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Weyl modules, Demazure modules, KR-modules, crystals, fusion products and limit constructions ☆

G. Fourier, P. Littelmann *

Mathematisches Institut der Universität zu Köln, Weyertal 86-90, 50931 Köln, Germany

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Abstract

We study finite-dimensional representations of current algebras, loop algebras and their quantized versions. For the current algebra of a simple Lie algebra of type ADE, we show that Kirillov–Reshetikhin modules and Weyl modules are in fact all Demazure modules. As a consequence one obtains an elementary proof of the dimension formula for Weyl modules for the current and the loop algebra. Further, we show that the crystals of the Weyl and the Demazure module are the same up to some additional label zero arrows for the Weyl module.

For the current algebra $\mathcal{C}\mathfrak{g}$ of an arbitrary simple Lie algebra, the fusion product of Demazure modules of the same level turns out to be again a Demazure module. As an application we construct the $\mathcal{C}\mathfrak{g}$-module structure of the Kac–Moody algebra $\hat{\mathfrak{g}}$-module $V(\ell\Lambda_0)$ as a semi-infinite fusion product of finite-dimensional $\mathcal{C}\mathfrak{g}$-modules.

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Contents

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* Corresponding author.
E-mail addresses: gfourier@math.uni-koeln.de (G. Fourier), littelma@math.uni-koeln.de (P. Littelmann).

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1. Introduction

Let $\mathfrak{g}$ be a semisimple complex Lie algebra. The theory of finite-dimensional representations of its loop algebra $\mathcal{L}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$, its quantized loop algebra $U_q(\mathcal{L}\mathfrak{g})$ and its current algebra $\mathcal{C}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t]$ have been the subject of many articles in the recent years. See for example [1,2,4, 6,7,9–12,17,19,20,25,27,36] for different approaches and different aspects of this subject.

The notion of a Weyl module in this context was introduced in [10] for the affine Kac–Moody algebra and its quantized version. These modules can be described in terms of generators and relations, and they are characterized by the following universal property: any finite-dimensional highest weight module which is generated by a one-dimensional highest weight space, is a quotient of a Weyl module. This notion can be naturally extended to the category of finite-dimensional representations of the current algebra [6,16]. Another intensively studied class of modules are the Kirillov–Reshetikhin modules, a name that originally refers to evaluation modules of the Yangian. In [5] Chari gave a definition of these modules for the current algebra in terms of generators and relations.

The current algebra is a subalgebra of a maximal parabolic subalgebra of the affine Kac–Moody algebra $\hat{\mathfrak{g}}$. Let Λ be a dominant weight for $\hat{\mathfrak{g}}$ and denote by $V(\Lambda)$ the associated (infinite-dimensional) irreducible $\hat{\mathfrak{g}}$-representation. Another natural class of finite-dimensional representations of the current algebra are provided by certain Demazure submodules of $V(\Lambda)$. Of particular interest for this paper are the twisted (see Section 2.2) $\mathcal{C}\mathfrak{g}$-stable Demazure submodules $D(m, \lambda)$ of $V(m\Lambda_0)$, where Λ_0 is the fundamental weight associated to the additional node of the extended Dynkin diagram of $\mathfrak{g}$.

If $\mathfrak{g}$ is simply laced, then we can identify the weight and the coweight lattice, so the Weyl modules as well as the twisted $\mathcal{C}\mathfrak{g}$-stable Demazure submodules of $V(m\Lambda_0)$ are classified by dominant weights $\lambda \in P^+$.

Theorem A. *For a simple complex Lie algebra of simply laced type, the Weyl module $W(\lambda)$ and the Demazure module $D(1, \lambda)$ are isomorphic as $\mathcal{C}\mathfrak{g}$-modules.*

Also the Demazure modules of higher level are related to an interesting class of finite-dimensional modules for $\mathcal{C}\mathfrak{g}$. Let $\mathfrak{g}$ be an arbitrary simple complex Lie algebra, the $\mathcal{C}\mathfrak{g}$-stable Demazure modules $D(m, \lambda^\vee)$ are classified by dominant coweights $\lambda \in \check{P}^+$.

Theorem B. *For a fundamental coweight $\omega_i^\vee$ let $d_i = 1, 2$ or 3 be such that $d_i\omega_i = \nu(\omega_i^\vee)$. The Kirillov–Reshetikhin module $KR(d_i m\omega_i)$ is, as $\mathcal{C}\mathfrak{g}$-module, isomorphic to the Demazure module $D(m, \omega_i^\vee)$. In particular, in the simply laced case all Kirillov–Reshetikhin modules are Demazure modules.*

Remark 1. The fact that $D(m, \omega_i^\vee)$ is a quotient of a Kirillov–Reshetikhin module has been already pointed out in [8]. In the same paper Chari and Moura have shown that $D(m, \omega_i^\vee)$ is isomorphic to $KR(d_i m\omega_i)$ for all classical groups using character calculations. Our proof is independent of the type of the algebra.

To stay inside the class of cyclic highest weight modules, the tensor product of cyclic $\mathcal{C}\mathfrak{g}$-modules is often replaced by the fusion product of modules [15].

Theorem C. *Let $\mathfrak{g}$ be a complex simple Lie algebra and let $\lambda^\vee = \lambda_1^\vee + \cdots + \lambda_r^\vee$ be a decomposition of a dominant coweight as a sum of dominant coweights. Then $D(m, \lambda^\vee)$ and the fusion product $D(m, \lambda_1^\vee) * \cdots * D(m, \lambda_r^\vee)$ are isomorphic as $\mathcal{C}\mathfrak{g}$-modules.*

Remark 2. The theorem shows in particular that the fusion product of Demazure modules of the same level is associative and independent of the parameters used in the fusion construction. In [1] it is shown that the fusion product of Kirillov–Reshetikhin modules of arbitrary levels is independent of the parameters.

As a consequence we obtain for the Weyl module $W(\lambda)$ in the simply laced case:

Corollary A. *Suppose $\mathfrak{g}$ is of simply laced type. Let $\lambda = a_1\omega_1 + \cdots + a_n\omega_n$ be a decomposition of a dominant weight $\lambda \in P^+$ as a sum of fundamental weights. Then the Weyl module $W(\lambda)$ for the current algebra is the fusion product of the fundamental Weyl modules*:

$$W(\lambda) \simeq \underbrace{W(\omega_1) * \cdots * W(\omega_1)}_{a_1} * \cdots * \underbrace{W(\omega_n) * \cdots * W(\omega_n)}_{a_n}.$$

The Weyl modules for the loop algebra are classified by n-tuples $\pi = (\pi_1, \ldots, \pi_n)$ of polynomials $\pi_j \in \mathbb{C}[u]$ with constant term 1 [10]. The associated dominant weight is $\lambda_\pi = \sum_i \deg \pi_i \omega_i$. Similarly, the Weyl modules for the quantized loop algebra are classified by n-tuples $\pi_q = (\pi_{q,1}, \ldots, \pi_{q,n})$ of polynomials $\pi_{q,j} \in \mathbb{C}(q)[u]$ with constant term 1, the associated weight λ_{π_q} is defined as above.

It was conjectured in [10] (and proved in the $\mathfrak{sl}_2$-case) that the dimension of the Weyl modules depend only λ_{π_q}, respectively λ_π. More precisely, they conjectured that the dimension is the (appropriate) product of the dimension of the “fundamental modules.” As Hiraku Nakajima has pointed out to us, the dimension conjecture can be deduced using the theory of global basis. The results of Kashiwara [24,25] imply that the Weyl modules are specializations (the $q = 1$ limit) of certain finite-dimensional quotients of the extremal weight modules for the quantum affine algebra. The results of Beck and Nakajima [2,36,37] imply that these quotients (and hence their specializations) have the correct dimension.

A different approach to prove the dimension conjecture was suggested in [10,11]. In fact, using the specialization and dimension arguments outlined there, in the simply laced case the dimension formula is an immediate consequence of Theorems A and C:

Corollary B. *Let $\mathfrak{g}$ be a simple Lie algebra of simply laced type, let $\lambda = \sum m_i\omega_i$ be a dominant weight (for $\mathfrak{g}$), let π (respectively π_q) be an n-tuple of polynomials in $\mathbb{C}[u]$ (respectively in $\mathbb{C}(q)[u]$) with constant term 1 such that $\lambda = \lambda_\pi = \lambda_{\pi_q}$. Then*

$$\dim W(\lambda) = \dim W(\pi) = \dim W_q(\pi_q) = \dim D\big(1, \lambda^\vee\big) = \prod_i \big(\dim W(\omega_i)\big)^{m_i}.$$

Remark 3. For $\mathfrak{g} = \mathfrak{sl}_n$, the connection between Demazure modules in $V(\Lambda_0)$ and Weyl modules had been already obtained by Chari and Loktev in [6]. The isomorphism between the Weyl module $W(\lambda)$ and the Demazure module $D(1,\lambda)$ has been conjectured in [17].

For a dominant weight $\lambda = \sum m_i\omega_i$ let $\pi_{\lambda,a}$ be the tuple having $(1-au)^{m_i}$ as ith entry. The quantum Demazure module $D_q(m,\lambda)$ has an associated crystal graph which is a subgraph of the crystal graph of the corresponding irreducible $U_q(\hat{\mathfrak{g}})$-representation. We conjecture that by adding appropriate label zero arrows, one gets the graph of an irreducible $U_q(\hat{\mathfrak{g}})$-representation. In the simply laced case and level one we have:

Proposition. *The crystal graph of $D_q(1,\lambda)$ is obtained from the crystal graph of $W_q(\pi_{q,\lambda,1})$ by omitting certain label zero arrows. More precisely, let $B(\lambda)_{cl}$ be the path model for $W_q(\pi_{q,\lambda,1})$ described in* [34]*, then the crystal graph of the Demazure module is isomorphic to the graph of the concatenation $\pi_{\Lambda_0} * B(\lambda)_{cl}$.*

In the simply laced case, the restriction of the loop Weyl module $W(\pi_{\lambda,a})$ to $\mathcal{C}\mathfrak{g}$ is (up to a twist by an automorphism) the Weyl module $W(\lambda)$. It follows:

Corollary C. *The Demazure module $D(m,\lambda)$ of level m can be equipped with the structure of a cyclic $U(\mathcal{L}\mathfrak{g})$-module such that the $\mathfrak{g}$-module structure coincides with the natural $\mathfrak{g}$-structure coming from the Demazure module construction.*

Let $V(m\Lambda_0)$, $m \in \mathbb{N}$, be the irreducible highest weight module of highest weight $m\Lambda_0$ for the affine Kac–Moody $\hat{\mathfrak{g}}$. In [18] we gave a description of the $\mathfrak{g}$-module structure of this representation in terms of a semi-infinite tensor product. Using Theorem C, we are able to lift this result to the level of modules for the current algebra. The theorem holds in a much more general setting (see Remark 19), but for the convenience of a uniform presentation, let Θ be the highest root of the root system of $\mathfrak{g}$.

Theorem D. *Let $D(m,n\Theta) \subset V(m\Lambda_0)$ be the Demazure module of level m corresponding to the translation at $-n\Theta$. Let $w \neq 0$ be a $\mathcal{C}\mathfrak{g}$-invariant vector of $D(m,\Theta)$. Let V_m^∞ be the direct limit*

$$D(m,\Theta) \hookrightarrow D(m,\Theta) * D(m,\Theta) \hookrightarrow D(m,\Theta) * D(m,\Theta) * D(m,\Theta) \hookrightarrow \cdots,$$

where the inclusions are given by $v \mapsto w \otimes v$.

Then $V(m\Lambda_0)$ *and* V_m^∞ *are isomorphic as* $U(\mathcal{C}\mathfrak{g})$*-modules.*

The semi-infinite fusion construction can be seen as an extension of the construction of Feigin and Feigin [14] ($\mathfrak{g} = \mathfrak{sl}_2$) and Kedem [27] ($\mathfrak{g} = \mathfrak{sl}_n$) to arbitrary simple Lie algebras. We conjecture (see Conjecture 2) that, as in [14] and [27], the semi-infinite fusion construction works for arbitrary dominant weights and not only for multiples of Λ_0.

Remark 4. Naito and Sagaki [33–35] gave a path model for the Weyl modules $W(\omega)$ for all fundamental weights and $\mathfrak{g}$ of arbitrary type. Since the Weyl modules coincide with the level-one Demazure modules provided $\mathfrak{g}$ is simply-laced, the semi-infinite limit construction above gives on the combinatorial side a combinatorial limit path model for the representation $V(\Lambda_0)$ as a semi-infinite concatenation of a finite path model, extending in this sense the approach of Magyar in [31].

After introducing some notation, we will recall in more detail the definition of Demazure and Weyl modules and fusions products. The proof of Theorems A–C, their corollaries and the proposition is given in Section 3 (see Theorems 4, 7 and 8). The proof of Theorem D is given in Section 4, see Theorem 9.

2. Notation and basics

2.1. *Affine Kac–Moody algebras*

In this section we fix the notation and the usual technical padding. Let $\mathfrak{g}$ be simple complex Lie algebra. We fix a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ and a Borel subalgebra $\mathfrak{b} \supseteq \mathfrak{h}$. Denote $\Phi \subseteq \mathfrak{h}^*$ the root system of $\mathfrak{g}$, and, corresponding to the choice of $\mathfrak{b}$, let Φ^+ be the set of positive roots and let $\Delta = \{\alpha_1, \dots, \alpha_n\}$ be the corresponding basis of Φ.

For a root $\beta \in \Phi$ let $\beta^\vee \in \mathfrak{h}$ be its coroot. The basis of the dual root system (also called the coroot system) $\Phi^\vee \subset \mathfrak{h}$ is denoted $\Delta^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\}$.

