SPECTRAL STUDY OF THE LAPLACE-BELTRAMI OPERATOR ARISING IN THE PROBLEM OF ACOUSTIC WAVE SCATTERING BY A QUARTER-PLANE

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Summary

The Laplace-Beltrami operator on a sphere with a cut arises when considering the problem of wave scattering by a quarter-plane. Recent methods developed for sound-soft (Dirichlet) and sound-hard (Neumann) quarter-planes rely on an a priori knowledge of the spectrum of the Laplace-Beltrami operator. In this paper we consider this spectral problem for more general boundary conditions, including Dirichlet, Neumann, real and complex impedance, where the value of the impedance varies like α/r, r being the distance from the vertex of the quarter-plane and α being constant, and any combination of these. We analyse the corresponding eigenvalues of the Laplace-Beltrami operator, both theoretically and numerically. We show in particular that when the operator stops being self-adjoint, its eigenvalues are complex and are contained within a sector of the complex plane, for which we provide analytical bounds. Moreover, for impedance of small enough modulus $|\alpha|$, the complex eigenvalues approach the real eigenvalues of the Neumann case.

1. Introduction

Scattering (or diffraction) problems involve studying the field resulting from a wave incident upon an obstacle. This can for example be an acoustic or an electromagnetic wave. In general, these are complicated time-dependent problems, but often a hypothesis of time harmonicity can be made, i.e the time dependency is simply a factor $e^{-i\Omega t}$, where Ω is the frequency of the incident wave, and the wave equation reduces to the Helmholtz equation. A class of scattering problems, the canonical scattering problems, is particularly important. It derives from studying scattering by simple obstacles with particular characteristics such as sharp edges or corners, often of infinite size. Although "simple", these geometries can be used to evaluate the scattered field from more complicated geometries with high frequency incident waves. The understanding of such canonical geometries represents the building blocks for Keller's Geometrical Theory of Diffraction (GTD) (**1**). There is a long history of mathematicians working on such canonical problems; one of the first was Sommerfeld, leading to his famous solution to the half-plane problem (**2**) and to the creation of the field of Mathematical Theory of Diffraction. Since then, some very ingenious mathematical methods have been developed to tackle such problems, the most famous being the Wiener-Hopf (**3**) and the Sommerfeld-Malyuzhinets (**4**) techniques. However, despite tremendous efforts in this field, some canonical problems remain open, in the sense that they have no clear analytical solution or rapidly convergent fast hybrid numerical-analytical scheme. In particular, this is the case for the canonical problem of scattering by a quarter-plane (see Figure 2.1).

In the last 50 years, the quarter-plane problem has attracted a lot of attention, and different approaches have been used. By considering the quarter-plane as a degenerated elliptic cone, the field can be expressed as a spherical-wave multipole series ((**5**), (**6**), (**7**)), but these series are poorly convergent in the far-field. A review of this approach and attempts to accelerate the series convergence are described in (**8**). Another approach ((**9**), (**10**)), elegantly based on the Wiener-Hopf technique in two complex variables, has been used, however it led to a solution that has proved to be erroneous ((**11**), (**12**)). It is also worth mentioning an unusual and interesting approach in (**13**), where diffraction theory and statistical analysis are used in order to obtain the corner diffraction coefficient for a restricted set of incidence and observation direction.

A different way of considering this problem (in fact the more general problem of arbitrarily shaped cones), based on the use of spherical Green's functions has been introduced in ((**14**), (**15**)) and led to an integral formula for the spherical diffraction coefficient. However, this solution is not valid for all incidence-observation directions and requires a numerical treatment and some regularisation of Abel-Poisson type in order to be evaluated (**16**). Building on this type of approach, a hybrid numerical-analytical method, which partially solves the acoustic quarter-plane problem in the Dirichlet case has been introduced in (**17**) and (**18**). The main advantage of this method compared to the one mentioned previously is that in this case the formulae giving the diffraction coefficient, the Modified Smyshlyaev Formulae (MSF) are "naturally convergent" in the sense that they do not require special treatment to regularise or accelerate their convergence. The method is based on edge and spherical edge Green's functions and on the theory of embedding formulae, introduced in (**19**) and further developed in (**20**) for example. This method has been extensively described, adapted to the Neumann case and implemented in (**21**). The solution obtained, though valid for all observer directions over a wide range of incidence directions, remain partial in general, since for a range of incidence directions, there exists observation points for which the MSF are not valid. However, the range of validity of the MSF is bigger than that of any other method, going beyond the first singularity regions caused by the primary edge diffracted waves. A reason behind the limits of the MSF validity is the existence of secondary edge diffracted waves in the quarter-plane problem. In (**22**), a rigorous definition of the notion of the diffraction coefficient for the quarter-plane problem was given and the first analytical expression for these secondary edge waves generated during the diffraction process was derived. More recently, the Sommerfeld-Malyuzhinets technique was applied to the quarter-plane problem in the acoustic (**23**) and the electromagnetic (**24**) case, giving the overall structure of the diffracted field and recovering the results of (**22**) regarding the secondary diffracted waves.

The motivation of the present work is two-fold. On one hand, it has been shown for example in (**21**) that the derivation and the numerical evaluation of the MSF relies strongly on a detailed knowledge of the spectrum of the Laplace-Beltrami operator on the unit sphere with a cut. On the other hand, it seems that most of the work regarding the quarter-plane problem has been carried out for Dirichlet or Neumann boundary conditions and that more general boundary conditions should now be considered. The case of constant impedance conditions on the surface of the quarter-plane cannot be treated in a similar way as the more usual Neumann and Dirichlet conditions. The reason behind this difference, is that the usual approach relies on the fact that such problems are separable in spherical coordinates. However, as shall be seen later, the problem of constant impedance on the face of the quarter-plane is not separable anymore. Important work regarding constant impedance for the diffraction by an arbitrary shaped cone has been carried out in (**25**, **26**, **27**), and alternative formulations have been provided. However, the problem becomes separable if one assumes that the value of the impedance depends on the radial variable r and can be written as α/r for some constant α. Therefore, it is reasonable to believe that in that case, an embedding (or MSF) approach should work. Once separated, the problem reduces to that of finding the eigenvalues of the Laplace-Beltrami operator on the unit sphere, with constant impedance boundary imposed on a cut. Hence, in order to make progress in this direction, it is first important to shed some light on the eigenvalues of the Laplace-Beltrami operator with impedance conditions imposed on a cut. Impedance boundary conditions in canonical scattering problems have led to some very interesting developments, such has the Malyuzhinets technique in the case of the impedance wedge and impedance half-plane ((**28**), (**29**)) or the Wiener-Hopf-Hilbert method for the impedance half-plane ((**30**)). There has also been some work on the spectrum of the Laplace-Beltrami operator on a sphere with a cut, such as (**31**), (**32**), (**33**), (**34**) and (**21**), but again, to the authors' knowledge, solely in the Dirichlet and Neumann cases.

In the present work, the Laplace-Beltrami eigenvalue problem on a sphere with a cut will be approached both theoretically and numerically for a wide range of boundary conditions including Dirichlet, Neumann, real and complex impedance, and any combination of these. One of the most interesting features comes from the fact that a change in boundary conditions can lead to a big theoretical change. Indeed, the Laplace-Beltrami operator with Dirichlet or Neumann boundary condition is a self-adjoint operator, for which the spectral theory is well understood and developed (see e.g. (**35**)), while for complex impedance boundary conditions this ceases to be the case and the operator becomes non-self-adjoint, resulting in a rich behaviour of the eigenvalues and necessitating a different theoretical approach (see e.g. (**36**) and (**37**)).

The rest of this paper is structured as follows. The problem is presented in Section 2. To this end, the problem of scattering by a quarter-plane is formulated in Subsection 2.1 and its link with the Laplace-Beltrami operator eigenvalue problem is explained in Subsection 2.2. In Subsection 2.3, we provide a short description of the impedance boundary conditions and their implementation.

In Section 3, we focus our attention on a set of boundary conditions, said to be of type I, corresponding to the Laplace-Beltrami operator being self-adjoint. In Subsection 3.1, we briefly summarise the relevant theoretical results in spectral theory of self-adjoint differential operators and, after having described important function sets in Subsection 3.2, we apply them to the case of the Laplace-Beltrami operator in Subsection 3.3, showing that in this case, as expected, there is an infinite set of real positive discrete eigenvalues. Moreover, the relative position of the eigenvalues on the real line is investigated for different sets of boundary conditions of type I.

In Section 4, we focus our attention on a set of boundary conditions, said to be of type II, corresponding to the Laplace-Beltrami operator being non-self-adjoint. The breakdown of self-adjointness is explained in Subsection 4.1. The appropriate theoretical framework for non-self-adjoint operators is presented in Subsection 4.2 and applied to the Laplace-Beltrami operator in Subsection 4.3, showing that in this case there is an infinite discrete set of complex eigenvalues that are contained within a sector of the complex plane. In Subsection 4.4, we show how to find an estimate of such sector.

Section 5 is dedicated to the numerical evaluation of the eigenvalues of the Laplace-Beltrami operator. The numerical method used (surface finite elements) is briefly described in Subsection 5.1 and the results are presented, starting with the self-adjoint case (boundary conditions of type I) in Subsection 5.2 and following with the non-self-adjoint case (boundary conditions of type II) in Subsection 5.3. We show that in this case, the complex eigenvalues are indeed contained within a sector of the complex plane, and that for small or large enough impedance, we may recover the self-adjoint eigenvalues. An explanation as why this is true on a theoretical level is provided in Appendix B.

2. Presentation of the problem

2.1 *The quarter-plane problem*

Throughout the paper we assume that the problem is harmonic in time with frequency Ω and a time dependency proportional to $e^{-i\Omega t}$. As illustrated in Figure 2.1, consider an incident acoustic plane wave u_{in} on a quarter-plane $\mathcal{P}$. Let us call $\mathcal{P}^+$ and P^- the upper and lower surfaces of the plane sector respectively. The problem of scattering can be summarised as follows. We can write $u = u_{\text{in}} + u_{\text{sc}} + u_{\text{re}}$, where the four quantities u (total field), u_{in} (incident field), u_{re} (reflected field) and u_{sc} (scattered field) satisfy the Helmholtz equation

$$\Delta u + k^2 u \quad = \quad 0, \tag{2.1}$$

and k is the wavenumber of the homogeneous media surrounding the quarter-plane.

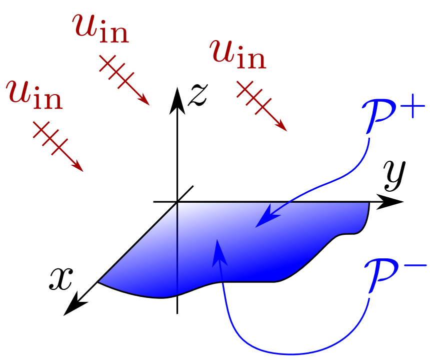

Fig. 2.1: (Colour online) The scattering problem by a quarter-plane

In order for the problem to be well-posed, some conditions need to be satisfied.

- u should satisfy the edge conditions, that is the energy of the system should remain bounded as we approach the edge. This can be expressed mathematically as $u \sim$ const $+ O(\rho^{1/2})$ near the edges, where ρ represents the distance to the edge. Physically, this also means that no sources should be located at the edges.

- u should satisfy the vertex condition, that is the energy of the system should remain bounded as we approach the vertex. This can be expressed mathematically by $\nabla u = o(r^{-3/2})$ as $r \to 0$, where r represents the distance to the vertex.
- The scattered field u_{sc} should satisfy a radiation condition. In other word, it should not exhibit any sources at infinity and u_{sc} should only consist of outgoing waves in the far-field. This radiation condition is an extension of the original Sommerfeld radiation condition (**38**), that is valid for infinite domains exhibiting edges or vertices. This extension follows the work by Rellich (**39**) and later Levine (**40**), as well as some uniqueness proof by Jones (**41**). For a comprehensive history of the evolution of the radiation condition, see (**42**).
- Finally, and most importantly, u has to satisfy some boundary conditions on $\mathcal{P}^+$ and $\mathcal{P}^-$. These conditions can for example be the Dirichlet boundary conditions ($u = 0$), the Neumann boundary conditions ($\boldsymbol{n} \cdot \nabla u = 0$, where $\boldsymbol{n}$ is a unit normal vector to $\mathcal{P}^\pm$) or varying Robin boundary conditions ($\boldsymbol{n} \cdot \nabla u + \frac{\alpha}{r} u = 0$, α being a specified constant).

