problem

Let there be a surface σ. The question is, under what conditions can the surface perform stigmatic imaging of some 3D region R? To answer this question, we will assume that the surface σ performs stigmatic imaging, and analyse what the consequences of this assumption are.

We formulate our argument for the case when R is an open set, that is a region of space that contains an open ball (the inside of a sphere of non-zero radius) around each of its points. R is therefore the inside of some 3D volume, and does not include the points on the volume's boundary. The assumption that R is open excludes certain types of regions such as those containing isolated points or lines; for such regions our argument does not necessarily apply. For regions R which are open, and to which our argument applies, it is physically irrelevant whether or not we include the boundary as any boundary point is arbitrarily close to at least one point in R.

Our argument is concerned only with the straight lines on which the light rays in object and image space travel immediately before and after intersecting the surface, respectively. If we consider all object-space straight lines through σ that intersect a point A in R, the object, then the corresponding image-space straight lines all intersect at another point, A′, which is the image of A. Note that A′ exists according to our assumption that σ performs stigmatic imaging. This assumption implies that an image position exists for any object point in R, and we will continue to denote such images by the name of the object point, primed. We restrict ourselves to direct imaging, i.e. we consider the effect of light rays intersecting σ only once. We will show that if the surface is a plane then there can in principle exist infinitely many different mappings between object and image positions. On the other hand, if the surface is curved, only the trivial mapping is possible by which each point is imaged to itself, and the surface either transmits without direction-change or retro-reflects all light rays.

Let us start with three non-collinear points A, B, C in the region R, positioned such that the straight lines AB, AC and BC intersect the surface σ. If we denote these intersection points by $\mathrm{P_{AB}}$, $\mathrm{P_{AC}}$ and $\mathrm{P_{BC}}$, respectively, then clearly all the points A, B, C, $\mathrm{P_{AB}}$, $\mathrm{P_{AC}}$ and $\mathrm{P_{BC}}$ lie in one plane. If any of the straight lines, for example the straight line through A and B, intersects with σ more than once, then $\mathrm{P_{AB}}$ can be any one of the intersection points. The intersection points of the straight lines A′B′, A′C′ and B′C′ with the surface σ are again $\mathrm{P_{AB}}$, $\mathrm{P_{AC}}$ and $\mathrm{P_{BC}}$, respectively, because each ray leaves σ from the same point where it intersected it. By the same argument then all the points A′, B′, C′, $\mathrm{P_{AB}}$, $\mathrm{P_{AC}}$ and $\mathrm{P_{BC}}$ lie in one plane.

Note that we do not make assumptions about whether object and image positions are real or virtual, i.e. whether it is the actual light rays that would intersect or their straight-line continuations. It is therefore not important for our argument if the light-ray direction is reversed, and so it does not matter from which side of the surface the object rays hit the surface and from which side the image rays leave — the surface can be transmissive or reflective, and it can even have unusual mathematical properties such as having only one side (like a Möbius strip).

3. Planar surface

First suppose that the surface σ is a plane. This case has already been examined in Ref. [8]; we review it here in a very concise form and as a way of introducing the type of argument we will use to proof the stigmatic imaging properties of curved surfaces.

We will show that the imaging transformation that assigns every point in region R its image is uniquely determined by just two points, A and B, and their images A′ and B′ (that must satisfy the condition that both the straight lines AB and A′B′ intersect σ at the same point $\mathrm{P_{AB}}$). We will first construct the image C′ of a point C that is non-collinear with A and B, but otherwise arbitrary. To do that, we find the intersection $\mathrm{P_{AC}}$ and $\mathrm{P_{BC}}$ of the straight lines AC and BC, respectively, with σ. The image C′ of C will then be at the intersection of the straight lines $\mathrm{P_{AC}A'}$ and $\mathrm{P_{BC}B'}$. For this to be the case, all the points A′, B′, $\mathrm{P_{AC}}$ and $\mathrm{P_{BC}}$ must lie in one plane, which they indeed do: it is the plane containing the straight lines $\mathrm{P_{AB}P_{AC}P_{BC}}$ and $\mathrm{P_{AB}A'B'}$ (Fig. 1(a)). Note that this plane is not necessarily perpendicular to σ. Note also that a different pair of object-image positions, e.g. A and C and the corresponding images A′ and C′, allows us to construct the image of points that are collinear with A and B.