We denote throughout the paper by $\Theta = \sum_{i=1}^n a_i\alpha_i$ the highest root of Φ and by $\Theta^\vee = \sum_{i=1}^n a_i^\vee \alpha_i^\vee$ its coroot. Note that $\Theta^\vee$ is in general not the highest root of $\Phi^\vee$. (For more details concerning the connection with the dual root system of the affine root system $\widehat{\Phi}$ see [23, Chapter 6].)

The Weyl group W of Φ is generated by the simple reflections $s_i = s_{\alpha_i}$ associated to the simple roots.

Let P be the weight lattice of $\mathfrak{g}$ and let P^+ be the subset of dominant weights. The group algebra of P is denoted $\mathbb{Z}[P]$, we write $\chi = \sum m_\mu e^\mu$ (finite sum, $\mu \in P, m_\mu \in \mathbb{Z}$) for an element in $\mathbb{Z}[P]$, where the embedding $P \hookrightarrow \mathbb{Z}[P]$ is defined by $\mu \mapsto e^\mu$.

We denote the coweight lattice by $\check{P}$, i.e., this is the lattice of integral weights for the dual root system. The dominant coweights are denoted $\check{P}^+$.

Corresponding to the enumeration of the simple roots let $\omega_1, \dots, \omega_n$ be the fundamental weights. Let $\mathfrak{h}_\mathbb{R}$ be the "real part" of $\mathfrak{h}$, i.e., $\mathfrak{h}_\mathbb{R}$ is the real span in $\mathfrak{h}$ of the coroots $\alpha_1^\vee, \dots, \alpha_n^\vee$, and let $\mathfrak{h}_\mathbb{R}^*$ be the real span of the fundamental weights $\omega_1, \dots, \omega_n$. Let $(\cdot,\cdot)$ be the unique invariant symmetric non-degenerate bilinear form on $\mathfrak{g}$ normalized such that the restriction to $\mathfrak{h}$ induces an isomorphism

$$\nu : \mathfrak{h}_{\mathbb{R}} \to \mathfrak{h}_{\mathbb{R}}^*, \quad \nu(h) : \begin{cases} \mathfrak{h} \to \mathbb{R} \\ h' \mapsto (h, h') \end{cases}$$

mapping $\Theta^\vee$ to Θ. With the notation as above it follows for the weight lattice $P^\vee$ of the dual root system $\Phi^\vee$ that

$$\nu\bigl(\alpha_i^\vee\bigr) = \frac{a_i}{a_i^\vee}\alpha_i \quad \text{and} \quad \nu\bigl(\omega_i^\vee\bigr) = \frac{a_i}{a_i^\vee}\omega_i, \quad \forall i = 1, \dots, n.$$

Let $\hat{\mathfrak{g}}$ be the affine Kac–Moody algebra corresponding to the extended Dynkin diagram of $\mathfrak{g}$ (see [23, Chapter 7]):

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}\bigl[t, t^{-1}\bigr] \oplus \mathbb{C}K \oplus \mathbb{C}d.$$

Here d denotes the derivation $d = t\frac{d}{dt}$, K is the canonical central element, and the Lie bracket is given by

$$\begin{aligned} &\bigl[t^m \otimes x + \lambda K + \mu d, t^n \otimes y + \nu K + \eta d\bigr] \\ &\quad = t^{m+n} \otimes [x, y] + \mu n t^n \otimes y + \eta m t^m \otimes x + m\delta_{m,-n}(x, y)K. \end{aligned} \tag{1}$$

The Lie algebra $\mathfrak{g}$ is naturally a subalgebra of $\hat{\mathfrak{g}}$. In the same way, the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and the Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ are subalgebras of the Cartan subalgebra $\hat{\mathfrak{h}}$ respectively the Borel subalgebra $\hat{\mathfrak{b}}$ of $\hat{\mathfrak{g}}$:

$$\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d, \qquad \hat{\mathfrak{b}} = \mathfrak{b} \oplus \mathbb{C}K \oplus \mathbb{C}d \oplus \mathfrak{g} \otimes_{\mathbb{C}} t\mathbb{C}[t]. \tag{2}$$

Denote by $\widehat{\Phi}$ the root system of $\hat{\mathfrak{g}}$ and let $\widehat{\Phi}^+$ be the subset of positive roots. The positive non-divisible imaginary root in $\widehat{\Phi}^+$ is denoted δ. The simple roots are $\widehat{\Delta} = \{\alpha_0, \alpha_1, \dots, \alpha_n\}$ where $\alpha_0 = \delta - \Theta$. We identify the root system Φ of $\mathfrak{g}$ with the root subsystem of $\widehat{\Phi}$ generated by the simple roots $\alpha_1, \dots, \alpha_n$.

Let $\Lambda_0, \dots, \Lambda_n$ be the corresponding fundamental weights, then for $i = 1, \dots, n$ we have

$$\Lambda_i = \omega_i + a_i^\vee \Lambda_0. \tag{3}$$

The decomposition of $\hat{\mathfrak{h}}$ in (2) has its corresponding version for the dual space $\hat{\mathfrak{h}}^*$:

$$\hat{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta. \tag{4}$$

Here the elements of $\mathfrak{h}^*$ are extended trivially, $\langle \Lambda_0, \mathfrak{h}\rangle = \langle \Lambda_0, d\rangle = 0$ and $\langle \Lambda_0, K\rangle = 1$, and $\langle \delta, \mathfrak{h}\rangle = \langle \delta, K\rangle = 0$ and $\langle \delta, d\rangle = 1$. Let $\widehat{\Delta}^\vee = \{\alpha_0^\vee, \alpha_1^\vee, \dots, \alpha_n^\vee\} \subset \hat{\mathfrak{h}}$ be the corresponding basis of the coroot system, then $\alpha_0^\vee = K - \Theta^\vee$. Recall that the positive affine roots are precisely the roots of the form

$$\widehat{\Phi}^+ = \bigl\{\beta + s\delta \mid \beta \in \Phi^+,\ s \geqslant 0\bigr\} \cup \bigl\{-\beta + s\delta \mid \beta \in \Phi^+,\ s > 0\bigr\} \cup \{s\delta \mid s > 0\}.$$

For a real positive root $\beta + s\delta$, respectively $-\beta + s\delta$, the corresponding coroot is

$$(\beta + s\delta)^\vee = \beta^\vee + s\frac{(\beta^\vee, \beta^\vee)}{2}K \quad \text{respectively } (-\beta + s\delta)^\vee = -\beta^\vee + s\frac{(\beta^\vee, \beta^\vee)}{2}K. \tag{5}$$

Let $\hat{\mathfrak{h}}^*_{\mathbb{R}}$ be the real span $\mathbb{R}\delta + \sum_{i=0}^n \mathbb{R}\Lambda_i$, note that by the decomposition (4) and by (3) we have $\mathfrak{h}^*_{\mathbb{R}} \subseteq \hat{\mathfrak{h}}^*_{\mathbb{R}}$. The affine Weyl group W^{aff} is generated by the reflections $s_0, s_1, \ldots, s_n$ acting on $\hat{\mathfrak{h}}^*_{\mathbb{R}}$. (We use again the abbreviation $s_i = s_{\alpha_i}$ for a simple root α_i.) The cone $\widehat{C} = \{\Lambda \in \hat{\mathfrak{h}}^*_{\mathbb{R}} \mid \langle \Lambda, \alpha_i^\vee \rangle \geqslant 0,\ i = 0, \ldots, n\}$ is the fundamental Weyl chamber for $\hat{\mathfrak{g}}$.

We keep the convention and put a $\widehat{\ }$ on (almost) everything related to $\hat{\mathfrak{g}}$. We denote by $\widehat{P}$ the weight lattice of $\hat{\mathfrak{g}}$ and by $\widehat{P}^+$ the subset of dominant weights. As before, let $\mathbb{Z}[\widehat{P}]$ be the group algebra of $\widehat{P}$, so an element in the algebra is a finite sum of the form $\sum m_\mu e^\mu$, $\mu \in \widehat{P}$ and $m_\mu \in \mathbb{Z}$. Recall the following special properties of the imaginary root δ (see for example [23, Chapter 6]):

$$\begin{aligned} \langle \delta, \alpha_i^\vee \rangle = 0 \quad \forall i = 0, \ldots, n, \qquad w(\delta) = \delta \quad \forall w \in W^{\mathrm{aff}}, \\ \langle \alpha_0, \alpha_i^\vee \rangle = -\langle \Theta, \alpha_i^\vee \rangle \quad \text{for } i \geqslant 1. \end{aligned} \tag{6}$$

Put $a_0 = a_0^\vee = 1$ and let $A = (a_{i,j})_{0 \leqslant i,j \leqslant n}$ be the (generalized) Cartan matrix of $\hat{\mathfrak{g}}$. We have a non-degenerate symmetric bilinear form $(\cdot,\cdot)$ on $\hat{\mathfrak{h}}$ defined by [23, Chapter 6]

$$\begin{cases} (\alpha_i^\vee, \alpha_j^\vee) = \frac{a_j}{a_j^\vee} a_{i,j} & i, j = 0, \ldots, \ell, \\ (\alpha_i^\vee, d) = 0 & i = 1, \ldots, \ell, \\ (\alpha_0^\vee, d) = 1 & (d, d) = 0. \end{cases} \tag{7}$$

The corresponding isomorphism $\nu : \hat{\mathfrak{h}} \to \hat{\mathfrak{h}}^*$ maps

$$\nu(\alpha_i^\vee) = \frac{a_i}{a_i^\vee}\alpha_i, \qquad \nu(K) = \delta, \qquad \nu(d) = \Lambda_0.$$

Since W^{aff} fixes δ, the affine Weyl W^{aff} can be defined as the subgroup of $GL(\mathfrak{h}^*_{\mathrm{sc},\mathbb{R}})$ generated by the induced reflections $s_0, \ldots, s_n$. Another well-known description of the affine Weyl group is the following. Let $M \subset \mathfrak{h}^*_{\mathbb{R}}$ be the lattice $M = \nu(\bigoplus_{i=1}^n \mathbb{Z}\alpha_i^\vee)$. If $\mathfrak{g}$ is simply laced, then M is the root lattice in $\mathfrak{h}^*_{\mathbb{R}}$, otherwise M is the lattice in $\mathfrak{h}^*_{\mathbb{R}}$ generated by the long roots.

An element $\Lambda \in \mathfrak{h}^*_{\mathrm{sc},\mathbb{R}}$ can be uniquely decomposed into $\Lambda = \lambda + b\Lambda_0$ such that $\lambda \in \mathfrak{h}^*_{\mathbb{R}}$. For an element $\mu \in M$ let $t_\mu \in GL(\mathfrak{h}^*_{\mathrm{sc},\mathbb{R}})$ be the map defined by

$$\Lambda = \lambda + b\Lambda_0 \mapsto t_\mu(\Lambda) = \lambda + b\Lambda_0 + b\mu = \Lambda + \langle \Lambda, K \rangle \mu. \tag{8}$$

Obviously we have $t_\mu \circ t_{\mu'} = t_{\mu+\mu'}$, denote t_M the abelian subgroup of $GL(\mathfrak{h}^*_{\mathrm{sc},\mathbb{R}})$ consisting of the elements t_μ, $\mu \in M$. Then W^{aff} is the semi-direct product $W^{\mathrm{aff}} = W \ltimes t_M$.

The *extended affine Weyl group* $\widetilde{W}^{\mathrm{aff}}$ is the semi-direct product $\widetilde{W}^{\mathrm{aff}} = W \ltimes t_L$, where $L = \nu(\bigoplus_{i=1}^n \mathbb{Z}\omega_i^\vee)$ is the image of the coweight lattice. The action of an element t_μ, $\mu \in L$, is defined as above in (8).

Let Σ be the subgroup of $\widetilde{W}^{\mathrm{aff}}$ stabilizing the dominant Weyl chamber $\widehat{C}$:

$$\Sigma = \{\sigma \in \widetilde{W}^{\mathrm{aff}} \mid \sigma(\widehat{C}) = \widehat{C}\}.$$

Then Σ provides a complete system of coset representatives of $\widetilde{W}^{\mathrm{aff}}/W^{\mathrm{aff}}$, so we can write in fact $\widetilde{W}^{\mathrm{aff}} = \Sigma \ltimes W^{\mathrm{aff}}$.

The elements $\sigma \in \Sigma$ are all of the form

$$\sigma = \tau_i t_{-\nu(\omega_i^\vee)} = \tau_i t_{-\omega_i},$$

where $\omega_i^\vee$ is a minuscule fundamental coweight. Further, set $\tau_i = w_0 w_{0,i}$, where w_0 is the longest word W and $w_{0,i}$ is the longest word in W_{ω_i}, the stabilizer of ω_i in W.

We extend the length function $\ell : W^{\mathrm{aff}} \to \mathbb{N}$ to a length function $\ell : \widetilde{W}^{\mathrm{aff}} \to \mathbb{N}$ by setting $\ell(\sigma w) = \ell(w)$ for $w \in W^{\mathrm{aff}}$ and $\sigma \in \Sigma$.

2.2. *Definition of Demazure modules*

For a dominant weight $\Lambda \in \widehat{P}^+$ let $V(\Lambda)$ be the (up to isomorphism) unique irreducible highest weight module of highest weight Λ.

Let $U(\mathfrak{I}_\mathfrak{b})$ be the enveloping algebra of the Iwahori subalgebra $\mathfrak{I}_\mathfrak{b} = \mathfrak{g} \otimes t\mathbb{C}[t] \oplus \mathfrak{b} \otimes 1$, and let $U(\hat{\mathfrak{n}})$ be the enveloping algebra of $\hat{\mathfrak{n}} = \mathfrak{n}^+ \otimes \mathbb{C}[t] \oplus \mathfrak{h} \otimes t\mathbb{C}[t] \oplus \mathfrak{n}^- \otimes t\mathbb{C}[t]$.