Using the geometric theory of diffraction (**22**) or the Sommerfeld integrals (**23**), it has been shown in the case of Dirichlet and Neumann boundary conditions that the far-field behaviour of the scattered field could be written as

$$u_{\mathrm{sc}} + u_{\mathrm{re}} \quad = \quad u_{\mathrm{sph}} + u_{\mathrm{co1}} + u_{\mathrm{co2}} + u_{\mathrm{co12}} + u_{\mathrm{co22}} + u_{\mathrm{re}}, \tag{2.2}$$

the subscript $_{\mathrm{sph}}$ refers to the spherical wave emanating from the vertex, the subscript $_{\mathrm{co}}$ refers to the different (primary and secondary) conical waves emanating from the edges and the subscript $_{\mathrm{re}}$ refers to the wave reflected by the illuminated surface of the quarter plane as explained in details in (**21**) and (**22**).

The structure of the conical waves is well understood (at least in the case of Dirichlet or Neumann boundary conditions), and some analytical expressions of their far-field structures are given in (**22**) and (**23**). The spherical wave is less understood. In order to reduce the problem, it is useful to introduce the diffraction coefficient $f(\omega, \omega_0)$, where ω represents the direction of observation and ω_0 the direction of incidence of u_{in}. The far-field behaviour of u_{sph} can be represented by

$$u_{\mathrm{sph}}(\omega, \omega_0, r) = 2\pi \frac{e^{ikr}}{kr} f(\omega, \omega_0) + O\left(\frac{e^{ikr}}{(kr)^2}\right) \quad \text{as} \quad kr \to \infty \ .$$

The evaluation of the diffraction coefficient $f(\omega, \omega_0)$ has been the subject of many studies. A recent way of approaching the problem, leading to a partial resolution of the problem in the case of Dirichlet (**17**) and Neumann (**21**), is to use the theory of embedding formulae to obtain an integral expression of $f(\omega, \omega_0)$. A spherical version of the embedded formulae, the so-called Modified Smyshlyaev formulae (MSF), are somehow easier to evaluate numerically, using the so-called coordinate equation ((**18**), (**21**)). Thanks to this method, the diffraction coefficient is now easily computable for many pairs (ω, ω_0). However, as emphasised in (**21**), because of the secondary diffracted waves there are still some regions where it is complicated to evaluate $f(\omega, \omega_0)$, and for this reason the problem of scattering by a quarter-plane is still considered as an open mathematical problem. In both cases (Dirichlet and Neumann), the definition and the numerical evaluation of the MSF relies heavily on knowledge of the eigenvalues of a certain linear differential operator: the Laplace-Beltrami operator

(LBO). Hence if one envisages the evaluation of the diffraction coefficient $f(\omega, \omega_0)$ for boundary conditions different from the pure Dirichlet or Neumann boundary conditions, it is important to be able to compute a priori the eigenvalues of the LBO associated to these boundary conditions. In the next subsection, we shall introduce the LBO in more detail.

2.2 *The Laplace Beltrami operator (LBO)*

Seeking a spherically separable solution to the Helmholtz equation (2.1) of the form $u(r,\theta,\varphi) = R(r)U(\theta,\varphi)$, where (r,θ,φ) are the usual spherical coordinates (see Figure 2.2a) leads to an equation for $R(r)$

$$r^2 R'' + 2rR' + ((kr)^2 - \lambda)R = 0,$$

with solution

$$R(r) = \frac{CJ_\nu(kr) + DH_\nu^{(1)}(kr)}{\sqrt{kr}},$$

where C and D are constants, J_ν and $H_\nu^{(1)}$ are the Bessel function and the Hankel function of type 1 of order ν respectively, and ν is defined by

$$\nu = (\lambda + 1/4)^{1/2}. \tag{2.3}$$

The separation of variables also leads to an equation for $U(\theta,\varphi)$:

$$-\tilde{\Delta} U = \lambda U, \tag{2.4}$$

where $\tilde{\Delta}$ is the LBO defined by

$$\tilde{\Delta} = \frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2(\theta)}\frac{\partial^2}{\partial\varphi^2} \tag{2.5}$$

This can be interpreted as the restriction of the spherical Laplace operator to the surface of the unit sphere. It is also useful to introduce $\tilde{\nabla}$ and $\tilde{\nabla}\cdot$ that represents the restriction of the spherical gradient operator and the spherical divergence operator to the surface of the sphere. More precisely, for a scalar function $U(\theta,\varphi)$ and a vector field $\boldsymbol{A} = (A_\theta, A_\varphi)$, we have

$$\tilde{\nabla} U = \begin{pmatrix} \dfrac{\partial U}{\partial \theta} \\ \dfrac{1}{\sin(\theta)}\dfrac{\partial U}{\partial\varphi} \end{pmatrix} \quad \text{and} \quad \tilde{\nabla}\cdot\boldsymbol{A} = \frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}(\sin(\theta)A_\theta) + \frac{1}{\sin(\theta)}\frac{\partial A_\varphi}{\partial\varphi}, \tag{2.6}$$

so that we have $\tilde{\Delta}U = \tilde{\nabla}\cdot\tilde{\nabla}U$. In fact, because of the geometry of the quarter plane, (2.4) is actually defined on the surface of a sphere with a slit (or cut) of length $\pi/2$ corresponding to the intersection between the unit sphere and the quarter-plane. As shall be seen in the next subsection, for the boundary conditions considered in this paper, the boundary condition equation is also separable. Hence, solving the problem on the unit sphere with a

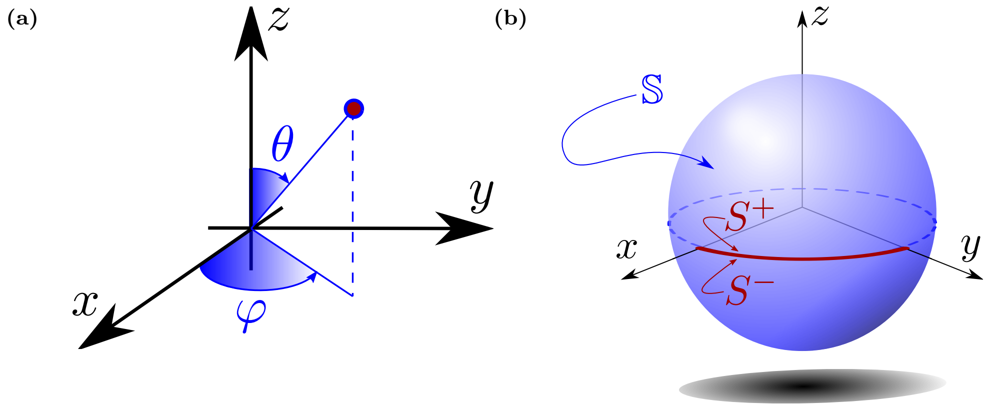

Fig. 2.2: (Colour online) (a) Spherical coordinates and (b) definition of $\mathbb{S}$, S^+ and S^-

cut is enough. Let us call $\mathbb{S}$ the corresponding domain (see Figure 2.2b) and S^+ and S^- the upper and lower part of the slit such that $\partial\mathbb{S} = S^+ \cup S^-$.

The aim of this work is the rigorous study of the eigenvalue problem (2.4), and that can be reformulated as

$$\text{Find all eigenvalues } \lambda \text{ such that } \exists U \neq 0 \text{ satisfying } \begin{cases} -\tilde{\Delta} U = \lambda U \text{ on } \mathbb{S} \\ \text{Boundary conditions on } S^+ \\ \text{Boundary conditions on } S^- \end{cases} \tag{2.7}$$

As already emphasised at the end of the previous subsection and shown in detail in (**21**), knowledge of the eigenvalues is crucial to the definition and evaluation of the MSF that involve complicated contours surrounding the said eigenvalues. Remember that ν and λ are linked by (2.3), and even if they are different, we may refer to both of them as eigenvalues of the problem (2.7). Moreover, the first eigenvalue is also useful in order to understand the behaviour of the total field at the vertex of the quarter-plane. Indeed, it can be shown (see (**34**)) that $u = O(r^{\nu_1 - 1/2})$ as r tends to zero, where ν_1, corresponding to the first eigenvalue of the Laplace-Beltrami operator, takes different values for different boundary conditions.

Many authors ((**33**), (**31**), (**32**), (**21**), (**34**)) have successfully considered this problem with different techniques (e.g., WKB or shooting methods) for the case of pure Dirichlet boundary conditions (i.e. Dirichlet on S^+ and Dirichlet on S^-) and pure Neumann boundary conditions (i.e. Neumann on S^+ and Neumann on S^-). In this paper, we would like to extend this knowledge to a broader range of boundary conditions. As we shall see in the following sections, the choice of boundary conditions may have a significant impact on the theory and the numerical methods necessary for the evaluation of the eigenvalues.

2.3 *On the Robin boundary condition*

Let us briefly summarise the definition and physical relevance of the Robin boundary conditions. Consider a locally reacting surface boundary surrounded by a homogeneous two- or three-dimensional space of density ρ, speed of sound c and wavenumber k. We can define the acoustic field by the pressure p and the acoustic velocity $\boldsymbol{v}$. Then the *acoustic* *impedance* or *normal acoustic impedance* z is defined as the ratio between the pressure and the normal fluid velocity at a point on the surface:

$$z = \frac{p}{\boldsymbol{v}\cdot\boldsymbol{n}} \quad \text{on the surface,} \tag{2.8}$$

where we consider $\boldsymbol{n}$ as the normal pointing into the surface. Let us now consider the potential ψ such that

$$p = -\rho\frac{\partial\psi}{\partial t} \quad \text{and} \quad \boldsymbol{v} = \nabla\psi\ . \tag{2.9}$$

Assuming that the acoustic field is harmonic with frequency Ω, we have $\Omega = kc$ and we can write

$$\psi(x,y,z,t) \;=\; \phi(x,y,z)e^{-i\Omega t}\ . \tag{2.10}$$

Now using (2.10) and (2.9) into (2.8), we obtain

$$\nabla\phi\cdot\boldsymbol{n} = ik\frac{\rho c}{z}\phi \quad \text{on the surface}\ . \tag{2.11}$$

Let us now introduce some notations/vocabulary. It is common to introduce the *characteristic acoustic impedance* z_0 of a medium defined by $z_0 = \rho c$. Note that this is totally independent of the boundary of the domain. One can then define the *specific acoustic impedance* ζ by $\zeta = z/z_0$. Looking at equation (2.11), it is also useful to define the *specific acoustic admittance* β by $\beta = z_0/z = 1/\zeta$. In general, β and ζ can be complex. It is quite useful to rewrite (2.11) in terms of β, which gives:

$$\frac{\partial\phi}{\partial n} = ik\beta\phi \quad \text{on the surface,} \tag{2.12}$$

where we have used the notation $\partial\phi/\partial n = \nabla\phi\cdot\boldsymbol{n}$. Physical boundaries have to be either passive or absorbent, that is that the energy flux which flows across a surface element ds of the boundary over an acoustic period $T_a = 2\pi/\Omega$ has to be positive. As emphasised in (**4**), using the energy flux density vector (or acoustic Poynting vector), it can be shown that for a surface to be either passive or absorbent, we need to have $\Re(\beta) \geqslant 0$. As it turns out, this condition is often required to prove uniqueness results of scattering problem on a surface with boundary conditions (2.12) see for example ((**4**), (**43**)).

In order to simplify the argument in the following sections, we shall introduce the quantity $\mathcal{A}$ defined by $\mathcal{A} = -ik\beta$, so that the impedance boundary condition on a surface can be rewritten as

$$\nabla\phi\cdot\boldsymbol{n} + \mathcal{A}\phi \;=\; 0.$$

Note that the absorbent condition (i.e. $\Re(\beta) \geqslant 0$) translates in a condition on $\mathcal{A}$ that is $\Im(\mathcal{A}) \leqslant 0$. In the rest of this paper, we shall refer to $\mathcal{A}$ as the impedance of our surface. Now, writing $\phi(r,\theta,\varphi) = R(r)U(\theta,\phi)$, this condition reduces to

$$\frac{1}{r}\boldsymbol{n}\cdot\tilde{\nabla}U + \mathcal{A}U \;=\; 0,$$

where $\boldsymbol{n}$ is now considered as a two-dimensional vector in the $(\boldsymbol{e}_\theta, \boldsymbol{e}_\varphi)$ basis. Hence, despite the fact that the Helmholtz equation is separable, this is not the case for the boundary equation if $\mathcal{A}$ is constant. However, if one considers the case when

$$\mathcal{A} = \mathcal{A}(r) = \frac{\alpha}{r},$$

for some constant α, then the boundary equation becomes separable and reduces to

$$\boldsymbol{n} \cdot \tilde{\nabla} U + \alpha U \quad = \quad 0.$$

Choosing such a varying impedance implies that the surface of the quarter-plane in that case is sound-soft at the vertex, and gradually becomes harder a we move away from the vertex. Note that uniqueness theorems for this type of boundary condition are studied in (**41**) and mentioned in (**40**). In what follows, when referring to impedance or Robin boundary conditions, we will refer to that situation.