We can also prove that the mapping is consistent. In other words, if we want to determine the position of the image of a fourth point, D, which lies outside the plane ABC, by the above construction, then employing the points A, B, A′ and B′ for this yields the same position of the point D′ as employing other known pairs of object and image positions, for example A, C, A′ and C′. Performing analogous steps to those above, we find that the point D′ must lie at the intersection of the straight lines $\mathrm{A'P_{AD}}$ and $\mathrm{B'P_{BD}}$, as well as on the intersection of the lines $\mathrm{A'P_{AD}}$ and $\mathrm{C'P_{CD}}$. What ensures that all the three straight lines $\mathrm{A'P_{AD}}$, $\mathrm{B'P_{BD}}$ and $\mathrm{C'P_{CD}}$ intersect at a single point? It is the fact that each pair of these straight lines lies in some plane by the same argument we used above for point C, and as there are three possible pairs of these lines, they lie in three planes. The intersection point of these three planes is then also the intersection point of these three straight lines: this is the unique position of the point D′. This completes the proof of consistency of the stigmatic imaging by the plane.

In Ref. [8] the mapping between object coordinates (x, y, z) and image coordinates (x', y', z') due to a surface

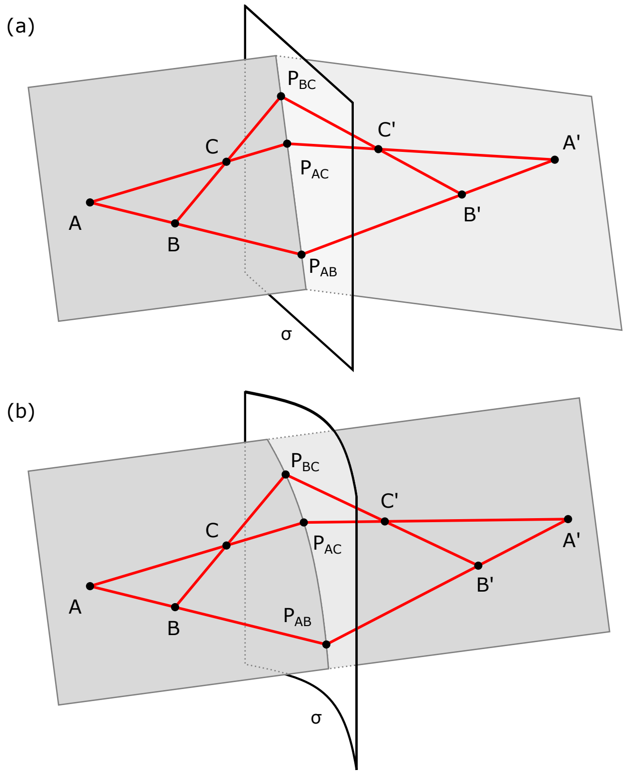

Fig. 1. Light-ray trajectories through pairs of object positions and through the corresponding image positions formed by a surface σ. A, B and C are the object positions, A′, B′ and C′ are the corresponding image positions. $\mathrm{P_{AB}}$ is the position where the light ray through A and B intersects the surface, after which it continues to pass through A′ and B′ (similarly for $\mathrm{P_{AC}}$ and $\mathrm{P_{BC}}$). (a) For a planar surface, the points $\mathrm{P_{AB}}$, $\mathrm{P_{AC}}$ and $\mathrm{P_{BC}}$ lie on the straight line where the plane containing the object positions and the plane containing the corresponding image positions intersect. (Both planes are shaded.) (b) For a curved surface, the points $\mathrm{P_{AB}}$, $\mathrm{P_{AC}}$ and $\mathrm{P_{BC}}$ do not in general lie on a straight line. As they have to lie on the line where the plane containing the object positions intersects the plane containing the corresponding image positions, those planes have to be identical.

in the z plane was found to be [12]

$$\frac{x'}{x} = \frac{f}{f+z}, \quad \frac{y'}{y} = \frac{f}{f+z}, \quad \frac{f}{-z} + \frac{g}{z'} = 1, \qquad (1)$$

where f and g are parameters of the surface. This is the mapping due to an ideal thin lens in which the object- and image-sided focal lengths, f and g, are in general different. Note that the required surface is, in general, inhomogeneous.