Given an element $w \in W^{\mathrm{aff}}/W_\Lambda$, fix a generator $v_{w(\Lambda)}$ of the line $V(\Lambda)_{w(\Lambda)} = \mathbb{C}v_{w(\Lambda)}$ of $\hat{\mathfrak{h}}$-eigenvectors in $V(\Lambda)$ of weight $w(\Lambda)$.

Definition 1. The $U(\hat{\mathfrak{b}})$-submodule $V_w(\Lambda) = U(\hat{\mathfrak{b}}) \cdot v_{w(\Lambda)}$ generated by $v_{w(\Lambda)}$ is called the *Demazure submodule of* $V(\Lambda)$ associated to w.

Remark 5. Since v is an $\hat{\mathfrak{h}}$-eigenvector, we can also view the Demazure module $V_w(\Lambda)$ as a cyclic $U(\mathfrak{I}_\mathfrak{b})$-module or a cyclic $U(\hat{\mathfrak{n}})$-module generated by $v_{w(\Lambda)}$:

$$V_w(\Lambda) = U(\mathfrak{I}_\mathfrak{b}) \cdot v_{w(\Lambda)} = U(\hat{\mathfrak{n}}) \cdot v_{w(\Lambda)}.$$

To associate more generally to every element $\sigma w \in \widetilde{W}^{\mathrm{aff}} = \Sigma \ltimes W^{\mathrm{aff}}$ a Demazure module, recall that elements in Σ correspond to automorphisms of the Dynkin diagram of $\hat{\mathfrak{g}}$, and thus define an associated automorphism of $\hat{\mathfrak{g}}$, also denoted σ. For a module V of $\hat{\mathfrak{g}}$ let V^σ be the module with the twisted action $g \circ v = \sigma^{-1}(g)v$. Then for the irreducible module of highest weight $\Lambda \in \widehat{P}^+$ we get $V(\Lambda)^\sigma = V(\sigma(\Lambda))$.

So for $\sigma w \in \widetilde{W}^{\mathrm{aff}} = \Sigma \ltimes W^{\mathrm{aff}}$ we set

$$V_{w\sigma}(\Lambda) = V_w\bigl(\sigma(\Lambda)\bigr) \quad \text{respectively } V_{\sigma w}(\Lambda) = V_{\sigma w \sigma^{-1}}\bigl(\sigma(\Lambda)\bigr). \tag{9}$$

Recall that for a simple root α the Demazure module $V_{w\sigma}(\Lambda)$ is stable for the associated subalgebra $\mathfrak{sl}_2(\alpha)$ if and only if $s_\alpha w\sigma \leqslant w\sigma \bmod W_\Lambda$ in the (extended) Bruhat order. In particular, $V_{w\sigma}(\Lambda)$ is a $\mathfrak{g}$-module if and only if $s_i w\sigma \leqslant w\sigma \bmod W_\Lambda^{\mathrm{aff}}$ for all $i = 1, \ldots, n$.

We are mainly interested in Demazure modules associated to the weight $\ell\Lambda_0$ for $\ell \geqslant 1$. In this case $W_\Lambda^{\mathrm{aff}} = W$, so $\widetilde{W}^{\mathrm{aff}}/W = L$. The Demazure module $V_{t_{\nu(\mu^\vee)}}(\Lambda_0)$ is a $\mathfrak{g}$-module if and only if $\mu^\vee$ is an anti-dominant coweight, or, in other words, $\mu^\vee = -\lambda^\vee$ for some dominant coweight.

Since we will mainly work with these $\mathfrak{g}$-stable Demazure modules, to simplify the notation, we write in the following

$$D\big(\ell,\lambda^\vee\big) \quad \text{for } V_{t_{-\nu(\lambda^\vee_*)}}(\ell\Lambda_0), \tag{10}$$

where $\lambda^\vee_* = -w_0(\lambda^\vee)$, the dual coweight of $\lambda^\vee$. This notation is justified by the fact that $D(\ell,\lambda^\vee)$ is, considered as $\mathfrak{g}$-module, far from being irreducible, but this $\mathfrak{g}$-module still has a unique maximal highest weight: $\ell\nu(\lambda^\vee)$, i.e., if $V(\mu)$ is an irreducible $\mathfrak{g}$-module of highest weight μ and $\mathrm{Hom}(V(\mu), D(\ell,\lambda)) \neq 0$, then necessarily we have $\ell\nu(\lambda^\vee)-\mu$ is a non-negative sum of positive roots. For more details on the $\mathfrak{g}$-module structure of $D(\ell,\lambda^\vee)$ see also Theorem 2, respectively [18].

2.3. Properties of Demazure modules

A description of Demazure modules in terms of generators and relations has been given by Joseph [22] (semisimple Lie algebras, characteristic zero) and Polo [38] (semisimple Lie algebras, characteristic free), and Mathieu [32] (symmetrizable Kac–Moody algebras). We give here a reformulation for the affine case.

Theorem 1. [32] *Let $\Lambda \in \widehat{P}^+$ and let w be an element of the affine Weyl group of $\hat{\mathfrak{g}}$. The Demazure module $V_w(\Lambda)$ is as a $U(\hat{\mathfrak{b}})$-module isomorphic to the following cyclic module, generated by $v \neq 0$ with the following relations: for all positive roots β of $\mathfrak{g}$ we have*

$$\big(X_\beta \otimes t^s\big)^{k_\beta+1}.v = 0 \quad \textit{where } s \geqslant 0,\ k_\beta = \max\big\{0, -\big\langle w(\Lambda), (\beta+s\delta)^\vee\big\rangle\big\},$$
$$\big(X^-_\beta \otimes t^s\big)^{k_\beta+1}.v = 0 \quad \textit{where } s > 0,\ k_\beta = \max\big\{0, -\big\langle w(\Lambda), (-\beta+s\delta)^\vee\big\rangle\big\}$$
$$\big(h \otimes t^s\big).v = 0, \quad \forall h \in \mathfrak{h},\ s > 0,$$
$$(h\otimes 1).v = w(\Lambda)(h)v, \quad \forall h \in \mathfrak{h}, \qquad d.v = w(\Lambda)(d).v, \qquad K.v = \mathit{level}(\Lambda)v.$$

Let $\lambda^\vee \in \check{P}^+$ be a dominant coweight. We reformulate now the description of the Demazure modules above for the Demazure modules $D(\ell,\lambda^\vee)$ we are interested in.

Corollary 1. *As a module for the current algebra $\mathcal{C}\mathfrak{g}$, $D(\ell,\lambda^\vee)$ is isomorphic to the cyclic $\mathcal{C}\mathfrak{g}$-module generated by a vector v subject to the following relations:*

$$\mathfrak{n}^+ \otimes \mathbb{C}[t].v = 0, \qquad \mathfrak{h}\otimes t\mathbb{C}[t].v = 0, \qquad h.v = \ell\nu\big(\lambda^\vee\big)(h)v \quad \textit{for all } h \in \mathfrak{h},$$

and for all positive roots $\beta \in \Phi^+$ one has

$$\big(X^-_\beta \otimes t^s\big)^{k_\beta+1}.v = 0 \quad \textit{where } s \geqslant 0 \textit{ and } k_\beta = \ell\max\big\{0, -\big\langle \Lambda_0 + \nu\big(\lambda^\vee\big), (-\beta+s\delta)^\vee\big\rangle\big\}. \tag{11}$$

Proof. Denote by M the cyclic $U(\mathcal{C}\mathfrak{g})$-module obtained by the relations above. Recall (see (8)) that $t_{\nu(\lambda^\vee)}(\ell\Lambda_0) = \ell\Lambda_0 + \ell\nu(\lambda^\vee)$ and set $\mu = t_{\nu(\lambda^\vee)}(\ell\Lambda_0)$. Write $t_{\nu(\lambda^\vee)} = w\sigma$ where $w \in W^{\mathrm{aff}}$ and $\sigma \in \Sigma$. Set $\Lambda = \sigma(\Lambda_0)$, then the highest weight $\hat{\mathfrak{g}}$-module $V(\ell\Lambda)$ has a unique line of $\hat{\mathfrak{h}}$-eigenvectors of weight μ, let v_μ be a generator. Fix also a generator $v_{w_0(\mu)}$ of weight $w_0(\mu)$.

Restricted to the current algebra, we have $\mu|_{\mathfrak{h}} = \ell\nu(\lambda^\vee)$. The submodule $U(\mathcal{C}\mathfrak{g}).v_\mu$ of $V(\ell\Lambda)$ is the Demazure module $D(\ell,\lambda^\vee)$ because:

$$D\big(\ell,\lambda^\vee\big) = V_{t_{-\nu(\lambda^\vee_*)}}(\ell\Lambda_0) = U(\hat{\mathfrak{n}})\cdot v_{w_0(\mu)} = U(\mathcal{C}\mathfrak{g}).v_\mu.$$

Since v_μ is an extremal weight vector, using $\mathfrak{sl}_2$-representation theory one verifies easily that v_μ satisfies the relations above. For example, if the root is of the form $\beta + s\delta$, where $s \geqslant 0$ and $\beta \in \Phi^+$ is a positive root, then the corresponding coroot is of the form $\beta^\vee + s'K$, $s' \geqslant 0$. It follows that

$$\big\langle \ell\Lambda_0 + \ell\nu\big(\lambda^\vee\big), \beta^\vee + s'K\big\rangle = \ell s' + \big\langle \ell\nu\big(\lambda^\vee\big), \beta^\vee\big\rangle \geqslant 0,$$

and hence $(\mathfrak{n}^+ \otimes \mathbb{C}[t])v_\mu = 0$. So we have an obvious surjective $\mathcal{C}\mathfrak{g}$-equivariant morphism $M \to D(\ell,\lambda^\vee)$, which maps the cyclic generator v to the cyclic generator v_μ.

To prove that this map is an isomorphism it suffices to prove: $\dim M \leqslant \dim D(\ell,\lambda^\vee)$. The module M is not trivial by the above, and the generator $v \in M$ is a highest weight vector for the Lie subalgebra $\mathfrak{g} \subset \mathcal{C}\mathfrak{g}$. In fact, the relations imply that the $\mathfrak{g}$-submodule $U(\mathfrak{g}).v \subseteq M$ is an irreducible, finite-dimensional highest weight $\mathfrak{g}$-module $V(\ell\nu(\lambda^\vee)) \subseteq M$. So we may replace for convenience the generator v by a generator $v' \in V(\ell\nu(\lambda^\vee))$ of weight $w_0(\ell\nu(\lambda^\vee))$, i.e., we replace a $\mathfrak{g}$-highest weight vector by a $\mathfrak{g}$-lowest weight vector. By construction, the following relations hold:

(1) $(X^+_\beta \otimes t^s)^{k_\beta+1}.v' = 0$ where $s \geqslant 0$; $k_\beta = \ell \max\{0, -\langle \Lambda_0 + w_0(\nu(\lambda^\vee)), (\beta + s\delta)^\vee\rangle\}$.
(2) $(h \otimes 1).v' = \ell\nu(w_0(\lambda^\vee))(h)v'$ where $h \in \mathfrak{h}$.
(3) $\mathfrak{h} \otimes t\mathbb{C}[t].v' = 0$.
(4) $\mathfrak{n}^- \otimes \mathbb{C}[t].v' = 0$.

Now in (4) we have roots of the form $-\beta + s\delta$, where β is a positive root and $s \geqslant 0$. It follows

$$\big\langle \Lambda_0 + w_0\big(\nu\big(\lambda^\vee\big)\big), -\beta^\vee + s'K\big\rangle = s' + \big\langle -w_0\big(\nu\big(\lambda^\vee\big)\big), \beta^\vee\big\rangle \geqslant 0,$$

so we can reformulate (4) in the following way:

(4′) $(X^-_\beta \otimes t^s)^{k_\beta+1}.v' = 0$ where $s \geqslant 0$; $k_\beta = \ell \max\{0, -\langle \Lambda_0 + w_0(\nu(\lambda^\vee)), (-\beta + s\delta)^\vee\rangle\}$.

Now (4) implies $M = U(\mathfrak{g}\otimes\mathbb{C}[t]).v' = U(\mathfrak{I}_\mathfrak{b}).v'$, and (1)–(3), (4′) show that the cyclic generator v' for M as $U(\mathfrak{I}_\mathfrak{b})$-module satisfies the same relations as the generator for the Demazure module $D(\ell,\lambda^\vee)$. Hence we have a surjective $U(\mathfrak{I}_\mathfrak{b})$-module homomorphism $D(\ell,\lambda^\vee) \to M$, which finishes the proof. □

Remark 6. We can easily extend the defining relations in Corollary 1 to an action of $\mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbb{C}K$ by letting K act by ℓ, the level of $\ell\Lambda_0$. This follows immediately from (1) since in the current algebra there are no elements of the form $x \otimes t^{-s}$, $s > 0$.

The $\mathfrak{g}$-module structure of these special Demazure modules has been investigated in [18]: let $\lambda^\vee = \lambda^\vee_1 + \cdots + \lambda^\vee_r$ be a sum of dominant integral coweights for $\mathfrak{g}$ and let $\ell \in \mathbb{N}$.

Theorem 2. [18] *As $\mathfrak{g}$-modules the following are isomorphic*

$$D(\ell,\lambda^\vee)\simeq D(\ell,\lambda_1^\vee)\otimes\cdots\otimes D(\ell,\lambda_r^\vee).$$

In this paper we will extend this isomorphism to an isomorphism of $\mathcal{C}\mathfrak{g}$-modules by replacing the tensor product by the fusion product.

