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Flow of fluids with pressure- and shear-dependent viscosity down an inclined plane

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In this paper we consider a fluid whose viscosity depends on both the mean normal stress and the shear rate flowing down an inclined plane. Such flows have relevance to geophysical flows. In order to make the problem amenable to analysis, we consider a generalization of the lubrication approximation for the flows of such fluids based on the development of the generalization of the Reynolds equation for such flows. This allows us to obtain analytical solutions to the problem of propagation of waves in a fluid flowing down an inclined plane. We find that the dependence of the viscosity on the pressure can increase the breaking time by an order of magnitude or more than that for the classical Newtonian fluid. In the viscous regime, we find both upslope and downslope travelling wave solutions, and these solutions are quantitatively and qualitatively different from the classical Newtonian solutions.

Key words: lubrication theory, shallow water flows, wave breaking

1. Introduction

The flow of fluids down an inclined plane has relevance to common occurrences such as the flow of rain water down the roofs and window panes of structures, to geophysical flows such as that of lava and that of rivers and seas due to the inclination of their beds, and to numerous industrial applications. Thus, it is not surprising that such flows and their stability have been studied in great detail, going back to the early studies by Nusselt (1916) and Jeffreys (1925) of the flow along a vertical wall. These pioneering studies have been followed by both theoretical and experimental works too numerous to cite, and thus we are content to mention some of the early works (Kirkbride 1934*a*,*b*; Keulegan & Patterson 1940; Friedman & Miller 1941; Grimley 1945; Dukler & Berkelin 1952; Yih 1955, 1963, 1965; Benjamin 1957; Binnie 1957; Ivanilov 1962; Gupta & Rai 1967) and some recent works that have a bearing on our study (Huppert 1982*a*,*b*; Silvi & Dussan 1985; Kondic & Diez 2001; Perazzo & Gratton 2003). Most of the studies concerning the flow down an inclined plane consider the fluid to be the classical Newtonian fluid. Recently, there have been a

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few studies wherein the fluid in question is a non-Newtonian power-law fluid (see for example Perazzo & Gratton 2003). An interesting observation concerning flows down inclined planes is the formation of waves and the breaking time associated with the waves. If the angle of inclination is small, it is found that the flow is stable to small disturbances, while if the angle of inclination is sufficiently large one observes the development of 'roll waves', wherein the flow is made up of waves interspersed with regions of laminar flow or 'slug flow', where the flow tends to be more chaotic (see Mayer 1959). There have been numerous studies concerning the stability of flows down an inclined plane and references to such studies can be found in the paper by Kondic & Diez (2001). There have also been several rigorous mathematical studies concerning the flows of fluids down an inclined plane (see for example Dressler 1949; Carasso & Shen 1977).

It has been well known ever since the seminal paper of Stokes (1845) that the viscosity of fluids depends on the mean normal stress. While the pressure dependence of the viscosity may be ignored in simple channel and pipe flows, when it comes to flows such as those encountered in elastohydrodynamics, or for that matter in flows wherein the range of pressures is large, one cannot ignore the effect of pressure on the viscosity. The manner in which the viscosity varies with pressure has been recorded in numerous experimental studies. The authoritative book by Bridgman (1931) documents most of the works until 1931 that attest to the fact that the viscosity can change by as much as 10^8 %! A paper by Barus (1893) proposed an exponential dependence of the viscosity on pressure. Later, the paper by Andrade (1934) suggested an explicit formula for how the viscosity changes with the density, temperature and pressure. However, the change in the density over a very large pressure range is of the order of 3–5 % (see Dowson & Higginson 1966; Rajagopal 2006) while the change in the viscosity is of the order of 10^8 %. Thus, it would be reasonable to consider most organic fluids as incompressible fluids with pressure-dependent viscosity. (An up-to-date list of the relevant experimental papers can be found in Bulicek, Malek & Rajagopal 2009.) Rajagopal and co-workers have studied issues concerning existence and uniqueness of flows as well as special flows of fluids with pressure-dependent viscosity. Of specific relevance is the paper by Rajagopal & Szeri (2003), who obtain the appropriate lubrication approximation, in the spirit of the early works by Reynolds, for the flows of such fluids, previous approximations being incorrect.

The mean normal stress in an incompressible nonlinear fluid need not be the 'pressure' in the fluid, if by 'pressure' we mean the reaction stress due to the constraint of incompressibility. It is worth noting that the popular theory of constraints that appeals to the notion, which can be traced back to the work of D'Alembert, Bernoulli and Lagrange, that constraint forces do no work does not allow the viscosity to depend on the Lagrange multiplier that enforces the constraint of incompressibility, namely the pressure. Such an assumption is not a necessity, as shown by Gauss (1829) in particle mechanics and Rajagopal & Srinivasa (2005) for continua. See also the discussion in Rajagopal & Saccomandi (2006). In this paper we consider a non-Newtonian fluid whose viscosity depends on both the mean normal stress (in the case of the model being considered, the pressure) and the shear rate (the fluid can shear-thin or shear-thicken). In a recent paper, Saccomandi & Vergori (2010), using the lubrication approximation appropriate to such fluids as developed by Rajagopal & Szeri (2003), studied in great detail the flow down an inclined plane of an incompressible fluid whose viscosity depends on pressure. They considered various flow regimes, namely flows wherein viscous effects, surface tension effects, etc, are predominant. They showed that in quasi-steady flow, due to the dependence of the viscosity on the pressure, the breaking time of the waves is delayed in comparison to the classical Newtonian fluid. In the viscous regime, in which the effect of the pressure gradient is balanced by the stresses due to the viscosity within the bulk, they found that if the fluid viscosity is affected by the pressure changes, then the travelling waves could be both qualitatively and quantitatively different from those occurring in a fluid with constant viscosity.

