Available online at www.sciencedirect.com
ScienceDirect
Journal of Algebra 309 (2007) 386–404
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Demazure structure inside Kirillov–Reshetikhin crystals

Ghislain Fourier [a], Anne Schilling [b,*,1], Mark Shimozono [c,2]

[a] *Mathematisches Institut der Universität zu Köln, Weyertal 86-90, 50931 Köln, Germany*
[b] *Department of Mathematics, University of California, One Shields Avenue, Davis, CA 95616-8633, USA*
[c] *Department of Mathematics, Virginia Tech, Blacksburg, VA 24061-0123, USA*

Received 18 May 2006
Available online 16 October 2006
Communicated by Masaki Kashiwara

Abstract

The conjecturally perfect Kirillov–Reshetikhin (KR) crystals are known to be isomorphic as classical crystals to certain Demazure subcrystals of crystal graphs of irreducible highest weight modules over affine algebras. Under some assumptions we show that the classical isomorphism from the Demazure crystal to the KR crystal, sends zero arrows to zero arrows. This implies that the affine crystal structure on these KR crystals is unique.

Keywords: Quantum affine algebras; Crystal bases; Demazure crystals; Kirillov–Reshetikhin crystals

1. Introduction

The irreducible finite-dimensional modules over a quantized affine algebra $U'_q(\mathfrak{g})$ were classified by Chari and Pressley [3,4] in terms of Drinfeld polynomials. We are interested in the subfamily of such modules which possess a global crystal basis. Kirillov–Reshetikhin (KR) modules are finite-dimensional $U'_q(\mathfrak{g})$-modules $W^{r,s}$ that were introduced in [7,8]. It is expected that

* Corresponding author.
E-mail addresses: gfourier@mi.uni-koeln.de (G. Fourier), anne@math.ucdavis.edu (A. Schilling), mshimo@math.vt.edu (M. Shimozono).
URLs: http://www.math.ucdavis.edu/~anne (A. Schilling), http://www.math.vt.edu/people/mshimo/ (M. Shimozono).
[1] The author was supported in part by NSF grant DMS-0200774 and DMS-0501101.
[2] The author was supported in part by NSF grant DMS-0401012.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2006.09.019

each KR module has a crystal basis $B^{r,s}$, and that every irreducible finite-dimensional $U_q'(\mathfrak{g})$-module with crystal basis, is a tensor product of the crystal bases of KR modules.

The KR modules $W^{r,s}$ are indexed by a Dynkin node r of the classical subalgebra (that is, the distinguished simple Lie subalgebra) $\mathfrak{g}_0$ of $\mathfrak{g}$ and a positive integer s. In general the existence of $B^{r,s}$ remains an open question. For type $A_n^{(1)}$ the crystal $B^{r,s}$ is known to exist [18] and its combinatorial structure has been studied [24]. In many cases, the crystals $B^{1,s}$ and $B^{r,1}$ for nonexceptional types are also known to exist and their combinatorics has been worked out in [16,18] and [9,14], respectively.

Viewed as a $U_q(\mathfrak{g}_0)$-module by restriction, $W^{r,s}$ is generally reducible; its decomposition into $U_q(\mathfrak{g}_0)$-irreducibles was conjectured in [7,8]. This was verified by Chari [1] for the nontwisted cases.

Kashiwara [13] conjectured that as classical crystals, many of the KR crystals (the ones conjectured to be perfect in [7,8]) are isomorphic to certain Demazure subcrystals of affine highest weight crystals. Kashiwara's conjecture was confirmed by Fourier and Littelmann [5] in the untwisted cases and Naito and Sagaki [22] in the twisted cases.

In this paper we prove that the classical isomorphism from the Demazure crystals to KR crystals sends zero arrows to zero arrows (see Theorem 4.4). It is not an affine crystal isomorphism but becomes an isomorphism after tensoring with an appropriate affine highest weight crystal. This recovers some of the isomorphisms given by the Kyoto path model. We emphasize this is accomplished without the assumption of perfectness of the KR crystals. The automorphisms on the crystals that are used in the definition of the ground state path in the Kyoto path model, come from affine Dynkin diagram automorphisms which can be calculated using the factorization of a translation element in the extended affine Weyl group in our setting. For the proof of our results we require the assumptions of regularity of KR crystals, the existence and uniqueness of a certain special element u in a KR crystal, and the existence of automorphisms on KR crystals coming from certain Dynkin automorphisms (see Assumption 1). We show that under these assumptions, the KR crystals admit a unique affine crystal structure (see Corollary 4.6), and we give an algorithm which shows that twofold tensor products of KR crystals are connected (see Corollary 6.1). We expect that Assumption 1 holds, that is, if the existence of the KR crystals were established these hypotheses could be removed.

In Section 2 we establish notation and review some results about the extended affine Weyl group. The definition of Demazure crystals and KR crystals is given in Section 3. Section 4 contains our main result stated in Theorem 4.4 showing that all zero arrows of the Demazure crystal are present in the KR crystal. In Section 5 we provide explicit sequences of lowering operators leading from the special element u of a KR crystal to all classical highest weight elements of the KR crystal. The connectedness of tensor products of KR crystals and an application regarding the algorithmic calculation of the combinatorial R-matrix can be found in Section 6.

2. Notation and basics

2.1. *Affine Kac–Moody algebras*

Let $\mathfrak{g}$ be an affine Kac–Moody algebra with Cartan subalgebra $\mathfrak{h}$, Dynkin node set $I = \{0, 1, \ldots, n\}$, Cartan matrix $A = (a_{ij})_{i,j\in I}$, realized by the set of linearly independent simple roots $\{\alpha_i \mid i \in I\} \subset \mathfrak{h}^*$ and simple coroots $\{\alpha_i^\vee \mid i \in I\} \subset \mathfrak{h}$, such that $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$ [10]. Let $d \in \mathfrak{h}$ be the scaling element, which is any element such that $\langle d, \alpha_i \rangle = 0$ for $i \in I \setminus \{0\}$ and $\langle d, \alpha_0 \rangle = 1$. Let $(a_i \mid i \in I)$ be the unique tuple of relatively prime positive integers that give a linear dependence relation among the columns of A, and let $(a_i^\vee \mid i \in I)$ be the tuple for the rows of A. Let $\delta = \sum_{i\in I} a_i\alpha_i$ be the null root, $\theta = \sum_{i\in I\setminus\{0\}} a_i\alpha_i$, and $c = \sum_{i\in I} a_i^\vee\alpha_i^\vee$ the canonical central element. We have $\langle d, \delta\rangle = a_0$. Let $\{\Lambda_i \mid i \in I\} \subset \mathfrak{h}^*$ be the fundamental weights, which, together with δ/a_0, are defined to the dual basis to the basis $\{\alpha_i^\vee \mid i \in I\} \cup \{d\}$ of $\mathfrak{h}$. In particular $\langle \alpha_i^\vee, \Lambda_j\rangle = \delta_{ij}$. Let $P = \bigoplus_{i\in I} \mathbb{Z}\Lambda_i \oplus \mathbb{Z}(\delta/a_0) \subset \mathfrak{h}^*$ be the weight lattice, $P^+ = \bigoplus_{i\in I} \mathbb{Z}_{\geqslant 0}\Lambda_i \oplus \mathbb{Z}(\delta/a_0) = \{\lambda \in P \mid \langle\alpha_i^\vee, \lambda\rangle \geqslant 0 \text{ for all } i \in I\}$ the set of dominant weights and $Q = \bigoplus_{i\in I} \mathbb{Z}\alpha_i \subset \mathfrak{h}^*$ the root lattice. The level of a weight $\lambda \in P$ is defined by $\langle c, \lambda\rangle$. Let W be the affine Weyl group, generated by the simple reflections $\{s_i \mid i \in I\}$. W acts on P by $s_i\lambda = \lambda - \langle \alpha_i^\vee, \lambda\rangle\alpha_i$.

Let $(\cdot \mid \cdot)$ be the nondegenerate W-invariant symmetric form on $\mathfrak{h}^*$; it is defined by $(\alpha_i \mid \alpha_j) = a_i^\vee a_i^{-1} a_{ij}$ for $i, j \in I$, $(\alpha_i \mid \Lambda_0) = 0$ for $i \in I \setminus \{0\}$, $(\alpha_0 \mid \Lambda_0) = a_0^{-1}$, and $(\Lambda_0 \mid \Lambda_0) = 0$. One may check that [10, (6.4.1)]

$$(\theta \mid \theta) = 2a_0 = \begin{cases} 4 & \text{for } A_{2n}^{(2)}, \\ 2 & \text{otherwise.} \end{cases} \tag{2.1}$$

The pairing $(\cdot \mid \cdot)$ induces an isomorphism $\nu : \mathfrak{h} \to \mathfrak{h}^*$ given by $\langle \nu(h), h'\rangle = (h \mid h')$ for all $h, h' \in \mathfrak{h}$. So $\nu(\alpha_i^\vee) = a_i(a_i^\vee)^{-1}\alpha_i$ for $i \in I$, $\nu(d) = a_0\Lambda_0$, and $\nu(c) = \delta$. Define $\theta^\vee \in \mathfrak{h}$ by $\nu(\theta^\vee) = 2\theta/(\theta \mid \theta) = \theta/a_0$.

Let $\mathfrak{g}_0 \subset \mathfrak{g}$ be the simple Lie subalgebra whose Dynkin node set is $I \setminus \{0\}$, with Weyl group $W_0 \subset W$, root lattice Q_0, weight lattice P_0, and fundamental weights $\{\omega_i \mid i \in I \setminus \{0\}\} \subset P_0$.

Let $P' = P/\mathbb{Z}(\delta/a_0)$. The natural projection $P' \to P_0$ has a section $P_0 \to P'$ defined by $\omega_i \mapsto \Lambda_i - a_i^\vee\Lambda_0$ for $i \in I \setminus \{0\}$. The image of this section is the set of elements in P' of level zero.