2.4. Weyl modules for the loop algebra

The Weyl modules for the loop algebra $\mathcal{L}\mathfrak{g}$ have been introduced in [10]. These modules are classified by n-tuples of polynomials $\pi=(\pi_1,\dots,\pi_n)$ with constant term 1, and they have the following universal property: every finite-dimensional cyclic $\mathcal{L}\mathfrak{g}$ highest weight module generated by a one-dimensional highest weight space is a quotient of some $W(\pi)$ (for a more precise formulation see [10]). So these can be considered as maximal finite-dimensional cyclic representations in this class. A special class of tuples of polynomials is defined as follows: fix $\lambda=\sum_{j=1}^n m_j\omega_j$ a dominant integral weight for $\mathfrak{g}$ and a non-zero complex number $a\in\mathbb{C}^*$, and set

$$\pi_{\lambda,a}=\big((1-au)^{m_1},\dots,(1-au)^{m_n}\big). \tag{12}$$

The Weyl modules $W(\pi_{\lambda,a})$ are of special interest because:

1. It has been shown in [10] that a Weyl module $W(\pi)$ is isomorphic to a tensor product $\bigotimes_j W(\pi_{\lambda_j,a_j})$ of Weyl modules corresponding to this special class of polynomials.
2. The defining relations for the Weyl module $W(\pi_{\lambda,a})$ reduce to (see [6]): $W(\pi_{\lambda,a})$ is the cyclic module generated by an element $w_{\lambda,a}$, subject to the relations

$$\big(\mathfrak{n}^+\otimes\mathbb{C}[t,t^{-1}]\big)w_{\lambda,a}=0,\qquad \big(h\otimes t^s\big)w_{\lambda,a}=a^s\lambda(h)w_{\lambda,a},\qquad \big(x_{\alpha_i}^-\otimes 1\big)^{m_i+1}w_{\lambda,a}=0$$

   for all $h\in\mathfrak{h}$, $1\leqslant i\leqslant n$, $s\in\mathbb{Z}$.

In the following we denote by $\lambda_\pi=\sum_i \deg\pi_i\omega_i$ the weight associated to a n-tuple of polynomials π.

2.5. Weyl modules for the current algebra

Let $\lambda=\sum m_i\omega_i$ be a dominant weight for $\mathfrak{g}$. A class of Weyl modules $W(\lambda)$ has also been introduced for the current algebra. In terms of generators and relations one has:

Definition 2. Let λ be a dominant integral weight of $\mathfrak{g}$, $\lambda=\sum m_i\omega_i$. Denote by $W(\lambda)$ the $\mathcal{C}\mathfrak{g}$-module generated by an element v with the relations:

$$\mathfrak{n}^+\otimes\mathbb{C}[t].v=0,\qquad \mathfrak{h}\otimes t\mathbb{C}[t].v=0,\qquad h.v=\lambda(h).v,\qquad \big(x_{\alpha_i}^-\otimes 1\big)^{m_i+1}.v=0$$

for all $h\in\mathfrak{h}$ and all simple roots α_i. This module is called the Weyl module for $\mathcal{C}\mathfrak{g}$ associated to $\lambda\in P^+$.

The same proofs as those in [10] show that $W(\lambda)$ exists, is finite-dimensional and has the same universal property (see also [6,7]).

Remark 7. It follows easily that for all positive roots β the following relation holds in $W(\lambda)$:

$$\left(X_\beta^- \otimes 1\right)^{k_\beta+1}.v = 0 \quad \text{for } k_\beta = \lambda\left(\beta^\vee\right).$$

For $a \in \mathbb{C}^*$ consider the Lie algebra homomorphism φ_a defined as follows:

$$\varphi_a : \mathcal{C}\mathfrak{g} \to \mathcal{C}\mathfrak{g}, \quad x \otimes t^m \mapsto x \otimes (t+a)^m.$$

Now $W(\pi_{\lambda,a})$ is module for the loop algebra and hence by restriction also a module for the current algebra. It has been shown in [7,10] that the twisted $\mathcal{C}\mathfrak{g}$-module ${\varphi_a}^*(W(\pi_{\lambda,a}))$, where the action is defined by

$$\left(x \otimes t^m\right) \circ_{\varphi_a} w = \left(x \otimes (t-a)^m\right) w,$$

is a cyclic $\mathcal{C}\mathfrak{g}$-module satisfying the relations in Definition 2, so:

Lemma 1. *As a $\mathcal{C}\mathfrak{g}$-module, ${\varphi_a}^* W(\pi_{\lambda,a})$ is a quotient of $W(\lambda)$.*

In [17] the so called higher level Weyl modules were introduced.

Definition 3. Let W be a cyclic $\mathcal{C}\mathfrak{g}$-module, with fixed generator w. We denote by $W^{[k]}$ the $\mathcal{C}\mathfrak{g}$ submodule of $W^{\otimes k}$ generated by $w^{\otimes k}$.

For a dominant integral weight λ let $W(\lambda)$ be the Weyl module for the current algebra. The Weyl module of level k corresponding to λ is defined as

$$W(\lambda)^{[k]}.$$

Remark 8. Let $V_w(\Lambda)$ denote the Demazure submodule in the irreducible highest weight $\hat{\mathfrak{g}}$-module $V(\Lambda)$ corresponding to the Weyl group element w. Then

$$V_w(\Lambda)^{[k]} = V_w(k\Lambda).$$

Remark 9. [17] Let V, W be cyclic $\mathcal{C}\mathfrak{g}$-modules, and suppose that V is a quotient of W. Then $V^{[k]}$ is a quotient of $W^{[k]}$.

2.6. Fusion products for the current algebra

In this section we recall some facts on tensor products and fusion products of cyclic $\mathcal{C}\mathfrak{g}$-modules. Let W be $\mathcal{C}\mathfrak{g}$-module and let a be a complex number. Let W_a be the $\mathcal{C}\mathfrak{g}$-module defined by the pullback $\varphi_a^* W$, so $x \otimes t^s$ acts as $x \otimes (t-a)^s$. The following is well known:

Lemma 2. [15] *Let $W^1, \ldots, W^r$ be cyclic graded, finite-dimensional $\mathcal{C}\mathfrak{g}$-modules with cyclic vectors $w_1, \ldots, w_r$ and let $C = \{c_1, \ldots, c_r\}$ be pairwise distinct complex numbers. Then $w_1 \otimes \cdots \otimes w_r$ generates $W^1_{c_1} \otimes \cdots \otimes W^r_{c_r}$.*

The Lie algebra $\mathcal{C}\mathfrak{g}$ has a natural grading and an associated natural filtration $F^\bullet(\mathcal{C}\mathfrak{g})$, where $F^s(\mathfrak{g} \otimes \mathbb{C}[t])$ is defined to be the subspace of $\mathfrak{g}$-valued polynomials with degree smaller or equal s. One has an induced filtration also on the enveloping algebra $U(\mathcal{C}\mathfrak{g})$. Let now W be a cyclic module and let w be a cyclic vector for W. Denote by W_s the subspace spanned by the vectors of the form $g.w$, where $g \in F^s(U(\mathcal{C}\mathfrak{g}))$, and denote the associated graded $\mathcal{C}\mathfrak{g}$-module by $\operatorname{gr}(W)$

$$\operatorname{gr}(W) = \bigoplus_{i \geqslant 0} W_s / W_{s-1} \quad \text{where } W_{-1} = 0.$$

As $\mathfrak{g}$-modules, W and $\operatorname{gr}(W)$ are naturally isomorphic, but in general not as $\mathcal{C}\mathfrak{g}$-modules.

Definition 4. [15] Let W^i and c_i as above in Lemma 2. The $\mathcal{C}\mathfrak{g}$-module

$$W^1 * \cdots * W^r := \operatorname{gr}_C\left(W^1_{c_1} \otimes \cdots \otimes W^r_{c_r}\right)$$

is called the fusion product.

Remark 10. It would be more appropriate to write $W^1_{c_1} * \cdots * W^r_{c_r}$ for the fusion product, since a priori the structure of the fusion product depends on the choice of C. It has been conjectured in fact in [15] that the fusion product (a) does not depend on the choice of the pairwise distinct complex numbers $C \in \mathbb{C}^r$, and (b) is associative. This has been proved in the case $\mathfrak{g} = \mathfrak{sl}_n$ for various fusion products. In this paper we will prove the independence and the associativity property for the fusion product of the Demazure modules $D(\ell, \lambda)$, which justifies the fact that we omit almost always the pairwise distinct complex numbers in the notation for the fusion product. In [1] it is shown that the fusion product of Kirillov–Reshetikhin modules of arbitrary levels is independent of the parameters.

Remark 11. The case $r = 1$ is of course not excluded. For example, let W be a graded cyclic $\mathcal{C}\mathfrak{g}$-module. Let $C = \{c\}$, where $c \in \mathbb{C}$, then $\operatorname{gr}_C(W) \simeq W$ as $\mathcal{C}\mathfrak{g}$-modules.

2.7. Kirillov–Reshetikhin modules

In [5] (see also [8]) for each multiple of a fundamental weight $m\omega_i$ a $\mathcal{C}\mathfrak{g}$-module $KR(m\omega_i)$ has been defined. These modules are called Kirillov–Reshetikhin module because in many cases (Lie algebras of simply laced type or of classical type, see [5]) they can also be obtained from the quantum Kirillov–Reshetikhin module by specialization and restriction to the current algebra.

Definition 5. Let $KR(m\omega_i)$ be the $\mathcal{C}\mathfrak{g}$-module generated by a vector $v \neq 0$ with relations

$$\left(\mathfrak{n}^+ \otimes \mathbb{C}[t]\right).v = 0, \qquad \left(\mathfrak{h} \otimes t\mathbb{C}[t]\right).v = 0, \qquad hv = m\omega_i(h), \quad h \in \mathfrak{h}, \tag{13}$$

$$\left(X^-_{\alpha_i}\right)^{m+1} v = \left(X^-_{\alpha_i} \otimes t\right) v = 0 \quad \text{and} \quad \left(X^-_{\alpha_j}\right) v = 0 \quad \text{for } j \neq i. \tag{14}$$

2.8. Quantum Weyl modules

Let $U_q(\mathcal{L}\mathfrak{g})$ be the quantum loop algebra over $\mathbb{C}(q)$, q an indeterminate, associated to $\mathfrak{g}$ (see [13]). As in the classical case, one can associate finite-dimensional modules of $U_q(\mathcal{L}\mathfrak{g})$ to n-tuples of polynomials π_q with constant term 1 and coefficients in $\mathbb{C}(q)$ (see [10]). These modules are called quantum Weyl modules. Again, the following universal property holds: every highest weight module generated by a one-dimensional highest weight space is a quotient of $W_q(\pi_q)$ for some n-tuple π_q (see [10]). Each such module has a unique irreducible quotient which we denote by $V_q(\pi_q)$.

For such a n-tuple $\pi_q = (\pi_{q,1}, \dots, \pi_{q,n})$ set $\lambda_{\pi_q} = \sum_i \deg \pi_{q,i}\omega_i$, and let $\pi_{q,\omega_i,1}$ be defined as in the classical case.

The connection with Demazure modules is given by a theorem due to Kashiwara. We state the theorem only for the simply laced type, but it holds in much more generality.

Theorem 3. [26] *Let $\mathfrak{g}$ be a simple Lie algebra of simply laced type, then*

$$\dim W_q(\pi_{q,\omega_i,1}) = \dim D_q\big(1, \omega_i^\vee\big)$$

and $W_q(\pi_{q,\omega_i,1})$ is irreducible.

Remark 12. The classical Demazure module $V_w(\Lambda)$ (respectively $D(m, \lambda^\vee)$) is the $q \to 1$ limit of the quantized Demazure module $V_{q,w}(\Lambda)$ (respectively $D_q(m, \lambda^\vee)$).

Definition 6. The n-tuple π_q is called integral if all coefficients are in $\mathcal{A}$, and if the coefficient of the highest degree term is in $\mathbb{C}^* q^{\mathbb{Z}}$.

Let $U_{\mathcal{A}}(\mathcal{L}\mathfrak{g})$ be the $\mathcal{A}$ subalgebra defined in [10]. It has been shown in [10] that for an integral n-tuple π_q the corresponding quantum Weyl module $W_q(\pi_q)$ admits a $U_{\mathcal{A}}(\mathcal{L}\mathfrak{g})$-stable $\mathcal{A}$-lattice $W_{\mathcal{A}}(\pi_q) \subset W_q(\pi_q)$.

Further, let $\mathbb{C}_1$ be the $\mathcal{A}$-module with q acting by 1, then $U(\mathcal{L}\mathfrak{g})$ is a quotient of $U_{\mathcal{A}}(\mathcal{L}\mathfrak{g}) \otimes \mathbb{C}_1$, and $\overline{W_q(\pi_q)} := W_{\mathcal{A}}(\pi_q) \otimes \mathbb{C}_1$ becomes in a natural way a $U(\mathcal{L}\mathfrak{g})$-module.

Let $\overline{\pi_q}$ be the n-tuple of polynomials obtained by setting $q = 1$, so the coefficients are in $\mathbb{C}$. The universality property of the Weyl modules implies [10]:

Lemma 3. *If π_q is integral, then the $U(\mathcal{L}\mathfrak{g})$-module $\overline{W_q(\pi_q)}$ is a quotient of the classical Weyl module $W(\overline{\pi_q})$.*

Actually, as outlined in the introduction, using results from global basis theory one can show that one has equality in the lemma above. But we need in the following only this weaker formulation.

It was already pointed out in [11] that the cyclicity result of Kashiwara ([26, Theorem 9.1], see [39] for the simply laced case) for twisted tensor products of cyclic modules implies a lower bound for the dimension:

$$\dim W_q(\pi_q) \geqslant \prod_i \big(\dim W_q(\pi_{q,\omega_i,1})\big)^{\deg \pi_{q,i}}. \tag{15}$$

3. Connections between the modules

3.1. Quotients

We have some obvious maps between the Weyl modules for the current algebra and certain Demazure modules.