However, the work of Saccomandi & Vergori (2010) lacks an accurate investigation of other non-Newtonian effects on the fluid flow. For this reason, in the present study, we carry out an analysis of the flow of a fluid with pressure- and shear-rate-dependent viscosity down an inclined plane within the context of the lubrication approximation. It is legitimate to ask where the pressure dependence of viscosity could become important within the context of the approximation that is carried out in this paper, namely the lubrication approximation. Thin film flows are ubiquitous in engineering, geophysics, biology and elsewhere, and low aspect ratios are often the basis for simplified fluid dynamical models. An important relevant application in geophysics is the flow of glaciers and ice sheets as well as rock glaciers. For instance, while the ice sheet covering Antarctica is several kilometres thick, it has a horizontal extent of several thousand kilometres, yielding a length-scale ratio ϵ of order 10^{-3} (see Schoof & Hindmarsh 2010). These glaciers clearly exhibit non-Newtonian characteristics in that their viscosities depend on the shear rate so that their flows are modelled using a shallow-ice approximation and Glen's flow law (Paterson 1994): in other words, as gravity currents with non-Newtonian (power-law) rheology. On the other hand, there are several papers that investigate the possibility of normal stress effects in the creep of polycrystalline ice (see e.g. McTigue, Passman & Jones 1985 and Man & Sun 1987). In particular, Jones & Chew (1983) have shown that hydrostatic pressure decreases the creep of polycrystalline ice slightly and, then, above 15 MPa, a minimum creep rate is reached followed by an increase in rate with increasing hydrostatic pressure. Therefore, in view of the depths of glaciers we would expect that the pressure would also influence the viscosity. As the viscosity depends on both the shear rate as well as the pressure, it is possible that these two effects could either compete against each other, thereby mitigating their effects, or join forces to enhance the qualitative and quantitative differences. As the fluid can shear-thin or shear-thicken, both possibilities may come to pass.

The organization of the paper is as follows. In § 2, we introduce the constitutive model for the fluid as well as the lubrication approximation that we shall employ. We consider two models for the way in which the viscosity depends on the pressure, an exponential form and a polynomial form. We then proceed to derive the non-dimensional equation that is appropriate for such flows and we delineate two flow regimes, the nearly uniform steady flow regime and the viscous regime. In § 3 we study the nearly uniform steady flow regime, and show that due to the dependence of viscosity on the pressure the breaking time could be increased by an order of magnitude. In fact, depending on the shear-thinning and piezo-viscous coefficients, the breaking time can increase by even larger values. In the final section we consider the viscous regime and, as the main result, we show how the pressure dependence of the viscosity might cause the occurrence of compressive shock waves.

2. Basic equations

We consider a fluid moving on an inclined plane, whose angle of inclination is α. Let $Oxyz$ be a Cartesian frame of reference with fundamental unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$, where the coordinate z is perpendicular to the plane, the x and y coordinates lie in the plane, y is horizontal and x increasing downward. We denote the components of the velocity $\boldsymbol{v}$ of the fluid in the directions x, y and z as u, v and w, respectively. The constraint of incompressibility and the equation of balance of linear momentum can be written as

$$\begin{cases} \dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}+\dfrac{\partial w}{\partial z}=0, \\ \rho\left(\dfrac{\partial u}{\partial t}+u\dfrac{\partial u}{\partial x}+v\dfrac{\partial u}{\partial y}+w\dfrac{\partial u}{\partial z}\right)=-\dfrac{\partial p}{\partial x}+\dfrac{\partial \sigma_{xx}}{\partial x}+\dfrac{\partial \sigma_{xy}}{\partial y}+\dfrac{\partial \sigma_{xz}}{\partial z}+\rho g\sin\alpha, \\ \rho\left(\dfrac{\partial v}{\partial t}+u\dfrac{\partial v}{\partial x}+v\dfrac{\partial v}{\partial y}+w\dfrac{\partial v}{\partial z}\right)=-\dfrac{\partial p}{\partial y}+\dfrac{\partial \sigma_{yx}}{\partial x}+\dfrac{\partial \sigma_{yy}}{\partial y}+\dfrac{\partial \sigma_{yz}}{\partial z}, \\ \rho\left(\dfrac{\partial w}{\partial t}+u\dfrac{\partial w}{\partial x}+v\dfrac{\partial w}{\partial y}+w\dfrac{\partial w}{\partial z}\right)=-\dfrac{\partial p}{\partial z}+\dfrac{\partial \sigma_{zx}}{\partial x}+\dfrac{\partial \sigma_{zy}}{\partial y}+\dfrac{\partial \sigma_{zz}}{\partial z}-\rho g\cos\alpha, \end{cases} \tag{2.1}$$

where ρ is the density of the fluid, p is the pressure or, to be more precise, the mean normal stress, and g is the acceleration due to gravity. We shall assume that the Cauchy stress in the fluid is given by

$$\boldsymbol{T}=-p\boldsymbol{I}+\boldsymbol{\sigma}=-p\boldsymbol{I}+2\mu(p,\|\boldsymbol{D}\|)\boldsymbol{D}, \tag{2.2}$$

where $\boldsymbol{D}=1/2[\nabla\boldsymbol{v}+(\nabla\boldsymbol{v})^T]$ is the symmetric part of the velocity gradient and $\|\boldsymbol{D}\|$ denotes its norm. The modulus μ occurring in (2.2) is the viscosity of the fluid and, in this paper, we shall assume it to be of the form

$$\mu(p,\|\boldsymbol{D}\|)=\gamma(p-p_0)\,\|\boldsymbol{D}\|^{(1-\lambda)/\lambda}, \tag{2.3}$$

with $\lambda>0$, p_0 the reference pressure and, as is reasonable to expect since the fluid viscosity increases as the pressure increases, γ is a positive function whose value increases with increasing pressure. Model (2.2), with a viscosity of the type (2.3), allows for a fluid that is capable of shear-thinning, when $\lambda>1$, or shear-thickening, when $\lambda\in(0,1)$. Here, for the sake of definiteness, we shall consider the following explicit forms for γ:

$$\text{(dependence of viscosity on pressure, type 1)}\quad \gamma(p-p_0)=\gamma_0 e^{\beta(p-p_0)}, \tag{2.4}$$

$$\text{(dependence of viscosity on pressure, type 2)}\quad \gamma(p-p_0)=\gamma_0+\beta\,(p-p_0)^n, \tag{2.5}$$

where $\gamma_0>0$, $\beta\geqslant 0$ and $n\geqslant 0$ are constants. In general, the material parameters that appear in (2.4) and (2.5) can be obtained by corroboration with experimental data. Here, in order to illustrate the effects due to the pressure dependence of viscosity, we merely carry out a parametric study.

We prescribe the following boundary conditions for the velocity and pressure fields:

$$\begin{cases} u=v=w=0 & \text{on } z=0, \\ \boldsymbol{T}\boldsymbol{n}=-p_0\boldsymbol{n} & \text{on } z=h(x,y,t), \end{cases} \tag{2.6}$$

where

$$\boldsymbol{n}=\frac{1}{\sqrt{1+\left(\dfrac{\partial h}{\partial x}\right)^2+\left(\dfrac{\partial h}{\partial y}\right)^2}}\left(-\frac{\partial h}{\partial x}\boldsymbol{i}-\frac{\partial h}{\partial y}\boldsymbol{j}+\boldsymbol{k}\right) \tag{2.7}$$

is the unit normal to the free surface of the current $z=h(x,y,t)$.