3. Self-adjoint operators and boundary conditions of type I

Before presenting the spectral theory of linear operators, let us specify the boundary conditions that will be considered in this section, the boundary conditions of type I.

Definition 3.1. A suitably smooth function u on $\mathbb{S}$ is said to satisfy boundary conditions of type I if it satisfies any one of the following boundary conditions on S^+ and S^-:

- Dirichlet: $u = 0$
- Neumann: $\boldsymbol{n}^\pm \cdot \tilde{\nabla} u = 0$ on $S^\pm$
- Robin: $\boldsymbol{n}^\pm \cdot \tilde{\nabla} u + \alpha^\pm u = 0$ on $S^\pm$ with $\alpha^\pm \in \mathbb{R}$ and $\alpha^\pm > 0$

The boundary conditions do not necessarily have to be the same on each side (S^+ or S^-) of the slit. The orientation of the normals $\boldsymbol{n}^\pm$ is specified in Figure 3.1a.

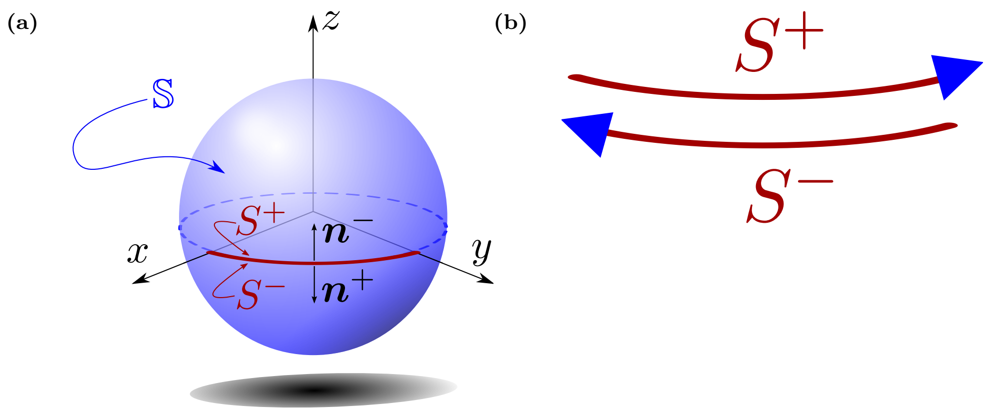

Fig. 3.1: (Colour online) Orientation of the normals $\boldsymbol{n}^+$ and $\boldsymbol{n}^-$ (a) and of the cut (b)

As we shall see in Subsection 3.3, the LBO is well behaved for these boundary conditions, in the sense that it is self-adjoint. Let us start by summarising the theoretical framework appropriate to the study of such operators.

3.1 *Theoretical framework*

We first recall some key definitions and results from (**35**) and (**36**). A linear operator T on a Hilbert space $(\boldsymbol{H}, \langle\cdot,\cdot\rangle)$ consists of a pair $(T, \mathrm{Dom}(T))$, where $\mathrm{Dom}(T) \subset \boldsymbol{H}$ is a dense linear subset of $\boldsymbol{H}$ for the Hilbert norm $\|\cdot\|_{\boldsymbol{H}}$. $\mathrm{Dom}(T)$ is the domain of definition of the linear map $T : \mathrm{Dom}(T) \to \boldsymbol{H}$ and is called the domain of the linear operator T. A linear operator T on $(\boldsymbol{H}, \langle\cdot,\cdot\rangle)$ is said to be symmetric if for any f, g in $\mathrm{Dom}(T)$, we have $\langle T(f), g\rangle = \langle f, T(g)\rangle$. It is said to be non-negative if for any $f \in \mathrm{Dom}(T)$, $\langle T(f), f\rangle \geqslant 0$. If S and T are two linear operators on $\boldsymbol{H}$ such that $\mathrm{Dom}(S) \subset \mathrm{Dom}(T)$ and $T(f) = S(f)$ for all $f \in \mathrm{Dom}(S)$, we say that T is an extension of S and S is a restriction of T, and we write $S \subset T$.

Given a linear operator T on $(\boldsymbol{H}, \langle\cdot,\cdot\rangle)$, we can define its adjoint operator T^*, with domain $\mathrm{Dom}(T^*)$ defined by

$$\mathrm{Dom}(T^*) \quad = \quad \{g \in \boldsymbol{H} \text{ s.t. } \exists h \in \boldsymbol{H} \text{ s.t. } \forall f \in \mathrm{Dom}(T), \langle T(f), g\rangle = \langle f, h\rangle\}\,,$$

and with the condition that $\forall f \in \mathrm{Dom}(T), \forall g \in \mathrm{Dom}(T^*), \langle T(f), g\rangle = \langle f, T^*(g)\rangle$. We say that a linear operator T on $\boldsymbol{H}$ is self-adjoint if T is symmetric and $\mathrm{Dom}(T) = \mathrm{Dom}(T^*)$.

The resolvent set $\rho(T)$ is defined as being the set of all $\zeta \in \mathbb{C}$ such that $\zeta I - T$ is invertible (I being the identity operator) and the resolvent operator $R(\zeta, T) = (\zeta I - T)^{-1}$ is bounded. A complex number λ is said to be an eigenvalue of T if there exists a non-zero f in $\mathrm{Dom}(T)$ such that $T(f) = \lambda f$. The spectrum of T, $\sigma(T)$, is defined by $\sigma(T) = \mathbb{C}\backslash\rho(T)$. In the finite dimensional case, $\sigma(T)$ consists purely of eigenvalues. However, in the infinite dimensional case (of interest here), this is not necessarily the case. The spectrum of T can be decomposed into the discrete spectrum of T, $\sigma_d(T)$ (the set of all isolated eigenvalues of finite multiplicity) and the essential spectrum of T, $\sigma_{\mathrm{ess}}(T)$ (the non-discrete part of the spectrum).

The main theoretical result that will be used in this section is the following spectral theorem.

Theorem 3.2. ***Spectral theorem*** *[Cor. 4.2.3 in (***35***)]* Let T be an unbounded self-adjoint non-negative linear operator on $\boldsymbol{H}$. The following are equivalent:

1. (Compact resolvent) The resolvent operator $R(1, -T) = (T + I)^{-1}$ is compact.
2. The operator T has empty essential spectrum.
3. There exists a complete orthonormal set of eigenvectors $\{\phi_n\}_{n=1}^{\infty}$ of T with corresponding eigenvalues $\lambda_n \geqslant 0$ satisfying $\lambda_n \to \infty$ as n tends to infinity.

In order to make some mathematical progress, it is very useful to think in terms of sesquilinear forms. In this paragraph, we shall recall a few facts and definitions about these. A sesquilinear form $\mathfrak{t}$ defined on a subspace $D(\mathfrak{t})$ of a Hilbert space $\boldsymbol{H}$ is a map $\mathfrak{t} : D(\mathfrak{t}) \times D(\mathfrak{t}) \to \mathbb{C}$ that is linear in its first argument and conjugate-linear in its second, which means that for any $u, v, w \in D(\mathfrak{t})$, and any $\alpha \in \mathbb{C}$, we have

$$\mathfrak{t}(\alpha u + v, w) = \alpha\mathfrak{t}(u, v) + \mathfrak{t}(v, w) \quad \text{and} \quad \mathfrak{t}(u, \alpha v + w) = \bar{\alpha}\mathfrak{t}(u, v) + \mathfrak{t}(u, w)\ ,$$

where $\overline{}$ denotes the complex conjugate. Concepts similar to the linear operators apply to sesquilinear forms. In particular, we say that a sesquilinear form $\mathfrak{t}$ is densely defined if $D(\mathfrak{t})$ is dense in $\boldsymbol{H}$. We say that a sesquilinear form $\mathfrak{t}$ is non-negative if for any u in $D(\mathfrak{t})$, we have $\mathfrak{t}(u,u) \geqslant 0$. A non-negative sesquilinear form $\mathfrak{t}$ is said to be closed if the normed space $(D(\mathfrak{t}), \|\cdot\|_{\mathfrak{t}})$ is complete, where $\|\cdot\|_{\mathfrak{t}}$ is the norm associated to $\mathfrak{t}$ defined for $u \in D(\mathfrak{t})$ by

$$\|u\|_{\mathfrak{t}} \quad = \quad (\mathfrak{t}(u,u) + \|u\|_{\boldsymbol{H}}^2)^{1/2} \ .$$

If this is the case, $(D(\mathfrak{t}), \|\cdot\|_{\mathfrak{t}})$ is called the Hilbert space associated to $\mathfrak{t}$. We can define extensions of sesquilinear forms similarly to the operator case. If $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are two non-negative sesquilinear forms such that $D(\mathfrak{t}_1) \subset D(\mathfrak{t}_2)$ and for any (u,v) in $D(\mathfrak{t}_1)$, we have $\mathfrak{t}_1(u,v) = \mathfrak{t}_2(u,v)$, we say that $\mathfrak{t}_2$ is an extension of $\mathfrak{t}_1$. A sesquilinear form $\mathfrak{t}$ is said to be closable if it admits a closed extension. The smallest closed extension of a closable form $\mathfrak{t}$ is called the closure of $\mathfrak{t}$. For a closed sesquilinear form $\mathfrak{t}$, a finite dimensional subspace $L \subset D(\mathfrak{t})$, we can introduce the functional $\Lambda^{\mathfrak{t}}(L)$ as follows:

$$\Lambda^{\mathfrak{t}}(L) \quad = \quad \sup\{\mathfrak{t}(u,u) \quad s.t. \quad u \in L \quad \text{and} \quad \|u\|_{\boldsymbol{H}} = 1\}$$

This allows us to define a quantity $\Lambda_n^{\mathfrak{t}}$ defined for each integer $n \geqslant 1$ by

$$\Lambda_n^{\mathfrak{t}} \quad = \quad \inf\left\{\Lambda^{\mathfrak{t}}(L) \quad s.t. \quad L \subset D(\mathfrak{t}) \quad \text{and} \quad \dim(L) = n\right\}$$

Note that when there is no ambiguity, we may just write $\Lambda(L)$ and Λ_n instead of $\Lambda^{\mathfrak{t}}(L)$ and $\Lambda_n^{\mathfrak{t}}$. Let us now state some useful results that link linear operators and sesquilinear forms.

THEOREM 3.3. ***Representation theorem*** *[Thm. VI.2.6 in (***36***), Thm. 4.4.2 in (***35***), Thm. B.1.6. in (***44***)]* A closed non-negative sesquilinear form $\mathfrak{t}$ with domain $D(\mathfrak{t})$ acting in a Hilbert space $\mathbf{H}$ gives rise to a non-negative self-adjoint operator T with domain $\mathrm{Dom}(T)$ defined by

$$\mathrm{Dom}(T) \quad = \quad \{u \in D(\mathfrak{t}) \quad s.t. \quad \exists h \in \boldsymbol{H} \quad s.t. \quad \forall v \in D(\mathfrak{t}), \quad \mathfrak{t}(u,v) = \langle h, v\rangle_{\boldsymbol{H}}\}$$

and such that for any $u, v \in \mathrm{Dom}(T)$, we have $\langle T(u), v\rangle_{\boldsymbol{H}} = \mathfrak{t}(u,v)$.

THEOREM 3.4. *[Direct corollary of Thm. 4.5.2 in (***35***)]* Let $\mathfrak{t}$ be a closed non-negative sesquilinear form on a Hilbert space $\boldsymbol{H}$, and T its associated self-adjoint operator (resulting from Theorem 3.3). If $\Lambda_n^{\mathfrak{t}} \to \infty$ as $n \to \infty$, then T has empty essential spectrum. And so by Theorem 3.2, T also has compact resolvent and an ordered discrete set of eigenvalues λ_n and these correspond exactly to Λ_n.

Finally, as a direct consequence of the definitions of Λ and Λ_n, it is possible to deduce the following theorem that will prove very useful in Subsection 3.3.

THEOREM 3.5. Let $(\mathfrak{t}_1, D(\mathfrak{t}_1))$ and $(\mathfrak{t}_2, D(\mathfrak{t}_2))$ be two closed sesquilinear forms. Then

i. If $D(\mathfrak{t}_1) \subset D(\mathfrak{t}_2)$ and $\mathfrak{t}_2|_{D(\mathfrak{t}_1)} = \mathfrak{t}_1$, then for any integer $n \geqslant 1$, $\Lambda_n^{\mathfrak{t}_2} \leqslant \Lambda_n^{\mathfrak{t}_1}$

ii. If $D(\mathfrak{t}_1) = D(\mathfrak{t}_2)$ and $\forall u \in D(\mathfrak{t}_1)$ we have $\mathfrak{t}_1(u,u) < \mathfrak{t}_2(u,u)$, then $\Lambda_n^{\mathfrak{t}_1} < \Lambda_n^{\mathfrak{t}_2}$.

