Power operations in K-theory completed at a prime

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Abstract

We describe the action of power operations on the p-completed cooperation algebras $K_0^\vee K = K_0(K)\widehat{_p}$ for K-theory at a prime p, and $K_0^\vee KO = K_0(KO)\widehat{_2}$. These results are used to identify the $K(1)$-local homotopy type of some E_∞ ring spectra obtained by killing elements of Hopf invariant 1.

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Introduction

Power operations in suitably completed (co)homology theories have been studied and used by several authors, for example Rezk [Rez09, Rez12, Rez17]; the paper of Barthel and Frankland [BF15] building on work of McClure [BMMS86] provides a convenient account of this, in particular for the case of p-complete K-theory. An important source on related mathematics is the article by Hopkins [Hop14], and indeed the volume [DFHH14] contains much that the reader may find helpful.

In the present paper we describe the action of the θ-operator (which we follow [BF15] in denoting by Q) on the p-completed cooperation algebra

$$K_0^\vee K = K_0(K)\widehat{_p} = \pi_0(L_{K(1)}(K \wedge K)),$$

where $K = KU$. We expect this to be of use in investigating the θ-action and its interaction with the $K_*^\vee(K)$-coaction on $K_*^\vee(A)$ for any $\mathcal{E}_\infty$ ring spectrum A. We also give some results on $K_0^\vee(KO)$ when $p = 2$ and on $K_*^\vee(\mathbb{P}X)$, where $\mathbb{P}X$ denotes the free commutative S-algebra on a spectrum X introduced in [EKMM97].

It is likely that some of our results are known to experts, but we have not found a published source, so we feel it worthwhile writing them down.

An obvious related problem to investigate is that of describing the actions of power operations on $K_0^\vee(BU)$ or equivalently on $K_0^\vee(MU)$ (these actions correspond under the Thom isomorphism). The $\mathcal{E}_\infty$ orientation of [Joa04] induces a morphism of θ-algebras $K_0^\vee(MU) \to K_0^\vee(K)$ but this

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is not injective on the image of the Hopf algebra primitives $\Pr K_0^\vee(BU)$, and this seems to make the determination of the action on primitives more delicate then in the case of ordinary mod p homology as carried out by Kochman [Koc73]. We may return to this in future work.

Conventions and notation: We will work with $\mathcal{E}_\infty$ ring spectra in the setting of commutative S-algebras of [EKMM97] and use these terms interchangeably. We will assume that KU and KO have their standard $\mathcal{E}_\infty$ ring structures as produced in [BR05] for example.

Throughout, p will be a fixed prime and $K = KU_{(p)}$ will denote the p-local 2-periodic complex K-theory ring spectrum; we will also denote the p-adic completion of K by $K_p^\wedge = KU_p^\wedge$. We will often denote (co)homology without brackets where appropriate by setting $K^*X = K^*(X)$ and $K_*X = K_*(X)$ for example, but include brackets where it improves readability.

1 L-complete modules

We will be working with p-complete K-theory for a prime p, and this takes values in the category of L-complete graded modules for the local ring $\mathbb{Z}_{(p)}$. The utility of working with such a category originated in work of Greenlees & May [GM92] and was made explicit by Hovey & Strickland [HS99]. The reader is also referred to Barthel & Frankland [BF15] for a more recent account.

A fundamental observations is that for any spectrum each p-completed K-theory group

$$K_n^\vee X = \pi_n(L_{K(1)}(K \wedge X))$$

is L-complete (with respect to $\mathbb{Z}_{(p)}$), i.e., $K_n^\vee X \cong L_0 K_n^\vee X$ where L_s ($s \geqslant 0$) is the left derived functor of p-adic completion on the category of $\mathbb{Z}_{(p)}$-modules. In fact L_s is trivial when $s > 1$.

When M is $\mathbb{Z}_{(p)}$-free or flat then $L_0 M = M_p^\wedge$ and $L_1 M = 0$ by [Bak09]. More generally, $L_* M$ can calculated by taking a free resolution

$$0 \leftarrow M \leftarrow F_0 \leftarrow F_1 \leftarrow 0$$

and taking homology of the induced complex

$$0 \leftarrow (F_0)_p^\wedge \leftarrow (F_1)_p^\wedge \leftarrow 0.$$

For $M = K_n(X)$ this allows us to induce up the effect of a natural transformation $\theta\colon K_n(-) \to K_n(-)$. To see how to do this we need some background.

Recall that a ring spectrum E satisfies the *Adams condition* of [Ada95] if it can be written as colimit $E = \operatorname{colim}_\alpha E_\alpha$ of dualisable spectra E_α. This condition ensures the existence of suitable resolutions for constructing Universal Coefficient spectral sequences.

In particular, KU and KO satisfy the Adams condition, see [Ada95, proposition 13.4]. The proof there uses even suspensions of skeleta of BU and BSp (with cells in even degrees); in fact these can be replaced by suspensions of skeleta of $\mathbb{C}\mathrm{P}^\infty$ and $\mathbb{H}\mathrm{P}^\infty$ by results of [AHS71].

Then the K_*-module $K_*(X)$ can be resolved using the following procedure due to Adams, see [Ada95, lemma 13.7]. Take a set of K_*-module generators of $K_*(X) = \operatorname{colim}_\alpha \pi_*(K_\alpha \wedge X)$ and form their adjoint maps $f\colon \Sigma^{n(f)} DE_\alpha \to X$ so that together these induce an epimorphism

$$\bigoplus_f K_*(DE_\alpha) = K_*\left(\bigvee_f DE_\alpha\right) \xrightarrow{\varepsilon} K_*X.$$

Here each $K_*(DE_\alpha)$ is a finitely generated free K_*-module and by work of Hovey [Hov08, theorem 3.3],

$$K_*^\vee\Big(\bigvee_f DE_\alpha\Big) \cong \Big(\bigoplus_f K_*(DE_\alpha)\Big)^{\wedge}_p.$$

which is pro-free. As K_* is a graded principal ideal domain, $\ker\varepsilon$ is also a free K_*-module, so $L_*K_*(X)$ can be calculated using the complex

$$0 \leftarrow K_*^\vee\Big(\bigvee_f DE_\alpha\Big) \leftarrow (\ker\varepsilon)^{\wedge}_p \leftarrow 0.$$