Lemma 4. *Let $\lambda^\vee$ be a dominant integral coweight of $\mathfrak{g}$. For all $m \geqslant 1$, the Demazure module $D(m, \lambda^\vee)$ is a quotient of the Weyl module $W(m\nu(\lambda^\vee))$.*

Proof. This follows immediately by comparing the relations for the Weyl module in Definition 2 and the relations for the Demazure module in Corollary 1. □

Lemma 5. *Let $\lambda_i^\vee$, $i = 1, \dots, r$ be dominant integral coweights, let $\lambda^\vee = \lambda_1^\vee + \dots + \lambda_r^\vee$, and let $a_1, \dots, a_n$ be pairwise distinct complex numbers. Then*

$$D\big(1, \lambda_1^\vee\big)_{a_1} * \dots * D\big(1, \lambda_r^\vee\big)_{a_r}$$

is a quotient of $W(\nu(\lambda^\vee))$.

Proof. Let $v_i \in D(1, \lambda_j^\vee)$ be the cyclic generator as in Definition 2 and let $\nu(\lambda_i^\vee) = \sum_j m_j^i \omega_j$, then the following relations hold:

$$\mathfrak{n}^+ \otimes \mathbb{C}[t].v_i = 0, \qquad \big(\mathfrak{h} \otimes t\mathbb{C}[t]\big)v_i = 0, \qquad h \otimes 1.v_i = \nu\big(\lambda_i^\vee\big)(h)v_i,$$
$$\big(x_{\alpha_j}^- \otimes 1\big)^{m_j^i+1}.v_i = 0.$$

Let $\nu(\lambda^\vee) = \sum_j m_j \omega_j$, then the following relations for the fusion product follow from the relations above:

$$\mathfrak{n}^+ \otimes \mathbb{C}[t].\big(v_i^{\otimes_{i=1}^r}\big) = 0, \qquad h \otimes 1.\big(v_i^{\otimes_{i=1}^r}\big) = \nu\big(\lambda^\vee\big)(h)\big(v_i^{\otimes_{i=1}^r}\big),$$
$$\big(x_{\alpha_j}^- \otimes 1\big)^{m_j^i+1}.\big(v_i^{\otimes_{i=1}^r}\big) = 0.$$

To see that all the relations of the Weyl module are satisfied in the fusion product, it remains to show that $\mathfrak{h} \otimes t\mathbb{C}[t]$ annihilates $v_1 \otimes \dots \otimes v_r$. So (recall that $(h \otimes t^k).v_i = 0$ for $k > 0$) we have for $n \geqslant 1$:

$$\begin{aligned}\big(h \otimes t^n\big).v_1 \otimes \dots \otimes v_r &= \sum_i v_1 \otimes \dots \otimes \big(h \otimes (t + c_j)^n\big)v_i \otimes \dots \otimes v_r \\ &= \sum_j c_j^n \nu\big(\lambda_j^\vee\big)(h) v_1 \otimes \dots \otimes v_r \\ &= \bigg(\sum_j c_j^n \nu\big(\lambda_j^\vee\big)(h)\bigg) v_1 \otimes \dots \otimes v_r.\end{aligned}$$

By definition, this an element in the nth part of the filtration, but obviously the vector $v_1 \otimes \cdots \otimes v_r$ in also 0th part of the filtration. Hence in the fusion product we have $(h \otimes t^n).v_1 \otimes \cdots \otimes v_r = 0$ for $n \geqslant 1$. It follows: $(\mathfrak{h} \otimes t\mathbb{C}[t]).v_1 \otimes \cdots \otimes v_r = 0$, which finishes the proof. $\square$

3.2. KR-modules

Theorem 4. *For a fundamental coweight $\omega_i^\vee$ let d_i such that $d_i\omega_i = \nu(\omega_i^\vee)$. The Kirillov–Reshetikhin module $KR(d_i m\omega_i)$ is, as $\mathcal{C}\mathfrak{g}$-module isomorphic to the Demazure module $D(m, \omega_i^\vee)$. In particular, in the simply laced case (i.e., the root system is of type $\mathtt{A}_n$, $\mathtt{D}_n$, $\mathtt{E}_n$) all Kirillov–Reshetikhin modules are Demazure modules.*

Remark 13. The fact that $D(m, \omega_i^\vee)$ is a quotient of a Kirillov–Reshetikhin module has been already pointed out in [8]. In the same paper Chari and Moura have shown that $D(m, \omega_i^\vee)$ isomorphic to $KR(d_i m\omega_i)$ for all classical groups using character calculations. Our proof is independent of these results, and holds for all types, and gives an alternative proof of the fact that these modules are finite-dimensional.

Proof. The fact that $D(m, \omega_i^\vee)$ is a quotient of $KR(d_i m\omega_i)$ is obvious by comparing the relations of the KR modules with the relations of the Demazure module from Corollary 1.

To show that the KR-modules above are quotients of Demazure modules, it remains to verify that the relations (14) above imply the relations (11) in Corollary 1. So let β be a positive root, set $\alpha = \alpha_i$, $d = d_i$ and $\omega = \omega_i$. Note that $[h \otimes t^k, X_\beta^- \otimes t^\ell] = \beta(h) X_\beta^- \otimes t^{\ell+k}$ implies:

$$\left(X_\beta^- \otimes t^s\right).v = 0 \quad \Rightarrow \quad \left(X_\beta^- \otimes t^r\right).v = 0, \quad \forall r \geqslant s. \tag{16}$$

The fact that (11) holds for the elements X_β^- follows by $\mathfrak{sl}_2$-theory. If $\langle \omega, \beta^\vee \rangle = 0$, then (11) holds for all elements $X_\beta^- \otimes t^s$, $s \geqslant 0$, by (16).

Assume now $\langle \omega, \beta^\vee \rangle > 0$ and consider an element of the form $X_\beta^- \otimes t^s$ for some $s \geqslant 1$. Let $\gamma \neq \alpha$ be a simple root, to verify the relation for $X_\beta^- \otimes t^s$ is equivalent to verify it for $X_{s_\gamma(\beta)}^- \otimes t^s$. By replacing β by $s_\gamma(\beta)$ if $\langle \beta, \gamma^\vee \rangle > 0$, without loss of generality we may assume that either $\beta = \alpha$, in which case the relations are satisfied, or $\beta \neq \alpha$ and α is the only simple root such that $\langle \beta, \alpha^\vee \rangle > 0$.

We have $\langle \beta, \alpha^\vee \rangle = j$, $j = 1, 2, 3$. So $\beta' = s_\alpha(\beta) = \beta - j\alpha$ and, if $t \geqslant j$, then, up to a scalar,

$$X_\beta^- \otimes t^s = \left[X_\alpha^- \otimes t, \left[\ldots, \left[X_\alpha^- \otimes t, X_{\beta'}^- \otimes t^{s-j}\right]\ldots\right]\right].$$

Except for the case where α, β' are two short roots in a root system of type $\mathtt{G}_2$, the elements $X_\alpha^- \otimes t$, $X_{\beta'}^- \otimes t^{s-j}$ generate the nilpotent part of a Lie algebra of type $\mathtt{A}_2$, $\mathtt{B}_2$ or $\mathtt{G}_2$.

We consider first the case α, β' are short roots in a root system of type $\mathtt{G}_2$. Let γ be the long simple root, then $\beta' = \gamma + \alpha$ and $\beta = \gamma + 2\alpha = \omega$. We have $X_\gamma^- v = 0$, $(X_\alpha^- \otimes t)v = 0$ and hence $(X_{\beta'}^- \otimes t)v = [X_\gamma^-, X_\alpha^- \otimes t]v = 0$. In the same way one concludes $(X_\beta^- \otimes t^2)v = 0$. Now using the commutation relations, one sees that $X_\beta.((X_\beta^- \otimes t)^k.v) = 0$ for all $k \geqslant 0$. So if $(X_\beta^- \otimes t)^k.v \neq 0$, then this a highest weight vector for the Lie algebra generated by X_β and X_β^-, and hence $(X_\beta^- \otimes t)^{3m+1}.v = 0$. It follows that the elements $X_{\beta'}^- \otimes t^s$, $X_\beta^- \otimes t^s$, $s \geqslant 0$, satisfy in this case the relations for the Demazure module $D(m, \omega^\vee)$.

Suppose now α, β' form a basis of a root system of type $\mathtt{X}_2$, $X = \mathtt{A}, \mathtt{B}, \mathtt{G}$. Using the higher order Serre relations (see for example [30, Corollary 7.1.7]), one sees by induction that $(X_\alpha^- \otimes t).v = 0$ implies for some constant $c \in \mathbb{C}$:

$$\begin{aligned}\left(X_\beta^- \otimes t^s\right)^m.v &= \left(\left[X_\alpha^- \otimes t, \left[\ldots, \left[X_\alpha^- \otimes t, X_{\beta'}^- \otimes t^{s-j}\right]\ldots\right]\right)^m.v \\ &= c \cdot \left(X_\alpha^- \otimes t\right)^{jm}\left(X_{\beta'}^- \otimes t^{s-j}\right)^m.v.\end{aligned}$$

Now if $X_{\beta'}^- \otimes t^{s-j}$ satisfies the relations for the Demazure module in (11), then so does $X_\beta^- \otimes t^s$.

In the simply laced case this finishes the proof since the arguments above provide an inductive method reducing the verification of the relations to the case either $\beta = \alpha$ or $s = 0$, and in both cases we know already that the relations hold. In the case $\mathfrak{g}$ is of type $\mathtt{B}_n$, $\mathtt{C}_n$ or $\mathtt{F}_4$, the procedure reduces the proof to the cases (1) $\beta = \alpha$, (2) $s = 0$ (now in these two cases the proof is finished), or (3) β is a long root, α is a simple short root and $\langle \beta, \alpha^\vee \rangle = 2$. In this case the relations have to be verified for the root vector $X_\beta^- \otimes t$.

Now except for one case (in type $\mathtt{F}_4$) the pair $(\alpha, \beta' = \beta - 2\alpha)$ is such that $\omega(\beta'^\vee) = 0$, so $X_{\beta'}^-.v = 0$ and hence $(X_\beta^- \otimes t^2).v = 0$. Now as above, using the commutation relations one sees that $X_\beta (X_\beta^- \otimes t)^k.v = 0$ for all k, so if $(X_\beta^- \otimes t)^k.v \neq 0$, then this is a highest weight vector for the Lie subalgebra generated by $X_\beta^\pm$. It follows $(X_\beta^- \otimes t)^{m+1}.v = 0$, and hence $X_\beta^- \otimes t$ satisfies the relations for the Demazure module.

In the remaining case in type $\mathtt{F}_4$ (using reflections by simple roots $\gamma \neq \alpha$) it suffices to consider the pair where $\alpha = \alpha_3$ and $\beta = \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4$ (notation as in [3]). Now β' is a positive root for which we already know that the relations for the Demazure module hold. (Using simple reflections s_γ, $\gamma \neq \alpha$, to verify the relations for β' is equivalent to verify them for $\alpha_2 + 2\alpha_3$, this is a positive root of the type discussed above.) One has $\omega(\beta'^\vee) = 2$, so $(X_{\beta'}^- \otimes t^2).v = 0$, $(X_{\beta'}^- \otimes t)^{m+1}.v = 0$ and $(X_{\beta'}^-)^{2m+1}.v = 0$, and the induction procedure shows $(X_\beta^- \otimes t^4).v = 0$, $(X_\beta^- \otimes t^3)^{m+1}.v = 0$ and $(X_\beta^- \otimes t^2)^{2m+1}.v = 0$. It remains to show that $(X_\beta^- \otimes t)^{3m+1}.v = 0$. Suppose $u = (X_\beta^- \otimes t)^{3m+1}.v \neq 0$, then, by $\mathfrak{sl}_2$-theory, $X_\beta^{2m+2} u \neq 0$. Now using the commutation relations, a simple induction procedure shows that $X_\beta^n (X_\beta^- \otimes t)^{3m+1}.v$ is a linear combination of terms of the form

$$\left(X_\beta^- \otimes t\right)^{3m+1-2n+j}\left(X_\beta^- \otimes t^2\right)^{n-2j}\left(X_\beta^- \otimes t^3\right)^j v.$$

For $n = 2m + 2$ one has $j \geqslant m + 2$, and hence all the terms vanish. It follows $u = 0$.

Now for $\mathfrak{g}$ of type $\mathtt{G}_2$ the induction procedure (respectively the arguments above for the two short roots) reduce the proof of the relations to the cases of the root vectors $X_\beta^- \otimes t$ and $X_\beta^- \otimes t^2$. Here α is the short simple root, β' is the long simple root and $\beta = s_\alpha(\beta')$. Since $(X_\beta^-)^{3m+1} v = 0$ and $(X_\beta^- \otimes t^3).v = 0$, the commutation relations as above show that $X_\beta^{m+2}(X_\beta^- \otimes t)^{2m+1} v = 0$ and hence, by $\mathfrak{sl}_2$-theory, $(X_\beta^- \otimes t)^{2m+1} v = 0$.