2.2. *Dynkin automorphisms*

Let X denote the affine Dynkin diagram and $\mathrm{Aut}(X)$ denote the group of automorphisms of X. By definition an element of $\mathrm{Aut}(X)$ is a permutation of the Dynkin node set I which preserves the kind of bonds between nodes. Observe that

$$\begin{aligned} a_{\tau(i)} &= a_i, \\ a_{\tau(i)}^\vee &= a_i^\vee, \end{aligned} \quad \text{for all } i \in I \text{ and } \tau \in \mathrm{Aut}(X). \tag{2.2}$$

There is an action of $\mathrm{Aut}(X)$ on P given by

$$\sigma(\Lambda_i) = \Lambda_{\sigma(i)}, \quad \text{for } i \in I,$$
$$\sigma(\delta) = \delta,$$

for $\sigma \in \mathrm{Aut}(X)$. By (2.2) this action restricts to an action of $\mathrm{Aut}(X)$ on P_0 called the level zero action.

2.3. *Translations*

For $\alpha \in P_0$, define the element $t_\alpha \in \mathrm{Aut}(P)$ by [10, (6.5.2)]

$$t_\alpha(\lambda) = \lambda + \langle c, \lambda\rangle\alpha - \left((\lambda \mid \alpha) + \frac{1}{2}(\alpha \mid \alpha)\langle c, \lambda\rangle\right)\delta. \tag{2.3}$$

The map $\alpha \mapsto t_\alpha$ defines an injective group homomorphism $P_0 \to \mathrm{Aut}(P)$ whose image shall be denoted $T(P_0)$. For any $w \in W_0$,

$$w t_\alpha w^{-1} = t_{w(\alpha)}. \tag{2.4}$$

Therefore, $W_0 \ltimes T(P_0)$ acts on P. There is an induced action of $W_0 \ltimes T(P_0)$ on P' that preserves the level of a weight. For every $m \in \mathbb{Z}$ there is an action of $W_0 \ltimes T(P_0)$ on P_0 called the level m action, given by $w *_m \mu = w(m\Lambda_0 + \mu) - m\Lambda_0$ for $\mu \in P_0$. Under the level one action, the element t_α is precisely translation by α.

2.4. Extended affine Weyl group

For each $i \in I \setminus \{0\}$, define $c_i = \max(1, a_i/a_i^\vee)$; these constants were introduced in [7]. Using the Kac indexing of the affine Dynkin diagrams [10, Table Fin, Aff1 and Aff2], we have $c_i = 1$ except for $c_i = 2$ for $\mathfrak{g} = B_n^{(1)}$ and $i = n$, $\mathfrak{g} = C_n^{(1)}$ and $1 \leqslant i \leqslant n-1$, $\mathfrak{g} = F_4^{(1)}$ and $i = 3, 4$, and $c_2 = 3$ for $\mathfrak{g} = G_2^{(1)}$. Consider the sublattices of P_0 given by

$$M = \bigoplus_{i \in I\setminus\{0\}} \mathbb{Z}c_i\alpha_i = \mathbb{Z}W_0 \cdot \theta/a_0,$$
$$\widetilde{M} = \bigoplus_{i \in I\setminus\{0\}} \mathbb{Z}c_i\omega_i.$$

It is easy to check that $M \subset \widetilde{M}$ and that the action of W_0 on P_0 restricts to actions on M and $\widetilde{M}$. Let $T(\widetilde{M})$ (respectively $T(M)$) be the subgroup of $T(P_0)$ generated by t_λ for $\lambda \in \widetilde{M}$ (respectively $\lambda \in M$).

There is an isomorphism [10, Proposition 6.5]

$$W \cong W_0 \ltimes T(M) \tag{2.5}$$

as subgroups of $\mathrm{Aut}(P)$. Under this isomorphism we have

$$s_0 = t_{\theta/a_0} s_\theta. \tag{2.6}$$

Define the extended affine Weyl group to be the subgroup of $\mathrm{Aut}(P)$ given by

$$\widetilde{W} = W_0 \ltimes T(\widetilde{M}). \tag{2.7}$$

When $\mathfrak{g}$ is of untwisted type, $M \cong Q^\vee$, $\widetilde{M} \cong P^\vee$, $c_i\omega_i = \nu(\omega_i^\vee)$, and $c_i\alpha_i = \nu(\alpha_i^\vee)$ for $i \in I \setminus \{0\}$.

Let $C \subset P \otimes_{\mathbb{Z}} \mathbb{R}$ be the fundamental chamber, the set of elements λ such that $\langle \alpha_i^\vee, \lambda\rangle \geqslant 0$ for all $i \in I$. Define the subgroup $\Sigma \subset \widetilde{W}$ to be the set of elements that send C into itself.

It follows from (2.4) and (2.5) that W is a normal subgroup of $\widetilde{W}$. Thus Σ acts on W by conjugation. Since the Weyl chambers adjacent to C are precisely those of the form $s_i(C)$ for $i \in I$, the element $\tau \in \Sigma$ induces a permutation (also denoted τ) of the set I given by

$$\tau s_i \tau^{-1} = s_{\tau(i)} \quad \text{for } i \in I. \tag{2.8}$$

Since the braid relations in W are preserved, Σ is a subgroup of $\mathrm{Aut}(X)$.

2.5. *Special automorphisms*

We identify the subgroup Σ explicitly. Say that an affine Dynkin node $i \in I$ is *special* if there is an automorphism $\tau \in \mathrm{Aut}(X)$ of the affine Dynkin diagram such that $\tau(i) = 0$. In the untwisted case, i is special if and only if $\omega_i^\vee$ is a minuscule coweight. Let $I^0 \subset I$ denote the set of special vertices. Explicitly, using the Kac labeling [10]:

$$I^0 = \begin{cases} \{0, 1, \ldots, n\} & \text{for } A_n^{(1)}, \\ \{0, 1\} & \text{for } B_n^{(1)},\ A_{2n-1}^{(2)}, \\ \{0, n\} & \text{for } C_n^{(1)},\ D_{n+1}^{(2)}, \\ \{0, 1, n-1, n\} & \text{for } D_n^{(1)}, \\ \{0, 1, 5\} & \text{for } E_6^{(1)}, \\ \{0, 6\} & \text{for } E_7^{(1)}, \\ \{0\} & \text{otherwise.} \end{cases}$$

Proposition 2.1. *For each $i \in I^0$ there is a unique element $\tau_i \in \Sigma$ such that $\tau_i(i) = 0$. Moreover, $\Sigma = \{\tau_i \mid i \in I^0\}$.*

We call τ_i the special automorphism associated with $i \in I^0$.

Note that every Dynkin automorphism is determined by its action on I^0. We describe the special automorphisms explicitly. τ_0 is the identity automorphism. If $\mathfrak{g}$ is of untwisted affine type and $i \in I^0$ then for all $j \in I^0$, $\tau_i(j) = k \in I^0$ where $-\omega_i + \omega_j \cong \omega_k \bmod Q_0$ and $\omega_0 = 0$ by convention. For $\mathfrak{g}$ of twisted type the only nonidentity (special) automorphisms are the elements of $\mathrm{Aut}(X)$ which on I^0 are given by $\tau_1 = (0, 1)$ in type $A_{2n-1}^{(2)}$ and $\tau_n = (0, n)$ in type $D_{n+1}^{(2)}$.

We now specify Σ explicitly as a subgroup of permutations of I^0. In all cases but $D_n^{(1)}$ and n even, Σ is a cyclic group. This determines τ_i and Σ completely except for types $A_n^{(1)}$ and $D_n^{(1)}$. For $A_n^{(1)}$, $\Sigma \cong \mathbb{Z}/(n+1)\mathbb{Z}$ where $\tau_i(j) = j - i \bmod (n+1)$ for all $i, j \in I^0$. For $D_n^{(1)}$ and n odd, Σ is cyclic with $\tau_{n-1} = (0, n, 1, n-1)$, $\tau_1 = (0, 1)(n-1, n)$ and $\tau_n = (0, n-1, 1, n)$ in cycle notation acting on I^0. For n even, $\Sigma \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with $\tau_1 = (0, 1)(n-1, n)$, $\tau_{n-1} = (0, n-1)(1, n)$ and $\tau_n = (0, n)(1, n-1)$.

Proposition 2.2. *$\Sigma \cong \widetilde{M}/M$ via $\tau_i \mapsto \omega_i + M$ for $i \in I^0$ and*

$$\widetilde{W} \cong W \rtimes \Sigma \tag{2.9}$$

as subgroups of $\mathrm{Aut}(P_0)$.

If $i \in I^0$ then $c_i = 1$ and we have

$$\tau_i = w_0^{\omega_i} t_{-\omega_i} \tag{2.10}$$

where, for $\lambda \in P_0^+$,

$$w_0^\lambda \in W_0 \quad \text{is the shortest element such that } w_0^\lambda \lambda \text{ is antidominant.} \tag{2.11}$$

2.6. Dynkin automorphisms revisited

Let X_0 be the Dynkin diagram for the classical subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}$.

Lemma 2.3. *There is a group homomorphism*

$$\begin{aligned} \mathrm{Aut}(X) &\to \mathrm{Aut}(X_0), \\ \sigma &\mapsto \sigma' \end{aligned} \tag{2.12}$$

where $\sigma'(i) = j$ if and only if $\sigma(\omega_i) \in W_0\omega_j$.