Notice also that the spectral sequence of [Hov, corollary 3.2] collapses to give a collection of short exact sequences

$$0 \to L_0K_n(X) \to K_*^\vee(X) \to L_1K_{n-1}(X) \to 0.$$

2 K-theory completed at a prime and power operations

We first recall some standard facts about the rings of p-local integers $\mathbb{Z}_{(p)}$ and p-adic integers $\mathbb{Z}_p$. By definition, if we give $\mathbb{Z}_{(p)}$ and $\mathbb{Z}_p$ the p-adic norm topologies then $\mathbb{Z}_{(p)} \subseteq \mathbb{Z}_p$ is a dense subring. The residue fields of $\mathbb{Z}_{(p)}$ and $\mathbb{Z}_p$ both agree with the finite field $\mathbb{F}_p$ which we give the discrete topology. There is a pullback square of topological multiplicative monoids

$$\begin{array}{ccc} \mathbb{Z}_{(p)}^\times & \hookrightarrow & \mathbb{Z}_{(p)} \\ \downarrow & & \downarrow \\ \mathbb{F}_p^\times & \hookrightarrow & \mathbb{F}_p \end{array}$$

and on p-adic completion this becomes the pullback square

$$\begin{array}{ccc} \mathbb{Z}_p^\times & \hookrightarrow & \mathbb{Z}_p \\ \downarrow & & \downarrow \\ \mathbb{F}_p^\times & \hookrightarrow & \mathbb{F}_p \end{array}$$

so $\mathbb{Z}_p^\times$ is the completion of $\mathbb{Z}_{(p)}^\times$ with respect to the p-adic norm.

It is known from [AHS71, AC77, Bak86, Bak00] that

$$K_0K \cong \{f(w) \in \mathbb{Q}[w, w^{-1}] : f(\mathbb{Z}_{(p)}^\times) \subseteq \mathbb{Z}_{(p)}\},$$

and K_0K is a free $\mathbb{Z}_{(p)}$-module. Since $\mathbb{Z}_{(p)}^\times$ is a dense subgroup of $\mathbb{Z}_p^\times$, we may interpret Laurent polynomials as continuous functions on $\mathbb{Z}_p^\times$ and obtain

$$K_0K \cong \{f(w) \in \mathbb{Q}[w, w^{-1}] : f(\mathbb{Z}_p^\times) \subseteq \mathbb{Z}_p\} \subseteq \mathrm{Cont}(\mathbb{Z}_p^\times, \mathbb{Z}_p),$$

where the latter is the p-adic Banach algebra of continuous maps $\mathbb{Z}_p^\times \to \mathbb{Z}_p$ equipped with the operator norm; it is known that this subring of $\mathrm{Cont}(\mathbb{Z}_p^\times, \mathbb{Z}_p)$ is dense. It follows that

$$K_0^\vee K = \pi_0((K \wedge K)\widehat{_p}) = (K_0 K)\widehat{_p},$$

where the p-adic topology involved in the completion agrees with p-adic norm topology inherited from $\mathrm{Cont}(\mathbb{Z}_p^\times, \mathbb{Z}_p)$. Therefore there is an isomorphism of p-adic Banach algebras

$$K_0^\vee K \cong \mathrm{Cont}(\mathbb{Z}_p^\times, \mathbb{Z}_p). \tag{2.1}$$

For $a \in \mathbb{Z}_{(p)}^\times$, the stable Adams operation

$$\psi^a \in K^0 K \cong \mathrm{Hom}_{\mathbb{Z}_{(p)}}(K_0 K, \mathbb{Z}_{(p)})$$

is determined by the pairing $\langle - | - \rangle : K^0 K \otimes K_0 K \to \mathbb{Z}_{(p)}$, i.e.,

$$\langle \psi^a | f(w) \rangle = f(a).$$

This extends to a continuous pairing given by

$$\langle \psi^a | f \rangle = f(a)$$

if $a \in \mathbb{Z}_p^\times$ and $f \in \mathrm{Cont}(\mathbb{Z}_p^\times, \mathbb{Z}_p)$; here ψ^a is best viewed as an element of the pro-group ring

$$\mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!] \cong (K^0 K)\widehat{_p}$$

For more details on $K_0(K)$ and $\mathrm{Cont}(\mathbb{Z}_p^\times, \mathbb{Z}_p)$, see [BR05, section 3]; for a broader overview of the connections with p-adic analysis see [Cla86].

We also recall that $K_0 K$ is a bicommutative $\mathbb{Z}_{(p)}$-Hopf algebra with coproduct Ψ given by

$$\Psi(f(w)) = f(w \otimes w)$$

and antipode χ given by

$$\chi(f(w)) = f(w^{-1}).$$

Using the linear pairing $\langle - | - \rangle$ we can obtain a left action of $K^0 K$ on $K_0 K$; for $\alpha \in K^0 K$, we write $\alpha f(w)$ for this. In particular, if $a \in \mathbb{Z}_{(p)}^\times$ this coincides with the action of the Adams operation ψ^a,

$$\psi^a f(w) = f(a^{-1} w).$$

The reason for the inverse is that we are using the standard left action of the dual of the Hopf algebra $K_0 K$ defined by

$$\alpha x = \sum_i \langle \alpha(\chi(x_i')) | x_i'' \rangle,$$

where $\Psi x = \sum_i x_i' \otimes x_i''$, $\Psi(g(w)) = g(w \otimes w)$ and $\chi(g(w)) = g(w^{-1})$.