3.2 *Important function sets on* $\mathbb{S}$

Let us start by defining different sets of smooth functions on $\mathbb{S}$

$$\begin{aligned} C^\infty(\mathbb{S}) &= \{\text{infinitely smooth functions } f \text{ on } \mathbb{S}\} \\ C^\infty(\bar{\mathbb{S}}) &= \left\{f \in C^\infty(\mathbb{S}), \text{ all of whose partial derivatives can be extended continuously to } \bar{\mathbb{S}}\right\} \end{aligned}$$

Let us also define different sets of compactly supported functions, where Supp(f) refers to the support of a function f.

$$\begin{aligned}
\mathcal{D}(\$) &= \left\{f \in C^{\infty}(\bar{\$}),\ \text{s.t. } \exists \text{ compact } K \subset \$, \text{ s.t. } \mathrm{Supp}(f) = K\right\} \\
\mathcal{D}^{+}(\$) &= \left\{f \in C^{\infty}(\bar{\$}),\ \text{s.t. } f \text{ vanishes in a neighbourhood of } S^{+}\right\} \\
\mathcal{D}^{-}(\$) &= \left\{f \in C^{\infty}(\bar{\$}),\ \text{s.t. } f \text{ vanishes in a neighbourhood of } S^{-}\right\},
\end{aligned}$$

where what is meant by such a neighbourhood is described in Figure 3.2.

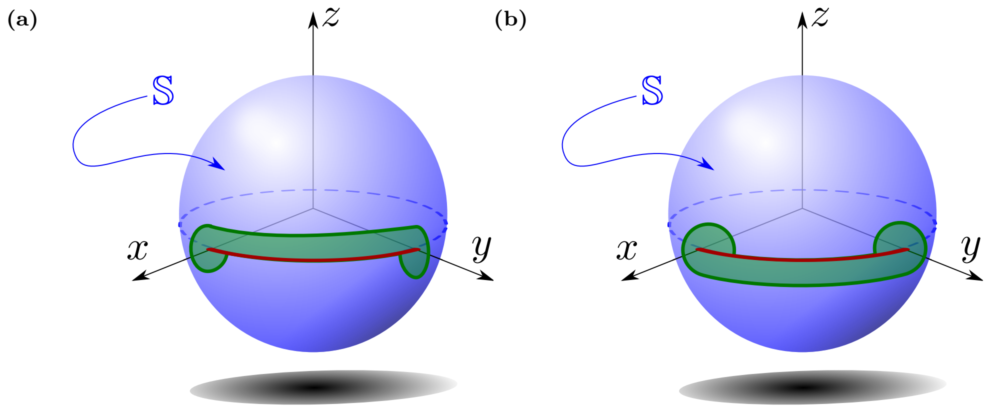

Fig. 3.2: (Colour online) Illustration of typical neighbourhood in the definition of (a) $\mathcal{D}^{+}(\$)$ and (b) $\mathcal{D}^{-}(\$)$

Let us also define the space $L_2(\$)$ as the Lebesgue space of square integrable functions on $\$$. $L_2(\$)$ is a Hilbert space for the inner product $\langle\cdot,\cdot\rangle_{L^2}$ and its associated norm $\|\cdot\|_{L^2}$ defined by

$$\forall u, v \in L_2(\$), \quad \langle u, v\rangle_{L^2} = \iint_{\$} u\bar{v}\mathrm{d}\$ \quad \text{and} \quad \|u\|_{L^2}^2 = \langle u, u\rangle_{L^2} = \iint_{\$} |u|^2\mathrm{d}\$,$$

where $\mathrm{d}\$ = \sin(\theta)\mathrm{d}\theta\mathrm{d}\varphi$. It is a classic result that $\mathcal{D}(\$)$ is dense in $L^2(\$)$ (for the norm $\|.\|_{L^2}$). Let us now introduce the classic Sobolev space $H^1(\$)$, also sometimes referred to as $W^{1,2}(\$)$, as

$$H^1(\$) \quad = \quad \left\{f \in L^2(\$) \text{ whose partial derivatives of order 1 are in } L^2(\$)\right\},$$

where here functions and derivatives are understood in the distributional sense. Note that $H^1(\$)$ is a Hilbert space for the inner product $\langle ., .\rangle_{H^1}$ and the associated norm $|||.|||$ defined by

$$\forall u, v \in H^1(\$), \quad \langle u, v\rangle_{H^1} = \langle u, v\rangle_{L^2} + \langle\tilde{\nabla}u, \tilde{\nabla}v\rangle_{L^2} \quad \text{and} \quad |||u||| = (\|u\|_{L^2}^2 + \|\tilde{\nabla}u\|_{L^2}^2)^{1/2}\ .$$

It is clear that $\mathcal{D}(\$), \mathcal{D}^{+}(\$)$ and $\mathcal{D}^{-}(\$)$ are all included in $H^1(\$)$. Hence, it is now possible to define the spaces $H^1_0(\$)$, $H^1_{0^+}(\$)$ and $H^1_{0^-}(\$)$ as the closure in H^1 (for the norm $|||.|||$) of the sets $\mathcal{D}(\$), \mathcal{D}^+(\$)$ and $\mathcal{D}^-(\$)$ respectively. It is important to note that

$$H^1_0(\$) \subset \left\{ \begin{array}{c} H^1_{0^+}(\$) \\ H^1_{0^-}(\$) \end{array} \right\} \subset H^1(\$).$$

3.3 *The LBO as a self-adjoint operator for boundary conditions of type I*

Consider the linear operator T acting in the Hilbert space $(L^2(\$), \langle .,.\rangle_{L^2}, \|.\|_{L^2})$ and formally defined by $T = -\tilde{\Delta}$ as in (2.5) with boundary conditions of type I. Let T be initially defined on the domain $\mathrm{Dom}(T)$ defined by

$$\mathrm{Dom}(T) \quad = \quad \left\{ f \in C^\infty(\bar{\$}), \quad \text{s.t. } f \text{ satisfies the correct boundary conditions of type I} \right\}.$$

The appropriate notation for each possible combination of boundary condition is specified in Table 3.1.

Boundary Condition on S^+	Boundary Condition on S^-	Operator T
Dirichlet	Dirichlet	T_{DD}
Neumann	Neumann	T_{NN}
Dirichlet	Neumann	T_{DN}
Neumann	Dirichlet	T_{ND}
Real positive Robin	Dirichlet	T_{RD}
Dirichlet	Real positive Robin	T_{DR}
Real positive Robin	Neumann	T_{RN}
Neumann	Real positive Robin	T_{NR}
Real positive Robin	Real positive Robin	T_{RR}

Table 3.1: The operator T for boundary conditions of type I

Because of the inclusion $\mathcal{D}(\$) \subset \mathrm{Dom}(T) \subset L^2(\$)$, and the fact that $\mathcal{D}(\$)$ is dense in $L^2(\$)$, it is automatic to see that $\mathrm{Dom}(T)$ is dense in $L^2(\$)$, the Hilbert space T is acting on. Hence $(T, \mathrm{Dom}(T))$ is a well-defined linear operator on $L^2(\$)$.

Now, using Green's identity on $\$$ (See Appendix A), we can write

$$\begin{aligned} \forall u, v \in \mathrm{Dom}(T), \quad \langle T(u), v\rangle_{L^2} &= \left\langle \tilde{\nabla} u, \tilde{\nabla} v \right\rangle_{L^2} - \int_{S^+} \bar{v} \left(\tilde{\nabla} u \cdot \boldsymbol{n}^+ \right) \mathrm{d}\ell - \int_{S^-} \bar{v} \left(\tilde{\nabla} u \cdot \boldsymbol{n}^- \right) \mathrm{d}\ell \\ \langle u, T(v)\rangle_{L^2} &= \left\langle \tilde{\nabla} u, \tilde{\nabla} v \right\rangle_{L^2} - \int_{S^+} u \left(\overline{\tilde{\nabla} v \cdot \boldsymbol{n}^+} \right) \mathrm{d}\ell - \int_{S^-} u \left(\overline{\tilde{\nabla} v \cdot \boldsymbol{n}^-} \right) \mathrm{d}\ell. \end{aligned}$$

When boundary conditions are applied, the right-hand side simplifies and the results are presented in Table 3.2, where what is meant by the line integrals along S^+ and S^- is specified in Appendix A by (A.6).

Since in this section, when considering Robin boundary conditions, we only consider real impedances $\alpha^\pm$, it is clear that $\overline{\alpha^\pm} = \alpha^\pm$, and that as such, using table 3.2 we have that

$$\forall u, v \in \mathrm{Dom}(T), \quad \langle T(u), v\rangle_{L^2} = \langle u, T(v)\rangle_{L^2},$$

Operator T	$\langle T(u),v\rangle_{L^2} =$	$\langle u,T(v)\rangle_{L^2} =$
T_{DD}, T_{NN}, T_{DN}, T_{ND},	$\langle\tilde{\nabla}u,\tilde{\nabla}v\rangle_{L^2}$	$\langle\tilde{\nabla}u,\tilde{\nabla}v\rangle_{L^2}$
T_{RD}, T_{RN}	$\langle\tilde{\nabla}u,\tilde{\nabla}v\rangle_{L^2}+\alpha^+\int_{S^+}u\bar{v}\,\mathrm{d}\ell$	$\langle\tilde{\nabla}u,\tilde{\nabla}v\rangle_{L^2}+\overline{\alpha^+}\int_{S^+}u\bar{v}\,\mathrm{d}\ell$
T_{DR}, T_{NR}	$\langle\tilde{\nabla}u,\tilde{\nabla}v\rangle_{L^2}+\alpha^-\int_{S^-}u\bar{v}\,\mathrm{d}\ell$	$\langle\tilde{\nabla}u,\tilde{\nabla}v\rangle_{L^2}+\overline{\alpha^-}\int_{S^-}u\bar{v}\,\mathrm{d}\ell$
T_{RR}	$\langle\tilde{\nabla}u,\tilde{\nabla}v\rangle_{L^2}+\alpha^+\int_{S^+}u\bar{v}\,\mathrm{d}\ell+\alpha^-\int_{S^-}u\bar{v}\,\mathrm{d}\ell$	$\langle\tilde{\nabla}u,\tilde{\nabla}v\rangle_{L^2}+\overline{\alpha^+}\int_{S^+}u\bar{v}\,\mathrm{d}\ell+\overline{\alpha^-}\int_{S^-}u\bar{v}\,\mathrm{d}\ell$

Table 3.2: Green's identity applied to T for boundary conditions of type I

which means that $(T,\mathrm{Dom}(T))$ is a symmetric linear operator. Now, using $v=u$ in Table 3.2, and the fact that $\alpha^\pm>0$, it is easy to see that

$$\forall u\in\mathrm{Dom}(T),\quad \langle T(u),u\rangle_{L^2}\geqslant 0,$$

which means that $(T,\mathrm{Dom}(T))$ is a non-negative linear operator. Note that the operator T gives rise to a sesquilinear form $\mathfrak{t}$ with $D(\mathfrak{t})=\mathrm{Dom}(T)$ defined in Table 3.3 for each type of boundary conditions.

Sesquilinear form $\mathfrak{t}$	$\mathfrak{t}(u,v)$	Associated norm $\|.\|_{\mathfrak{t}}$
$\mathfrak{t}_{\mathrm{DD}}$, $\mathfrak{t}_{\mathrm{NN}}$, $\mathfrak{t}_{\mathrm{DN}}$, $\mathfrak{t}_{\mathrm{ND}}$	$\langle\tilde{\nabla}u,\tilde{\nabla}v\rangle_{L^2}$	$\vert\vert\vert.\vert\vert\vert$
$\mathfrak{t}_{\mathrm{RD}}$, $\mathfrak{t}_{\mathrm{RN}}$	$\langle\tilde{\nabla}u,\tilde{\nabla}v\rangle_{L^2}+\alpha^+\int_{S^+}u\bar{v}\,\mathrm{d}\ell$	$\left(\vert\vert\vert.\vert\vert\vert^2+\alpha^+\int_{S^+}\vert u\vert^2\mathrm{d}\ell\right)^{1/2}$
$\mathfrak{t}_{\mathrm{DR}}$, $\mathfrak{t}_{\mathrm{NR}}$	$\langle\tilde{\nabla}u,\tilde{\nabla}v\rangle_{L^2}+\alpha^-\int_{S^-}u\bar{v}\,\mathrm{d}\ell$	$\left(\vert\vert\vert.\vert\vert\vert^2+\alpha^-\int_{S^-}\vert u\vert^2\mathrm{d}\ell\right)^{1/2}$
$\mathfrak{t}_{\mathrm{RR}}$	$\langle\tilde{\nabla}u,\tilde{\nabla}v\rangle_{L^2}+\alpha^+\int_{S^+}u\bar{v}\,\mathrm{d}\ell+\alpha^-\int_{S^-}u\bar{v}\,\mathrm{d}\ell$	$\left(\vert\vert\vert.\vert\vert\vert^2+\alpha^+\int_{S^+}\vert u\vert^2\mathrm{d}\ell+\alpha^-\int_{S^-}\vert u\vert^2\mathrm{d}\ell\right)^{1/2}$

Table 3.3: Definition of the sesquilinear form $\mathfrak{t}$ arising from T for boundary conditions of type I

Now it is possible to define a closed sesquilinear form $(\tilde{\mathfrak{t}},D(\tilde{\mathfrak{t}}))$ that is an extension of $(\mathfrak{t},D(\mathfrak{t}))$, i.e, the formal definition of $\tilde{\mathfrak{t}}$ and its associated norm $\|.\|_{\tilde{\mathfrak{t}}}$ is the same as that given for $\mathfrak{t}$ in Table 3.3, but its domain $D(\tilde{\mathfrak{t}})$ is bigger. The domain $D(\tilde{\mathfrak{t}})$ for different boundary conditions is given in Table 3.4.