Proof. We first claim that there is a group action of $\mathrm{Aut}(X)$ on $W_0 \setminus P_0$ defined by $\sigma(W_0\lambda) = W_0\sigma\lambda$ where $\mathrm{Aut}(X)$ acts on P_0 via the level zero action. The level zero action of s_0 on P_0 is the same as that of $s_\theta \in W_0$, by (2.6) and (2.3). Thus for the level zero action, $W\lambda = W_0\lambda$ for $\lambda \in P_0$. By (2.8), $\sigma W_0\sigma^{-1} \subset W$ as it is generated by $s_{\sigma(i)}$ for $i \in I \setminus \{0\}$. Thus we have $W_0\sigma W_0\tau\lambda = W_0(\sigma W_0\sigma^{-1})\sigma\tau\lambda = W_0\sigma\tau\lambda$. Therefore, $\mathrm{Aut}(X)$ acts on $W_0 \setminus P_0$.

Next we show that this action restricts to an action on $F \subset W_0 \setminus P_0$ where F is the set of W_0-orbits of fundamental weights ω_i for $i \in I \setminus \{0\}$. Due to the above group action we need only that $\sigma F \subset F$ for generators σ of $\mathrm{Aut}(X)$. By (2.2) we have $\sigma(\omega_r) = \omega_{\sigma(r)} - a_r^\vee \omega_{\sigma(0)}$ where we write $\omega_i = \Lambda_i - a_i^\vee \Lambda_0$ for all $i \in I$. Using this one may straightforwardly check the lemma for each affine root system. $\square$

$\mathrm{Aut}(X_0)$ is trivial except in the following cases, where the homomorphism is described explicitly. The elements of $\mathrm{Aut}(X)$ and $\mathrm{Aut}(X_0)$ are given by their action as permutations of I^0 and $I^0 \setminus \{0\}$, respectively.

(1) $\mathrm{Aut}(A_n)$ is generated by the involution $i \mapsto n+1-i$ for $i \in I \setminus \{0\}$. In this case $\mathrm{Aut}(A_n^{(1)})$ is the dihedral group $D_{2(n+1)}$. For $\sigma \in \mathrm{Aut}(A_n^{(1)})$, σ' is the nontrivial element in $\mathrm{Aut}(A_n)$ if and only if σ reverses orientation.
(2) $\mathrm{Aut}(D_n)$ is generated by $(n-1, n)$ when $n > 4$. In this case $\mathrm{Aut}(D_n^{(1)})$ is generated by $(0, 1)$, $(n-1, n)$ and $(0, n)(1, n-1)$. All these map to the nontrivial element of $\mathrm{Aut}(D_n)$ except in the case that n is even, when $(0, n)(1, n-1)$ maps to the identity.
(3) $\mathrm{Aut}(D_4)$ is the symmetric group on the three “satellite” vertices $\{1, 3, 4\}$. $\mathrm{Aut}(D_4^{(1)})$ is the symmetric group on the vertices $\{0, 1, 3, 4\}$ and is generated by $(0, i)$ for $i \in \{1, 3, 4\}$. The generator $(0, i)$ is sent to the element (j, k) in $\mathrm{Aut}(D_4)$ where $\{0, i, j, k\} = \{0, 1, 3, 4\}$ as sets.
(4) $\mathrm{Aut}(E_6)$ is generated by $(1, 5)$. $\mathrm{Aut}(E_6^{(1)})$ is isomorphic to the S_3 that permutes the special vertices $\{0, 1, 5\}$. Then each of the elements of order two in $\mathrm{Aut}(E_6^{(1)})$ is sent to the nontrivial element of $\mathrm{Aut}(E_6)$.

Remark 1. In all cases, for all $\tau \in \Sigma$, τ' is the identity in $\mathrm{Aut}(X_0)$. However for $\sigma = (0, 1) \in \mathrm{Aut}(D_n^{(1)})$ we have $\sigma' = (n-1, n) \in \mathrm{Aut}(D_n)$.

3. Crystals

3.1. Definition of crystals

A P-weighted I-crystal is a set B, equipped with Kashiwara operators $e_i, f_i : B \to B \sqcup \{\emptyset\}$, and weight function $\mathrm{wt}: B \to P$ such that $e_i(f_i(b)) = b$ if $f_i(b) \neq \emptyset$, $f_i(e_i(b)) = b$ if $e_i(b) \neq \emptyset$, $\mathrm{wt}(f_i(b)) = \mathrm{wt}(b) - \alpha_i$ if $f_i(b) \neq \emptyset$, $\mathrm{wt}(e_i(b)) = \mathrm{wt}(b) + \alpha_i$ if $e_i(b) \neq \emptyset$, and $\langle \alpha_i^\vee, \mathrm{wt}(b)\rangle = \varphi_i(b) - \varepsilon_i(b)$ where $\varphi_i(b) = \min\{m \mid f_i^m(b) \neq \emptyset\}$ and $\varepsilon_i(b) = \min\{m \mid e_i^m(b) \neq \emptyset\}$ are assumed to be finite for all $b \in B$ and $i \in I$. If $f_i(b) \neq \emptyset$ we draw an arrow colored i from b to $f_i(b)$. The connected components of the graph obtained by removing all arrows from B except the arrows colored i, are called the i-strings of B. We write $\varepsilon(b) = \sum_{i \in I} \varepsilon_i(b)\Lambda_i$ and $\varphi(b) = \sum_{i \in I} \varphi_i(b)\Lambda_i$.

An I-crystal B is *regular* if, for each subset $K \subset I$ with $|K| = 2$, each K-component of B is isomorphic to the crystal basis of an irreducible integrable highest weight $U_q'(\mathfrak{g}_K)$-module where $\mathfrak{g}_K$ is the subalgebra of $\mathfrak{g}$ with simple roots α_i for $i \in K$.

The crystal reflection operator $S_i : B \to B$ is defined by the property that $S_i(b)$ is the unique element in the i-string of b such that $\varepsilon_i(S_i(b)) = \varphi_i(b)$ or equivalently $\varphi_i(S_i(b)) = \varepsilon_i(b)$. This defines an action of the Weyl group W on B if B is regular [12].

If B and B' are P-weighted I-crystals, their tensor product $B \otimes B'$ is a P-weighted I-crystal as follows (we use the opposite of Kashiwara's convention). As a set $B \otimes B'$ is just the Cartesian product $B \times B'$ where traditionally one writes $b \otimes b'$ instead of (b, b'). The Kashiwara operators are given by

$$f_i(b \otimes b') = \begin{cases} f_i(b) \otimes b' & \text{if } \varepsilon_i(b) \geqslant \varphi_i(b'), \\ b \otimes f_i(b') & \text{if } \varepsilon_i(b) < \varphi_i(b'), \end{cases}$$

$$e_i(b \otimes b') = \begin{cases} e_i(b) \otimes b' & \text{if } \varepsilon_i(b) > \varphi_i(b'), \\ b \otimes e_i(b') & \text{if } \varepsilon_i(b) \leqslant \varphi_i(b'). \end{cases}$$

Given any P-weighted I-crystal B and Dynkin automorphism σ, there is a P-weighted I-crystal B^σ whose vertex set is written $\{b^\sigma \mid b \in B\}$ and whose edges are given by $f_i(b) = b'$ in B if and only if $f_{\sigma(i)}(b^\sigma) = (b')^\sigma$. The weight function satisfies $\mathrm{wt}(b^\sigma) = \sigma(\mathrm{wt}(b))$ where the second σ is the automorphism of P defined by σ. A similar statement holds for P_0-weighted I-crystals, using the level zero action of σ on P_0 defined in Section 2.2.

Given any P-weighted I-crystal B, define the *contragradient dual* crystal $B^\vee = \{b^\vee \mid b \in B\}$ with $\mathrm{wt}(b^\vee) = -\mathrm{wt}(b)$ and $f_i(b) = b'$ if and only if $e_i(b^\vee) = b'^\vee$.

3.2. Branching

The following ideas have been applied extensively (in [18,25], for example) to identify the 0-arrows in KR crystals. We shall use them here for the same purpose.

Let B be the crystal graph of a $U_q'(\mathfrak{g})$-module and $K \subset I$. A *K-component* of B is a connected component of the graph obtained from B by removing all i-edges for $i \notin K$. A *K-highest weight vector* is an element $b \in B$ such that $\varepsilon_i(b) = 0$ for all $i \in K$. Suppose K is a proper subset of I. Since the subalgebra of $\mathfrak{g}$ with simple roots $\{\alpha_i \mid i \in K\}$ is semisimple, each K-component of B has a unique K-highest weight vector. When $K = I \setminus \{0\}$ we call the K-components and K-highest weight vectors *classical* components and highest weight vectors.

Suppose σ is a Dynkin automorphism that fixes K and induces an automorphism (also denoted σ) on B that sends i-arrows to $\sigma(i)$-arrows for all $i \in I$. Then by definition σ preserves

i-arrows for all $i \in K$. There is a projection from the classical weight lattice to that of the subalgebra with simple roots α_i for $i \in K$; we refer to the latter as the *K-weight lattice*. In particular σ permutes the collection of K-components, sending K-highest weight vectors to those with the same K-weight (that is, $\varphi_i \circ \sigma = \varphi_i$ for $i \in K$).