In the p-complete setting, (stable) Adams operations are indexed by the p-adic units $\mathbb{Z}_p^\times \subseteq \mathbb{Z}_p$. It follows that there is a continuous action

$$\mathbb{Z}_p^\times \times K_r(X)\widehat{_p} \to K_r(X)\widehat{_p}; \quad (\alpha, x) \mapsto \psi^\alpha(x).$$

We use notation from [BMMS86, chapter IX] and the more recent [BF15]. For an $\mathcal{E}_\infty$ ring spectrum A there is a natural power operation $\mathrm{Q}\colon K_0^\vee A \to K_0^\vee A$ (sometimes also called θ) satisfying properties that can be deduced from those listed in [BMMS86, theorem IX.3.3] for the homology theories $K_*(-;p^r)$ with coefficients, and are discussed in [BF15, section 6], although the version there is for $\mathbb{Z}/2$-graded K-theory. However, as we are mainly interested in the case of $K_*^\vee K$ which is concentrated in even degrees, we work mostly with $K_0^\vee(-)$ but sometimes need to relate this to $K_{2n}^\vee(-)$ for an integer n.

The operation Q is neither additive nor multiplicative, but it satisfies the identities

$$\mathrm{Q}(x+y) = \mathrm{Q}\,x + \mathrm{Q}\,y + \frac{1}{p}\Big(x^p + y^p - (x+y)^p\Big), \tag{2.2a}$$

$$\mathrm{Q}(xy) = y^p\,\mathrm{Q}\,x + x^p\,\mathrm{Q}\,y + p\,\mathrm{Q}\,x\,\mathrm{Q}\,y, \tag{2.2b}$$

or equivalently the operation $\widehat{\mathrm{Q}}$ defined by

$$\widehat{\mathrm{Q}}\,x = p\,\mathrm{Q}\,x + x^p$$

is additive and multiplicative,

$$\widehat{\mathrm{Q}}(x+y) = \widehat{\mathrm{Q}}\,x + \widehat{\mathrm{Q}}\,y,$$
$$\widehat{\mathrm{Q}}(xy) = \widehat{\mathrm{Q}}\,x\,\widehat{\mathrm{Q}}\,y.$$

We also have $\mathrm{Q}\,1 = 0$, hence $\widehat{\mathrm{Q}}\,1 = 1$ and $\widehat{\mathrm{Q}}$ is a (unital) ring homomorphism. Finally, if $a \in \mathbb{Z}_{(p)}$ and $u \in \mathbb{Z}_{(p)}^\times$,

$$\mathrm{Q}(ax) = a\,\mathrm{Q}(x) + \frac{(a-a^p)}{p}x^p,$$
$$\widehat{\mathrm{Q}}(ax) = a\,\widehat{\mathrm{Q}}\,x,$$
$$\psi^u\,\mathrm{Q}(x) = \mathrm{Q}(\psi^u x).$$

When $K_r^\vee(A) = K_r(A)^\wedge_p$, the operations Q and $\widehat{\mathrm{Q}}$ are continuous with respect to the p-adic topology. This allows us to extend these identities to the case where $\alpha \in \mathbb{Z}_p^\times$,

$$\mathrm{Q}(\alpha x) = \alpha\,\mathrm{Q}(x) + \frac{(\alpha-\alpha^p)}{p}x^p,$$
$$\psi^\alpha\,\mathrm{Q}(x) = \mathrm{Q}(\psi^\alpha x),$$
$$\widehat{\mathrm{Q}}(\alpha x) = \alpha\,\widehat{\mathrm{Q}}\,x,$$
$$\psi^\alpha\,\widehat{\mathrm{Q}}(x) = \widehat{\mathrm{Q}}(\psi^\alpha x).$$

Suppose that X is an infinite loop space (and so $\Sigma_+^\infty X$ is an $\mathcal{E}_\infty$ ring spectrum). If $K_0(\Sigma_+^\infty X)$ is $\mathbb{Z}_{(p)}$-free so that $K_0^\vee(\Sigma_+^\infty X) = K_0(\Sigma_+^\infty X)^\wedge_p$ is pro-free, the diagonal map on X induces a coalgebra structure on $K_0(\Sigma_+^\infty X)$ and a topological coalgebra structure on $K_0^\vee(\Sigma_+^\infty X)$. In that situation, $\widehat{\mathrm{Q}}$ is a coalgebra morphism; in particular, $\widehat{\mathrm{Q}}$ preserves coalgebra primitives.

We also mention a useful fact about Adams operations. Let $\alpha \in \mathbb{Z}_p^\times$ and suppose that $\psi^\alpha x = \alpha^d x$. Since ψ^α is a ring homomorphism,

$$\begin{aligned}\psi^\alpha \widehat{\mathrm{Q}}\, x &= p\, \mathrm{Q}(\psi^\alpha x) + (\psi^\alpha x)^p \\ &= p\, \mathrm{Q}(\alpha^d x) + (\alpha^d x)^p \\ &= \widehat{\mathrm{Q}}(\alpha^d x),\end{aligned}$$

giving the identity

$$\psi^\alpha \widehat{\mathrm{Q}}\, x = \alpha^d \widehat{\mathrm{Q}}\, x.$$

3 Power operations on $K_0^\vee K$ and on $K_0^\vee KO$ for $p = 2$

For the case of $K_0^\vee K$ we continue to assume that p is an arbitrary prime.

We begin with the action of Q on the basic element $w \in K_0 K \subseteq K_0^\vee K$. For $a \in \mathbb{Z}_{(p)}^\times$,

$$\psi^a \mathrm{Q}(w) = \mathrm{Q}(\psi^a w) = \mathrm{Q}(a^{-1} w).$$

We write $\mathrm{Q}(w) = f_0(w)$ where $f_0 \in \mathrm{Cont}(\mathbb{Z}_p^\times, \mathbb{Z}_p)$ is the function given by $x \mapsto f_0(x)$, so we are identifying w with the inclusion function $\mathbb{Z}_p^\times \to \mathbb{Z}_p$ under the isomorphism (2.1).