The fact that $\tilde{\mathfrak{t}}$ is closed follows directly for boundary conditions of type I not involving Robin conditions, since for these (see Table 3.3), we have $\|.\|_{\tilde{\mathfrak{t}}}=|||.|||$ and by definition the associated $D(\tilde{\mathfrak{t}})$ defined as a closure for $|||.|||$ is complete for that norm. In order to show that $\tilde{\mathfrak{t}}$ is closed in the Robin case as well, we need to show that there exists a constant C

Sesquilinear form $\mathfrak{t}$	Closed extension $\tilde{\mathfrak{t}}$	$D(\tilde{\mathfrak{t}})$
$\mathfrak{t}_{\mathrm{DD}}$	$\tilde{\mathfrak{t}}_{\mathrm{DD}}$,	$H_0^1(\$)$
$\mathfrak{t}_{\mathrm{DN}}$, $\mathfrak{t}_{\mathrm{DR}}$	$\tilde{\mathfrak{t}}_{\mathrm{DN}}$, $\tilde{\mathfrak{t}}_{\mathrm{DR}}$	$H_{0^+}^1(\$)$
$\mathfrak{t}_{\mathrm{ND}}$, $\mathfrak{t}_{\mathrm{RD}}$	$\tilde{\mathfrak{t}}_{\mathrm{ND}}$, $\tilde{\mathfrak{t}}_{\mathrm{RD}}$	$H_{0^-}^1(\$)$
$\mathfrak{t}_{\mathrm{NN}}$, $\mathfrak{t}_{\mathrm{RN}}$, $\mathfrak{t}_{\mathrm{NR}}$, $\mathfrak{t}_{\mathrm{RR}}$	$\tilde{\mathfrak{t}}_{\mathrm{NN}}$, $\tilde{\mathfrak{t}}_{\mathrm{RN}}$, $\tilde{\mathfrak{t}}_{\mathrm{NR}}$, $\tilde{\mathfrak{t}}_{\mathrm{RR}}$	$H^1(\$)$

Table 3.4: Domain of the closed extension $\tilde{\mathfrak{t}}$ for boundary conditions of type I

such that $\|.\|_{\tilde{\mathfrak{t}}} \leqslant C \,|||\,.|||$. This proof will be omitted at this point, but will be shown to be a direct consequence of the analysis in Subsection 4.3, as will be explained in Remark 4.9. This implies that the norms $\|.\|_{\tilde{\mathfrak{t}}}$ and $|||.|||$ are equivalent and hence it is clear that $\tilde{\mathfrak{t}}$ is the closure of $\mathfrak{t}$. Now, by theorem 3.3, the closed sesquilinear form $\tilde{\mathfrak{t}}$ gives rise to a non-negative self-adjoint operator $\tilde{T}$ with domain $\mathrm{Dom}(\tilde{T})$ defined by

$$\mathrm{Dom}(\tilde{T}) \quad = \quad \left\{u \in D(\tilde{\mathfrak{t}}) \quad s.t. \quad \exists h \in \boldsymbol{H} \quad s.t. \quad \forall v \in D(\tilde{\mathfrak{t}}), \quad \tilde{\mathfrak{t}}(u,v) = \langle h, v\rangle_{L^2}\right\}$$

and such that for any $u, v \in \mathrm{Dom}(\tilde{T})$, we have $\langle \tilde{T}(u), v\rangle_{L^2} = \tilde{\mathfrak{t}}(u,v)$. It is easy to see that $\tilde{T}$ is an extension of T. We shall refer to $\tilde{T}$ as the Friedrich's extension of T, and when referring to the Laplace-Beltrami Operator with boundary conditions of type I, we will from now on mean $\tilde{T}$.

Theorem 3.6. For boundary conditions of type I, the self-adjoint Laplace-Beltrami operator $\tilde{T}$ has an empty essential spectrum and a discrete set of real positive eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}^*}$ enumerated in increasing order and repeated in accordance with multiplicity. In particular, if $\{\lambda_n^{\mathrm{DD}}\}_{n\in\mathbb{N}^*}$ and $\{\lambda_n^{\mathrm{NN}}\}_{n\in\mathbb{N}^*}$ are the corresponding eigenvalues for the pure Dirichlet and pure Neumann operators $\tilde{T}_{\mathrm{DD}}$ and $\tilde{T}_{\mathrm{NN}}$, then we have that

$$\forall n \in \mathbb{N}^*, \quad \lambda_n^{\mathrm{NN}} \leqslant \lambda_n \leqslant \lambda_n^{\mathrm{DD}}\ , \tag{3.1}$$

where $\mathbb{N}^*$ is the set of non-zero natural numbers.

Proof. Let $\tilde{T}$ be the non-negative self-adjoint Laplace-Beltrami operator with some boundary condition of type I and let $\tilde{\mathfrak{t}}$ its associated closed sesquilinear form.

Let us start by using the fact that in (**31**) it has been shown that the pure Neumann operator $\tilde{T}_{\mathrm{NN}}$ has eigenvalues (and hence also the associated Λ_n, denoted Λ_n^{NN}) that tend to infinity and so by Theorem 3.4 $\tilde{T}_{\mathrm{NN}}$ has compact resolvent and a discrete set of real positive eigenvalues $\{\lambda_n^{\mathrm{NN}}\}_{n\in\mathbb{N}^*}$ enumerated in increasing order and repeated in accordance with multiplicity.

Now let us remark that we have $D(\tilde{\mathfrak{t}}_{\mathrm{RN,NR,RR}}) = D(\tilde{\mathfrak{t}}_{\mathrm{NN}})$ (see table 3.4), and that for any $u \in D(\tilde{\mathfrak{t}}_{\mathrm{NN}})$ we have $\tilde{\mathfrak{t}}_{\mathrm{RN,NR,RR}}(u,u) > \tilde{\mathfrak{t}}_{\mathrm{NN}}(u,u)$ (see table 3.3, and the fact that $\alpha^{\pm} > 0$), hence by Theorem 3.5 *(ii)* we have for any $n \geqslant 1$ that

$$\Lambda_n^{\mathrm{RN,NR,RR}} \quad > \quad \Lambda_n^{\mathrm{NN}}. \tag{3.2}$$

Note also that $D(\tilde{\mathfrak{t}}_{\mathrm{DD,DN,ND}}) \subset D(\tilde{\mathfrak{t}}_{\mathrm{NN}})$ (see Table 3.4) and that $\tilde{\mathfrak{t}}_{\mathrm{NN}}|_{D(\tilde{\mathfrak{t}}_{\mathrm{DD,DN,ND}})} =$

$\tilde{\mathfrak{t}}_{\text{DD,DN,ND}}$ (see Table 3.3). Hence, by Theorem 3.5 *(i)*, we have for any $n \geqslant 1$ that

$$\Lambda_n^{\text{DD,DN,ND}} \quad \geqslant \quad \Lambda_n^{\text{NN}}. \tag{3.3}$$

The last two cases remaining are treated in a similar way by noting that $D(\tilde{\mathfrak{t}}_{\text{RD}}) = D(\tilde{\mathfrak{t}}_{\text{ND}})$ and $D(\tilde{\mathfrak{t}}_{\text{DR}}) = D(\tilde{\mathfrak{t}}_{\text{DN}})$ (see Table 3.4) and that for any $u \in D(\tilde{\mathfrak{t}}_{\text{DN}})$ and $v \in D(\tilde{\mathfrak{t}}_{\text{ND}})$, we have $\tilde{\mathfrak{t}}_{\text{DR}}(u,u) > \tilde{\mathfrak{t}}_{\text{DN}}(u,u)$ and $\tilde{\mathfrak{t}}_{\text{RD}}(v,v) > \tilde{\mathfrak{t}}_{\text{ND}}(v,v)$ (see Table 3.3). Hence, by Theorem 3.5 *(ii)*, we have for any $n \geqslant 1$ that $\Lambda_n^{\text{RD}} > \Lambda_n^{\text{ND}}$ and $\Lambda_n^{\text{DR}} > \Lambda_n^{\text{DN}}$ and so by using (3.3), we have that

$$\Lambda_n^{\text{RD}} > \Lambda_n^{\text{NN}} \quad \text{and} \quad \Lambda_n^{\text{DR}} > \Lambda_n^{\text{NN}} \ . \tag{3.4}$$

All the different cases have been treated and so, using the fact that $\Lambda_n^{\text{NN}} \to \infty$ as $n \to \infty$ and the inequalities (3.2),(3.3) and (3.4), we have that $\Lambda_n \to \infty$ as $n \to \infty$ and so by Theorem 3.4, $\tilde{T}$ has compact resolvent and a discrete set of real positive eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}^*}$ enumerated in increasing order and repeated in accordance with multiplicity corresponding exactly to $\{\Lambda_n\}_{n\in\mathbb{N}^*}$. From the inequalities (3.2),(3.3) and (3.4), we also know that for any $n \geqslant 1$

$$\lambda_n^{\text{NN}} \quad \leqslant \quad \lambda_n \ ,$$

and that along the way we have proved that $\lambda_n^{\text{RD}} > \lambda_n^{\text{ND}}$ and $\lambda_n^{\text{DR}} > \Lambda_n^{\text{DN}}$.

In order to conclude the proof, we just need to show the second part of the inequality (3.1), and we shall proceed in a similar way. In order to do that, let us note that $D(\tilde{\mathfrak{t}}_{\text{DD}}) \subset D(\tilde{\mathfrak{t}})$ (see Table 3.4) and that moreover we have $\tilde{\mathfrak{t}}\Big|_{D(\tilde{\mathfrak{t}}_{\text{DD}})} = \tilde{\mathfrak{t}}_{\text{DD}}$. This comes from the fact that the line integrals that may occur in the definition of $\tilde{\mathfrak{t}}$ (see Table 3.3) are always zero when $u, v \in D(\tilde{\mathfrak{t}}_{\text{DD}}) = H_0^1(\$)$. Hence, by Theorem 3.5 (i), we have that

$$\lambda_n \quad \leqslant \quad \lambda_n^{\text{DD}},$$

which completes the proof. Note also that with the same reasoning, and realising that $D(\tilde{\mathfrak{t}}_{\text{DN}}) \subset D(\tilde{\mathfrak{t}}_{\text{RN}})$ and $\tilde{\mathfrak{t}}_{\text{RN}}|_{D(\tilde{\mathfrak{t}}_{\text{DN}})} = \tilde{\mathfrak{t}}_{\text{DN}}$, we can show that $\lambda_n^{\text{RN}} \leqslant \lambda_n^{\text{DN}}$. Similarly, we have that $\lambda_n^{\text{NR}} \leqslant \lambda_n^{\text{ND}}$. Moreover, using the fact that $D(\tilde{\mathfrak{t}}_{\text{RR}}) = D(\tilde{\mathfrak{t}}_{\text{RN}})$ and $\tilde{\mathfrak{t}}_{\text{RR}}(u,u) > \tilde{\mathfrak{t}}_{\text{RN}}(u,u)$, we can conclude by Theorem 3.5 (ii) that $\lambda_n^{\text{RR}} > \lambda_n^{\text{RN}}$. Finally, using the fact that $D(\tilde{\mathfrak{t}}_{\text{RD}}) \subset D(\tilde{\mathfrak{t}}_{\text{RR}})$ and $\tilde{\mathfrak{t}}_{\text{RR}}|_{D(\tilde{\mathfrak{t}}_{\text{RD}})} = \tilde{\mathfrak{t}}_{\text{RD}}$, we can conclude by Theorem 3.5 (i) that $\lambda_n^{\text{RR}} \leqslant \lambda_n^{\text{RD}}$.