3.3. Demazure modules and crystals

Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra and $U_q(\mathfrak{g})$ its quantized universal enveloping algebra. For a dominant weight Λ denote by $V(\Lambda)$ the irreducible integrable highest weight $U_q(\mathfrak{g})$-module with highest weight Λ. Write $B(\Lambda)$ for its crystal basis. Let $\mathfrak{b}$ be a Borel Lie subalgebra of $\mathfrak{g}$. For $\mu \in W \cdot \Lambda$ let u_μ be a generator of the line of weight μ in $V(\Lambda)$. Write $\mu = w\Lambda$ where w is shortest in its coset wW^Λ and $W^\Lambda = \{w \in W \mid w\Lambda = \Lambda\}$. When writing an element $w\Lambda \in W \cdot \Lambda$ we shall always assume w is of minimum length. Define the Demazure module

$$V_w(\Lambda) := U_q(\mathfrak{b}) \cdot u_{w(\Lambda)}.$$

It is known that $V_w(\Lambda)$ has a crystal base $B_w(\Lambda)$ [11]; it is the full subgraph of $B(\Lambda)$ whose vertex set consists of the elements in $B(\Lambda)$ that are reachable by raising operators, from the unique element $u_{w\Lambda} \in B(\Lambda)$ of weight $w\Lambda$. We shall make use of the following result. By abuse of notation let

$$f_w(b) = \left\{ f_{i_N}^{m_N} \cdots f_{i_1}^{m_1}(b) \mid m_k \in \mathbb{Z}_{\geqslant 0} \right\} \tag{3.1}$$

where $w = s_{i_N} \cdots s_{i_1}$ is any fixed reduced decomposition of w. It is known [15,20,21] that as sets,

$$B_w(\Lambda) = f_w(u_\Lambda). \tag{3.2}$$

For $\mathfrak{g}$ affine, let $w \in \widetilde{W}$. By (2.9) we may express it uniquely as $w = z\tau$ where $z \in W$ and $\tau \in \Sigma$. We define the Demazure module to be

$$V_w(\Lambda) := V_z\big(\tau(\Lambda)\big).$$

Its crystal graph is denoted $B_w(\Lambda) = B_z(\tau\Lambda)$. For a dominant $\lambda \in \widetilde{M}$, let $\lambda^* = -w_0(\lambda)$, where w_0 is the longest element in W_0. Define $D(\lambda, s) = V_{t_{-\lambda^*}}(s\Lambda_0)$ and by abuse of notation, $D(\lambda, s) = B_{t_{-\lambda^*}}(s\Lambda_0)$. For any $\sigma \in \mathrm{Aut}(X)$ let $D^\sigma(\lambda, s) = B_{t_{-\sigma(\lambda)^*}}(s\Lambda_{\sigma(0)})$; it is obtained from $D(\lambda, s)$ by changing every i arrow into a $\sigma(i)$ arrow.

3.4. KR crystals

Kirillov–Reshetikhin (KR) modules $W^{r,s}$, labeled by $(r, s) \in I \setminus \{0\} \times \mathbb{Z}_{>0}$, are finite-dimensional $U_q'(\mathfrak{g})$-modules. See [7] for the precise definition. It is conjectured that $W^{r,s}$ has a global crystal basis $B^{r,s}$.

In [7] a conjecture is given for the decomposition of each Kirillov–Reshetikhin (KR) module $W^{r,c_r s}$ into its $\mathfrak{g}_0$-components. Chari [1] proved this conjecture for the nonexceptional untwisted algebras and for the exceptional cases for the nodes r such that either $r \in I^0$ or ω_r is the highest root. Recently the G_2 case was treated in full [2]. In [5], the $\mathfrak{g}_0$-structure of the Demazure modules was calculated for the same cases as in [1], and it was verified that the Demazure and KR modules agree as $\mathfrak{g}_0$-modules. In addition, it was shown in [6] that no matter what the precise $\mathfrak{g}_0$-structure is, the Demazure and the KR modules agree as $\mathfrak{g}_0$-modules for all untwisted algebras. Naito and Sagaki [22] proved the conjectures of [7] on the level of crystals for the twisted cases under the assumption that the KR crystals for the untwisted algebras exist. In unpublished work, Naito and Sagaki did the same construction for the twisted cases on the Demazure modules.

Remark 2. Assuming that $B^{r,c_r s}$ exists, the Demazure crystal $D(c_r\omega_r, s)$ and the KR crystal $B^{r,c_r s}$ have the same classical crystal structure.

In this paper we assume that the KR crystal $B^{r,c_r s}$ has the properties of Assumption 1, which we expect to hold if the KR crystals exist. In the next section we will see that with these assumptions the Demazure crystal sits inside the KR crystal (see Theorem 4.4) and that the KR crystal is unique (see Corollary 4.6). For types $B_n^{(1)}$, $D_n^{(1)}$, and $A_{2n-1}^{(2)}$ let σ be the Dynkin automorphism exchanging the Dynkin nodes 0 and 1 and fixing all others. For types $C_n^{(1)}$ and $D_{n+1}^{(2)}$ let σ be the Dynkin automorphism defined by $i \mapsto n-i$ for all $i \in I$. We also write σ for the induced automorphism of P.

Assumption 1. The KR crystal $B^{r,c_r s}$ has the following properties:

(1) $B^{r,c_r s}$ is regular.
(2) There is a unique element $u \in B^{r,c_r s}$ such that

$$\varepsilon(u) = s\Lambda_0 \quad \text{and} \quad \varphi(u) = s\Lambda_{\tau(0)},$$

where $t_{-c_r\omega_r} = w\tau$ with $w \in W$ and $\tau \in \Sigma$.
(3) For all types different from $A_{2n}^{(2)}$, $B^{r,c_r s}$ admits the automorphism corresponding to σ (also denoted σ) such that

$$\varepsilon \circ \sigma = \sigma \circ \varepsilon, \qquad \varphi \circ \sigma = \sigma \circ \varphi. \tag{3.3}$$

For type $A_{2n}^{(2)}$ we assume that $B^{r,c_r s}$ is given explicitly by the virtual crystal construction in [23].

4. Relation between Demazure and KR crystals

In this section we show that the Demazure crystal sits inside the KR crystals in Theorem 4.4 and, assuming their existence, that the KR crystals are unique in Corollary 4.6.

The main technique that we use in the proof is a decomposition of the translation elements $t_{-c_r\omega_r}$ that ends in a word for the subalgebra associated to the nodes $\{0, 1, \ldots, r-1\}$ of the Dynkin diagram in analogy to the results of [5].

Proposition 4.1. *Let $\mathfrak{g}$ be of nonexceptional affine type, $r \in I \setminus I^0$ and $t_{-c_r\omega_r} = w\tau$ for $w \in W$ and $\tau \in \Sigma$. Then a reduced word for the minimum length coset representative w_2 in $W_0 w$ is given by*

$$w_2 = \begin{cases} \prod_{k=i}^{1} s_0(s_2 s_3 \cdots s_{2k-1})(s_1 s_2 \cdots s_{2k-2}) & \text{for } r = 2i \text{ and } B_n^{(1)},\ D_n^{(1)},\ A_{2n-1}^{(2)}, \\ \prod_{k=i}^{1} s_0(s_2 s_3 \cdots s_{2k})(s_1 s_2 \cdots s_{2k-1}) & \text{for } r = 2i+1 \text{ and } B_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, \\ \prod_{k=i}^{1} s_0(s_1 s_2 \cdots s_{k-1}) & \text{for } r = i \text{ and } C_n^{(1)},\ A_{2n}^{(2)},\ D_{n+1}^{(2)}, \end{cases} \tag{4.1}$$

where the index k decreases as the product is formed from left to right.

Proof. All nodes for $A_n^{(1)}$ are special so we may assume $\mathfrak{g}$ is not of this type.

Applying the sequence of reflections in (4.1) to $\Lambda_{\tau(0)}$, we see that each reflection s_j changes the weight by a positive multiple of α_j, and the final weight is $\Lambda_0 + c_r\omega_r - i\delta$. It follows that (4.1) yields a reduced decomposition of some element $w_2 \in W$.

Using (2.3), in all cases we have

$$w\Lambda_{\tau(0)} = t_{-c_r\omega_r}\tau^{-1}\Lambda_{\tau(0)} = \Lambda_0 - c_r\omega_r - i\delta/a_0.$$

Since $r \notin I^0$ we have $w_0^{\omega_r}\omega_r = -\omega_r$ where $w_0^{\omega_r}$ is defined in (2.11). Moreover, $w_0^{\omega_r}$ is also the shortest element of W_0 sending $\Lambda_0 + c_r\omega_r - i\delta/a_0$ to $\Lambda_0 - c_r\omega_r - i\delta/a_0$. It follows that $w = w_0^{\omega_r} w_2$ is a length-additive factorization and that w_2 is the minimum length coset representative in $W_0 w$. □

Remark 3. Let $K = \{0, 1, \ldots, r-1\} \subset I$, $\mathfrak{g}_K \subset \mathfrak{g}$ the simple subalgebra with Dynkin nodes K, $\{\widetilde{\omega}_j \mid j \in K\}$ the fundamental weights for $\mathfrak{g}_K$, and $W_K = \langle s_j \mid j \in K\rangle \subset W$ the Weyl group of $\mathfrak{g}_K$. This given, we have $w_2 = w_0^{\widetilde{\omega}_{\tau(0)}}$ where $w_0^{\widetilde{\omega}_j} \in W_K$ is defined with respect to $\mathfrak{g}_K$.