By [BMMS86, theorem IX.3.3(vi)], for $k \in \mathbb{Z}$,

$$\mathrm{Q}(kw) = k\, \mathrm{Q}(w) + \frac{(k - k^p)}{p} w^p,$$

so as $\mathbb{Z}_{(p)}^\times \subseteq \mathbb{Z}_p^\times$ is dense, this defines a continuous function

$$\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times \to \mathbb{Z}_p; \quad (x, y) \mapsto x f_0(y) + \frac{(x - x^p)}{p} y^p.$$

Taking $y = 1$, this restricts to the continuous function

$$\mathbb{Z}_p^\times \to \mathbb{Z}_p; \quad x \mapsto x f_0(1) + \frac{(x - x^p)}{p},$$

and as $f_0(1) = 0$, we have

$$f_0(x) = \frac{(x - x^p)}{p}.$$

Hence we have

$$\mathrm{Q}\, w = f_0(w) = \frac{(w - w^p)}{p}. \tag{3.1}$$

For $n \in \mathbb{N}$, by [BMMS86, theorem IX.3.3(vii)]

$$\mathrm{Q}(w^{n+1}) = w^p\, \mathrm{Q}(w^n) + w^{np}\, \mathrm{Q}(w) + p\, \mathrm{Q}(w^n)\, \mathrm{Q}(w)$$

and an easy induction gives the general formula

$$\mathrm{Q}(w^n) = \frac{(w^n - w^{np})}{p}$$

for all natural numbers. We also have

$$0 = \mathrm{Q}(1) = \mathrm{Q}(w^n w^{-n}) = w^{np}\,\mathrm{Q}(w^{-n}) + w^{-np}\,\mathrm{Q}(w^n) + p\,\mathrm{Q}(w^n)\,\mathrm{Q}(w^{-n})$$

and so

$$\mathrm{Q}(w^{-n}) = \frac{w^{-n} - w^{-np}}{p}.$$

Therefore for all $n \in \mathbb{Z}$,

$$\mathrm{Q}(w^n) = \frac{w^n - w^{np}}{p}. \tag{3.2}$$

The operation $\widehat{\mathrm{Q}}$ is given by

$$\widehat{\mathrm{Q}}(w^n) = \widehat{\mathrm{Q}}(w)^n,$$

so for any $g \in \mathrm{Cont}(\mathbb{Z}_p^\times, \mathbb{Z}_p)$ we have

$$\widehat{\mathrm{Q}}(g(w)) = g(\widehat{\mathrm{Q}}\, w) = g(w),$$

and therefore

$$\mathrm{Q}(g(w)) = \frac{1}{p}(g(w) - g(w)^p).$$

This shows that the sequence of polynomial functions defined recursively by $\theta_0(w) = w$ and for $n \geqslant 1$,

$$\theta_n(w) = \frac{1}{p}(\theta_{n-1}(w) - \theta_{n-1}(w)^p),$$

is also given by

$$\theta_n(w) = \mathrm{Q}(\theta_{n-1}(w)). \tag{3.3}$$

It is known that a (topological) $\mathbb{Z}_p$-basis for $K_0^\vee K$ can be made using monomials in the $\theta_n(w)$, see [Bak86] for example. One interpretation of what we have shown is the following result which seems to have been long known to Mike Hopkins *et al*, but we do not know a published source; a referee has drawn our attention to Mark Behrens' article [DFHH14, chapter 12, section 6] which contains a related moduli-theoretic interpretation of such θ-algebras which may lead to similar results. We interpret the operation Q as a realisation of an action of θ and therefore $K_0^\vee K$ becomes a p-complete $\mathbb{Z}_p$-θ-algebra [Bou96, BF15].

Proposition 3.1. The p-complete $\mathbb{Z}_p$-θ-algebra $K_0^\vee K$ is generated by the element w. Hence $K_0^\vee K$ is a quotient of the free p-complete $\mathbb{Z}_p$-θ-algebra $K_0^\vee(\mathbb{P}S^0)$, namely

$$K_0^\vee K \cong \mathbb{Z}_p[\theta^s(w) : s \geqslant 0]^\wedge_p \Big/ ((\theta^s(w)^p - \theta^s(w) + p\theta^{s+1}(w) : s \geqslant 0)).$$

Here the quotient is taken with respect to the p-adic closure of the ideal generated by the stated elements, indicated by the use of $((-))$ rather than $(-)$. This shows that apart from the p-adic completion involved, $K_0^\vee K$ is a colimit of Artin-Schreier extensions of the form

$$\mathbb{Z}_p[X]/(X^p - X + pa)$$

whose mod p reduction is the étale $\mathbb{F}_p$-algebra

$$\mathbb{F}_p[X]/(X^p - X) \cong \prod_{0\leqslant r\leqslant p-1} \mathbb{F}_p.$$

Our discussion also shows that the antipode of $K^\vee_0(K)$, χ satisfies

$$\chi\,\mathrm{Q} = \mathrm{Q}\,\chi. \tag{3.4}$$

Suppose that A is an $\mathcal{E}_\infty$ ring spectrum (or a $K(1)$-local $\mathcal{E}_\infty$ ring spectrum). Then we may consider $K^\vee_\bullet(A)$ where $K^\vee_\bullet(-)$ denotes the $\mathbb{Z}/2$-graded p-complete theory. The power operation Q intertwines with the coaction as described in [Bak15, (2.5)], giving

$$\Psi\,\mathrm{Q}\,x = \mathrm{Q}(\Psi x) \tag{3.5}$$

since the antipode χ satisfies (3.4) and we have a simpler situation compared to ordinary mod p homology where the dual Steenrod algebra supports two distinct Dyer-Lashof structures related by the antipode.

We now give a brief description of the modification required to describe power operations in $K^\vee_0 KO$ at the prime $p = 2$. For $KO_*KO_{(2)}$, results of [AC77, AHS71] give

- for all $m \in \mathbb{Z}$, $KO_mKO_{(2)} \cong KO_m \otimes KO_0KO_{(2)}$;
- $KO_0KO_{(2)}$ is a countable free $\mathbb{Z}_{(2)}$-module;
- $KO_0KO_{(2)} = \{f(w) \in \mathbb{Q}[w^2, w^{-2}] : f(\mathbb{Z}_2^\times) \subseteq \mathbb{Z}_2\}$.