And so we can summarise our findings by

$$\lambda_n^{\text{NN}} < \left\{ \begin{array}{c} \lambda_n^{\text{RN}} \\ \lambda_n^{\text{NR}} \end{array} \right\} \leqslant \left\{ \begin{array}{c} \lambda_n^{\text{DN}} \\ \lambda_n^{\text{ND}} \end{array} \right\} < \left\{ \begin{array}{c} \lambda_n^{\text{DR}} \\ \lambda_n^{\text{RD}} \end{array} \right\} \leqslant \lambda_n^{\text{DD}} \quad \text{and} \quad \left\{ \begin{array}{c} \lambda_n^{\text{RN}} \\ \lambda_n^{\text{NR}} \end{array} \right\} < \lambda_n^{\text{RR}} \leqslant \left\{ \begin{array}{c} \lambda_n^{\text{DR}} \\ \lambda_n^{\text{RD}} \end{array} \right\}. \tag{3.5}$$

Remark 3.7. Note of course that the eigenvalues related to the cases with Robin boundary conditions do depend on α^+ and α^-, and with the exact same method as that used in the proof of 3.6, using Theorem 3.5 (ii), we can show that $\lambda_n^{\text{RN}}(\alpha^+)$ and $\lambda_n^{\text{RD}}(\alpha^+)$ are strictly increasing functions of α^+, that $\lambda_n^{\text{NR}}(\alpha^-)$ and $\lambda_n^{\text{DR}}(\alpha^-)$ are strictly increasing functions of

α^- and that $\lambda_n^{\mathrm{RR}}(\alpha^+,\alpha^-)$ is a "strictly increasing function of α^+ and α^-" in the sense that if $\alpha_1^+ < \alpha_2^+$ and $\alpha_1^- < \alpha_2^-$, then $\lambda_n^{\mathrm{RR}}(\alpha_1^+,\alpha_1^-) < \lambda_n^{\mathrm{RR}}(\alpha_2^+,\alpha_2^-)$. It results from the boundedness and the strict increasing character of these eigenvalues, that they converge to a finite limit as $\alpha^\pm \to 0$ or $\alpha^\pm \to \infty$. And that we have

$$\lim_{\alpha^+\to 0} \lambda_n^{\mathrm{RN}}(\alpha^+) = \lambda_n^{\mathrm{NN}}, \quad \lim_{\alpha^-\to 0} \lambda_n^{\mathrm{NR}}(\alpha^-) = \lambda_n^{\mathrm{NN}}, \quad \lim_{\alpha^\pm\to 0} \lambda_n^{\mathrm{RR}}(\alpha^+,\alpha^-) = \lambda_n^{\mathrm{NN}},$$

as can be verified by assuming† continuity of the eigenvalues as functions of $\alpha^\pm$ and setting $\alpha^\pm = 0$. We can also infer that

$$\lim_{\alpha^+\to \infty} \lambda_n^{\mathrm{RN}}(\alpha^+) = \lambda_n^{\mathrm{DN}}, \quad \lim_{\alpha^-\to \infty} \lambda_n^{\mathrm{NR}}(\alpha^-) = \lambda_n^{\mathrm{ND}}, \quad \lim_{\alpha^\pm\to \infty} \lambda_n^{\mathrm{RR}}(\alpha^+,\alpha^-) = \lambda_n^{\mathrm{DD}}.$$

in a similar way. For $\alpha^\pm \neq 0$, divide the Robin boundary condition through by $\alpha^\pm$ to get a condition of the type $\boldsymbol{n}^\pm \cdot \nabla u/\alpha^\pm + u = 0$. The equality can then be verified by assuming continuity of the eigenvalues as functions of $1/\alpha^\pm$ and setting $1/\alpha^\pm = 0$.

4. Non-self-adjoint operators and boundary conditions of type II

4.1 *Breakdown of self-adjointness for boundary conditions of type II*

Let us now consider the same problem as in the previous section, but with a different type of boundary conditions.

Definition 4.1. A suitably smooth function on $\mathbb{S}$ is said to satisfy boundary conditions of type II if it satisfies the complex Robin boundary condition on at least one of S^+ or S^-:

$$\boldsymbol{n}^\pm \cdot \tilde{\nabla} u + \alpha^\pm u = 0 \quad \text{s.t.} \quad \alpha^\pm \in \mathbb{C} \text{ and } \Im(\alpha^\pm) \neq 0$$

The condition on the other face can either be of type I or type II.

Remark 4.2. Note that to be physically relevant, as mentioned at the end of Subsection 2.3, we should have $\Im(\alpha) \leqslant 0$. However, mathematically, the sign of $\Im(\alpha)$ is not relevant to obtain the eigenvalues. In fact if (u_1, λ_1) is solution to the eigenvalue problem associated with $\alpha_1^\pm$, then $(u_2,\lambda_2) = (\overline{u_1}, \bar{\lambda}_1)$ is solution to the eigenvalue problem associated with $\alpha_2^\pm = \overline{\alpha_1^\pm}$. Hence we can restrict our study to $\Im(\alpha) > 0$, the cases with negative imaginary part can be directly obtained by symmetry.

Consider the linear operator H acting in the Hilbert space $(L^2(\mathbb{S}), \langle .,. \rangle_{L^2}, \|.\|_{L^2})$ and formally defined by $H = -\tilde{\Delta}$ as in (2.5) with boundary conditions of type II. Let H be initially defined on the domain $\mathrm{Dom}(H)$ defined by

$$\mathrm{Dom}(H) \;=\; \left\{ f \in C^\infty(\bar{\mathbb{S}}), \quad \text{s.t. } f \text{ satisfies the correct boundary conditions of type II} \right\}. \tag{4.1}$$

Note that once again $\mathrm{Dom}(H)$ is dense in $L^2(\mathbb{S})$ and so H is a well defined linear operator. The appropriate notation for each possible combination of boundary condition of type II is specified in in Table 4.1.

As in Subsection 3.3, we can use Green's identity on $\mathbb{S}$ in order to obtain expressions for $\langle H(u), v\rangle_{L^2}$ and $\langle u, H(v)\rangle_{L^2}$, a summary of which is displayed in Table 4.2.

† Here, for brevity, we do not justify this assumption, however, in order to do so, one could use the same arguments of perturpation theory as those developed in a more complicated case in Appendix B.

Boundary Condition on S^+	Boundary Condition on S^-	Operator H
Complex Robin	Dirichlet	H_{RD}
Dirichlet	Complex Robin	H_{DR}
Complex Robin	Neumann	H_{RN}
Neumann	Complex Robin	H_{NR}
Complex Robin	Complex Robin	H_{RR}

Table 4.1: The operator H for boundary conditions of type II

Operator H	$\langle H(u), v\rangle_{L^2} =$	$\langle u, H(v)\rangle_{L^2} =$
H_{RD}, H_{RN}	$\langle\tilde{\nabla}u, \tilde{\nabla}v\rangle_{L^2} + \alpha^+ \int_{S^+} u\bar{v}\,\mathrm{d}\ell$	$\langle\tilde{\nabla}u, \tilde{\nabla}v\rangle_{L^2} + \overline{\alpha^+} \int_{S^+} u\bar{v}\,\mathrm{d}\ell$
H_{DR}, H_{NR}	$\langle\tilde{\nabla}u, \tilde{\nabla}v\rangle_{L^2} + \alpha^- \int_{S^-} u\bar{v}\,\mathrm{d}\ell$	$\langle\tilde{\nabla}u, \tilde{\nabla}v\rangle_{L^2} + \overline{\alpha^-} \int_{S^-} u\bar{v}\,\mathrm{d}\ell$
H_{RR}	$\langle\tilde{\nabla}u, \tilde{\nabla}v\rangle_{L^2} + \alpha^+ \int_{S^+} u\bar{v}\,\mathrm{d}\ell + \alpha^- \int_{S^-} u\bar{v}\,\mathrm{d}\ell$	$\langle\tilde{\nabla}u, \tilde{\nabla}v\rangle_{L^2} + \overline{\alpha^+} \int_{S^+} u\bar{v}\,\mathrm{d}\ell + \overline{\alpha^-} \int_{S^-} u\bar{v}\,\mathrm{d}\ell$

Table 4.2: Green's identity applied to H for boundary conditions of type II

Note however that this time we have $\alpha^{\pm} \neq \overline{\alpha^{\pm}}$ and so, as such, we have $\langle H(u), v\rangle_{L^2} \neq \langle u, H(v)\rangle_{L^2}$ and the operator H is not symmetric, and hence it will not be possible to find a self-adjoint extension of H. Note that we also lose the non-negative property here since $\langle H(u), u\rangle_{L^2}$ is not even real. Hence the theoretical framework of the previous section cannot be used in this case and a new approach must be taken.

4.2 *Theoretical framework*

Let us consider again a densely defined linear operator H on a Hilbert space $\boldsymbol{H}$. The numerical range of H, $\Theta(H)$, is defined by $\Theta(H) = \{\langle H(f), f\rangle \text{ s.t. } f \in \mathrm{Dom}(H) \text{ and } \|f\|_{\boldsymbol{H}} = 1\}$. We say that H is m-accretive if $\mathrm{Re}(\Theta(H)) \geqslant 0$ and 1 belongs to $\rho(-H)$. We say that H is quasi-m-accretive if there exists $\alpha \in \mathbb{R}$ such that $H + \alpha$ is m-accretive. H is said to be sectorial with vertex $\gamma \in \mathbb{R}$ and semi-angle ϑ, if its numerical range is included in a sector of the complex plane with vertex $\gamma \in \mathbb{R}$ and semi-angle $\vartheta \in [0, \pi/2]$, as illustrated in Figure 4.1. Finally, we say that a linear operator H is m-sectorial if it is sectorial and quasi-m-accretive.

In order to get some information about the spectrum of a non-self-adjoint linear operator, it is, once again, very useful to think in terms of sesquilinear forms. The numerical range of a sesquilinear form $\mathfrak{h}$ is defined by $\Theta(\mathfrak{h}) = \{\mathfrak{h}(u, u) \text{ s.t. } u \in D(\mathfrak{h}) \text{ and } \|u\|_{\boldsymbol{H}} = 1\}$. We say that $\mathfrak{h}$ is accretive if $\mathrm{Re}(\Theta(\mathfrak{h})) \geqslant 0$ and sectorial with vertex $\gamma \in \mathbb{R}$ and semi-angle ϑ if its numerical range is included in a sector of the complex plane with vertex $\gamma \in \mathbb{R}$ and semi-angle $\vartheta \in [0, \pi/2]$. We say that a sectorial sesquilinear form $\mathfrak{h}$ is closed if the normed space $(D(\mathfrak{h}), \|\cdot\|_{\mathfrak{h},\gamma})$ is complete, where $\|\cdot\|_{\mathfrak{h},\gamma}$ is the norm associated to $\mathfrak{h}$ defined for $u \in D(\mathfrak{h})$ by

$$\|u\|_{\mathfrak{h},\gamma} \quad = \quad (\Re(\mathfrak{h}(u, u)) - \gamma + \|u\|^2_{\boldsymbol{H}})^{1/2} \;.$$

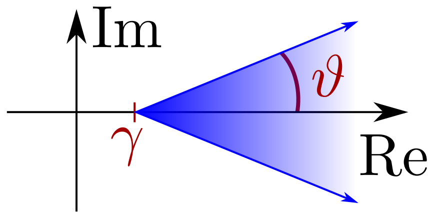

Fig. 4.1: Typical sector of the complex plane with vertex γ and semi-angle ϑ

If this is the case, $(D(\mathfrak{h}), \|\cdot\|_{\mathfrak{h},\gamma})$ is called the Hilbert space associated to $\mathfrak{h}$. A sectorial sesquilinear form $\mathfrak{h}$ is said to be closable if it admits a closed extension. The smallest closed extension of a closable form $\mathfrak{h}$ is called the closure of $\mathfrak{h}$.

Let us now present three important results from (**36**), that we will use in this paper to link m-sectorial linear operators and sectorial sesquilinear forms

THEOREM 4.3. *[Thm. VI.2.1 in (**36**)]* Let $\mathfrak{h}$ be a densely defined, closed, sectorial sesquilinear form in $\boldsymbol{H}$. Then there exists an m-sectorial linear operator H such that

1. $\mathrm{Dom}(H) \subset D(\mathfrak{h})$ and $\forall u \in \mathrm{Dom}(H), \forall v \in D(\mathfrak{h}), \mathfrak{h}(u,v) = \langle H(u), v\rangle$
2. The resulting operator H is uniquely determined by this construction

Theorem 4.3 is important in the sense that it allows us to associate an m-sectorial operator to a given (closed and sectorial) sesquilinear form. The following theorem makes use of this association and will prove extremely important when trying to describe the non-self-adjoint operators arising in Subsection 4.3.