Lemma 4.2. *All of the weights of $B^{r,c_r s}$ are in the convex hull of the W_0-orbit $W_0 \cdot c_r s\omega_r$. Moreover, for every $\mu \in W_0 \cdot c_r s\omega_r$, there is a unique element $u_\mu \in B(c_r s\omega_r) \subset B^{r,c_r s}$ of the extremal weight μ.*

Proof. By [5,22] the classical decomposition of $D(c_r\omega_r, s)$ agrees with that specified in [7]. In every case the above condition holds. □

Lemma 4.3. *Let $\mathfrak{g}$ be of nonexceptional affine type, $r \in I \setminus I^0$, $s \in \mathbb{Z}_{>0}$, $k < r$ where $B(c_r s\omega_k)$ occurs in $B^{r,c_r s}$, and $b = u_{c_r s\omega_k} \in B(c_r s\omega_k) \subset B^{r,c_r s}$. Define*

$$y = \begin{cases} S_2 \cdots S_{k+1} S_1 \cdots S_k(b) & \text{for } B_n^{(1)},\ D_n^{(1)},\ A_{2n-1}^{(2)}, \\ S_1 \cdots S_k(b) & \text{for } C_n^{(1)},\ D_{n+1}^{(2)},\ A_{2n}^{(2)}. \end{cases}$$

Then

$$f_0^s(y) = \begin{cases} u_{c_r s\omega_{k+2}} & \text{for } B_n^{(1)},\ D_n^{(1)},\ A_{2n-1}^{(2)}, \\ u_{c_r s\omega_{k+1}} & \text{for } C_n^{(1)},\ D_{n+1}^{(2)},\ A_{2n}^{(2)}. \end{cases} \tag{4.2}$$

Proof. By definition the element y is an extremal weight vector within the classical crystal $B(c_r s\omega_k)$. By weight considerations one may check that

$$y = \begin{cases} f_2^s \cdots f_k^s f_{k+1}^s f_1^s f_2^s \cdots f_{k-1}^s f_k^s(b) & \text{for } B_n^{(1)},\ D_n^{(1)},\ A_{2n-1}^{(2)}, \\ f_1^{c_r s} f_2^{c_r s} \cdots f_k^{c_r s}(b) & \text{for } C_n^{(1)},\ D_{n+1}^{(2)},\ A_{2n}^{(2)}. \end{cases}$$

We claim that

$$\begin{aligned} &\varepsilon(y) = s(\Lambda_0 + \Lambda_2), && \varphi(y) = s(\Lambda_0 + \Lambda_{k+2}), && \text{for } B_n^{(1)},\ D_n^{(1)},\ A_{2n-1}^{(2)},\ k > 0, \\ &\varepsilon(y) = s(\Lambda_0 + c_r\Lambda_1), && \varphi(y) = s(\Lambda_0 + c_r\Lambda_{k+1}), && \text{for } C_n^{(1)},\ A_{2n}^{(2)},\ D_{n+1}^{(2)},\ k > 0, \\ &\varepsilon(y) = s\Lambda_0, && \varphi(y) = s\Lambda_0, && \text{for } k = 0. \end{aligned}$$

By extremality and Lemma 4.2, y is in the indicated position within its i-strings for $i \in I \setminus \{0\}$. It remains to show that $\varepsilon_0(y) = \varphi_0(y) = s$ and (4.2) holds. In each case we shall use Assumption 1(3) either for the existence of a crystal automorphism σ on $B^{r,c_r s}$ or, in type $A_{2n}^{(2)}$, for the virtual crystal construction of $B^{r,c_r s}$.

We begin with type $D_n^{(1)}$. We have $c_r = 1$ and $\mu := \mathrm{wt}(y) = (0^2, s^k, 0^{n-k-2})$. Here we realize $P_0 \subset ((1/2)\mathbb{Z})^n$ with ith standard basis element ϵ_i, with $\omega_i = (1^i, 0^{n-i})$ for $1 \leqslant i \leqslant n-2$ (we do not need the spin weights) and $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \leqslant i \leqslant n-1$. Let $b' = u_{s\omega_{k+2}} \in B(s\omega_{k+2}) \subset B^{r,s}$. We have $\varphi_0(b') = 0$, for otherwise $f_0(b') \in B^{r,s}$ has weight contradicting Lemma 4.2. Since $\langle \alpha_0^\vee, \mathrm{wt}(b')\rangle = 2s$, we have $\varepsilon_0(b') = 2s$.

For type $D_n^{(1)}$, the automorphism σ of $B^{r,c_r s}$ satisfies $e_0 = \sigma \circ e_1 \circ \sigma$. Define $z = e_1^s(\sigma(b'))$. It suffices to show that

$$y = \sigma(z).$$

Let $K = \{2, 3, \ldots, n\} \subset I$. The subalgebra of $\mathfrak{g}$ with simple roots α_i for $i \in K$, is of type D_{n-1}. For this reason we shall refer to D_{n-1}-components and D_{n-1}-highest weight vectors instead of K-components and K-highest weight vectors. Our proof rests on the following fact:

> $B^{r,s}$ contains a unique element of weight μ that satisfies $\varepsilon_1 = 0$ and whose associated D_{n-1}-highest weight vector has D_n-weight $\lambda := (0, s^k, 0^{n-k-1})$.

For the classical components of $B^{r,s}$ that contain D_{n-1}-components of weight λ, are precisely those of the form $B((s-t)\omega_k + t\omega_{k+2})$ for $0 \leqslant t \leqslant s$, and only for $t = 0$ does the classical component contain an element of weight μ with $\varepsilon_1 = 0$ (and by extremality $B(s\omega_k)$ contains a unique element of weight μ).

y clearly satisfies the above property. It suffices to show that $\sigma(z)$ does also.

$\sigma(b')$ is a D_{n-1}-highest weight vector with $\mathrm{wt}(\sigma(b')) = (-s, s^{k+1}, 0^{n-k-2})$. So $\mathrm{wt}(z) = \mu$. By weight considerations and Lemma 4.2, $z' = S_{k+1} \cdots S_2(z)$ is a D_{n-1}-highest weight vector of weight λ. Therefore, $\sigma(z)$ has weight $\sigma(\mu) = \mu$ and has associated D_{n-1}-highest weight vector $\sigma(z')$, which has weight $\sigma(\lambda) = \lambda$. Since the Dynkin nodes 0 and 1 are nonadjacent we have $\varepsilon_1(\sigma(z)) = \varepsilon_1(e_0^s(b')) = \varepsilon_1(b') = 0$. Thus $\sigma(z)$ fulfills the above criteria and so must be equal to y.

The proof is analogous for types $B_n^{(1)}$ and $A_{2n-1}^{(2)}$ using the same set K, which defines subalgebras of types B_{n-1} and C_{n-1}, respectively.

For type $C_n^{(1)}$ we have $c_r = 2$ for all $1 \leqslant r \leqslant n-1$. Let $K = \{1, 2, \ldots, n-1\}$; the associated subalgebra is of type A_{n-1}. Here we realize $P_0 \cong \mathbb{Z}^n$ with $\omega_i = (1^i, 0^{n-i})$ for $1 \leqslant i \leqslant n$ and $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \leqslant i \leqslant n-1$ and $\alpha_n = 2\epsilon_n$. Our argument uses the fact that

> $B^{r,2s}$ contains a unique element of weight $\mu := (0, (2s)^k, 0^{n-k-1})$ such that $\varepsilon_n = 0$ and whose associated A_{n-1}-highest weight vector has C_n-weight $2s\omega_k$.

For the classical components in $B^{r,2s}$ that contain such an A_{n-1}-component, are precisely those of the form $B(2(s-t)\omega_k + 2t\omega_{k+1})$ for $0 \leqslant t \leqslant s$, and among these, only for $t = 0$ does the classical component contain an element of weight μ for which $\varepsilon_n = 0$ (and by extremality $B(2s\omega_k)$ contains a unique element of weight μ).

By construction y satisfies this property. It suffices to show that $\sigma(z)$ does also, where $z = e_n^s \circ \sigma(b')$ and $b' = u_{2s\omega_{k+1}} \in B(2s\omega_{k+1}) \subset B^{r,s}$.

We have $\varphi_0(b') = 0$ for otherwise $f_0(b') \in B^{r,2s}$ would have weight contradicting Lemma 4.2. Since $\langle \alpha_0^\vee, \mathrm{wt}(b')\rangle = 2s$ we have $\varepsilon_0(b') = 2s$.

$\sigma(b')$ is an A_{n-1}-highest weight vector of weight $\sigma(2s\omega_{k+1}) = (0^{n-k-1}, (-2s)^{k+1})$. Therefore, z has weight $(0^{n-k-1}, (-2s)^k, 0)$ and associated A_{n-1}-highest weight vector $z' = S_{n-k} \cdots S_{n-1}(z)$, which has weight $(0^{n-k}, (-2s)^k)$. It follows that $\sigma(z)$ has weight μ and its associated A_{n-1}-highest weight vector has weight $2s\omega_k$. Now $\varepsilon_n(\sigma(z)) = \varepsilon_n(e_0^s(b')) = \varepsilon_n(b') = 0$ since the Dynkin nodes 0 and n are nonadjacent. We have shown that $\sigma(z)$ satisfies the above criteria and so must be equal to y.

Type $D_{n+1}^{(2)}$ is similar to type $C_n^{(1)}$.

For type $A_{2n}^{(2)}$, the above kind of argument is not available since $A_{2n}^{(2)}$ admits no nontrivial Dynkin automorphism. Instead we apply virtual crystals. Under Assumption 1(3), by [23] the crystal $B^{r,s}$ is realized as the subset of $V^{r,s} = B_A^{2n-r,s} \otimes B_A^{r,s}$ of type $A_{2n-1}^{(1)}$ generated from $u_{s\omega_{2n-r}} \otimes u_{s\omega_r}$ by the virtual crystal operators $\hat{f}_i = f_i f_{2n-i}$ for $1 \leqslant i \leqslant n$ and $\hat{f}_0 = f_0^2$ where f_i are the crystal operators of the $A_{2n-1}^{(1)}$-crystal $V^{r,s}$. Denote the virtualization by $v : B^{r,s} \hookrightarrow V^{r,s}$. We perform explicit computations using the tableau realization of $U_q(A_{2n-1})$-crystals in [19] and 0-arrows given by [24]. We have

$$\begin{aligned}
v(b) &= (2n-k)^s \cdots (r+2)^s (r+1)^s k^s \cdots 2^s 1^s \otimes r^s \cdots 2^s 1^s, \\
v(y) &= (2n)^s (2n-k-1)^s \cdots (r+2)^s (r+1)^s (k+1)^s \cdots 3^s 2^s \otimes r^s \cdots 2^s 1^s, \\
v\big(f_0^s y\big) &= (2n-k-1)^s \cdots (r+2)^s (r+1)^s (k+1)^s \cdots 2^s 1^s \otimes r^s \cdots 2^s 1^s = v(u_{s\omega_{k+1}}). \qquad \square
\end{aligned}$$

The next theorem is the main result of this paper. It shows that under the isomorphism between the Demazure and the KR crystals as classical crystals zero arrows map to zero arrows. In addition it yields the isomorphism (4.3) without the assumption that the KR crystal $B^{r,c_r s}$ is perfect.