Passing to $K^\vee_0 KO$, recalling that the squaring homomorphism

$$\mathbb{Z}_2^\times = \{\pm 1\} \times (1 + 4\mathbb{Z}_2) \to 1 + 8\mathbb{Z}_2 \subseteq \mathbb{Z}_2^\times$$

is surjective, the natural $\mathcal{E}_\infty$ morphism $KO \to KU$ induces a monomorphism of 2-complete θ-algebras $K^\vee_0(KO) \to K^\vee_0(K)$ coinciding with the inclusion of the continuous functions factoring through $(-)^2$.

It is clear that Q restricts to $K^\vee_0 KO$ and is given by

$$\mathrm{Q}(f) = \frac{(f - f^2)}{2}.$$

The following elements defined inductively provide a topological basis for $K^\vee_0 KO$:

$$\Theta_0(w) = \frac{1 - w^2}{8}, \qquad \Theta_n(w) = \frac{\Theta_{n-1}(w) - \Theta_{n-1}(w)^2}{2} \quad (n \geqslant 1).$$

Then the distinct monomials $\Theta_0(w)^{\varepsilon_0}\Theta_1(w)^{\varepsilon_1} \cdots \Theta_\ell(w)^{\varepsilon_\ell}$ with $\varepsilon_j = 0, 1$ form a topological basis. Here is the analogue of Proposition 3.1.

Proposition 3.2. The 2-complete $\mathbb{Z}_2$-θ-algebra $K^\vee_0 KO$ is a quotient of the free 2-complete $\mathbb{Z}_2$-θ-algebra generated by the element $\Theta_0(w)$, i.e.,

$$K^\vee_0 KO \cong \mathbb{Z}_2[\Theta_s(w) : s \geqslant 0]^\wedge_2 / \left(\!\left(\Theta_s(w)^2 - \Theta_s(x) + 2\Theta_{s+1}(x) : s \geqslant 0\right)\!\right).$$

4 The completed K-theory of free algebras

In this section we will describe $K_0^\vee(\mathbb{P}X)$, at least for spectra X for which $K_0^\vee X$ is suitably restricted. For our purposes, it will suffice to assume that X is a CW spectrum with only finitely many even dimensional cells. It will be useful to examine how $K_0^\vee(\mathbb{P}X)$ behaves for such complexes.

Suppose that the $(n-1)$-skeleton $X^{[n-1]}$ of X is defined. Then the n-skeleton $X^{[n]}$ is a pushout defined by a diagram of the form

$$\begin{array}{ccc} \bigvee_i S^{n-1} & \longrightarrow & \bigvee_i D^n \\ \downarrow & \ulcorner & \downarrow \\ X^{[n-1]} & \longrightarrow & X^{[n]} \end{array}$$

for a finite wedge of spheres $\bigvee_i S^{n-1}$. Similarly there is a pushout diagram of commutative S-algebras

$$\begin{array}{ccc} \mathbb{P}(\bigvee_i S^{n-1}) & \longrightarrow & \mathbb{P}(\bigvee_i D^n) \\ \downarrow & \ulcorner & \downarrow \\ \mathbb{P}(X^{[n-1]}) & \longrightarrow & \mathbb{P}(X^{[n]}) \end{array}$$

so $(\mathbb{P}X)^{\langle n\rangle} = \mathbb{P}(X^{[n]})$ is the $\mathcal{E}_\infty$ n-skeleton of the CW commutative S-algebra $\mathbb{P}X$.

If the cells of X are all even dimensional, we only encounter pushout diagrams of the form

$$\begin{array}{ccc} \mathbb{P}(\bigvee_i S^{2m-1}) & \longrightarrow & \mathbb{P}(\bigvee_i D^{2m}) \\ \downarrow & \ulcorner & \downarrow \\ (\mathbb{P}X)^{\langle 2m-2\rangle} & \longrightarrow & (\mathbb{P}X)^{\langle 2m\rangle} \end{array}$$

where

$$(\mathbb{P}X)^{\langle 2m\rangle} \cong (\mathbb{P}X)^{\langle 2m-2\rangle} \wedge_{\mathbb{P}(\bigvee_i S^{2m-1})} \mathbb{P}(\bigvee_i D^{2m}).$$

To calculate $K_*^\vee((\mathbb{P}X)^{\langle 2m\rangle})$ we may use a Künneth spectral sequence of the form

$$\mathrm{E}^2_{s,t} = \mathrm{Tor}_{s,t}^{K_*^\vee(\mathbb{P}(\bigvee_i S^{2m-1}))}(K_*^\vee((\mathbb{P}X)^{\langle 2m-2\rangle}), K_*) \implies K_{s+t}^\vee((\mathbb{P}X)^{\langle 2m\rangle}), \tag{4.1}$$

where the internal t grading is in $\mathbb{Z}/2$, i.e., it is an integer modulo 2. This is essentially described in [EKMM97], but we will require its multiplicativity, and also the fact that it inherits an action of power operations. The latter structure is constructed in a similar fashion to the mod p Dyer-Lashof operations in [LM75].

Proposition 4.1. The spectral sequence (4.1) collapses at E^2 to give

$$K_{s+t}^\vee((\mathbb{P}X)^{\langle 2m\rangle}) = K_{s+t}^\vee((\mathbb{P}X)^{\langle 2m-2\rangle})[\mathrm{Q}^s x_i : s \geqslant 0,\ i\,]_p^\wedge,$$

where each x_i is in even degree.

Proof. Recall from [BF15] that

$$K^\vee_*\left(\mathbb{P}\left(\bigvee_i S^{2m-1}\right)\right) = \Lambda(z_i)\widehat{_p},$$

the p-completed exterior algebra on odd degree generators $z_i \in K^\vee_1(\mathbb{P}(\bigvee_i S^{2m-1}))$, each of which originates on a wedge summand.