Let H and L denote the characteristic thickness and characteristic length along the plane of the current-free surface $h = h(x, y, t)$, respectively. The main assumption in lubrication approximation (see Szeri 1998) is that the length-scale ratio H/L is small. Here, as we are interested in fluids whose viscosity depends on the pressure, we assume that the ratio H/L is small, though H is large enough to have a significant dependence of the viscosity on the pressure.

As a consequence of the smallness of the length-scale ratio H/L, the component of the velocity parallel to the plane is much larger than the normal component, so that

$$\sqrt{u^2 + v^2} \gg |w|. \tag{2.8}$$

We call U, V and W the characteristic velocities along the x, y and z directions, respectively. Hence, $U_\parallel = \sqrt{U^2 + V^2}$ and W are the characteristic velocities parallel and perpendicular to the inclined plane, respectively. From $(2.1)_1$ and (2.8) we find that $W = HU_\parallel/L$.

There are many ways of transforming the governing equations (2.1) and boundary conditions (2.6) into dimensionless expressions. Here we introduce a scaling which is similar to that introduced by Ancey (2007):

$$\begin{cases} \boldsymbol{x}^* = \dfrac{1}{L}(x\boldsymbol{i} + y\boldsymbol{j}) + \dfrac{z}{H}\boldsymbol{k}, \quad \boldsymbol{v}^* = \dfrac{1}{U_\parallel}(u\boldsymbol{i} + v\boldsymbol{j}) + \dfrac{w}{W}\boldsymbol{k}, \quad h^* = \dfrac{h}{H}, \\ W = \dfrac{H}{L}U_\parallel, \quad t^* = \dfrac{U_\parallel}{L}t, \quad p^* = \dfrac{p - p_0}{\rho g \cos\alpha H}, \quad \gamma^* = \dfrac{\gamma}{\gamma_0}, \end{cases} \tag{2.9}$$

where γ_0 is the value of γ at the reference pressure p_0.

Substituting the dimensionless quantities (2.9) into (2.1), (2.2) and (2.7) and into the boundary conditions (2.6) leads to (omitting all asterisks)

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0, \\ \epsilon Re\left(\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} + w\dfrac{\partial u}{\partial z}\right) = \epsilon\dfrac{Re}{Fr^2}\left(\dfrac{\tan\alpha}{\epsilon} - \dfrac{\partial p}{\partial x}\right) + \epsilon\dfrac{\partial \sigma_{xx}}{\partial x} \\ \qquad\qquad + \epsilon\dfrac{\partial \sigma_{xy}}{\partial y} + \dfrac{\partial \sigma_{xz}}{\partial z}, \\ \epsilon Re\left(\dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} + w\dfrac{\partial v}{\partial z}\right) = -\epsilon\dfrac{Re}{Fr^2}\dfrac{\partial p}{\partial y} + \epsilon\dfrac{\partial \sigma_{yx}}{\partial x} + \epsilon\dfrac{\partial \sigma_{yy}}{\partial y} + \dfrac{\partial \sigma_{yz}}{\partial z}, \\ \epsilon^2 Re\left(\dfrac{\partial w}{\partial t} + u\dfrac{\partial w}{\partial x} + v\dfrac{\partial w}{\partial y} + w\dfrac{\partial w}{\partial z}\right) = -\dfrac{Re}{Fr^2}\left(1 + \dfrac{\partial p}{\partial z}\right) + \epsilon\dfrac{\partial \sigma_{zx}}{\partial x} + \epsilon\dfrac{\partial \sigma_{zy}}{\partial y} + \dfrac{\partial \sigma_{zz}}{\partial z}, \end{cases} \tag{2.10}$$

$$\begin{cases} u = v = w = 0 & \text{on } z = 0, \\ (-p\boldsymbol{I} + \boldsymbol{\sigma})\left(-\epsilon\dfrac{\partial h}{\partial x}\boldsymbol{i} - \epsilon\dfrac{\partial h}{\partial y}\boldsymbol{j} + \boldsymbol{k}\right) = \boldsymbol{0} & \text{on } z = h(x, y, t), \end{cases} \tag{2.11}$$

where

$$\epsilon = \frac{H}{L} \ll 1, \tag{2.12}$$

$$
\begin{aligned}
\boldsymbol{\sigma} = \gamma(p) & \left[\epsilon^2\left(\frac{\partial u}{\partial x}\right)^2 + \epsilon^2\left(\frac{\partial v}{\partial y}\right)^2 + \epsilon^2\left(\frac{\partial w}{\partial z}\right)^2 + \frac{\epsilon^2}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 \right.\\
& \left. + \frac{1}{2}\left(\frac{\partial u}{\partial z} + \epsilon^2\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial z} + \epsilon^2\frac{\partial w}{\partial y}\right)^2\right]^{(1-\lambda)/2\lambda} \\
& \times \left[2\epsilon\left(\frac{\partial u}{\partial x}\boldsymbol{i}\otimes\boldsymbol{i} + \frac{\partial v}{\partial y}\boldsymbol{j}\otimes\boldsymbol{j} + \frac{\partial w}{\partial z}\boldsymbol{k}\otimes\boldsymbol{k}\right) + \epsilon\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)(\boldsymbol{i}\otimes\boldsymbol{j} + \boldsymbol{j}\otimes\boldsymbol{i}) \right.\\
& \left. + \left(\frac{\partial u}{\partial z} + \epsilon^2\frac{\partial w}{\partial x}\right)(\boldsymbol{i}\otimes\boldsymbol{k} + \boldsymbol{k}\otimes\boldsymbol{i}) + \left(\frac{\partial v}{\partial z} + \epsilon^2\frac{\partial w}{\partial y}\right)(\boldsymbol{j}\otimes\boldsymbol{k} + \boldsymbol{k}\otimes\boldsymbol{j})\right],
\end{aligned}
\tag{2.13}
$$

where the symbol '$\otimes$' denotes the tensor product. In this framework the dimensionless version of γ is an increasing function such that $\gamma(0) = 1$. In particular, (2.4) and (2.5) become, respectively,

$$\gamma(p) = \mathrm{e}^{\omega p} \quad \text{with } \omega = \beta\rho g \cos\alpha H, \tag{2.14}$$

and

$$\gamma(p) = 1 + \omega p^n \quad \text{with } \omega = \beta\left(\rho g \cos\alpha H\right)^n. \tag{2.15}$$

The dimensionless quantities

$$Re = \frac{\rho U_\parallel^{(2\lambda-1)/\lambda} H^{1/\lambda}}{\gamma_0} \quad \text{and} \quad Fr = \frac{U_\parallel}{\sqrt{g\cos\alpha H}} \tag{2.16}$$

are the Reynolds and Froude numbers, respectively.