The special case of a homogeneous surface was treated in [9]. If such a homogeneous surface is the $z = 0$ plane, then the mapping is

$$x' = x - z\,t_x, \quad y' = y - z\,t_y, \quad z' = \eta\,z, \qquad (2)$$

where t_x, t_y and η are parameters that can be freely chosen.

Of course, the explicit relationship between object and image coordinates means that any light ray that reaches a point on the surface from an object position must be redirected such that it subsequently passes through the corresponding image position. This then fixes the generalised law of refraction at the surface point. In Ref. [9], this was shown to be the light-ray-direction change performed by a subset of a class of micro-structured surfaces called generalised confocal lenslet arrays (gCLAs) [13]. But the homogeneous surface from Ref. [13] was reached as the limit of an arbitrary point on the inhomogeneous surface from Ref. [8] stretched out over the entire surface, which implies that the inhomogeneous planar surfaces from Ref. [8] must also refract according to the light-ray-direction change in gCLAs.

4. Curved surface

Suppose now that the surface σ is curved. We again choose three non-collinear points A, B, and C in R such that the straight lines through each pair of the points A, B and C intersect σ, i.e. such that corresponding points $\mathrm{P_{AB}}$, $\mathrm{P_{AC}}$, $\mathrm{P_{BC}}$ exist. The points $\mathrm{P_{AB}}$, $\mathrm{P_{AC}}$, $\mathrm{P_{BC}}$ will lie on the intersection of the plane ABC with σ. In the case of a planar surface σ discussed above, this intersection is a straight line, so $\mathrm{P_{AB}}$, $\mathrm{P_{AC}}$, $\mathrm{P_{BC}}$ are collinear. However, now σ is curved, so it is in general no longer true that the points $\mathrm{P_{AB}}$, $\mathrm{P_{AC}}$, $\mathrm{P_{BC}}$ are collinear (Fig. 1(b)). Instead they lie in a uniquely defined plane in which the points A′, B′ and C′ must also lie [14], but as this plane also contains the object positions A, B and C, this means that the points A, B, C, A′, B′ and C′ all lie in the same plane.

Before we proceed with the argument, we must mention that it might be possible to choose A, B and C such that the points $\mathrm{P_{AB}}$, $\mathrm{P_{AC}}$, $\mathrm{P_{BC}}$ are collinear even for a curved surface σ. Clearly, this can happen for surfaces that contain planar regions and for surfaces comprising straight lines, such as ruled surfaces; more generally it can be the case for any surface σ that intersects with at least one straight line in three or more points. However, thanks to the fact that the points A, B, C can be positioned anywhere within a spherical subregion of R — one of the open balls surrounding a point —, each of the straight lines through pairs of these points can point in any direction, and therefore intersect any point on σ. This, in turn, means that we can position A, B and C such that the points $\mathrm{P_{AB}}$, $\mathrm{P_{AC}}$, $\mathrm{P_{BC}}$ are not collinear, as the opposite would necessarily imply that σ is planar, which would be a contradiction with our assumption.

Now we consider four points, A, B, C and D, in R. D can be an any arbitrary point in R; we choose the points A, B and C such that they are non-collinear, that the plane ABC does not contain D, and that all straight lines through any pair of the points A, B, C and D intersect σ (at positions $\mathrm{P_{AB}}$, $\mathrm{P_{AC}}$, $\mathrm{P_{BC}}$, $\mathrm{P_{AD}}$, $\mathrm{P_{BD}}$ and $\mathrm{P_{CD}}$). Because we can position the points A, B, C anywhere in a spherical subregion of R, namely the open ball around D, we can ensure that *all* triples of the intersection points $\mathrm{P_{AB}}$, $\mathrm{P_{AC}}$, $\mathrm{P_{BC}}$, $\mathrm{P_{AD}}$, $\mathrm{P_{BD}}$ and $\mathrm{P_{CD}}$ are non-collinear.