The E^2-term is a divided power algebra over $K^\vee_*((\mathbb{P}X)^{\langle 2m-2\rangle})$ on generators of bidegree $(1,1)$, each represented in the cobar complex by $[\mathrm{Q}^s\, z_i]$. We will write $\gamma_r([\mathrm{Q}^s\, z_i])$ for the r-th divided power of this element and recall that the particular elements $\gamma_{(r)}([\mathrm{Q}^s\, z_i]) = \gamma_{p^r}([\mathrm{Q}^s\, z_i])$ generate the algebra subject to relations of the form

$$\gamma_{(r)}([\mathrm{Q}^s\, z_i])^p = \binom{p^{r+1}}{p^r,\ldots,p^r}\gamma_{(r+1)}([\mathrm{Q}^s\, z_i]),$$

where the multinomial coefficient satisfies

$$\binom{p^{r+1}}{p^r,\ldots,p^r} = pt$$

for some integer t not divisble by p. For degree reasons there can only be trivial differentials, so the only issue still to be resolved is that of the multiplicative structure.

We follow a line of argument similar to that of [LM75]. In the spectral sequence we have

$$\mathrm{Q}[z_i] = [\mathrm{Q}\, z_i],$$

so it only remains to relate this element to a p-th power in the target of the spectral sequence. By [BMMS86, chapter IX, theorem 3.3(viii)], if Z_i is represented by $[z_i]$, then $Z_i^p + p\,\mathrm{Q}\,Z_i$ is represented by $[\mathrm{Q}\, z_i]$, therefore Z_i^p is represented by

$$(1-p)[\widehat{\mathrm{Q}}\, z_i] \equiv [\widehat{\mathrm{Q}}\, z_i] \pmod{p}.$$

It follows that each such Z_i has non-trivial p-th power also represented in the 1-line. By induction this can be extended to show that each $\gamma_{(r)}([\mathrm{Q}^s\, z_i])$ represents an element with non-trivial p-th power. Finally, an easy argument shows that the target is a completed polynomial algebra as stated. Q.E.D.

It is also useful to generalise this to the case of a CW spectrum Y with chosen 0-cell $S^0 \to Y$, where $S^0 \xrightarrow{\sim} S$ is the functorial cofibrant replacement of S in the model category of S-modules. We may then consider the reduced free commutative S-algebras $\widetilde{\mathbb{P}}Y$ which is defined as the homotopy pushout of the diagram of solid arrows

$$\begin{array}{ccc} \mathbb{P}S^0 & \longrightarrow & \mathbb{P}Y \\ \downarrow & \ulcorner & \vdots \\ S & \cdots\cdots\to & \widetilde{\mathbb{P}}Y \end{array}$$

where the vertical map is the canonical multiplicative extension of $S^0 \to S$; see [Bak12] for more on this construction. As a particular case, we can consider a map $f\colon S^{2m-1} \to S^0$ and form its mapping cone $C_f = S^0 \cup_f D^{2m}$. Then take $S/\!/f = \widetilde{\mathbb{P}}C_f$ to be a homotopy pushout for the diagram

$$\begin{array}{ccc} \mathbb{P}S^0 & \longrightarrow & \mathbb{P}C_f \\ \downarrow & \ulcorner & \downarrow \\ S & \dashrightarrow & S/\!/f \end{array}$$

and there is an associated Künneth spectral sequence

$$\mathrm{E}^2_{s,t} = \mathrm{Tor}^{K^\vee_*(\mathbb{P}S^0)}(K_*, K^\vee_*(\mathbb{P}C_f)) \implies K^\vee_{s+t}(S/\!/f). \tag{4.2}$$

It is easily seen that

$$K^\vee_*(\mathbb{P}S^0) = \mathbb{Z}_p[\mathrm{Q}^s\, x_0 : s \geqslant 0]^\wedge_p$$

is a subalgebra of

$$K^\vee_*(\mathbb{P}C_f) = \mathbb{Z}_p[\mathrm{Q}^s\, x_0, \mathrm{Q}^s\, x_{2m} : s \geqslant 0]^\wedge_p,$$

and the spectral sequence has

$$\mathrm{E}^2_{0,*} = K_* \otimes_{K^\vee_*(\mathbb{P}S^0)} K^\vee_*(\mathbb{P}C_f) = \mathbb{Z}_p[\mathrm{Q}^s\, x_{2m} : s \geqslant 0]^\wedge_p, \qquad \mathrm{E}^2_{r,*} = 0 \quad (r \geqslant 1).$$

This discussion establishes

Proposition 4.2. We have

$$K^\vee_*(S/\!/f) = \mathbb{Z}_p[\mathrm{Q}^s\, x_{2m} : s \geqslant 0]^\wedge_p.$$

Provided we know the coaction for $K^\vee_*(C_f)$, that for $K^\vee_*(S/\!/f)$ follows formally. In general we have only the following possible form of coaction,

$$\Psi(x_{2m}) = w^m \otimes x_{2m} + c(f)(1 - w^m),$$

where $c(f)$ is a certain kind of rational number. Then

$$\Psi(\mathrm{Q}^s\, x_{2m}) = \mathrm{Q}^s(\Psi x_{2m})$$

which involves iterated application of Q.

5 Some examples based on elements of Hopf invariant 1

Throughout this section we assume that $p = 2$.

We will consider the examples $S/\!/\eta$ and $S/\!/\nu$ previously discussed in [Bak18]. Similar considerations apply to other examples constructed using elements in the image of the J-homomorphism at an arbitrary prime. In order to study these examples, it is necessary to determine the $K^\vee_0 K$-coaction on $K^\vee_0(S/\!/f)$. Our goal is to explain why the following algebraic results holds.