Theorem 4.4. *Let $(r,s) \in I \setminus \{0\} \times \mathbb{Z}_{>0}$. Suppose that $r \in I^0$, or $c_r\omega_r = \theta$, or $\mathfrak{g}$ is of nonexceptional affine type. Write $t_{-c_r\omega_r^*} = w\tau$ with $w \in W$ and $\tau \in \Sigma$. Then there is an affine crystal isomorphism*

$$B(s\Lambda_{\tau(0)}) \cong B^{r,c_r s} \otimes B(s\Lambda_0),$$
$$u_{s\Lambda_{\tau(0)}} \mapsto u' := u \otimes u_{s\Lambda_0} \tag{4.3}$$

where u is the element specified by Assumption 1(2). *It restricts to an isomorphism*

$$D(c_r\omega_r, s) \cong B^{r,c_r s} \otimes u_{s\Lambda_0} \tag{4.4}$$

where both sides of (4.4) *are regarded as full subcrystals of their respective sides in* (4.3).

Proof. Let w_2 be the minimum length coset representative in $W_0 w$. Then $w = w_1 w_2$ is a length-additive factorization with $w_1 = w w_2^{-1} \in W_0$. We choose a reduced word of w by concatenating reduced words of w_1 and w_2. We claim that it suffices to establish the following assertions.

(A1) There is a bijection

$$B_{w_2}(s\Lambda_{\tau(0)}) \to B' := f_{w_2}(u'),$$
$$u_{s\Lambda_{\tau(0)}} \mapsto u' \tag{4.5}$$

that preserves all arrows in f_{w_2}.

(A2) $B' \subset B^{r,c_r s} \otimes u_{s\Lambda_0}$.

Suppose (A1) and (A2) hold. Since $w_1 \in W_0$, $B_{w_2}(s\Lambda_{\tau(0)})$ contains all the classical highest weight vectors of $D(c_r\omega_r, s)$. By (A1) these classical highest weight vectors correspond to the classical highest weight vectors in B'. Let $B'' \subset B^{r,c_r s} \otimes B(s\Lambda_0)$ be the classical subcrystal generated by B'; by (A2) $B'' \subset B^{r,c_r s} \otimes u_{s\Lambda_0}$. By Demazure theory for highest weight modules over simple Lie algebras, the bijection (4.5) extends uniquely to a classical crystal isomorphism $D(c_r\omega_r, s) \cong B''$. By Assumption 1 and Remark 2 we have $B'' = B^{r,c_r s} \otimes u_{s\Lambda_0}$. So we have a bijection

$$D(c_r\omega_r, s) \cong B^{r,c_r s} \otimes u_{s\Lambda_0} \tag{4.6}$$

which is an isomorphism of classical crystals that extends the bijection (4.5). It follows that $B^{r,c_r s} \otimes u_{s\Lambda_0}$ and, therefore, $B^{r,c_r s} \otimes B(s\Lambda_0)$, have a unique affine highest weight vector, namely, u'. By [17, Proposition 2.4.4] there is an affine crystal isomorphism (4.3). It must extend the bijection (4.6), and the theorem follows.

We prove (A1) and (A2) by cases.

If $r \in I^0$ then by (2.10) w_2 is the identity, $B_{w_2}(s\Lambda_{\tau(0)}) = \{u_{s\Lambda_{\tau(0)}}\}$, $B' = \{u'\}$, $c_r = 1$, and $B^{r,s} \cong B(s\omega_r)$ as a classical crystal with classical highest weight vector u. In this case (A1) and (A2) are immediate. This is the only case where $\omega_r^* \neq \omega_r$.

If $c_r\omega_r = \theta$ then τ is the identity, $w_1 = s_\theta$ and $w_2 = s_0$. By Assumption 1(2), $B_{w_2}(s\Lambda_0)$ and B' are the 0-strings of $u_{s\Lambda_0}$ and u', respectively. The elements are at the dominant ends of their respective 0-strings, which both have length s. This gives (A1). (A2) follows by the signature rule and Assumption 1(2).

Otherwise we assume that $\mathfrak{g}$ is of nonexceptional affine type and $r \in I \setminus I^0$. Then w_2 is given in Proposition 4.1. We use the notation of Remark 3 throughout the rest of the proof. Since $K \subsetneq I$, $\mathfrak{g}_K$ is a simple Lie algebra and Assumption 1(1) implies that $B^{r,c_r s}$ decomposes into a direct sum of K-components, each of which is isomorphic to the crystal graph of an irreducible highest weight module for $U_q(\mathfrak{g}_K)$. We have the K-crystal isomorphisms

$$B_{w_2}(s\Lambda_{\tau(0)}) \cong B_{w_2}(s\widetilde{\omega}_{\tau(0)}) = B(s\widetilde{\omega}_{\tau(0)}) \cong B'. \tag{4.7}$$

The first isomorphism holds by restriction from an I-crystal to a K-crystal. The equality holds by Remark 3 and Demazure theory for the simple Lie algebra $\mathfrak{g}_K$. We have $B_{w_2}(s\widetilde{\omega}_{\tau(0)}) \cong B'$, since both sides are generated by f_{w_2} (with $w_2 \in W_K$) applied to K-highest weight vectors of K-weight $s\widetilde{\omega}_{\tau(0)}$; see Assumption 1(2). This establishes (A1).

For types $D_n^{(1)}$, $B_n^{(1)}$, $A_{2n-1}^{(2)}$ we have $c_r = 1$ for all r and $\tau = \tau_0$ or $\tau = \tau_1$ (and $\tau(0) = 0$ or $\tau(0) = 1$) according as r is even or odd. Here $u = u_{s\omega_{\tau(0)}} \in B(s\omega_{\tau(0)}) \subset B^{r,c_r s}$, where $\omega_0 = 0$ by convention.

We consider the decomposition of $B^{r,c_r s}$ into K-components, which we call D_r-components. Note that 0 and 1 are the spinor nodes in D_r. Now $u_{c_r s\omega_r} \in B^{r,c_r s}$ is a D_r-lowest weight vector of D_r-weight $-2s\widetilde{\omega}_0$. Therefore, there is a D_r-crystal embedding

$$\begin{gathered} B(2s\widetilde{\omega}_0) \otimes u_{s\Lambda_0} \to B(s\widetilde{\omega}_{\tau(0)})^{\otimes 2} \otimes B(s\Lambda_0), \\ u_{c_r s\omega_r} \otimes u_{s\Lambda_0} \mapsto u_{-s\widetilde{\omega}_0}^{\otimes 2} \otimes u_{s\Lambda_0}. \end{gathered}$$

But by Lemma 4.3 there is a D_r-path from u' to $u_{c_r s\omega_r} \otimes u_{s\Lambda_0}$ that never changes the right-hand tensor factor. Therefore, there is a D_r-embedding

$$\begin{gathered} B' \to B(s\widetilde{\omega}_{\tau(0)})^{\otimes 2} \otimes B(s\Lambda_0), \\ u' = u \otimes u_{s\Lambda_0} \mapsto u_{s\widetilde{\omega}_{\tau(0)}} \otimes u_{-s\widetilde{\omega}_0} \otimes u_{s\Lambda_0}. \end{gathered}$$

The image of u' is uniquely determined by Assumption 1(2) since $u_{s\widetilde{\omega}_{\tau(0)}} \otimes u_{-s\widetilde{\omega}_0}$ is the unique element of $B(s\widetilde{\omega}_{\tau(0)})^{\otimes 2}$ with $\varepsilon = s\Lambda_0$ and $\varphi = s\Lambda_{\tau(0)}$.

The form of the image of u' now clearly shows that when f_{w_2} is applied to u' it only acts on the left-hand tensor factor. This implies (A2).

Next let us consider type $C_n^{(1)}$ for $r \notin I^0$; for such r, $c_r = 2$ and τ is the identity. Here u is the unique element in the one-dimensional C_n-crystal in $B^{r,2s}$. We decompose $B^{r,2s}$ as a K-crystal, which is a C_r-crystal in this case. All other arguments go through as for type $D_n^{(1)}$.

Types $D_{n+1}^{(2)}$ and $A_{2n}^{(2)}$ follow in the same fashion. In this case the decomposition of $B^{r,c_r s}$ as a K-crystal is a B_r crystal. □

Remark 4. We expect Theorem 4.4 to hold for any affine algebra $\mathfrak{g}$ and any Dynkin node $r \in I \setminus \{0\}$. Our proof requires a special property, that the minimum length coset representative w_2 of Proposition 4.1 has a certain form, namely, in the notation of (2.11), $w_2 = w_0^\lambda$ where λ is a fundamental weight for some subalgebra $\mathfrak{g}_K$ where $K \subsetneq I$. This property of w_2 does not hold for the trivalent node in type $E_6^{(1)}$. For such nodes a different strategy is required.

Remark 5. In the notation of Lemma 2.3 we expect that for any affine algebra $\mathfrak{g}$ with affine Dynkin diagram X and any $\sigma \in \mathrm{Aut}(X)$, there is a bijection $\sigma : B^{r,c_r s} \to B^{\sigma'(r),c_r s}$ such that (3.3) holds. In particular, for any $\sigma \in \mathrm{Aut}(X)$, we expect that there is an automorphism σ on $B^{r,c_r s}$

satisfying (3.3) if and only if $\sigma'(r)=r$. By Remark 1 this means that every special Dynkin automorphism $\sigma\in\Sigma$ should induce an automorphism of each $B^{r,c_r s}$. In contrast, for the nonspecial automorphism $\sigma=(0,1)$ of $D_n^{(1)}$, $\sigma'=(n-1,n)$ is not the identity and σ induces a bijection $B^{n-1,s}\to B^{n,s}$ satisfying (3.3).