Note that the root vectors X_β^- and $X_{\beta-\alpha}^-$ commute. Since $(X_{\beta-\alpha}^- \otimes t^2) v = 0$, we see that $X_\alpha (X_\beta^- \otimes t^2)^k.v = 0$ for all $k \geqslant 0$. So if $(X_\beta^- \otimes t^2)^k.v$ is non-zero, then this is a highest weight vector for the Lie subalgebra generated by $X_\alpha^\pm$. Since $\langle 3m\omega - k\beta, \alpha^\vee \rangle = 3m - 3k$, it follows that $(X_\beta^- \otimes t^2)^{m+1} v = 0$. □

3.3. The $\mathfrak{sl}_2$-case

Before we come to the proof of the main results, let us recall the case $\mathfrak{g} = \mathfrak{sl}_2$. Note that for $\mathfrak{sl}_2$ we have $\nu(\lambda^\vee) = \lambda = \lambda_*$. Recall:

Theorem 5. [10] *Let $\lambda = m\omega$, then* $\dim W(\lambda) = 2^m$.

As immediate consequence one obtains (already proved in [10], see also [6]):

Theorem 6. *For $\mathfrak{g} = \mathfrak{sl}_2$ one has $W(\lambda) \simeq D(1, \lambda)$ as $\mathcal{C}\mathfrak{g}$-modules.*

Proof. The Demazure module is a quotient of the Weyl module, and by [18] and the theorem above one knows that $\dim D(1, \lambda) = (\dim D(1, \omega))^m = 2^m = \dim W(\lambda)$. $\square$

3.4. The simply laced case

In this section let $\mathfrak{g}$ be a simple simply laced Lie algebra, so $\mathfrak{g}$ is of type $\mathtt{A}_n$, $\mathtt{D}_n$ or $\mathtt{E}_n$. Note that in this case $\nu(\lambda^\vee) = \lambda$. We are now ready to prove

Theorem 7. *Let $\mathfrak{g}$ be simply laced. Let $\lambda^\vee \in \check{P}^+$ be a dominant integral coweight for $\mathfrak{g}$. The $\mathcal{C}\mathfrak{g}$-Weyl module $W(\lambda)$ is isomorphic to the Demazure module $D(1, \lambda^\vee)$.*

Remark 14. In [6] the result has been proved for $\mathfrak{sl}_n$ by showing that the dimension conjecture of [10] is true for the classical Weyl module for $\mathfrak{g} = \mathfrak{sl}_n$. Our approach is different and uses the relations defining a Demazure module. On the other hand, we obtain a proof of the dimension conjecture of [10] for the simply laced case by combining the result above with Theorem 2, see Proposition 1.

Proof. We know already that the Demazure module is a quotient of the Weyl module. By comparing the defining relations in Corollary 1 and in Definition 2, we see that to prove that this map is an isomorphism, it is sufficient to show for the Weyl modules that the following set of relations hold: for all positive roots $\beta \in \Phi^+$ and all $s \geqslant 0$ one has

$$\left(X_\beta^- \otimes t^s\right)^{k_\beta+1}.v = 0 \quad \text{where } s \geqslant 0 \text{ and } k_\beta = \max\left\{0, -\left\langle \Lambda_0 + \nu\left(\lambda^\vee\right), (-\beta + s\delta)^\vee\right\rangle\right\}. \tag{17}$$

Let β be a positive root of $\mathfrak{g}$, let $s \in \mathbb{N}$ be a non-negative integer and set

$$k = \max\left\{0, \lambda\left(\beta^\vee\right) - s\right\}.$$

Let $w_\lambda \in W(\lambda)$ be a generator of weight λ. To prove (17), we have to show

$$\left(X_\beta^- \otimes t^s\right)^{k+1}.w_\lambda = 0.$$

Let $\mathfrak{sl}_\beta$ be the Lie subalgebra generated by $X_\beta^-, X_\beta, \beta^\vee$. Let V be the $\mathfrak{sl}_\beta \otimes \mathbb{C}[t]$-submodule of $W(\lambda)$ generated by w_λ, i.e., $V = U(\mathfrak{sl}_\beta \otimes \mathbb{C}[t]).w_\lambda$. Then V satisfies obviously the defining relations for the $\mathfrak{sl}_\beta \otimes \mathbb{C}[t]$-Weyl module $W_\beta(\lambda(\beta^\vee))$ (see Remark 7), so V is a quotient of this Weyl module $W_\beta(\lambda(\beta^\vee))$. By Theorem 6 we know for the current algebra $\mathfrak{sl}_\beta \otimes \mathbb{C}[t]$ that the Weyl module $W_\beta(\lambda(\beta^\vee))$ is the same as the Demazure module $D_\beta(1, \lambda(\beta^\vee))$. In particular, the defining relations of $D_\beta(1, \lambda(\beta^\vee))$ hold for the corresponding generator of $W_\beta(\lambda(\beta^\vee))$, and hence also for the corresponding generator of V. It follows: $(X_\beta^- \otimes t^s)^{k+1}.w_\lambda = 0$. $\square$

The following proposition is an immediate consequence of Theorems 2 and 7.

Proposition 1. *Let $\mathfrak{g}$ be a simple, simply laced Lie algebra and let $\lambda^\vee = \sum m_i \omega_i^\vee$ be a dominant integral coweight. The dimension of $W(\lambda)$ is*

$$\dim W(\lambda) = \prod \bigl(\dim W(\omega_i)\bigr)^{m_i} = \prod \bigl(\dim D\bigl(1, \omega_i^\vee\bigr)\bigr)^{m_i}.$$

We can now describe the current algebra module $\varphi_a^*(W(\pi_{\lambda,a}))$ obtained as a pull back from the Weyl module for the loop algebra. Here $\lambda = \sum m_i \omega_i$ and $\pi_{\lambda,a}$ is the n-tuple of polynomials as in Section 2.4.

Proposition 2. *Let λ be a dominant, integral weight for $\mathfrak{g}$ of simply laced type. Then*

$$\varphi_a^*\bigl(W(\pi_{\lambda,a})\bigr) \simeq W(\lambda).$$

Proof. We know that $\varphi_a^*(W(\pi_{\lambda,a}))$ is a quotient of $W(\lambda)$, so it suffices to show that $\dim W(\pi_{\lambda,a}) \geqslant \dim W(\lambda)$. We have already seen that the specialization $\overline{W_q(\pi_{q,\lambda,a})}$ at $q = 1$ of a quantum Weyl module is a quotient of the Weyl module $W(\pi_{\lambda,a})$ (see Lemma 3). By Theorem 3, inequality (15) and Proposition 1 it follows hence:

$$\begin{aligned}\dim W(\pi_{\lambda,a}) &\geqslant \dim W_q(\pi_{q,\lambda,a}) \geqslant \prod \bigl(\dim W_q(\pi_{q,\omega_i,1})\bigr)^{m_i} = \prod \bigl(\dim D\bigl(1, \omega_i^\vee\bigr)\bigr)^{m_i} \\ &= \prod \bigl(\dim W(\omega_i)\bigr)^{m_i} = \dim W(\lambda). \qquad \square\end{aligned}$$

As an immediate and simple consequence we see:

Corollary 2. *Let $\mathfrak{g}$ be a simple Lie algebra of simply laced type, let λ be a dominant weight (for $\mathfrak{g}$), let π (respectively π_q) be an n-tuple of polynomials in $\mathbb{C}[u]$ (respectively in $\mathbb{C}(q)[u]$) with constant term 1 such that $\lambda = \lambda_\pi = \lambda_{\pi_q}$.*

(1) $\dim W(\lambda) = \dim W(\pi) = \dim W_q(\pi_q) = \dim D(1, \lambda^\vee) = \prod_i (\dim W(\omega_i))^{m_i}$.
(2) *If π_q is integral, then $\overline{W_q(\pi_q)} \simeq W(\overline{\pi_q})$ as $U(\mathcal{L}\mathfrak{g})$-modules.*
(3) *The quantum Weyl module $W_q(\pi)$ is irreducible (note, the π_i have complex coefficients), and its specialization at $q = 1$ is the Weyl module $W(\pi)$ for the classical loop algebra.*

Remark 15. This proof of the dimension conjecture (in the simply laced case) can be seen as a more elementary alternative to the proof outlined in the introduction using results from global basis theory.

Proof. The first claim follows from Theorem 7, Propositions 1, 2, the tensor product decomposition property (see Section 2.4) and the specialization arguments outlined in [11] (see Section 2.8).

Now (2) is an immediate consequence of (1). To prove (3), let m_{a^j} be the multiplicity of the root $a^j \in \mathbb{C}^*$ of the polynomial $\pi_j(u)$. The tensor product $W_q = \bigotimes_{j,a^j} W_q(\omega_j, a^j)^{\otimes m_{a^j}}$ over all j and all roots a^j of $\pi_j(u)$ is irreducible by [25, Theorem 9.2], it is again a highest weight module associated to the right n-tuple of polynomials, and has the right dimension by (1), so it follows $W_q = W_q(\pi)$. The rest of the claim follows from (2). $\square$

We can now also prove the first step of Conjecture 1 in [18] for $\mathfrak{g}$ of simply laced type. In the case of a multiple of a fundamental weight, this provides a method to reconstruct the *KR*-module structure for $U(\mathcal{L}\mathfrak{g})$ from the $U(\mathcal{C}\mathfrak{g})$-structure on the Demazure module.

Corollary 3. *Let $D(m, \lambda^\vee)$ be a Demazure module of level m, corresponding to $\lambda^\vee$. Then $D(m, \lambda^\vee)$ can be equipped with the structure of a $U(\mathcal{L}\mathfrak{g} \oplus \mathbb{C}K)$-module such that the $\mathfrak{g}$-module structure of $D(m, \lambda^\vee)$ coming from the construction of the Demazure module and the $\mathfrak{g}$-module structure of $D(m, \lambda^\vee)$ obtained by the restriction of the $U(\mathcal{L}\mathfrak{g} \oplus \mathbb{C}K)$-module structure coincide.*

Proof. As a $\mathcal{C}\mathfrak{g}$-module, $D(m, \lambda^\vee)$ is a quotient of $W(m\lambda)$ (Lemma 4). Let $N(m\lambda)$ be the kernel of the map, so $D(m, \lambda^\vee) \simeq W(m\lambda)/N(m\lambda)$ as $\mathcal{C}\mathfrak{g}$-modules. By Corollary 2, we know that $W(m\lambda)$ is isomorphic to $\varphi_1^* W(\pi_{m\lambda,1})$ as module for the current algebra.

Let $N_1(m\lambda) = \varphi_{-1}^* N(m\lambda)$ be the submodule of $W(\pi_{m\lambda,1})$ corresponding to $N(m\lambda)$. Using [7, Proposition 3.3] (see also [10]), one can show that $x \otimes t^{-s}$ operates as a linear combination of elements of $U(\mathcal{C}\mathfrak{g})$ on $W(\pi_{m\lambda,1})$. So a $U(\mathcal{C}\mathfrak{g})$-submodule of $W(\pi_{m\lambda,1})$ is actually a $U(\mathcal{L}\mathfrak{g})$-submodule. Since K is central (and operates trivially), we conclude that $N_1(m\lambda)$ is a $U(\mathcal{L}\mathfrak{g} \oplus \mathbb{C}K)$-submodule of $W(\pi_{m\lambda,1})$.

So the quotient $W(\pi_{m\lambda,1})/N_1(m\lambda)$ is a $U(\mathcal{L}\mathfrak{g} \oplus \mathbb{C}K)$-module, isomorphic to the Demazure module $D(m, \lambda^\vee)$ as vector space. Further, since φ^* does not change the $\mathfrak{g}$-structure of a $\mathcal{C}\mathfrak{g}$-module, we see that the $\mathfrak{g}$-module structure on $D(m, \lambda^\vee)$ and on the quotient $W(\pi_{m\lambda,1})/N_1(m\lambda)$ are identical. $\square$

We conjecture that the corresponding statement also holds for the quantum algebras and that the module admits a crystal basis as $U_q(\mathcal{L}\mathfrak{g})$-module. Its crystal graph should be obtained from the crystal graph of the quantum Demazure module just by adding certain arrows with label zero.

In the level 1 case we know that we can identify $W_q(\pi_{q,\lambda,1})$ with $D_q(1, \lambda)$. To compare the crystals, let $P_{cl} = P/\mathbb{Z}\delta$ be the quotient of the weight lattice by the imaginary root and let $\psi : P \otimes_{\mathbb{Z}} \mathbb{R} \to P_{cl} \otimes_{\mathbb{Z}} \mathbb{R}$ be the projection of the associated real spaces. For a weight ν let $\pi_\nu : [0, 1] \to P \otimes_{\mathbb{Z}} \mathbb{R}$, $t \mapsto t\nu$, be the straight line path joining the origin with ν, and let $\psi(\pi_\nu)$ be the image of the path in $P_{cl} \otimes_{\mathbb{Z}} \mathbb{R}$.

Proposition 3. *The crystal graph of $D_q(1, \lambda)$ is obtained from the crystal graph of $W_q(\pi_{q,\lambda,1})$ by omitting certain arrows with label zero. More precisely, let $B(\lambda)_{cl}$ be the path model for $W_q(\pi_{q,\lambda,1})$ described in* [34]*, then the crystal graph of the Demazure module is isomorphic to the graph of the concatenation $\psi(\pi_{\Lambda_0}) * B(\lambda)_{cl}$.*

Proof. Write $t_{-\lambda_*}$ as $w\sigma$, so $D_q(1, \lambda)$ is the Demazure submodule $V_{q,w}(\sigma(\Lambda_0))$. The path model theory (see [29]) is independent of the choice of an initial path, we are going to choose an appropriate path. Instead of the straight line $\pi_{\sigma(\Lambda_0)}$ joining 0 and $\sigma(\Lambda_0)$, consider the two straight line paths π_{Λ_0} and $\pi_{-\lambda_*}$ joining the origin with Λ_0, respectively $-\lambda_*$, in $P \otimes_{\mathbb{Z}} \mathbb{R}$. Let $\eta = \pi_{\Lambda_0} * \pi_{-\lambda_*}$ be the concatenation of these two and denote by $B(\eta)$ the set of paths generated by applying the root operators to η. By [29, Section 6], η is linked for arbitrary L to the straight line path $\pi_{\Lambda_0 - \lambda_*}$, which is an LS-path of shape $\sigma(\Lambda_0)$. It follows that the two path models are isomorphic, and hence: (a) the crystal associated to the set of paths $B(\eta)$ is isomorphic to the crystal of $V_q(\sigma(\Lambda_0))$, and (b) in $B(\eta)$ there exists a unique path π_0 contained in the dominant Weyl chamber and ending in $\sigma(\Lambda_0)$. Denote by $B(\eta)_{cl}$ the image of this set of paths under the projection ψ. The root operators e_α, f_α, α a simple root for $\hat{\mathfrak{g}}$, are still well defined on paths in $P_{cl} \otimes_{\mathbb{Z}} \mathbb{R}$ since δ vanishes on all coroots. In fact, the operators commutes with the map ψ. So the uniqueness of η (as path contained in the dominant Weyl chamber) implies that ψ induces a bijection between the crystals $B(\eta)$ and $B(\eta)_{cl}$.