We have shown above that the images A′, B′ and C′ lie in the same plane as A, B and C themselves, and we can now repeat this argument for each of the triples of points {A,B,D}, {A,C,D} and {B,C,D}. We see that each of the triples $\{P_{AB}, P_{AD}, P_{BD}\}$, $\{P_{AC}, P_{AD}, P_{CD}\}$, and $\{P_{BC}, P_{BD}, P_{CD}\}$ is non-collinear and thus spans a unique plane. It then follows by the above argument that the image positions A′, B′, and D′ lie in the plane ABD, A′, C′, and D′ lie in the plane ACD, and B′, C′, and D′ lie in the plane BCD. Specifically, the point D′ lies in all three planes ABD, ACD and BCD, i.e. at their intersection. This intersection is unique because A, B, C are non-collinear and we have chosen D to lie outside the plane ABC. The point D also lies at the intersection of these planes, and so we come to the conclusion that the points D′ and D coincide. As the point D can be any point in R, this shows that the curved surface σ can image R only trivially, imaging each point in R to itself. This implies that σ either transmits any light ray from R without altering its direction, or that it retro-reflects such light rays — either way, the light ray continues along the same straight line.

5. Indirect imaging

The argument we used above does not apply to indirect imaging, i.e. imaging due to multiple transmissions through a surface. Such indirect imaging is not the main focus of this paper; nevertheless, we briefly observe the following.

Fig. 2(b) shows a diagram of a spherical surface that changes the direction of a light ray by 90° in the plane of incidence (which includes the ray and the surface normal), so the light-ray-direction change is according to the generalised law of refraction

$$\theta' = \begin{cases} \theta - 90^\circ \text{ if } \theta > 0, \\ \theta + 90^\circ \text{ if } \theta < 0, \end{cases} \tag{3}$$

where θ is the angle of incidence and θ' is the angle of refraction. For reasons that will become clear, we call a surface that refracts according to Eqn (3) an *Eaton-lens surface*. Our argument about direct imaging with curved surfaces shows that there can be no images after a single passage through the surface. However, after two passages through the surface, such a sphere would look identical to an Eaton lens [16], which images each point P to a point P′ on the opposite side of the lens centre [17] (Fig. 2(c)). We have therefore discovered an example of a curved, homogeneous, surface, that performs perfect stigmatic imaging after two intersections.

6. Conclusions

In conclusion, we have found that non-trivial direct, stigmatic, imaging is restricted to planar surfaces. Our result is relevant to a number of areas of research, including generalised refraction and transformation optics.

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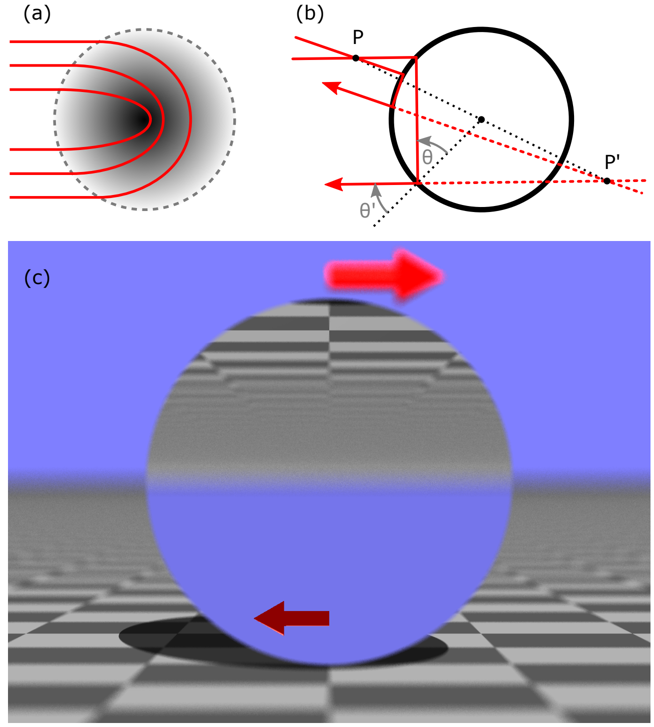