Remark 5 comes into play in Section 6 and the following theorem.

Theorem 4.5. *For the cases in Assumption* 1(3) *where σ is defined, there exist unique maps*

$$\Psi : D(\omega_r,s)\hookrightarrow B^{r,c_r s}\quad \textit{and}\quad \Psi^\sigma : D^\sigma(\omega_r,s)\hookrightarrow B^{r,c_r s}.$$

The maps are induced by $\Psi(u_{s\Lambda_0})=u$ and $\Psi^\sigma(u_{s\Lambda_{\sigma(0)}})=\sigma(u)$.

Proof. The map Ψ^σ is obtained by applying σ to everything in sight. □

Corollary 4.6. *The affine structure of $B^{r,c_r s}$ is uniquely determined.*

Theorem 4.7. *Suppose that $\lambda=\sum_{r\in I\setminus\{0\}} m_r c_r\omega_r$ with $m_r\in\mathbb{Z}_{\geqslant 0}$ and $m_r>0$ only when r is as in Theorem* 4.4. *Write $t_{-\lambda^*}=w\tau$ for $w\in W$ and $\tau\in\Sigma$. Assume that for each $k\in I^0$ and every $r\in I\setminus\{0\}$ with $m_r>0$, the special Dynkin automorphism $\tau_k\in\Sigma$ induces an automorphism of $B^{r,c_r s}$ that sends i-arrows to $\tau_k(i)$-arrows. Then for every $r'\in I^0$ there is an isomorphism*

$$B(s\Lambda_{\tau(r')})\cong\left(\bigotimes_{r\in I\setminus\{0\}}\left(B^{r,c_r s}\right)^{\otimes m_r}\right)\otimes B(s\Lambda_{r'})$$

which restricts to an isomorphism of full subcrystals

$$B_{\tau_{r'}^{-1}w\tau_{r'}}(s\Lambda_{\tau(r')})\cong\left(\bigotimes_{r\in I\setminus\{0\}}\left(B^{r,c_r s}\right)^{\otimes m_r}\right)\otimes u_{s\Lambda_{r'}}.$$

Proof. Induction allows a straightforward reduction to the case of one KR tensor factor. Applying a special Dynkin automorphism allows the reduction to the case $r'=0$, which is Theorem 4.4. □

Corollary 4.8. *Let λ be as in Theorem* 4.7. *Then the Demazure crystal $D(\lambda,s)$ can be extended to a full affine crystal by adding* 0*-arrows.*

Remark 6. This proves Conjecture 1 in [5] on the level of crystals. However it is not yet clear whether there exists a global basis of the Demazure module, whose corresponding crystal basis is the one given in Theorem 4.7. For level $s=1$, Theorem 4.7 was proved using the Littelmann path model in [6, Proposition 3].

5. Reaching the classical highest weight vectors of a KR crystal

In the proof of Lemma 4.3, explicit paths in the KR crystal were given, from the element u to certain classical highest weight vectors in the KR crystal. For $\mathfrak{g}$ of nonexceptional affine type and for each KR crystal $B^{r,c_r s}$, we shall give (without proof) an explicit way to reach each classical highest weight vector in $B^{r,c_r s}$ from the element u of Assumption 1.

If $r \in I^0$ then the KR crystal $B^{r,c_r s}$ is connected as a classical crystal and the problem is trivial. This includes all $r \in I \setminus \{0\}$ for $A_n^{(1)}$.

So we now assume $r \notin I^0$.

We shall use the standard realizations of the weight lattices of B_n, C_n, D_n by sublattices of $((1/2)\mathbb{Z})^n$. We let $\omega_i = (1^i, 0^{n-i})$ for $i \in I \setminus \{0\}$ nonspin. Since $r \notin I^0$ the only spin weight we need is $\omega_n = (1/2)(1^n)$ in type B_n, and in that case $c_n = 2$. Thus all the weights we must consider, correspond to partitions, elements in $\mathbb{Z}^n_{\geqslant 0}$ consisting of weakly decreasing sequences. Moreover, for the nonexceptional affine algebras the KR crystals are multiplicity-free as classical crystals.

For $\mathfrak{g}$ of type $B_n^{(1)}$, $D_n^{(1)}$, or $A_{2n-1}^{(2)}$, $B(\lambda)$ occurs in $B^{r,c_r s}$ if and only if the diagram of the partition corresponding to λ, is obtained from the $r \times s$ rectangular partition by removing vertical dominoes. Let $t = 0$ or $t = 1$ according as r is even or odd. We have

$$u_\lambda = \left(\prod_{i=(r-t)/2}^{1} f_0^{\lambda_{2i}} \left(f_2^{\lambda_{2i}} f_3^{\lambda_{2i}} \cdots f_{2i-1+t}^{\lambda_{2i}} \right) \left(f_1^{\lambda_{2i}} f_2^{\lambda_{2i}} \cdots f_{2i-2+t}^{\lambda_{2i}} \right) \right) u$$

where the product is formed from left to right using decreasing indices i.

Example 1. Let $\mathfrak{g}$ be of type $D_7^{(1)}$, $(r, s) = (5, 4)$ and λ be the weight $\omega_5 + \omega_3 + 2\omega_1$. Then $t = 1$, λ is the partition $(4, 2, 2, 1, 1)$, and the sequence of lowering operators is $(f_0 f_2 f_3 f_4 f_1 f_2 f_3)(f_0^2 f_2^2 f_1^2)$. This is applied to the classical highest weight vector of weight given by the partition (4), and the parenthesized subexpressions successively yield classical highest weight vectors corresponding to the partitions $(4, 2, 2)$, and $(4, 2, 2, 1, 1)$, respectively.

For $\mathfrak{g}$ of type $C_n^{(1)}$, $A_{2n}^{(2)}$ or $D_{n+1}^{(2)}$, the partitions corresponding to classical highest weights in $B^{r,c_r s}$ are precisely those of the form $c_r\lambda = (c_r\lambda_1, c_r\lambda_2, \ldots)$ where λ runs over the partitions contained in the $r \times s$ rectangle. We have

$$u_{c_r\lambda} = \left(\prod_{i=r}^{1} f_0^{c_r\lambda_i} f_1^{c_r\lambda_i} \cdots f_{i-1}^{c_r\lambda_i} \right) u$$

where the product of operators is formed from left to right as i decreases.

Example 2. Let $\mathfrak{g}$ be of type $C_3^{(1)}$, $(r, s) = (2, 3)$, and $\lambda = \omega_2 + 2\omega_1$. Then we have $c_r = 2$, the partition $\lambda = (3, 1)$, and the sequence of lowering operators $(f_0^2 f_1^2)(f_0^6)$. This is applied to the classical highest weight vector of weight 0 (corresponding to the empty partition). After f_0^6 the classical weight is given by the partition (6) and after $f_0^2 f_1^2$ one has the partition $(6, 2) = 2\lambda$.

6. Connectedness

Theorem 4.4 shows that the KR crystals $B^{r,c_r s}$ are connected. In this section we show that the tensor product of two KR crystals is also connected by providing an algorithm which for any given element in the crystal yields a string of operators e_i (or f_i) to reach a given special element.

This algorithm is also useful in defining crystal morphisms such as the combinatorial R-matrix. Since KR crystals and their tensor products are not highest weight crystals, it is not completely obvious which sequence of raising operators e_i will yield a given special element.

Here we give a construction on how to reach $u_1 \otimes u_2 \in B^{r_1,c_{r_1}s_1} \otimes B^{r_2,c_{r_2}s_2}$ where u_1 is the unique element of $B^{r_1,c_{r_1}s_1}$ with $\varepsilon(u_1) = s_1\Lambda_0$ and $\varphi(u_1) = s_1\Lambda_{\tau_1(0)}$ as required in Assumption 1(2), and u_2 is the unique element in $B^{r_2,c_{r_2}s_2}$ with $\varepsilon(u_2) = s_2\Lambda_{\tau_2^{-1}(0)}$ and $\varphi(u_2) = s_2\Lambda_0$ as required in Assumption 1(2) and Remark 5.

By Theorems 4.4 and 4.5 we have the following isomorphism of affine crystals

$$\begin{gathered}B^{r_1,c_{r_1}s_1} \otimes B^{r_2,c_{r_2}s_2} \otimes B(s_2\Lambda_{\tau_2^{-1}(0)}) \cong B^{r_1,c_{r_1}s_1} \otimes B(s_2\Lambda_0),\\ u_1 \otimes u_2 \otimes u_{s_2\Lambda_{\tau_2^{-1}(0)}} \mapsto u_1 \otimes u_{s_2\Lambda_0}.\end{gathered}$$

Assume that $s_1 \geqslant s_2$. Acting with raising operators e_i with $i \in I$ one can bring any element $b_1 \otimes b_2 \otimes u_{s_2\Lambda_{\tau_2^{-1}(0)}}$ into the form $c_1 \otimes u_2 \otimes u_{s_2\Lambda_{\tau_2^{-1}(0)}}$ since by the tensor product rule the e_i will eventually act on the right tensor factors and by Theorem 4.4 $b_2 \otimes u_{s_2\Lambda_{\tau_2^{-1}(0)}}$ is connected to $u_2 \otimes u_{s_2\Lambda_{\tau_2^{-1}(0)}}$. Once such an element is reached, tensor from the right by $u_{(s_1-s_2)\Lambda_0} \in B((s_1 - s_2)\Lambda_0)$ to obtain

$$\begin{aligned}&B^{r_1,c_{r_1}s_1} \otimes B^{r_2,c_{r_2}s_2} \otimes B(s_2\Lambda_{\tau_2^{-1}(0)}) \otimes B\big((s_1 - s_2)\Lambda_0\big)\\ &\quad\cong B^{r_1,c_{r_1}s_1} \otimes B(s_2\Lambda_0) \otimes B\big((s_1 - s_2)\Lambda_0\big)\end{aligned}$$

under which $c_1 \otimes u_2 \otimes u_{s_2\Lambda_{\tau_2^{-1}(0)}} \otimes u_{(s_1-s_2)\Lambda_0}$ maps to $c_1 \otimes u_{s_2\Lambda_0} \otimes u_{(s_1-s_2)\Lambda_0}$. The latter element is the image of the vector $c_1 \otimes u_{s_1\Lambda_0}$ under the embedding of affine crystals $B^{r_1,c_{r_1}s_1} \otimes B(s_1\Lambda_0) \to B^{r_1,c_{r_1}s_1} \otimes B((s_1 - s_2)\Lambda_0) \otimes B(s_2\Lambda_0)$.