Let $B(\lambda)$ be the set of all LS-paths of shape λ and denote by $B(\lambda)_{cl}$ the image of this set under the projection ψ. Combining part 3 of Corollary 2 with the result of Naito and Sagaki in [34], we see that $B(\lambda)_{cl}$ is a combinatorial model for the crystal of the Weyl module $W_q(\pi_{q,\lambda,1})$. The concatenation $\psi(\pi_{\Lambda_0}) * B(\lambda)_{cl}$ in $P_{cl} \otimes_{\mathbb{Z}} \mathbb{R}$ provides a set of paths stable under all root operators e_α, α a simple root, and f_{α_i}, $i = 1, \dots, n$.

To describe in $B(\eta)_{cl}$ the set of path corresponding to the Demazure module $D(1, \lambda)$, recall that the latter is the union of all paths of the form $f_{\alpha_{j_1}}^{n_1} \cdots f_{\alpha_{j_t}}^{n_t} \pi_0$, where $t_{-\lambda_*} = w\sigma$. $w = s_{\alpha_{j_1}} \cdots s_{\alpha_{j_t}}$ is a reduced decomposition and $n_i \in \mathbb{N}$. Recall that π_0 is of the form $\pi_{\Lambda_0} * \pi'$, where $\pi' \in B(\lambda)$. Since we work modulo δ and λ is a level zero weight, $e_\alpha \pi_0 = 0$ for all simple roots of $\hat{\mathfrak{g}}$ means that π' is a path completely contained in the fundamental alcove of the root system of $\mathfrak{g}$. This path is obtained from the straight line path π_λ by folding it successively back into the alcove (in the same way as in [28], proof of the PRV-conjecture). Next consider the sequence of turning points.

If λ is regular and generic (i.e., $\lambda \neq m\mu$ for some $m \geqslant 2$, $\mu \in P^+$), then this are exactly the points where π_0 meets the codimension one faces of the fundamental alcove Δ_f, and the corresponding product of the simple reflections is exactly a reduced decomposition of w', where $t_\lambda = w'\sigma$. We get a reduced decomposition of w (recall, $t_{-\lambda_*} = w\sigma$) by multiplying the given reduced decompositions with appropriate simple reflection s_{α_i}, $i \geqslant 1$. By the choice of the reduced decomposition above, the paths $f_{\alpha_{j_1}}^{n_1} \cdots f_{\alpha_{j_t}}^{n_t} \pi_0$ are all of the form $\psi(\pi_{\Lambda_0}) * \pi'$, where $\pi' \in B(\lambda)$.

The same holds also in the general case, only that the turning points are not anymore associated to just one simple reflection an element of maximal length in a coset W'/W'' of subgroups of W^{aff}. Here W', W'' are associated to the turning point and the path π_0, for details in terms of galleries see for example [21, Example 4].

So the set of paths in the path crystal of $V_q(\sigma(\Lambda_0))$ corresponding to the subcrystal of $D(1, \lambda)$ is a subset of $\psi(\pi_{\Lambda_0}) * B(\lambda)_{cl}$. By the equality of the number of elements, the two sets have in fact to be equal. □

3.5. Demazure modules as fusion modules

In this section let $\mathfrak{g}$ be a simple Lie algebra of arbitrary type. So, unless it is explicitly mentioned, in this section we do not assume that $\mathfrak{g}$ is necessarily simply laced.

Theorem 8. *Let $\lambda^\vee = \sum_{i=1}^s \lambda_i^\vee$ be a sum of dominant integral coweights and let $c_1, \dots, c_s$ be pairwise distinct complex numbers, then*

$$D(1, \lambda^\vee) \simeq D(1, \lambda_1^\vee) * \cdots * D(1, \lambda_s^\vee)$$

as modules for the current algebras $\mathfrak{g} \otimes \mathbb{C}[t]$.

In the simply laced case we have of course equivalently:

Corollary 4. *Let $\mathfrak{g}$ be a simple simply laced Lie algebra. For $\lambda = \sum_{i=1}^{s} \lambda_i$, λ_i dominant integral coweights, and $c_1, \ldots, c_s$ pairwise distinct complex numbers*:

$$W(\lambda) \simeq W(\lambda_1) * \cdots * W(\lambda_s)$$

as $\mathcal{C}\mathfrak{g}$-modules.

Remark 16. For $\mathfrak{g} = \mathfrak{sl}_n$ and the λ_i, $i = 1, \ldots, s$, all fundamental weights, the theorem above (and its corollary) was proved by Chari and Loktev in [6].

Corollary 5. *Let $\mathfrak{g}$ be again a simple Lie algebra of arbitrary type and let $\lambda^\vee = \sum_{i=1}^{s} \lambda_i^\vee$ be a sum of dominant integral coweights and let $c_1, \ldots, c_s$ be pairwise distinct complex numbers, then for all $k \geqslant 1$*

$$D\big(k, \lambda^\vee\big) \simeq D\big(k, \lambda_1^\vee\big) * \cdots * D\big(k, \lambda_s^\vee\big)$$

as modules for the current algebras $\mathfrak{g} \otimes \mathbb{C}[t]$.

As obvious consequences we have:

Corollary 6. (1) *The fusion product of the Demazure modules $D(k, \lambda_j)$ is associative and independent of the choice of the pairwise distinct complex numbers $\{c_1, \ldots, c_s\}$.*

(2) *Let d_i be as in Theorem 4, then $KR(md_i\omega_i)$ is the m-fold fusion product $KR(d\omega_i)^{*m}$.*

Proof of Corollary 5. It follows from Remark 8 and Theorem 8 that

$$D\big(k, \lambda^\vee\big) = D\big(1, \lambda^\vee\big)^{[k]} \simeq \big(D\big(1, \lambda_1^\vee\big) * \cdots * D\big(1, \lambda_s^\vee\big)\big)^{[k]}.$$

By Proposition 2.10 in [17] the latter is a quotient of $D(1, \lambda_1^\vee)^{[k]} * \cdots * D(k, \lambda_s^\vee)^{[k]} = D(k, \lambda_1^\vee) * \cdots * D(k, \lambda_s^\vee)$. The dimension formula (Theorem 2) implies again that the map is an isomorphism. □

Proof of Theorem 8. In the simply laced case the result follows immediately from the equality of Demazure and Weyl modules: the right-hand side is a Weyl module by Theorem 7, and the left-hand side is a quotient of this Weyl module by Lemma 5. Now by Theorem 2 the dimension of both modules is equal, which finishes the proof.

In the general case we need to use the defining equations for Demazure module (see Corollary 1). In the proof of Lemma 5 we have already seen that the fusion module:

$$D\big(1, \lambda_1^\vee\big) * \cdots * D\big(1, \lambda_r^\vee\big)$$

is a quotient of the Weyl module and hence satisfies the relations:

$$\mathfrak{n}^+ \otimes \mathbb{C}[t].\big(v_i^{\otimes_{i=1}^r}\big) = 0, \qquad h \otimes 1.\big(v_i^{\otimes_{i=1}^r}\big) = \nu\big(\lambda^\vee\big)(h)\big(v_i^{\otimes_{i=1}^r}\big) \quad \text{and} \quad \mathfrak{h} \otimes t\mathbb{C}[t]\big(v_i^{\otimes_{i=1}^r}\big) = 0.$$

Let now $\beta \in \Phi^+$ be a positive root. The following lemma implies that the fusion product is a quotient of the Demazure module. Since both have the same dimension by Theorem 2, they are isomorphic, which finishes the proof. □

Lemma 6.

$$\big(X_\beta^- \otimes t^s\big)^{k_\beta+1}\big(v_i^{\otimes_{i=1}^r}\big) = 0 \quad \textit{for } k_\beta = \max\big\{0, \big\langle \Lambda_0 + \nu\big(\lambda^\vee\big), (-\beta + s\delta)^\vee\big\rangle\big\}. \tag{18}$$

The proof of Lemma 6 is by reduction to the $\widehat{\mathfrak{sl}}_2$-case. Note that in this case we know already that Theorem 8 and Corollary 5 hold.

We fix first some notation. For a positive root $\beta \in \Phi^+$ let $Z_\beta \subset \hat{\mathfrak{g}}$ be the Lie subalgebra generated by the root spaces $\hat{\mathfrak{g}}_{\pm\beta+s\delta}$, $s \in \mathbb{Z}$, the elements in the Cartan subalgebra $(\pm\beta \pm s\delta)^\vee$, and the derivation d. Then Z_β is an affine Kac–Moody algebra isomorphic to $\widehat{\mathfrak{sl}}_2$ with Cartan subalgebra $\hat{\mathfrak{h}}_\beta = \langle \beta^\vee, \epsilon K, d\rangle_{\mathbb{C}}$, where $\epsilon = (\beta^\vee, \beta^\vee)/2$ (see Eq. (5)) is 1 if β and Θ have the same length, and $\epsilon = 2$ or 3 if β is a short root. Set

$$\hat{\mathfrak{n}}_\beta^\pm = \hat{\mathfrak{n}}^\pm \cap Z_\beta \quad \text{and} \quad \mathfrak{sl}_2(\beta) \otimes \mathbb{C}[t] = Z_\beta \cap \mathfrak{g} \otimes \mathbb{C}[t].$$

Write $t_{\nu(\lambda^\vee)} = w\sigma$ (see Eqs. (9) and (10)), so $w \in W^{\mathrm{aff}}$. Set $\mu = w(\sigma(\Lambda_0))$ and let $v_\mu \in V(\sigma(\Lambda_0))$ be an extremal weight vector of weight μ. The submodule $M = U(Z_\beta)v_\mu \subset V(\sigma(\Lambda_0))$ is an irreducible (since v_μ is an extremal weight vector) Z_β-submodule, say $M = V^\beta(\Omega)$ is the Z_β-representation of highest weight Ω. The subspace

$$M\big(\nu\big(\lambda^\vee\big)\big) := U\big(\mathfrak{sl}_2(\beta) \otimes \mathbb{C}[t]\big).v_\mu = U\big(\hat{\mathfrak{n}}_\beta^+\big)s_\beta(v_\mu) \tag{19}$$

is then a Demazure module, stable under $U(\mathfrak{sl}_2(\beta) \otimes \mathbb{C}[t])$. Now $V(\sigma(\Lambda_0))$ is a level one module for $\hat{\mathfrak{g}}$, but the irreducible Z_β-submodule V_β is a level ϵ-module for the affine Kac–Moody algebra $Z_\beta \simeq \widehat{\mathfrak{sl}}_2$ (recall, the canonical central element of Z_β is ϵK). We need the following more precise statement:

Lemma 7. *As $\mathfrak{sl}_2(\beta) \otimes \mathbb{C}[t]$-module, the submodule $M(\nu(\lambda^\vee))$ is isomorphic to $D(\epsilon, m\omega^\beta)$, where $m = \nu(\lambda^\vee)(\beta^\vee)/\epsilon$ and ω^β denotes the fundamental weight for the Lie algebra $\mathfrak{sl}_2(\beta)$.*

Proof. The first step is to show that the highest weight Ω is a multiple of a fundamental weight for Z_β. The only non-trivial case is when Θ and β have different lengths. We show first that in this case:

$$\nu\big(\lambda^\vee\big)\big(\beta^\vee\big) \equiv 0 \mod \epsilon\mathbb{Z}. \tag{20}$$

To prove this, recall that for $\lambda^\vee = \sum_i m_i\omega_i^\vee$ one has $\nu(\lambda^\vee) = \sum_i m_i \frac{a_i}{a_i^\vee}\omega_i$. If α_i is a short root, then $\frac{a_i}{a_i^\vee} = \epsilon$, so $\frac{a_i}{a_i^\vee}\omega_i(\beta^\vee) \equiv 0 \bmod \epsilon\mathbb{Z}$. Now a case by case consideration shows that if α_i is a long simple root and β is a short positive root, then again $\frac{a_i}{a_i^\vee}\omega_i(\beta^\vee) \equiv 0 \bmod \epsilon\mathbb{Z}$.