Depending on the values considered for the characteristic scales, different types of flow regime occur. In this paper we shall focus on the following two types of flow regimes.

(i) The *nearly steady uniform regime*, where the viscous contribution is comparable to the gravitational effect. In this case, we have

$$U_\parallel = \left[\frac{\rho g \sin\alpha H^{(\lambda+1)/\lambda}}{\gamma_0}\right]^\lambda \tag{2.17}$$

and $Fr^2 = O(Re)$. Inertial terms and pressure gradient terms must be negligible, which means $\epsilon Re \ll 1$. Therefore, from (2.10) and (2.13) the approximate equations are found to be given by

$$
\begin{cases}
\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0,\\[2mm]
\dfrac{\partial}{\partial z}\left\{\gamma(p)\left[\left(\dfrac{\partial u}{\partial z}\right)^2 + \left(\dfrac{\partial v}{\partial z}\right)^2\right]^{(1-\lambda)/2\lambda}\dfrac{\partial u}{\partial z}\right\} + 2^{(1-\lambda)/2\lambda} = 0,\\[2mm]
\dfrac{\partial}{\partial z}\left\{\gamma(p)\left[\left(\dfrac{\partial u}{\partial z}\right)^2 + \left(\dfrac{\partial v}{\partial z}\right)^2\right]^{(1-\lambda)/2\lambda}\dfrac{\partial v}{\partial z}\right\} = 0,\\[2mm]
\dfrac{\partial p}{\partial z} + 1 = 0.
\end{cases}
\tag{2.18}
$$

(ii) The *viscous regime*, where the effect of the pressure gradient is balanced by stresses induced due to the viscosity within the bulk. In this case, we have

$$U_{\parallel} = \left[\frac{\rho g \cos \alpha H^{(2\lambda+1)/\lambda}}{\gamma_0 L} \right]^{\lambda} \tag{2.19}$$

and consequently $Fr^2 = \epsilon Re$. Inertial terms must be small compared to the effect of the pressure gradient and the slope must be gentle ($\tan\alpha = O(\epsilon)$). This imposes the following constraint: $\epsilon Re \ll 1$. From (2.10) and (2.13) we deduce the approximate equations

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0, \\ \dfrac{\partial}{\partial z} \left\{ \gamma(p) \left[\left(\dfrac{\partial u}{\partial z} \right)^2 + \left(\dfrac{\partial v}{\partial z} \right)^2 \right]^{(1-\lambda)/2\lambda} \dfrac{\partial u}{\partial z} \right\} + 2^{(1-\lambda)/2\lambda} \left(\dfrac{\tan\alpha}{\epsilon} - \dfrac{\partial p}{\partial x} \right) = 0, \\ \dfrac{\partial}{\partial z} \left\{ \gamma(p) \left[\left(\dfrac{\partial u}{\partial z} \right)^2 + \left(\dfrac{\partial v}{\partial z} \right)^2 \right]^{(1-\lambda)/2\lambda} \dfrac{\partial v}{\partial z} \right\} - 2^{(1-\lambda)/2\lambda} \dfrac{\partial p}{\partial y} = 0, \\ \dfrac{\partial p}{\partial z} + 1 = 0. \end{cases} \tag{2.20}$$

Moreover, from (2.11) and (2.13), by virtue of the smallness of ϵ, the boundary conditions (2.6) approximate to

$$\begin{cases} u = v = w = 0 & \text{on } z = 0, \\ p = 0 & \text{on } z = h(x, y, t), \\ \dfrac{\partial u}{\partial z} = \dfrac{\partial v}{\partial z} = 0 & \text{on } z = h(x, y, t). \end{cases} \tag{2.21}$$

Finally, we derive the evolution equation for the free surface $z = h(x, y, t)$. We first integrate the constraint of incompressibility over the flow depth to obtain, by means of boundary condition $(2.21)_1$,

$$\int_0^h \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \mathrm{d}z = \frac{\partial}{\partial x} \int_0^h u \, \mathrm{d}z + \frac{\partial}{\partial y} \int_0^h v \, \mathrm{d}z - u_{|z=h} \frac{\partial h}{\partial x} - v_{|z=h} \frac{\partial h}{\partial y} - w_{|z=h}. \tag{2.22}$$

But, obviously,

$$w_{|z=h} = \frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\partial h}{\partial t} + u_{|z=h} \frac{\partial h}{\partial x} + v_{|z=h} \frac{\partial h}{\partial y}. \tag{2.23}$$

Therefore, combining (2.22) and (2.23) gives the required equation for h,

$$h_t + \frac{\partial (h\bar{u})}{\partial x} + \frac{\partial (h\bar{v})}{\partial y} = 0, \tag{2.24}$$

where we have introduced the depth-averaged variables defined as

$$\bar{\varphi}(x, y, t) = \frac{1}{h(x, y, t)} \int_0^{h(x,y,t)} \varphi(x, y, z, t) \, \mathrm{d}z. \tag{2.25}$$

3. Nearly steady uniform regime

It is easy to verify that system (2.18) with boundary conditions (2.21) admits the solution

$$\begin{cases} u = 2^{(1-\lambda)/2} \displaystyle\int_0^z \left[\frac{h-\zeta}{\gamma(h-\zeta)} \right]^\lambda \mathrm{d}\zeta, \\ v = 0, \\ w = -2^{(1-\lambda)/2} \dfrac{\partial}{\partial x} \displaystyle\int_0^z \left\{ \int_0^{\zeta_1} \left[\frac{h-\zeta}{\gamma(h-\zeta)} \right]^\lambda \mathrm{d}\zeta_2 \right\} \mathrm{d}\zeta_1, \\ p = h - z. \end{cases} \tag{3.1}$$

Therefore

$$h\bar{u} = 2^{(1-\lambda)/2} \int_0^h \xi \left[\frac{\xi}{\gamma(\xi)} \right]^\lambda \mathrm{d}\xi =: F(h), \tag{3.2}$$

and (2.24) becomes

$$\frac{\partial h}{\partial t} + F'(h) \frac{\partial h}{\partial x} = 0, \tag{3.3}$$

where the prime denotes differentiation with respect to h.