Theorem 5.1. There are continuous epimorphisms of 2-complete $\mathbb{Z}_2$-θ-algebras

$$K^\vee_0(S/\!/\eta) \to K^\vee_0 K, \quad K^\vee_0(S/\!/\nu) \to K^\vee_0 K,$$

where in each case the domain is a free θ-algebra. Moreover, these are induced by morphisms of $\mathcal{E}_\infty$ ring spectra $S/\!/\eta \to K$ and $S/\!/\nu \to K$.

Proof. We give the ingredients required for the case of η, the other being similar.

We will use the following elements $\Phi_s = \Phi_s(w)$ $(s \geqslant 0)$ of $K_0^\vee K$:

$$\Phi_0 = \frac{(1-w)}{2}, \qquad \Phi_n = \frac{(\Phi_{n-1} - \Phi_{n-1}^2)}{2} \quad (n \geqslant 1). \tag{5.1}$$

By results of [Bak86], $K_0^\vee K$ has a topological basis consisting of the monomials

$$\Phi_0^{\varepsilon_0}\Phi_1^{\varepsilon_1}\cdots\Phi_\ell^{\varepsilon_\ell} \quad (\varepsilon_i = 0,1). \tag{5.2}$$

If we view these as continuous functions on $\mathbb{Z}_2^\times$, then for a 2-adic unit α expressed as

$$\alpha = 1 - (2a_0 + 2^2a_1 + \cdots + 2^{r+1}a_r + \cdots)$$

with $a_r = 0, 1$, in $\mathbb{Z}_2$ we have

$$\Phi_r(\alpha) \equiv a_r \pmod 2.$$

We also know that $\mathrm{Q}\,\Phi_s = \Phi_{s+1}$, hence $\Phi_s = \mathrm{Q}^s\,\Phi_0$.

In the case where $f = \eta$, we can take the generator x_2 to have coaction

$$\Psi(x_2) = \Phi_0 \otimes 1 + w \otimes x_2 = \Phi_0 + wx_2, \tag{5.3}$$

where we suppress the tensor product symbols when the meaning seems clear without them. For the coproduct in $K_0^\vee K$ we have

$$\Psi\Phi_0 = \Phi_0 \otimes 1 + w \otimes \Phi_0,$$

and also

$$\Psi\,\mathrm{Q}\,x_2 = w\,\mathrm{Q}\,x_2 + w\Phi_0x_2^2 - w\Phi_0x_2 + \Phi_1.$$

Without further calculation we see that there is a homomorphism of topological comodule algebras

$$\mathbb{Z}_2[x_2]\widehat{_2} \to K_0^\vee K; \quad x_2 \mapsto \Phi_0.$$

This is induced from a morphism of $\mathcal{E}_\infty$ ring spectra $S/\!/\eta \to K$ arising from the fact that the composition of $\eta\colon S^1 \to S$ with the unit $S \to K$ is null homotopic. Therefore there is an extension to a continuous epimorphism

$$K_0^\vee(S/\!/\eta) \to K_0^\vee K; \quad \mathrm{Q}^s\,x_2 \mapsto \Phi_s.$$

This displays $K_0^\vee K$ as a quotient of the free θ-algebra $K_0^\vee(S/\!/\eta)$ as in Proposition 3.1. Q.E.D.

Theorem 5.2. There is a $K(1)$-local equivalence

$$S/\!/\eta \xrightarrow{\sim} \prod_{j\geqslant 0} K.$$

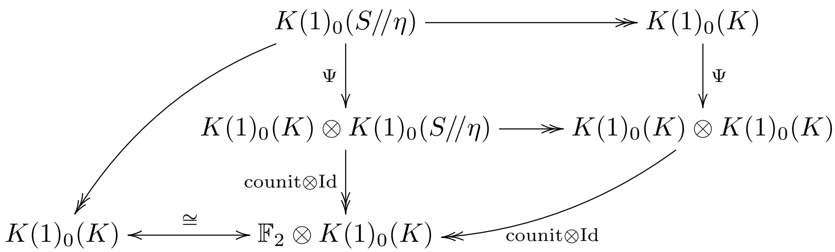
Outline of Proof. We will use the homology theory $K(1)_*(-)$, i.e., mod 2 K-theory. For the spectra we are considering, odd degree groups are trivial so we can consider the ungraded $\mathbb{F}_2$-vector spaces obtained from $K(1)_0(-)$. This functor takes values in the category of $K(1)_0(K)$-comodules, where

$K(1)_0(K) \subseteq K(1)_0(K(1))$ is the subHopf algebra called the Morava stabiliser (Hopf) algebra and often denoted (rather confusingly) $K(1)_0K(1)$ in the literature.

Using the basis of (5.2), we see that the group-like element $w = 1 - 2\Theta_0 \in K_0^\vee(K)$ reduces mod 2 to 1 and this is the only group-like element of $K(1)_0(K)$. The reductions mod 2 of this basis give a basis for $K(1)_0(K)$ and the increasing coradical filtration $F_kK(1)_0(K)$ $(k \geqslant 0)$ defined by Laures & Schuster [LS19, section 2] has

$$F_kK(1)_0(K) = \mathbb{F}_2\{1, \Phi_0, \ldots, \Phi_{k-1}\}.$$

The epimorphism $K_0^\vee(S/\!/\eta) \to K_0^\vee(K)$ gives rise to a commutative diagram of $K(1)_0(K)$-comodule algebras of the following shape.