Now $s_\beta(\nu(\lambda^\vee)|_{\hat{\mathfrak{h}}_\beta}) \equiv t_\eta(\Omega) \bmod \mathbb{Z}\Lambda_0$ (respectively $t_\eta(\sigma(\Omega)) \bmod \mathbb{Z}\Lambda_0$) for some $\mathfrak{sl}_2(\beta)$-weight η. Since $t_\eta(\Omega) = \Omega + \epsilon\eta$, respectively $t_\eta(\sigma(\Omega)) = \sigma(\Omega) + \epsilon\eta$, it follows

$$\Omega\big(\beta^\vee\big) \equiv 0 \mod \epsilon\mathbb{Z}, \quad \text{respectively} \quad \sigma(\Omega)\big(\beta^\vee\big) \equiv 0 \mod \epsilon\mathbb{Z}. \tag{21}$$

But this is only possible if $\Omega = \epsilon\Lambda_0^\beta$ or $\Omega = \epsilon\Lambda_1^\beta$ as highest weight for the irreducible $Z_\beta \simeq \widehat{\mathfrak{sl}}_2$-representation M, and hence $M(\nu(\lambda^\vee)) \simeq D(\epsilon, m\omega^\beta)$ for some m. Since $\nu(\lambda^\vee)(\beta^\vee) = (\epsilon\Lambda_0^\beta + \epsilon m\omega^\beta)(\beta^\vee)$, it follows that $m = \nu(\lambda^\vee)(\beta^\vee)$. □

Proof of Lemma 6. For each of the Demazure modules $D(1,\lambda_i^\vee)$ denote by $M(\nu(\lambda_i^\vee))$ the Z_β-Demazure submodule generated by v_i, as in (19). By the lemma above we have $M(\nu(\lambda_i^\vee)) \simeq D(\epsilon, m_i\omega^\beta)$, where $m_i = \nu(\lambda_i^\vee)(\beta^\vee)/\epsilon$. Taking the tensor product, we get an embedding

$$M\big(\nu\big(\lambda_1^\vee\big)\big) \otimes \cdots \otimes M\big(\nu\big(\lambda_s^\vee\big)\big) \hookrightarrow D\big(1,\lambda_1^\vee\big) \otimes \cdots \otimes D\big(1,\lambda_s^\vee\big).$$

Now the filtration on $M(\nu(\lambda_1^\vee)) \otimes \cdots \otimes M(\nu(\lambda_s^\vee))$ as $\mathfrak{sl}_2(\beta) \otimes \mathbb{C}[t]$-module is compatible with the filtration of $D(1,\lambda_1^\vee) \otimes \cdots \otimes D(1,\lambda_s^\vee)$ as $\mathfrak{g} \otimes \mathbb{C}[t]$-module, so we get an induced map

$$M\big(\nu\big(\lambda_1^\vee\big)\big) * \cdots * M\big(\nu\big(\lambda_s^\vee\big)\big) \longrightarrow D\big(1,\lambda_1^\vee\big) * \cdots * D\big(1,\lambda_s^\vee\big).$$

Since we are in the simply laced case, we know by Corollary 5 that the left $\mathfrak{sl}_2(\beta) \otimes \mathbb{C}[t])$-module is isomorphic to $M(\nu(\lambda^\vee))$. By construction, the generator of this module satisfies Eq. (18), and hence also the image $v_i^{\otimes_{i=1}^s}$ satisfies Eq. (18). □

Remark 17. It is not anymore true that $W(\nu(\lambda^\vee)) \simeq D(1,\lambda^\vee)$. As a counter example consider $\mathfrak{g}$ of type $\mathtt{C}_2$ and take $\lambda^\vee = \omega_1^\vee$. Note that $\nu(\omega_1^\vee) = 2\omega_1$. By [18], $D(1,\omega_1^\vee)$ has dimension 11, and by [5], $KR(\omega_1)$ has dimension 4. The fusion product $KR(\omega_1) * KR(\omega_1)$ is a quotient of $W(2\omega_1)$, so $\dim W(2\omega_1) \geqslant 16 > \dim D(1,\omega_1^\vee)$.

We conjecture that Corollary 4 also holds in the non-simply laced case:

Conjecture 1. *Let $\lambda = \sum \lambda_i$ be a sum of dominant integral weights, $c_1,\dots,c_n$ be pairwise distinct complex numbers, then*

$$W(\lambda) \simeq W(\lambda_1) * \cdots * W(\lambda_n).$$

4. Limit constructions

In this section we start with a simple Lie algebra $\mathfrak{g}$ of arbitrary type. We want to reconstruct the $\mathcal{C}\mathfrak{g}$-module structure of the irreducible highest weight $U(\hat{\mathfrak{g}})$-module $V(l\Lambda_0)$ as a direct limit of fusion products of Demazure modules.

In [18] we have given such a construction of the $\mathfrak{g}$-module structure of $V(l\Lambda_0)$ as a semi-infinite tensor product of finite-dimensional $\mathfrak{g}$-module. In this section we want to extend this construction to the $U(\mathcal{C}\mathfrak{g})$-module structure by replacing the tensor product by the fusion product.

We need first a few facts about inclusions of Demazure modules. Set $\tilde{\mathfrak{b}} = \mathfrak{h} \oplus \hat{\mathfrak{n}}^+ \oplus \mathbb{C}K$, and, as before, we denote by W^{aff} the affine Weyl group. Let Λ be an integral dominant weight for $\hat{\mathfrak{g}}$. We fix for all $w \in W^{\mathrm{aff}}/W^{\mathrm{aff}}_\Lambda$ a generator v_w of the line of weight $w(\Lambda) \subset V(\Lambda)$. Denote $V_w(\Lambda) = U(\hat{\mathfrak{b}}).v_w$ the Demazure module and let $\iota_w : V_w(\Lambda) \hookrightarrow V(\Lambda)$ the inclusion.

Lemma 8. *Let Λ be an integral dominant weight for $\hat{\mathfrak{g}}$. Given $w \in W^{\mathrm{aff}}/W^{\mathrm{aff}}_\Lambda$, there is a unique* (*up to scalar multiplication*) *non-trivial morphism of $U(\tilde{\mathfrak{b}})$-modules*

$$V_w(\Lambda) \to V(\Lambda).$$

In fact, this morphism is, up to scalar multiples, the canonical embedding of the Demazure module.

Proof. We want to prove that, up to scalar multiples, $\iota_w : V_w(\Lambda) \to V(\Lambda)$ is the only non-trivial morphism of $U(\tilde{\mathfrak{b}})$-modules. The proof is by induction on the length of w.

For $w = id$, the Demazure module is one-dimensional. The generator v is killed by $U(\hat{\mathfrak{n}}^+)$, so its image in $V(\Lambda)$ is a highest weight vector. But such a vector is unique (up to scalar multiple) in $V(\Lambda)$, and hence there exists, up to a scalar multiples, only one such morphism.

Suppose now $\ell(w) \geqslant 1$, and let $\tau = s_\alpha w$, α a simple root, be such that $\tau < w$, and let $\varphi : V_w(\Lambda) \to V(\Lambda)$ be a non-trivial $U(\tilde{\mathfrak{b}})$-equivariant morphism. Let v_w be a generator of the weight space in $V_w(\Lambda)$ corresponding to the weight $w(\Lambda)$, and set $m_\alpha = w(\Lambda)(\alpha^\vee)$. Then $(x_\alpha)^{m_\alpha}.v_w \neq 0$, but $(x_\alpha)^{m_\alpha+1}.v_w = 0$.

Now φ is an $U(\tilde{\mathfrak{b}})$-morphism, so the image $\varphi(v_w) \in V(\Lambda)$ is again an eigenvector for $\mathfrak{h} \oplus \mathbb{C}K$ of weight $w(\Lambda)|_{\mathfrak{h}\oplus\mathbb{C}K}$. Since $V(\Lambda)$ is a $\hat{\mathfrak{g}}$-module, $\mathfrak{sl}_2$-representation theory implies $(x_\alpha)^{m_\alpha}.\varphi(v_w) \neq 0$, and since φ is an $U(\tilde{\mathfrak{b}})$-morphism, we have $(x_\alpha)^{m_\alpha+1}.\varphi(v_w) = 0$.

Now $(x_\alpha)^{m_\alpha}.v_w$ is a generator of the Demazure module $V_\tau(\Lambda) \subset V_w(\Lambda)$, so $\varphi|_{V_\tau(\Lambda)}$ provides a non-trivial $U(\tilde{\mathfrak{b}})$-morphism, which by induction can only be a non-zero scalar multiple of the standard inclusion. Hence $(x_\alpha)^{m_\alpha}.\varphi(v_w)$ is a non-zero multiple of v_τ. Further, by weight reasoning and $\mathfrak{sl}_2$-representation theory, it follows that $x_{-\alpha}\varphi(v_w) = 0$. By the usual exchange relation we get

$$x_{-\alpha}^{m_\alpha}(x_\alpha)^{m_\alpha}\varphi(v_w) = c\varphi(v_w),$$

for some non-zero complex number c, and hence $\varphi(v_w)$ is an extremal weight vector of weight $w(\Lambda)$, which finishes the proof. $\square$

Corollary 7. *Let $\tau < w$, then there exists* (*up to scalar multiples*) *a unique morphism of $U(\tilde{\mathfrak{b}})$-modules $V_\tau(m\Lambda_0) \to V_w(m\Lambda_0)$.*

Consider the Demazure module $D(m, n\Theta) = V_{-n\Theta}(m\Lambda_0)$. We fix a generator $w \neq 0$ of the unique $U(\mathcal{C}\mathfrak{g})$-fixed line in $D(m, \Theta)$. Note (see [18]) that w spans the line of the highest weight vectors for $\hat{\mathfrak{g}}$ in $V(m\Lambda_0)$. By Theorem 8 we have for $c_1 \neq c_2$ an isomorphism

$$D\big(m, (n+1)\Theta\big) \simeq D(m, \Theta)_{c_2} * D(m, n\Theta)_{c_1}.$$

We extend this to an isomorphism of $U(\mathcal{C}\mathfrak{g} \oplus \mathbb{C}K)$-modules by letting K operate on $D(m, (n+1)\Theta)$ by the level m, and letting K act on the second module by 0 on the first factor and on the second factor by the level m. Define the map

$$\tilde{\varphi} : D(m, n\Theta)_{c_1} \to D(m, \Theta)_{c_2} \otimes D(m, n\Theta)_{c_1}$$

by $\tilde{\varphi}(v) = w \otimes v$. This map is an $U(\mathcal{C}\mathfrak{g})$-module morphism because w is $U(\mathcal{C}\mathfrak{g})$-invariant, which extends, as above, to a $U(\mathcal{C}\mathfrak{g} \oplus \mathbb{C}K)$-module morphism.

The map respects the filtrations up to a shift: let $v_2 \in D(m, \Theta)$ be a generator and let q be minimal such that $w \otimes v_2 \in F^q(D(m, \Theta)_{c_2} \otimes D(m, n\Theta)_{c_1})$. By the $U(\mathcal{C}\mathfrak{g})$-equivariance it follows that

$$\tilde{\varphi}\big(F^j\big(D(m, n\Theta)_{c_1}\big)\big) \subseteq F^{j+q}\big(D(m, \Theta)_{c_2} \otimes D(m, n\Theta)_{c_1}\big).$$

So we get an induced $U(\mathcal{C}\mathfrak{g} \oplus \mathbb{C}K)$-morphism φ between the associated graded modules by $\varphi(\bar{v}) = \overline{w \otimes v}$. φ is non-trivial and so by Corollary 7 it is (up to multiplication by a scalar) the embedding of Demazure modules ι. We proved:

Lemma 9. *The map $\varphi : D(m, n\Theta) \to D(m, \Theta)_{c_2} * D(m, n\Theta)_{c_1} \simeq D(m, (n+1)\Theta)$ induced by $\varphi(v) = \overline{w \otimes v}$ is an embedding of $U(\tilde{\mathfrak{b}})$-modules.*

One knows that $V(m\Lambda_0) = \lim_{n\to\infty} D(m, n\Theta)$ as $U(\mathcal{C}\mathfrak{g})$-modules, and also as $U(\tilde{\mathfrak{b}})$-modules. It follows by the above:

Lemma 10. *Let $\mathfrak{g}$ be a simple Lie algebra.*

The following is a commutative diagram of $U(\mathcal{C}\mathfrak{g})$-modules

$$\begin{tikzcd}
D(m, n\Theta) \arrow[r, hook, "\iota"] \arrow[d, "\wr"] & D(m, (n+1)\Theta) \arrow[d, "\wr"] \\
D(m, n\Theta) \arrow[r, "\varphi"] \arrow[d, "\wr"] & D(m, \Theta)_{c_{n+1}} * D(m, n\Theta)_c \arrow[d, "\wr"] \\
D(m, \Theta)_{c_n} * \cdots * D(m, \Theta)_{c_1} \arrow[r, "\varphi"] & D(m, \Theta)_{c_{n+1}} * D(m, \Theta)_{c_n} * \cdots * D(m, \Theta)_{c_1}
\end{tikzcd}$$

where the down arrows are the isomorphism of Corollary 5.

Theorem 9. *Let $\mathfrak{g}$ be a simple Lie algebra. As $U(\mathcal{C}\mathfrak{g})$-module, $V(m\Lambda_0)$ is isomorphic to the semi-infinite fusion product*

$$V(m\Lambda_0) \simeq \lim_{n\to\infty} D(m, \Theta) * \cdots * D(m, \Theta).$$

We expect the following to hold:

Conjecture 2. *Let $\Lambda = m\Lambda_0 + \lambda$ be a dominant integral weight for $\hat{\mathfrak{g}}$, then $V(\Lambda)$ and*

$$\lim_{n\to\infty} D(m, \Theta) * \cdots * D(m, \Theta) * V(\lambda)$$

are isomorphic as $\mathcal{C}\mathfrak{g}$-modules.

Remark 18. This isomorphism holds for the $\mathfrak{g}$-module structure, see [18].

Remark 19. As in [18], the limit construction above works in a much more general setting. Let $D(m, \mu^\vee)$ be a Demazure module with the property that for some k the fusion product

$W = D(m, \mu^\vee) * \cdots * D(m, \mu^\vee) \simeq D(m, k\mu^\vee)$ contains a highest weight vector of weight $m\Lambda_0$. Instead of $D(m, \Theta)$ one can then use the module W in the direct limit construction above.

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