Equation (3.3) is a quasilinear first-order partial differential equation whose general solution can be found by the method of characteristics. If $f(\xi)$ is an initial profile, then the corresponding solution is given by

$$h = f(x - F'(h)t). \tag{3.4}$$

The wave (3.4) could break, i.e. its profile could become multivalued, at time $t_B = -\left[F''(f(\xi_B))\,\mathrm{d}f/\mathrm{d}\xi(\xi_B)\right]^{-1}$ at the point $x_B = \xi_B + F'(f(\xi_B))t_B$, where ξ_B has to be determined by means of the conditions

$$\begin{cases} F''(f(\xi_B)) \dfrac{\mathrm{d}f}{\mathrm{d}\xi}(\xi_B) < 0, \\ |F''(f(\xi_B))f'(\xi_B)| = \max \left| \dfrac{\mathrm{d}\,F'(f(\xi))}{\mathrm{d}\xi} \right|. \end{cases} \tag{3.5}$$

Since γ is a positive increasing function, from (3.5) we deduce that the pressure dependence of the viscosity has the effect of delaying the time at which the wave could break. To quantify this delaying effect we consider γ of the form (2.14) and assume that $h(x,0) = f(x) = 1 - x^2$. If the fluid is Newtonian with a constant viscosity μ_0 (i.e. $\lambda = 1$ and $\gamma(p - p_0) = \mu_0$ in (2.3)), it is easy to show that the wave breaks at time $t_{BN} = 3\sqrt{3}/8$. In order to make the differences between the non-Newtonian case that is being considered and the classical Newtonian case more evident, we have plotted the ratio between the breaking time t_B in the non-Newtonian case and t_{BN} as a function of λ and as a function of the non-dimensional piezo-viscous coefficient ω (see figure 1). Furthermore, the solutions to the wave equation (3.3) with $\lambda = 0.5$ and $\lambda = 1.5$ (figure 2) are plotted at different times together with the profiles of the free surface $z = h(x,t)$ in the classical Newtonian case. We find that the solutions are qualitatively similar, though quantitatively different.

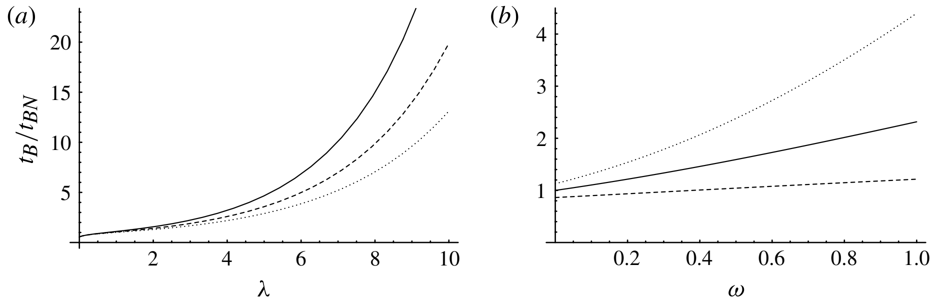

FIGURE 1. Ratio t_B/t_{BN} as a function of (*a*) λ and (*b*) the piezo-viscous coefficient ω. γ is assumed to be of the form (2.14) and the initial profile considered is $h(x, 0) = 1 - x^2$.

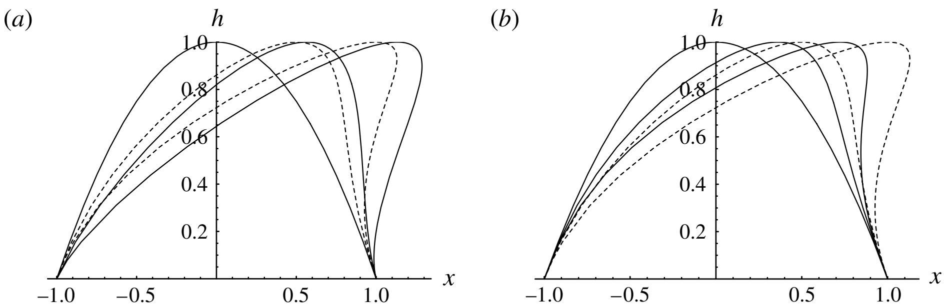

FIGURE 2. Solutions of (3.3) with an initial profile $f(x) = (1 - x^2)$ at $t = 0$, $t = 0.5$, $t = 1$. The dashed line represents the solution in the classical Newtonian case, whereas the solid line represents the solution in the case in which γ depends on the pressure according to the law $\gamma(p) = \mathrm{e}^{0.1p}$ and (*a*) $\lambda = 0.5$ and (*b*) $\lambda = 1.5$.

Finally, in order to look for self-similar solutions of (3.3), we need to know whether F' is invertible. The invertibility of F' is linked with the equation

$$\lambda h\gamma'(h) - (\lambda + 1)\gamma(h) = 0. \tag{3.6}$$

Indeed, if (3.6) admits positive roots, the least of which we denote by $\hat{h}$, then F' is invertible in $[0, \hat{h}[$. On the contrary, if (3.6) does not admit positive roots, then F' is invertible in $[0, +\infty[$. In any case F'^{-1} is continuous and increasing. It is interesting to show that some time after the initiation of the flow, no matter what the initial shape, the solution tends to the unique self-similar solution of (3.3), i.e.

$$h(x, t) \to F'^{-1}\left(\frac{x}{t}\right) \quad \text{as } t \to +\infty. \tag{3.7}$$

In order to prove (3.7), from (3.3) we deduce that h is constant along the characteristics given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F'(h). \tag{3.8}$$

Thus, if initially $h(x, 0) = f(x)$, the characteristics are straight lines

$$x = x_0 + F'[f(x_0)]t, \tag{3.9}$$

x_0 being the initial value of the characteristic. The solution of (3.3) is then

$$h(x,t)=F'^{-1}\left(\frac{x-x_0}{t}\right)\rightarrow F'^{-1}\left(\frac{x}{t}\right)\quad \text{as } t\rightarrow+\infty. \tag{3.10}$$

If the viscosity does not depend on the pressure, (3.7) reduces to the self-similar solution found by Perazzo & Gratton (2003), which in turn is the non-Newtonian counterpart of the self-similar solution derived by Huppert (1982*a*) for Newtonian fluids.