Now from $c_1 \otimes u_{s_1\Lambda_0} \in B^{r_1,c_{r_1}s_1} \otimes B(s_1\Lambda_0)$ one can reach $u_1 \otimes u_{s_1\Lambda_0}$ using e_i with $i \in I$.

If $s_1 < s_2$ we tensor from the left with the dual crystals. Explicitly,

$$B^\vee(s_1\Lambda_{\tau_1(0)}) \otimes B^{r_1,c_{r_1}s_1} \otimes B^{r_2,c_{r_2}s_2} \cong B^\vee(s_1\Lambda_0) \otimes B^{r_2,c_{r_2}s_2}.$$

The lowest weight element $u^\vee_{s_1\Lambda_0} \in B^\vee(s_1\Lambda_0)$ corresponds to $u^\vee_{s_1\Lambda_{\tau_1(0)}} \otimes u_1 \in B^\vee(s_1\Lambda_{\tau_1(0)}) \otimes B^{r_1,c_{r_1}s_1}$. Acting with lowering operators f_i with $i \in I$ one can bring any element $u^\vee_{s_1\Lambda_{\tau_1(0)}} \otimes b_1 \otimes b_2$ into the form $u^\vee_{s_1\Lambda_{\tau_1(0)}} \otimes u_1 \otimes c_2$. Once this element is reached, tensor on the left by $u^\vee_{(s_2-s_1)\Lambda_0} \in B^\vee((s_2 - s_1)\Lambda_0)$, obtaining the element $u^\vee_{(s_2-s_1)\Lambda_0} \otimes u^\vee_{s_1\Lambda_{\tau_1(0)}} \otimes u_1 \otimes c_2$, which can be identified with $u^\vee_{s_2\Lambda_0} \otimes c_2 \in B^\vee(s_2\Lambda_0) \otimes B^{r_2,c_{r_2}s_2}$. Now move down to the lowest weight vector $u^\vee_{s_2\Lambda_0} \otimes u_2$ using f_i with $i \in I$.

As a result of the above construction we obtain the following corollary:

Corollary 6.1. *The tensor product $B^{r_1,c_{r_1}s_1} \otimes B^{r_2,c_{r_2}s_2}$ of KR crystals is connected.*

The combinatorial R-matrix is a crystal morphism. More precisely

$$R : B^{r_1,c_{r_1}s_1} \otimes B^{r_2,c_{r_2}s_2} \to B^{r_2,c_{r_2}s_2} \otimes B^{r_1,c_{r_1}s_1}$$

satisfies $R \circ e_i = e_i \circ R$ and $R \circ f_i = f_i \circ R$ for all $i \in I$. There exists a unique element $u_{c_{r_k} s_k \omega_{r_k}} \in B^{r_k, c_{r_k} s_k}$ and by weight considerations R must map $R(u_{c_{r_1} s_1 \omega_{r_1}} \otimes u_{c_{r_2} s_2 \omega_{r_2}}) = u_{c_{r_2} s_2 \omega_{r_2}} \otimes u_{c_{r_1} s_1 \omega_{r_1}}$. Assume that $s_1 \geqslant s_2$. Then for any element $b_1 \otimes b_2 \in B^{r_1, c_{r_1} s_1} \otimes B^{r_2, c_{r_2} r_2}$ the above algorithm provides a sequence $e_{\{i\}} := e_{i_1} e_{i_2} \cdots e_{i_\ell}$ such that $e_{\{i\}}(b_1 \otimes b_2) = u_1 \otimes u_2$. In particular, $e_{\{j\}}(u_{c_{r_1} s_1 \omega_{r_1}} \otimes u_{c_{r_2} s_2 \omega_{r_2}}) = u_1 \otimes u_2$. Set $f_{\{\leftarrow i\}} := f_{i_\ell} \cdots f_{i_1}$. Then

$$R(b_1 \otimes b_2) = f_{\{\leftarrow i\}} e_{\{j\}} (u_{c_{r_2} s_2 \omega_{r_2}} \otimes u_{c_{r_1} s_1 \omega_{r_1}}).$$

For the case $s_1 < s_2$ a similar construction works where f_i and e_i are interchanged.

Acknowledgments

We thank Philip Sternberg for many stimulating discussion regarding KR crystals of type $D_n^{(1)}$.

References

[1] V. Chari, On the fermionic formula and the Kirillov–Reshetikhin conjecture, Int. Math. Res. Not. 12 (2001) 629–654.

[2] V. Chari, A. Moura, Kirillov–Reshetikhin modules associated to G_2, preprint, math.RT/0604281.

[3] V. Chari, A. Pressley, Quantum affine algebras and their representations, in: Representations of Groups, Banff, AB, 1994, in: CMS Conf. Proc., vol. 16, Amer. Math. Soc., Providence, RI, 1995, pp. 59–78.

[4] V. Chari, A. Pressley, Twisted quantum affine algebras, Comm. Math. Phys. 196 (1998) 461–476.

[5] G. Fourier, P. Littelmann, Tensor product structure of affine Demazure modules and limit constructions, Nagoya Math. J. 182 (2006) 171–198.

[6] G. Fourier, P. Littelmann, Weyl modules, Demazure modules, KR-modules, crystals, fusion products and limit constructions, preprint, math.RT/0509276.

[7] G. Hatayama, A. Kuniba, M. Okado, T. Takagi, Z. Tsuboi, Paths, crystals and fermionic formulae, in: MathPhys Odyssey, 2001, in: Prog. Math. Phys., vol. 23, Birkhäuser Boston, Boston, MA, 2002, pp. 205–272.

[8] G. Hatayama, A. Kuniba, M. Okado, T. Takagi, Y. Yamada, Remarks on fermionic formula, in: Recent Developments in Quantum Affine Algebras and Related Topics, Raleigh, NC, 1998, in: Contemp. Math., vol. 248, Amer. Math. Soc., Providence, RI, 1999, pp. 243–291.

[9] N. Jing, K.C. Misra, M. Okado, q-Wedge modules for quantized enveloping algebras of classical type, J. Algebra 230 (2) (2000) 518–539.

[10] V.G. Kac, Infinite-Dimensional Lie Algebras, third ed., Cambridge Univ. Press, Cambridge, 1990.

[11] M. Kashiwara, The crystal base and Littelmann's refined Demazure character formula, Duke Math. J. 71 (3) (1993) 839–858.

[12] M. Kashiwara, Crystal bases of modified quantized enveloping algebra, Duke Math. J. 73 (2) (1994) 383–413.

[13] M. Kashiwara, On level-zero representation of quantized affine algebras, Duke Math. J. 112 (1) (2002) 117–195.

[14] Y. Koga, Level one perfect crystals for $B_n^{(1)}$, $C_n^{(1)}$, and $D_n^{(1)}$, J. Algebra 217 (1) (1999) 312–334.

[15] S. Kumar, Demazure character formula in arbitrary Kac–Moody setting, Invent. Math. 89 (1987) 395–423.

[16] S.-J. Kang, M. Kashiwara, K.C. Misra, Crystal bases of Verma modules for quantum affine Lie algebras, Compos. Math. 92 (1994) 299–325.

[17] S.-J. Kang, M. Kashiwara, K.C. Misra, T. Miwa, T. Nakashima, A. Nakayashiki, Affine crystals and vertex models, in: Infinite Analysis, Parts A, B, Kyoto, 1991, in: Adv. Ser. Math. Phys., vol. 16, World Scientific, River Edge, NJ, 1992, pp. 449–484.

[18] S.-J. Kang, M. Kashiwara, K.C. Misra, T. Miwa, T. Nakashima, A. Nakayashiki, Perfect crystals of quantum affine Lie algebras, Duke Math. J. 68 (3) (1992) 499–607.

[19] M. Kashiwara, T. Nakashima, Crystal graphs for representations of the q-analogue of classical Lie algebras, J. Algebra 165 (2) (1994) 295–345.

[20] P. Littelmann, A Littlewood–Richardson rule for symmetrizable Kac–Moody algebras, Invent. Math. 116 (1–3) (1994) 329–346.

[21] O. Mathieu, Formules de caractères pour les algèbres de Kac–Moody générales, Astérisque 159–160 (1988).

[22] S. Naito, D. Sagaki, Construction of perfect crystals conjecturally corresponding to Kirillov–Reshetikhin modules over twisted quantum affine algebras, Comm. Math. Phys. 263 (3) (2006) 749–787.

[23] M. Okado, A. Schilling, M. Shimozono, Virtual crystals and fermionic formulas of type $D_{n+1}^{(2)}$, $A_{2n}^{(2)}$, and $C_n^{(1)}$, Represent. Theory 7 (2003) 101–163.

[24] M. Shimozono, Affine type A crystal structure on tensor products of rectangles, Demazure characters, and nilpotent varieties, J. Algebraic Combin. 15 (2) (2002) 151–187.

[25] P. Sternberg, A. Schilling, Finite-dimensional crystals $B^{2,s}$ for quantum affine algebras of type $D_n^{(1)}$, J. Algebraic Combin. 23 (2006) 317–354.