THEOREM 4.4. *[Thm. VI.3.4 in (**36**)]* Let $\mathfrak{h}_1$ be a densely defined, closed, accretive and sectorial sesquilinear form. Let H_1 be the m-sectorial linear operator associated to $\mathfrak{h}_1$. Let $\mathfrak{h}_2$ be a sesquilinear form bounded with respect to $\mathfrak{h}_1$ in the sense that $D(\mathfrak{h}_1) \subset D(\mathfrak{h}_2)$ and for $u \in D(\mathfrak{h}_1)$, we have $|\mathfrak{h}_2(u,u)| \leqslant A\|u\|^2_{\boldsymbol{H}} + B\Re(\mathfrak{h}_1(u,u))$, where A and B are non-negative real numbers and $B < 1$.

Then the sesquilinear form $\mathfrak{h}$ defined by $\mathfrak{h} = \mathfrak{h}_1 + \mathfrak{h}_2$ is also closed and sectorial. Let H be the m-sectorial operator associated to $\mathfrak{h}$. If H_1 has compact resolvent, then H has compact resolvent too.

REMARK 4.5. At this stage, it is useful to remark that from the definitions it is clear that a non-negative self-adjoint operator with compact resolvent is an m-sectorial operator and a closed non-negative sesquilinear form is a closed, accretive and sectorial sesquilinear form.

THEOREM 4.6. *[Thm. III.6.29 in (**36**)]* Let H be an m-sectorial operator with compact resolvent. Then the spectrum of H, $\sigma(H)$, consists entirely of isolated eigenvalues with finite multiplicities and is included in its numerical range, i.e. $\sigma(H) \subset \Theta(H)$. In order to avoid confusion with the eigenvalues described in Subsection 3.3, we shall denote these eigenvalues $\{\mu_n\}_{n\in\mathbb{N}^*}$.

Proof. The last part of this theorem, i.e. $\sigma(H) \subset \Theta(H)$, is not directly included in Theorem 6.29 p187 in (**36**), however it is relatively straightforward. Indeed, let $\mu \in \sigma(H)$, and u a corresponding eigenvector in Dom(H). We have $H(u) = \mu u$ and so $\langle H(u), u\rangle_{\boldsymbol{H}} = \mu\|u\|^2_{\boldsymbol{H}}$. Now let $v = u/\|u\|_{\boldsymbol{H}}$. It is clear that $v \in$ Dom(H), that $\|v\|_{\boldsymbol{H}} = 1$ and that $\langle H(v), v\rangle_{\boldsymbol{H}} = \mu$. Hence, by definition of the numerical range, $\mu \in \Theta(H)$.

4.3 *The LBO as m-sectorial operator for boundary conditions of type II*

Note that the operator H defined in Subsection 4.1 for boundary conditions of type II gives rise to a densely defined sesquilinear form $\mathfrak{h}$ with $D(\mathfrak{h}) = \mathrm{Dom}(H)$ defined in table 4.3 for each type of boundary conditions.

Sesquilinear form $\mathfrak{h}$	$\mathfrak{h}(u,v)$	Sesquilinear form $\tilde{\mathfrak{h}}_2$	$\tilde{\mathfrak{h}}_2(u,v)$
$\mathfrak{h}_{\mathrm{RD}}$, $\mathfrak{h}_{\mathrm{RN}}$	$\langle\tilde{\nabla}u,\tilde{\nabla}v\rangle_{L^2} + \alpha^+\int_{S^+} u\bar{v}\,\mathrm{d}\ell$	$\tilde{\mathfrak{h}}_2^{\mathrm{DR}}$, $\tilde{\mathfrak{h}}_2^{\mathrm{NR}}$	$\alpha^+\int_{S^+} u\bar{v}\,\mathrm{d}\ell$
$\mathfrak{h}_{\mathrm{DR}}$, $\mathfrak{h}_{\mathrm{NR}}$	$\langle\tilde{\nabla}u,\tilde{\nabla}v\rangle_{L^2} + \alpha^-\int_{S^-} u\bar{v}\,\mathrm{d}\ell$	$\tilde{\mathfrak{h}}_2^{\mathrm{RD}}$, $\tilde{\mathfrak{h}}_2^{\mathrm{RN}}$	$\alpha^-\int_{S^-} u\bar{v}\,\mathrm{d}\ell$
$\mathfrak{h}_{\mathrm{RR}}$	$\langle\tilde{\nabla}u,\tilde{\nabla}v\rangle_{L^2} + \alpha^+\int_{S^+} u\bar{v}\,\mathrm{d}\ell + \alpha^-\int_{S^-} u\bar{v}\,\mathrm{d}\ell$	$\tilde{\mathfrak{h}}_2^{\mathrm{RR}}$	$\alpha^+\int_{S^+} u\bar{v}\,\mathrm{d}\ell + \alpha^-\int_{S^-} u\bar{v}\,\mathrm{d}\ell$

Table 4.3: Expression of the sesquilinear form $\mathfrak{h}$ arising from H and the sesquilinear form $\tilde{\mathfrak{h}}_2$ for boundary conditions of type II.

Let us now consider the extension $\tilde{\mathfrak{h}}$ of $\mathfrak{h}$ defined as $\mathfrak{h}$ in Table 4.3, but with a bigger domain $D(\tilde{\mathfrak{h}})$ that contains $D(\mathfrak{h})$ and given in Table 4.4.

Sesquilinear form $\mathfrak{h}$	Extension $\tilde{\mathfrak{h}}$	$D(\tilde{\mathfrak{h}})$
$\mathfrak{h}_{\mathrm{DR}}$	$\tilde{\mathfrak{h}}_{\mathrm{DR}}$	$H^1_{0^+}(\$)$
$\mathfrak{h}_{\mathrm{RD}}$	$\tilde{\mathfrak{h}}_{\mathrm{RD}}$	$H^1_{0^-}(\$)$
$\mathfrak{h}_{\mathrm{RN}}$, $\mathfrak{h}_{\mathrm{NR}}$, $\mathfrak{h}_{\mathrm{RR}}$	$\tilde{\mathfrak{h}}_{\mathrm{RN}}$, $\tilde{\mathfrak{h}}_{\mathrm{NR}}$, $\tilde{\mathfrak{h}}_{\mathrm{RR}}$	$H^1(\$)$

Table 4.4: Domain of the extension $\tilde{\mathfrak{h}}$ for boundary conditions of type II

It is now possible to define two new sesquilinear forms $\tilde{\mathfrak{h}}_1$ and $\tilde{\mathfrak{h}}_2$ with $D(\tilde{\mathfrak{h}}_1) = D(\tilde{\mathfrak{h}}_2) = D(\tilde{\mathfrak{h}})$, such that

$$\forall u, v \in D(\tilde{\mathfrak{h}}), \quad \tilde{\mathfrak{h}}_1(u,v) = \langle\tilde{\nabla}u,\tilde{\nabla}v\rangle_{L^2} \quad \text{and} \quad \tilde{\mathfrak{h}}(u,v) = \tilde{\mathfrak{h}}_1(u,v) + \tilde{\mathfrak{h}}_2(u,v)$$

The exact expression of $\tilde{\mathfrak{h}}_2(u,v)$ is given in table 4.3 for different boundary conditions of type II. The crucial point at this stage is to realise that $\tilde{\mathfrak{h}}_1$, by its definition and its domain $D(\tilde{\mathfrak{h}}_1)$ corresponds exactly to one of the sesquilinear forms $\tilde{\mathfrak{t}}_{\mathrm{DN}}$, $\tilde{\mathfrak{t}}_{\mathrm{ND}}$ or $\tilde{\mathfrak{t}}_{\mathrm{NN}}$ defined in Subsection 3.3, as specified in Table 4.5. But we know from Subsection 3.3 that $\tilde{\mathfrak{t}}_{\mathrm{DN}}$, $\tilde{\mathfrak{t}}_{\mathrm{ND}}$ and $\tilde{\mathfrak{t}}_{\mathrm{NN}}$ are closed and non-negative sesquilinear forms, and so, by Remark 4.5, they are also closed, accretive and sectorial forms and hence, so is $\tilde{\mathfrak{h}}_1$. Hence we can conclude that $\tilde{\mathfrak{h}}_1$ is associated uniquely to an m-sectorial linear operator $\tilde{H}_1$ in the sense of Theorem 4.3. Moreover, we know from Subsection 3.3 that $\tilde{\mathfrak{t}}_{\mathrm{DN}}$, $\tilde{\mathfrak{t}}_{\mathrm{ND}}$ and $\tilde{\mathfrak{t}}_{\mathrm{NN}}$ are associated in the same way to the non-negative self-adjoint operators with compact resolvent $\tilde{T}_{\mathrm{DN}}$, $\tilde{T}_{\mathrm{ND}}$ and

$\tilde{T}_{\text{NN}}$. By Remark 4.5, we know that $\tilde{T}_{\text{DN}}$, $\tilde{T}_{\text{ND}}$ and $\tilde{T}_{\text{NN}}$ are also m-sectorial, and so, by uniqueness, $\tilde{H}_1$ is exactly equal to one of these operators as specified in Table 4.5 and has compact resolvent.

Sesq. Form $\tilde{\mathfrak{h}}_1$	Corresponding Sesq. Form $\tilde{\mathfrak{t}}$	Operator $\tilde{H}_1$	Corresponding Operator $\tilde{T}$
$\tilde{\mathfrak{h}}_1^{\text{DR}}$	$\tilde{\mathfrak{t}}_{\text{DN}}$	$\tilde{H}_1^{\text{DR}}$	$\tilde{T}_{\text{DN}}$
$\tilde{\mathfrak{h}}_1^{\text{RD}}$	$\tilde{\mathfrak{t}}_{\text{ND}}$	$\tilde{H}_1^{\text{RD}}$	$\tilde{T}_{\text{ND}}$
$\tilde{\mathfrak{h}}_1^{\text{RN}}$, $\tilde{\mathfrak{h}}_1^{\text{NR}}$, $\tilde{\mathfrak{h}}_1^{\text{RR}}$	$\tilde{\mathfrak{t}}_{\text{NN}}$	$\tilde{H}_1^{\text{RN}}$, $\tilde{H}_1^{\text{NR}}$, $\tilde{H}_1^{\text{RR}}$	$\tilde{T}_{\text{NN}}$

Table 4.5: Correspondence between $\tilde{\mathfrak{h}}_1$ and $\tilde{\mathfrak{t}}$ as well as between $\tilde{H}_1$ and $\tilde{T}$

We are now almost in a position to apply the Theorem 4.4 to $\tilde{\mathfrak{h}} = \tilde{\mathfrak{h}}_1 + \tilde{\mathfrak{h}}_2$. The last remaining hypothesis that we need to verify is that $\tilde{\mathfrak{h}}_2$ is bounded with respect to $\tilde{\mathfrak{h}}_1$.

LEMMA 4.7. $\tilde{\mathfrak{h}}_2$ is bounded with respect to $\tilde{\mathfrak{h}}_1$

Proof. We will focus on the most general case of Robin conditions on both sides of the cut (the other boundary conditions of type II can be treated in exactly the same way), so in this proof we shall focus only on $\tilde{\mathfrak{h}}_2^{\text{RR}}$ and $\tilde{\mathfrak{h}}_1^{\text{RR}}$ and will drop the superscript for the duration of the proof only. In order to prove that $\tilde{\mathfrak{h}}_2$ is bounded with respect to $\tilde{\mathfrak{h}}_1$, we need to show that there exists two positive real constants A and B, with $B<1$ such that

$$\begin{aligned} |\tilde{\mathfrak{h}}_2(u,u)| &\leqslant A\|u\|_{L^2}^2 + B\Re(\tilde{\mathfrak{h}}_1(u,u)) \\ &\leqslant A\|u\|_{L^2}^2 + B\|\tilde{\nabla}u\|_{L^2}^2, \end{aligned}$$

for u belonging to a dense subset of $D(\tilde{\mathfrak{h}}_1) = H^1(\$)$. Here we shall choose u to be in the dense subset $\text{Dom}(H)$ defined in (4.1). Before embarking into the proof, let us state an intermediate lemma deriving from ((**36**), Eqn. IV.(1.19)) and from the fact that for any real x and y, we have $(x+y)^2 \leqslant 2(x^2+y^2)$.