The coaction for $K(1)_0(S/\!/\eta) = \mathbb{F}_2[\mathrm{Q}^s : s \geqslant 0]$ is computable recursively, for example (5.3) gives

$$\Psi(x_2) = \Phi_0 \otimes 1 + 1 \otimes x_2.$$

and

$$\Psi(\mathrm{Q}\, x_2) = \Phi_1 \otimes 1 + \Phi_0 \otimes (x_2 + x_2^2) + 1 \otimes \mathrm{Q}\, x_2.$$

Now we can use an appropriate version of the classic Milnor-Moore Theorem of [MM65], see for example Laures & Schuster [LS19, theorem 2.8], to deduce that

$$K(1)_0(S/\!/\eta) \cong K(1)_0(K) \otimes \mathrm{Prim}_{K(1)_0(K)} K(1)_0(S/\!/\eta),$$

where $\mathrm{Prim}_{K(1)_0(K)} K(1)_0(S/\!/\eta) \subseteq K(1)_0(S/\!/\eta)$ is the subalgebra of primitives. To use this, we need to determine filtration

$$F_kK(1)_0(S/\!/\eta) = \Psi^{-1}(F_kK(1)_0(K) \otimes K(1)_0(S/\!/\eta)) \quad (k \geqslant 0)$$

associated with the coradical filtration. By induction we find that

$$F_kK(1)_0(S/\!/\eta) = \mathbb{F}_2[x_2, \ldots, \mathrm{Q}^{k-1}\, x_2].$$

We need to check the condition that the surjection $K(1)_0(S/\!/\eta) \to K(1)_0(K)$ is a $\star$-isomorphism as in [LS19, definition 2.6] (note that as we are working with *left* comodules we need to consider graded *right* primitives). Using an induction on k, we find that the k-graded right primitive subspace is $F_kK(1)_0(S/\!/\eta)$ and this maps onto $F_kK(1)_0(K)$ which is the k-graded right primitive subspace of $K(1)_0(K)$.

Dualising and taking care with the inherent linearly compact topologies and completed tensor products involved, we obtain an isomorphism of left topological $K(1)^0(K)$-modules

$$K(1)^0(S/\!/\eta) \cong K(1)^0(K)\widehat{\otimes}(\mathrm{Prim}_{K(1)_0(K)} K(1)_0(S/\!/\eta))^\dagger,$$

where $V^\dagger$ denotes the set of functionals supported on finite dimensional subspaces of the vector space V. Choosing a topological basis $\{b_\alpha : \alpha \in A\}$ for $(\mathrm{Prim}_{K(1)_0(K)} K(1)_0(S/\!/\eta))^\dagger$, we may lift each b_α to an element $\widetilde{b_\alpha} \in K^0(S/\!/\eta)$ since $K(1)_1(S/\!/\eta) = 0$. This gives a map $S/\!/\eta \to \prod_{\alpha\in A} K$ which induces a $K(1)$-isomorphism, hence it is a $K(1)$-local equivalence. In fact A can be taken to be countable, so we might as well index on the natural numbers. Q.E.D.

Notice that there is an $\mathcal{E}_\infty$ morphism $S/\!/\eta \to kU$ which induces a surjection on $\pi_*(-)$ but not on $H_*(-;\mathbb{F}_2)$. Hence kU cannot be a retract of $S/\!/\eta$ 2-locally or after 2-completion. However, multiplication by the Bott map induces a cofibre sequence

$$\Sigma^2 kU \to kU \to H\mathbb{Z}$$

where $KU \wedge H\mathbb{Z}$ is rational. Therefore $\Sigma^2 kU \to kU$ is a $K(1)$-local equivalence, so it induces an isomorphism on $K^\vee(-)$.

Notice that

$$w^2 = (1-2\Phi_0)^2 = 1-4(\Phi_0-\Phi_0^2) = 1-8\Phi_1,$$

so

$$1-w^2 = 8\Phi_1.$$

Similarly,

$$w^4 = 1-16(\Phi_1-\Phi_1^2)+48\Phi_1^2,$$

and therefore

$$1-w^4 = 16(\Phi_1-\Phi_1^2)-48\Phi_1^2 = 32\Phi_2-48\Phi_1^2.$$

Such identities allow us to describe the groups

$$\mathrm{Ext}^{1,2n}_{K_*K}(K_*,K_*) = \mathrm{Pr}\, K_{2n}K/(\eta_{\mathrm{L}}-\eta_{\mathrm{R}})K_{2n}$$

that detect the 2-primary part of image of the J-homomorphism through the e-invariant. Here Pr denotes the subgroup of primitive elements which satisfy

$$\Psi(x) = 1\otimes x + x\otimes 1,$$

and $\eta_{\mathrm{L}}, \eta_{\mathrm{R}}$ denote the left and right units respectively. When $n = 1, 2, 4$, these groups are cyclic with the following orders and generators:

- 2, generator represented by $u\Phi_0$;
- 8, generator represented by $u^2\Phi_1$;
- 16, generator represented by $u^4(2\Phi_2 - 3\Phi_1^2)$.

Here we write $u \in K_2$ for the Bott generator. In the first and last cases, a generator of $(\operatorname{im} J)_{2n-1}$ maps to the generator, but in the middle case only the multiples of $2u^2\Phi_1$ are hit; for details see [MRW77, Rav78].

For $S/\!/\nu$ and $S/\!/\sigma$,

$$K_0^\vee(S/\!/\nu) = \mathbb{Z}_2[\mathrm{Q}^s\, x_4 : s \geqslant 0]^{\widehat{}}_2, \quad K_0^\vee(S/\!/\sigma) = \mathbb{Z}_2[\mathrm{Q}^s\, x_8 : s \geqslant 0]^{\widehat{}}_2,$$

we have the coactions

$$\Psi x_4 = w^2 \otimes x_4 + 2\Phi_1, \quad \Psi x_8 = w^4 \otimes x_8 + 2\Phi_2 - 3\Phi_1^2.$$

Finally, we note that there is an $\mathcal{E}_\infty$ morphism $S/\!/\nu \to kO$ inducing an epimorphism on $\pi_*(-)$ which is not an epimorphism on $H_*(-;\mathbb{F}_2)$. The composition $S/\!/\nu \to kO \to KO$ induces a $K(1)$-local splitting whose proof is similar to that of Theorem 5.2.

Theorem 5.3. There is a $K(1)$-local equivalence

$$S/\!/\nu \xrightarrow{\sim} \prod_{j\geqslant 0} \Sigma^{4\rho(j)} KO,$$

for some numerical function ρ taking values in $\{0, 1\}$.

Remark 5.4. The case of $S/\!/\sigma$ should also be amenable to a similar analysis, however we have not found convenient way to formalise an argument for this case.

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