4. Viscous regime

A lengthy but straightforward algebraic manipulation allows us to obtain the solution to the boundary-value problem (2.20)–(2.21):

$$\begin{cases} u=2^{(1-\lambda)/2}\left(\dfrac{\tan\alpha}{\epsilon}-\dfrac{\partial h}{\partial x}\right)\left[\left(\dfrac{\partial h}{\partial x}-\dfrac{\tan\alpha}{\epsilon}\right)^2+\left(\dfrac{\partial h}{\partial y}\right)^2\right]^{(\lambda-1)/2}\displaystyle\int_0^z\left[\frac{h-\zeta}{\gamma\,(h-\zeta)}\right]^{\lambda}\mathrm{d}\zeta, \\ v=-2^{(1-\lambda)/2}\dfrac{\partial h}{\partial y}\left[\left(\dfrac{\partial h}{\partial x}-\dfrac{\tan\alpha}{\epsilon}\right)^2+\left(\dfrac{\partial h}{\partial y}\right)^2\right]^{(\lambda-1)/2}\displaystyle\int_0^z\left[\frac{h-\zeta}{\gamma\,(h-\zeta)}\right]^{\lambda}\mathrm{d}\zeta, \\ w=-2^{(1-\lambda)/2}\nabla_s\cdot\left\{\left|\dfrac{\tan\alpha}{\epsilon}\boldsymbol{i}-\nabla_s h\right|^{\lambda-1}\left(\dfrac{\tan\alpha}{\epsilon}\boldsymbol{i}-\nabla_s h\right)\right. \\ \qquad\left.\times\displaystyle\int_0^z\left[\int_0^{\zeta_1}\left(\frac{h-\zeta_2}{\gamma(h-\zeta_2)}\right)^{\lambda}\mathrm{d}\zeta_2\right]\mathrm{d}\zeta_1\right\}, \\ p=h-z, \end{cases} \tag{4.1}$$

where ∇_s is the two-dimensional gradient

$$\nabla_s\varphi=\frac{\partial\varphi}{\partial x}\boldsymbol{i}+\frac{\partial\varphi}{\partial y}\boldsymbol{j}. \tag{4.2}$$

Then

$$h\bar{u}=F(h)\left(\frac{\tan\alpha}{\epsilon}-\frac{\partial h}{\partial x}\right)\left[\left(\frac{\partial h}{\partial x}-\frac{\tan\alpha}{\epsilon}\right)^2+\left(\frac{\partial h}{\partial y}\right)^2\right]^{(\lambda-1)/2}, \tag{4.3}$$

$$h\bar{v}=-F(h)\frac{\partial h}{\partial y}\left[\left(\frac{\partial h}{\partial x}-\frac{\tan\alpha}{\epsilon}\right)^2+\left(\frac{\partial h}{\partial y}\right)^2\right]^{(\lambda-1)/2}, \tag{4.4}$$

so that (2.24) becomes

$$\frac{\partial h}{\partial t}+\nabla_s\cdot\left\{F(h)\left|\frac{\tan\alpha}{\epsilon}\boldsymbol{i}-\nabla_s h\right|^{\lambda-1}\left(\frac{\tan\alpha}{\epsilon}\boldsymbol{i}-\nabla_s h\right)\right\}=0. \tag{4.5}$$

Now let us make the further assumption that the flow depends only on the x coordinate. Then $\partial h/\partial y=0$ (so that $v=0$) and (4.5) reduces to

$$\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}\left\{F(h)\left|\frac{\tan\alpha}{\epsilon}-\frac{\partial h}{\partial x}\right|^{\lambda-1}\left(\frac{\tan\alpha}{\epsilon}-\frac{\partial h}{\partial x}\right)\right\}=0. \tag{4.6}$$

To find travelling wave solutions we assume that h depends on the single variable $s\equiv x-ct$, where c is a constant which represents the wave speed. Then (4.6) can be

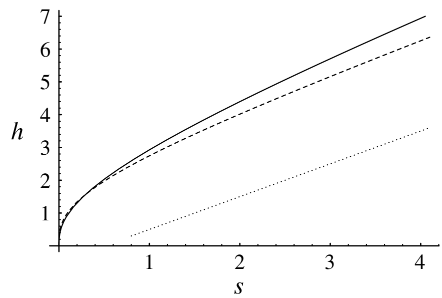

FIGURE 3. Profiles of upslope travelling waves. The solid line represents the travelling wave solution when $\lambda = 1.5$ and $\gamma(p) = 1 + 0.2p$, whereas the dashed line represents the travelling wave solution in the classical Newtonian case. We have assumed $\tan\alpha/\epsilon = 1$ and $c = -1$. Observe that the solutions tend to be parallel to the horizontal plane (dotted line) as $s \to +\infty$. Since $h' \to +\infty$ as $s \to 0$, the upslope travelling waves do not satisfy the lubrication approximation near the front.

integrated once to obtain

$$\left|\frac{\tan\alpha}{\epsilon} - \frac{\mathrm{d}h}{\mathrm{d}s}\right|^{\lambda-1}\left(\frac{\tan\alpha}{\epsilon} - \frac{\mathrm{d}h}{\mathrm{d}s}\right) = \frac{c_1 + ch}{F(h)}, \tag{4.7}$$

c_1 being an integration constant. Let $c_1 = 0$ in (4.7). Then, (4.7) may be written as

$$\frac{\mathrm{d}h}{\mathrm{d}s} = \frac{\tan\alpha}{\epsilon} - \mathrm{sign}(c)\left\{\frac{|c|h}{F(h)}\right\}^{1/\lambda}. \tag{4.8}$$

If $c < 0$, the right-hand side of (4.8) never vanishes, and hence (4.7) may be integrated numerically over the range $(0, \bar{h})$ for any $\bar{h} > 0$. In this case the general solution is an increasing function defined over the interval $(c_2, +\infty)$, c_2 being an integration constant, and tends to $+\infty$ as $s \to +\infty$. Therefore, these solutions do not satisfy the lubrication approximation as the length-scale ratio fails to be small as $s \to +\infty$.

Assume now that the incline is not infinite, and in the valley the plane is horizontal. Furthermore, since we are considering upslope travelling waves, we suppose that the fluid flows from the horizontal to the incline. From (4.8) it follows that the fluid free surface tends to the horizontal as $s \to +\infty$ if and only if there exist $\omega > 0$ and $n \in [0, 1 + 1/\lambda[$ such that $\gamma(p) \simeq 1 + \omega p^n$ for large pressure p. In such a case the travelling waves meet the lubrication approximation, as the dimensionless thickness of the fluid remains finite as $s \to +\infty$ (figure 3).