Fig. 2. Eaton lens and spherical Eaton-lens surface. (a) An Eaton lens is a sphere (dashed line) filled with a spherically symmetric refractive-index distribution that bends light-ray trajectories (solid lines) as shown. The light rays emerge with a direction opposite to that with which they entered. (b) A spherical Eaton-lens surface (thick solid circle) refracts light rays such that light rays exit the surface from the same position and with the same direction as if they had passed through an Eaton lens. From outside, the surface looks identical to an Eaton lens. (c) Ray-tracing simulation of imaging with a spherical Eaton-lens surface. The object is an arrow pointing to the right, visible at the top of the image. The virtual camera is focussed to the distance where the surface is expected to produce the image of the arrow, and indeed the image, an arrow pointing to the left which is visible close to the bottom of the sphere, can be seen sharply (while the object and the sphere are blurred). The simulation was performed with the scientific raytracer Dr TIM [15].

References

[1] U. Leonhardt and T. Tyc, "Broadband Invisibility by Non-Euclidean Cloaking," Science **323**, 110–112 (2009).

[2] H. Chen and B. Zheng, "Broadband polygonal invisibility cloak for visible light," Scientific Reports **2**, 255 (2012).

[3] J. Courtial and T. Tyc, "Generalised laws of refraction that can lead to wave-optically forbidden light-ray fields," J. Opt. Soc. Am. A **29**, 1407–1411 (2012).

[4] N. Yu, P. Genevet, M. A. Kats, F. Aieta, J.-P. Tetienne, F. Capasso, and Z. Gaburro, "Light propagation with

phase discontinuities: Generalized laws of reflection and refraction," Science **334**, 333–337 (2011).

[5] Y. Xu, K. Yao, and H. Chen, "Generalized laws of reflection and refraction from transformation optics," Europhys. Lett. **99**, 44002 (2012).

[6] A. C. Hamilton and J. Courtial, "Metamaterials for light rays: ray optics without wave-optical analog in the ray-optics limit," New J. Phys. **11**, 013042 (2009).

[7] J. Courtial and T. Tyc, "METATOYs and optical vortices," J. Opt. **13**, 115704 (2011).

[8] J. Courtial, "Geometric limits to geometric optical imaging with infinite, planar, non-absorbing sheets," Opt. Commun. **282**, 2480–2483 (2009).

[9] S. Oxburgh and J. Courtial, "Perfect imaging with planar interfaces," J. Opt. Soc. Am. A **30**, 2334–2338 (2013).

[10] B. van Brunt, "Mathematical possibility of certain systems in geometrical optics," J. Opt. Soc. Am. A **11**, 2905–2914 (1994).

[11] F. Duerr, P. Benítez, J. C. Minano, Y. Meuret, and H. Thienpont, "Analytic design method for optimal imaging: coupling three ray sets using two free-form lens profiles," Opt. Express **20**, 5576–5585 (2012).

[12] Following Ref. [9], we have not written these equations in terms of object and image distances, but in terms of a common z coordinate.

[13] A. C. Hamilton and J. Courtial, "Generalized refraction using lenslet arrays," J. Opt. A: Pure Appl. Opt. **11**, 065502 (2009).

[14] Unless all three image positions coincide, which is only possible if the surface performs many-to-one direction mappings, a case we do not consider here.

[15] S. Oxburgh, T. Tyc, and J. Courtial, "Dr TIM: Ray-tracer TIM, with additional specialist capabilities," Comp. Phys. Commun. **185**, 1027–1037 (2014).

[16] J. E. Eaton, "On spherically symmetric lenses," Trans. IRE Antennas Propag. **4**, 66–71 (1952).

[17] T. Tyc, L. Herzánová, M. Šarbort, and K. Bering, "Absolute instruments and perfect imaging in geometrical optics," New J. Phys. **13**, 115004 (2011).