In order to discuss the integrability of (4.8) with $c > 0$, we have to find the positive roots of the equation

$$\left(\frac{\tan\alpha}{\epsilon}\right)^{\lambda} F(h) - ch = 0. \tag{4.9}$$

The roots of (4.9) may be found numerically. Nevertheless, we can deduce the number of positive roots of (4.9) by studying the function $\mathscr{F}(h) := (\tan\alpha/\epsilon)^{\lambda} F(h)/h$. $\mathscr{F}$ is a continuous differentiable function that tends to zero as $h \to 0$, whose derivative may be written as

$$\mathscr{F}'(h) = \left(\frac{\tan\alpha}{\epsilon}\right)^{\lambda} \frac{1}{h^2} \int_0^h \xi F''(\xi)\,\mathrm{d}\xi$$
$$= 2^{(1-\lambda)/2} \left(\frac{\tan\alpha}{\epsilon}\right)^{\lambda} \frac{1}{h^2} \int_0^h \left(\frac{\xi}{\gamma(\xi)}\right)^{\lambda+1} [(\lambda+1)\gamma(\xi) - \lambda\xi\gamma'(\xi)]\,\mathrm{d}\xi. \quad (4.10)$$

From (4.10) it follows that $\mathscr{F}'$ is positive in a neighbourhood of $h = 0$, but it might change sign away from zero if (3.6) admits positive roots. Here, for the sake of simplicity, we shall limit our analysis to the constitutive functions γ for which (3.6) admits at most one positive root. It is easy to recognize that models (2.14) and (2.15) meet this requirement.

We are now able to say how many positive roots (4.9) admits. In fact:

(i) if $[\gamma(h)/h]^{\lambda}$ has linear growth as $h \to +\infty$, then $\mathscr{F}$ is increasing and tends to $l > 0$ as $h \to +\infty$ so that (4.9) with $c \in]0, l[$ admits only one positive root, whereas it does not admit a positive root for $c \geqslant l$;

(ii) if $[\gamma(h)/h]^{\lambda}$ has sublinear growth as $h \to +\infty$, then $\mathscr{F}$ is increasing and tends to $+\infty$ as $h \to +\infty$ so that, for any $c > 0$, (4.9) admits a unique positive root;

(iii) if $[\gamma(h)/h]^{\lambda}$ has superlinear growth as $h \to +\infty$, then $\mathscr{F}$ attains its absolute maximum at $h = h^* > 0$ and tends to zero as $h \to +\infty$ so that (4.9) admits two positive root if $c \in]0, \mathscr{F}(h_*)[$, only one positive root if $c = \mathscr{F}(h_*)$, and no positive root for $c > \mathscr{F}(h_*)$.

According to the number of positive roots of (4.9), one, two or three families of solutions to (4.7) may arise.

If (4.9) does not admit a positive root, then (4.7) may be numerically integrated over the range $(0, \bar{h})$ for all $\bar{h} > 0$. In this case the general solution is a decreasing function defined over the interval $(-\infty, c_2)$, c_2 being an integration constant, and tends to $+\infty$ as $s \to -\infty$. Therefore, we do not consider these solutions as they do not satisfy the lubrication approximation.

If (4.9) admits only one positive root h_m, then two families of solutions to (4.7) arise. The first is formed by bounded decreasing functions defined over the range $(-\infty, c_2)$ satisfying the inequality $0 \leqslant h \leqslant h_m$. For these solutions we have $h \to h_m$ as $s \to -\infty$. Then they represent travelling waves behind a front running downslope that, far behind the front ($s \to -\infty$), tends to the steady downslope flow $h = h_m$ (see figure 4). The other family is formed by increasing functions bounded from below for which $h \geqslant h_m$. These solutions represents downslope travelling waves with no front, for which $h \to h_m$ as $s \to -\infty$ and $h \to +\infty$ as $s \to +\infty$. Therefore, they do not satisfy the lubrication approximation.

As before, assume that the incline is not infinite, and in the valley the plane is horizontal. But, since we are now considering downslope travelling waves, we suppose that the fluid flows from the incline to the horizontal. In such a case the downslope travelling waves with no front meet the lubrication approximation if and only if, for large pressure, γ is of the form (2.15) with $n \in [0, 1 + 1/\lambda[$ (figure 5).

If (4.7) admits two positive roots, $h_m < h_M$, then, as well as the downslope travelling waves behind a front, two other families of solutions to (4.7) arise, representing downslope travelling waves with no front (see figure 5). The former is constituted by bounded increasing functions satisfying the inequality $h_m < h < h_M$ and for which we have $h \to h_m$ as $s \to -\infty$ and $h \to h_M$ as $s \to +\infty$. The latter is formed by decreasing functions that are bounded from below as they satisfy the inequality $h \geqslant h_M$ and for

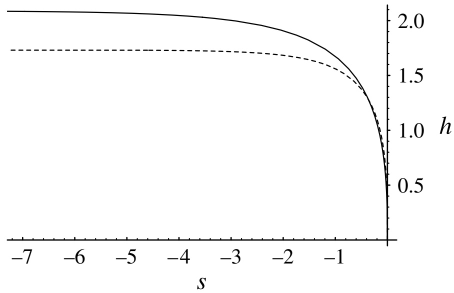

FIGURE 4. Profiles of downslope travelling waves behind a front. The solid line represents the travelling wave solution when $\lambda = 1.5$ and $\gamma(p) = 1 + 0.2p$, whereas the dashed line represents the travelling wave solution in the classical Newtonian case. We have considered $\tan\alpha/\epsilon = 1$ and $c = 1$. Since $h' \to -\infty$ as $s \to 0$, the downslope travelling waves behind a front do not satisfy the lubrication approximation near the front.

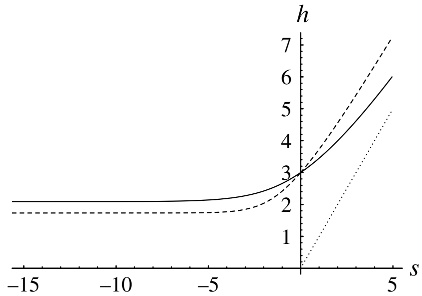

FIGURE 5. Profiles of downslope travelling waves with no front. The solid line represents the travelling wave solution when $\lambda = 1.5$ and $\gamma(p) = 1 + 0.2p$, whereas the dashed line represents the travelling wave solution in the classical Newtonian case. We have considered $\tan\alpha/\epsilon = 1$ and $c = 1$. Observe that the solutions tend to be parallel to the horizontal plane (dotted line) as $s \to +\infty$.

which we have $h \to +\infty$ as $s \to -\infty$ and $h \to h_M$ as $s \to +\infty$. We disregard the travelling wave solutions belonging to this family as the length-scale ratio fails to be small as $s \to -\infty$. On the contrary, we discuss in detail the class of downslope travelling waves with no front that satisfies the lubrication approximation.

From (4.6), it is easy to recognize that, for $|x|$ large enough, these downslope travelling waves behave like the solutions of the one-dimensional wave equation

$$\frac{\partial h}{\partial t} + \left(\frac{\tan\alpha}{\epsilon}\right)^{\lambda} \frac{\partial F(h)}{\partial x} = 0, \tag{4.11}$$

with characteristic speed

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \left(\frac{\tan\alpha}{\epsilon}\right)^{\lambda} F'(h). \tag{4.12}$$

Since $\mathscr{F}$ attains its maximum at $h_* \in]h_m, h_M[$, we deduce that $\mathscr{F}'(h_M) < 0 < \mathscr{F}'(h_m)$, by which

$$\left(\frac{\tan\alpha}{\epsilon}\right)^{\lambda} F'(h_M) < c < \left(\frac{\tan\alpha}{\epsilon}\right)^{\lambda} F'(h_m). \tag{4.13}$$

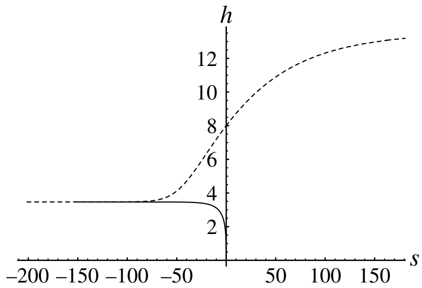

FIGURE 6. Profiles of downslope travelling wave solutions for $\lambda = 1.1$, $\gamma(p) = 1 + 0.05p^3$, $\tan\alpha/\epsilon = 0.5$ and $c = 1$. In the case we are considering two families of downslope travelling wave solutions that satisfy the lubrication approximation arise (see the text): downslope travelling waves behind a front (solid line) and compressive shock waves (dashed line).

Furthermore, in virtue of the conditions satisfied by h as $s \to -\infty$,

$$\lim_{s\to-\infty} h = h_m \quad \text{and} \quad \lim_{h\to-\infty} \frac{\mathrm{d}h}{\mathrm{d}s} = 0, \tag{4.14}$$

the integration constant c_1 in (4.7) may be calculated to obtain

$$\frac{\mathrm{d}h}{\mathrm{d}s} = \frac{\tan\alpha}{\epsilon} - \left[\frac{\left(\frac{\tan\alpha}{\epsilon}\right)^{\lambda} F(h_m) + c(h - h_m)}{F(h)}\right]^{1/\lambda}. \tag{4.15}$$

In the limit as $s \to +\infty$, (4.15) yields

$$\left(\frac{\tan\alpha}{\epsilon}\right)^{\lambda} [F(h_M) - F(h_m)] = c(h_M - h_m). \tag{4.16}$$

Therefore, in view of (4.16) and (4.13), the travelling waves at issue satisfy both the Rankine–Hugoniot condition, and the Lax entropy condition and hence they represent *compressive shock waves* (Bertozzi & Shearer 2000).

On taking into account (4.16), we linearize (4.15) around its equilibria $h = h_e$, $e = m, M$, and obtain

$$\frac{\mathrm{d}h}{\mathrm{d}s} = \frac{1}{\lambda}\left(\frac{\tan\alpha}{\epsilon}\right)^{1-\lambda} \frac{\left(\frac{\tan\alpha}{\epsilon}\right)^{\lambda} F'(h_e) - c}{F(h_e)}(h - h_e) \quad e = m, M, \tag{4.17}$$

by means of which, with the aid of (4.13), we deduce that $h = h_m$ is unstable whereas $h = h_M$ is stable. We may conclude that the compressive shock waves we have derived may be viewed as heteroclinic orbits connecting the two equilibria of (4.8) (see figure 6).

Finally, observe that

$$F(h) \simeq 2^{(\lambda-1)/2} \frac{h^{\lambda+2}}{\lambda+2} \quad \text{as } h \to 0. \tag{4.18}$$

Therefore, near the wave front, where the effects of pressure can be neglected, the solution to (4.7) is approximated by that found by Perazzo & Gratton (2003), namely

$$\frac{\tan\alpha}{\epsilon}(s-c_2)$$
$$= h - h\,{}_2F_1\left[\frac{\lambda}{\lambda+1}, 1, \frac{2\lambda+1}{\lambda+1}, \mathrm{sign}(c)\frac{\tan\alpha}{\epsilon}\left(\frac{|c|(\lambda+2)}{2^{(1-\lambda)/2}}\right)^{-1/\lambda} h^{1+1/\lambda}\right]. \quad (4.19)$$

Here ${}_2F_1(a, b, c, d)$ is the hypergeometric function. From (4.19) we deduce that h' tends to infinity as $s \to c_2$. Hence, near the wave front, the component of the fluid velocity normal to the incline is not small with respect to the parallel component and thus the solution does not satisfy the lubrication approximation.

5. Concluding remarks

In this paper the flow of a fluid with pressure- and shear-dependent viscosity over an inclined plane has been studied. In particular, two different flow regimes are investigated: the nearly steady uniform regime in which the viscous and gravitational effects are of the same order of magnitude, and the viscous regime in which the pressure gradient is balanced by the viscous stresses within the bulk. In the former case we have derived a quasilinear first-order partial differential equation that governs the flow of the current-free surface. Integrating this equation under a prescribed initial datum gets a wave that breaks whenever the initial profile satisfies suitable conditions. The breaking time, if a break in the wave occurs, delays both as the piezo-viscous coefficient increases and as the parameter that determines the shear-thickening or shear-thinning behaviour of the fluid increases. It is worth noting that the pressure dependence of the viscosity always delays the occurrence of a break but this effect is more evident in fluids that are capable of shear-thinning than in those that are capable of shear-thickening. In the viscous regime we have derived travelling wave solutions of the evolution equation for the current-free surface. It is interesting to note that in this case the pressure dependence of the viscosity influences the solutions not only quantitatively but also qualitatively, as the particular type of dependence of the viscosity on the pressure determines the number of families of downslope travelling wave solutions. In Newtonian and power-law non-Newtonian liquids, two families of downslope travelling wave solutions satisfying the lubrication approximation always occur (see Perazzo & Gratton 2003); in non-Newtonian fluids with pressure- and shear-dependent viscosity one, two or three families of downslope travelling waves may occur according to the values of the material parameters of the fluid, but not all the solutions satisfy the lubrication approximation. Moreover, the dependence of viscosity on the pressure may cause the occurrence of downslope compressive shock waves connecting two constant equilibrium profiles of the current-free surface. To be more precise, downslope compressive shock waves may occur only if the constitutive function accounting for the pressure dependence of the fluid viscosity has growth exponent greater than $1 + 1/\lambda$ as the pressure tends to infinity, λ being the parameter that determines the shear-thinning or shear-thickening behaviour of the non-Newtonian fluid.

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