LEMMA 4.8. Let (a,b) be an open segment of $\mathbb{R}$ with $b>a$. Then for any $n>0$, any c in the closed segment $[a,b]$ and any $f \in C^\infty((a,b))$, we have:

$$|f(c)|^2 \leqslant \frac{2(b-a)}{2n+3}\int_a^b |f'(x)|^2 \mathrm{d}x + \frac{2(n+1)^2}{(b-a)(2n+1)}\int_a^b |f(x)|^2 \mathrm{d}x.$$

We are now well equipped to start the proof. Let us try to bound $\tilde{\mathfrak{h}}_2(u,u)$ by first noting that

$$\begin{aligned} |\tilde{\mathfrak{h}}_2(u,u)| &\leqslant |\alpha^+|\int_{\varphi=0}^{\pi/2}\left|u\left(\frac{\pi}{2}^-,\varphi\right)\right|^2 \mathrm{d}\varphi + |\alpha^-|\int_{\varphi=0}^{\pi/2}\left|u\left(\frac{\pi}{2}^+,\varphi\right)\right|^2 \mathrm{d}\varphi \\ &= |\alpha^+|I_1 + |\alpha^-|I_2, \end{aligned} \tag{4.2}$$

where

$$I_1 = \int_{\varphi=0}^{\pi/2}\left|u\left(\frac{\pi}{2}^-,\varphi\right)\right|^2 \mathrm{d}\varphi \quad \text{and} \quad I_2 = \int_{\varphi=0}^{\pi/2}\left|u\left(\frac{\pi}{2}^+,\varphi\right)\right|^2 \mathrm{d}\varphi.$$

Let us try to bound I_1 first. Making use of the lemma 4.8, by choosing $a = \pi/2 - L$, $b = \pi/2$ and $c = \pi/2$, for $0 < L < \pi/2$, we can show that

$$\left| u\left(\frac{\pi}{2}^{-}, \varphi\right)\right|^2 \leqslant \frac{2L}{(2n+3)} \int_{\theta=\pi/2-L}^{\pi/2} \left|\frac{\partial u}{\partial \theta}(\theta,\varphi)\right|^2 \mathrm{d}\theta + \frac{2(n+1)^2}{L(2n+1)} \int_{\theta=\pi/2-L}^{\pi/2} |u(\theta,\varphi)|^2 \,\mathrm{d}\theta. \tag{4.3}$$

Hence we can use (4.3) to show that

$$\begin{aligned} I_1 &\leqslant \frac{2L}{(2n+3)} \int_{\varphi=0}^{\pi/2} \int_{\theta=\pi/2-L}^{\pi/2} \left|\frac{\partial u}{\partial \theta}(\theta,\varphi)\right|^2 \mathrm{d}\theta\mathrm{d}\varphi + \frac{2(n+1)^2}{L(2n+1)} \int_{\varphi=0}^{\pi/2} \int_{\theta=\pi/2-L}^{\pi/2} |u(\theta,\varphi)|^2 \,\mathrm{d}\theta\mathrm{d}\varphi \\ &\leqslant \frac{2L}{(2n+3)} \int_{\varphi=0}^{\pi/2} \int_{\theta=\pi/2-L}^{\pi/2} |\tilde{\nabla} u(\theta,\varphi)|^2 \,\mathrm{d}\theta\mathrm{d}\varphi + \frac{2(n+1)^2}{L(2n+1)} \int_{\varphi=0}^{\pi/2} \int_{\theta=\pi/2-L}^{\pi/2} |u(\theta,\varphi)|^2 \,\mathrm{d}\theta\mathrm{d}\varphi. \end{aligned}$$

Now noting that when $\theta \in [\pi/2 - L, \pi/2]$, we have $\dfrac{\sin(\theta)}{\cos(L)} \geqslant 1$, we can deduce that

$$\begin{aligned} I_1 &\leqslant \frac{2L}{\cos(L)(2n+3)} \int_{\varphi=0}^{\pi/2} \int_{\theta=\pi/2-L}^{\pi/2} |\tilde{\nabla} u(\theta,\varphi)|^2 \sin(\theta) \,\mathrm{d}\theta\mathrm{d}\varphi \\ &\quad + \frac{2(n+1)^2}{\cos(L)L(2n+1)} \int_{\varphi=0}^{\pi/2} \int_{\theta=\pi/2-L}^{\pi/2} |u(\theta,\varphi)| \sin(\theta)\mathrm{d}\theta\mathrm{d}\varphi \\ &\leqslant \frac{2L}{\cos(L)(2n+3)} \iint_{\mathcal{R}^+} |\tilde{\nabla} u|^2 \,\mathrm{d}\$ + \frac{2(n+1)^2}{\cos(L)L(2n+1)} \iint_{\mathcal{R}^+} |u|^2 \mathrm{d}\$ \\ &\leqslant \frac{2L}{\cos(L)(2n+3)} \|\tilde{\nabla} u\|_{L^2}^2 + \frac{2(n+1)^2}{\cos(L)L(2n+1)} \|u\|_{L^2}^2, \end{aligned} \tag{4.4}$$

because the region $\mathcal{R}^+ = \{(\theta,\varphi) : \theta \in [\pi/2 - L, \pi/2] \text{ and } \varphi \in [0, \pi/2]\}$ described in Figure 4.2a is a subset of \$. It is useful to introduce the coefficient A_n and B_n defined by

$$B_n = B_n(L) = \frac{2L}{\cos(L)(2n+3)} \quad \text{and} \quad A_n = A_n(L) = \frac{2(n+1)^2}{\cos(L)L(2n+1)}\,. \tag{4.5}$$

Similarly, but this time using an intermediate region $\mathcal{R}^- = \{(\theta,\varphi) : \theta \in [\pi/2, \pi/2 + L] \text{ and } \varphi \in [0, \pi/2]\}$, also described in Figure 4.2b, and choosing $a = \pi/2$, $b = \pi/2 + L$ and $c = \pi/2$, we can also show that

$$I_2 \leqslant B_n(L)\|\tilde{\nabla} u\|_{L^2}^2 + A_n(L)\|u\|_{L^2}^2. \tag{4.6}$$

Hence, using (4.4) and (4.6) in (4.2), we obtain

$$|\tilde{\mathfrak{h}}_2(u,u)| \leqslant (|\alpha^+| + |\alpha^-|)B_n(L)\|\tilde{\nabla} u\|_{L^2}^2 + (|\alpha^+| + |\alpha^-|)A_n(L)\|u\|_{L^2}^2, \tag{4.7}$$

and note that we can always have $(|\alpha^+| + |\alpha^-|)B_n(L) < 1$ upon choosing n big enough. Hence by choosing

$$B = (|\alpha^+| + |\alpha^-|)B_n(L) \quad \text{and} \quad A = (|\alpha^+| + |\alpha^-|)A_n(L), \tag{4.8}$$

we have proved that $\tilde{\mathfrak{h}}_2$ is bounded with respect to $\tilde{\mathfrak{h}}_1$.

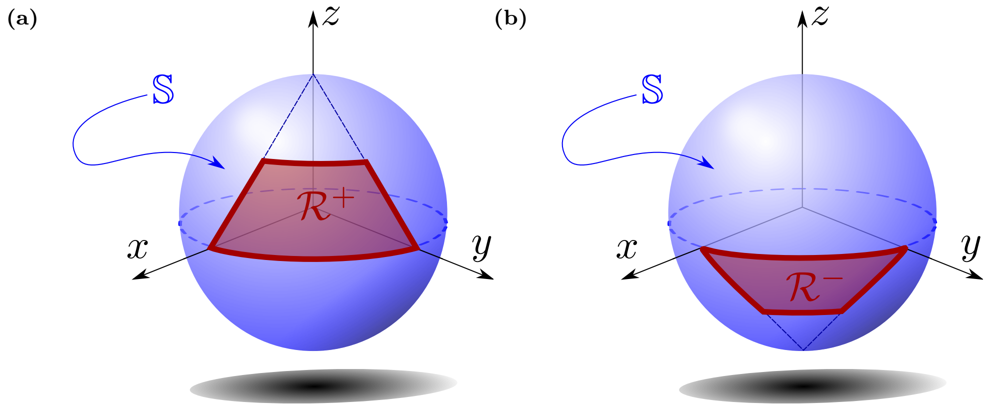

Fig. 4.2: (Colour online) Illustration of the intermediate domains $\mathcal{R}^+$ (a) and $\mathcal{R}^-$ (b)

REMARK 4.9. Note that if α^+ and α^- are real and positive as in Subsection 3.3, the inequality (4.7) implies that we have

$$\|u\|^2_{\tilde{\mathfrak{t}}_{RR}} \leqslant |||u|||^2 + B\|\tilde{\nabla} u\|^2_{L^2} + A\|u\|^2_{L^2} \leqslant (1+\max(A,B))\,|||u|||^2$$

and hence upon choosing $C = \sqrt{1+\max(A,B)}$, we have $\|.\|_{\tilde{\mathfrak{t}}_{RR}} \leqslant C\,|||.|||$, as requested in Subsection 3.3.

To summarise, we now know that $\tilde{\mathfrak{h}} = \tilde{\mathfrak{h}}_1 + \tilde{\mathfrak{h}}_2$, with $D(\tilde{\mathfrak{h}}) = D(\tilde{\mathfrak{h}}_1) = D(\tilde{\mathfrak{h}}_2)$, where $\tilde{\mathfrak{h}}_1$ is a densely defined closed accretive and sectorial sesquilinear form associated to a linear operator $\tilde{H}_1$ that is m-sectorial and has compact resolvent. Moreover the sesquilinear form $\tilde{\mathfrak{h}}_2$ is bounded with respect to $\tilde{\mathfrak{h}}_1$. Hence, we can apply the Theorem 4.4 to conclude that the sesquilinear form $\tilde{\mathfrak{h}}$ is sectorial and closed and is associated with an m-sectorial operator $\tilde{H}$ that has compact resolvent†. From now on, when referring to the Laplace-Beltrami Operator with boundary conditions of type II, we will mean $\tilde{H}$. This allows us to obtain the main result of this subsection:

THEOREM 4.10. For boundary conditions of type II, the spectrum of the Laplace-Beltrami operator $\tilde{H}$ consists entirely of isolated eigenvalues with finite multiplicities denoted $\{\mu_n\}_{n\in\mathbb{N}^*}$. Moreover, there exists $\gamma \in \mathbb{R}$ and $\vartheta \in [0,\pi/2]$ such that the eigenvalues are contained in the sector of the complex plane with vertex γ and semi-angle ϑ.

Proof. We know that $\tilde{H}$ is m-sectorial, so its numerical range $\Theta(\tilde{H})$ is contained in a sector of the complex plane with vertex $\gamma \in \mathbb{R}$ and semi-angle $\vartheta \in [0,\pi/2]$. Moreover, $\tilde{H}$ has compact resolvent and so by Theorem 4.6, its spectrum $\sigma(\tilde{H})$ consists entirely of isolated eigenvalues with finite multiplicities, and we have $\sigma(\tilde{H}) \subset \Theta(\tilde{H})$.

4.4 *Estimating an appropriate sector*

So far, we know that the eigenvalues of the LBO lie within a sector of the complex plane, but we do not have a constructive method of finding the characteristic of this sector. In

† In fact since we have shown in Subsection 3.3 that $\tilde{\mathfrak{h}}_1$ is the closure of $\mathfrak{h}_1$, the relatively boundedness of $\tilde{\mathfrak{h}}_2$ implies that $\tilde{\mathfrak{h}}$ is the closure of $\mathfrak{h}$ by Thm. VI.1.33 in (**36**).

order to do so, let us start by noting that, following exactly the same technique as that used in the proof of Lemma 4.7, we obtain

$$|\Re(\tilde{\mathfrak{h}}_2(u,u))| \leqslant A_R(B_n\|\tilde{\nabla}u\|^2_{L^2} + A_n\|u\|^2_{L^2}) \quad (4.9)$$
$$|\Im(\tilde{\mathfrak{h}}_2(u,u))| \leqslant A_I(B_n\|\tilde{\nabla}u\|^2_{L^2} + A_n\|u\|^2_{L^2}), \quad (4.10)$$

where

$$A_R = |\Re(\alpha^+)| + |\Re(\alpha^-)| \quad \text{and} \quad A_I = |\Im(\alpha^+)| + |\Im(\alpha^-)|. \quad (4.11)$$

Finding a pair (γ,ϑ) such that $\Theta(\tilde{\mathfrak{h}})$ is included in the sector with vertex γ and semi-angle ϑ is equivalent to finding a pair (γ,ϑ) such that we have

$$\Re(\tilde{\mathfrak{h}}(u,u)) \geqslant \gamma\|u\|^2_{L_2} \quad (4.12)$$
$$|\Im(\tilde{\mathfrak{h}}(u,u))| \leqslant \tan(\vartheta)(\Re(\tilde{\mathfrak{h}}(u,u)) - \gamma\|u\|^2_{L_2}). \quad (4.13)$$

In order to obtain an inequality of the type (4.12), we need to work with the real part of $\tilde{\mathfrak{h}}$. Let us proceed as follows:

$$\begin{aligned}
\Re(\tilde{\mathfrak{h}}(u,u)) &= \Re(\tilde{\mathfrak{h}}_1(u,u)) + \Re(\tilde{\mathfrak{h}}_2(u,u)) \\
&= \tilde{\mathfrak{h}}_1(u,u) + \Re(\tilde{\mathfrak{h}}_2(u,u)) \\
&= \|\tilde{\nabla}u\|^2_{L^2} + \Re(\tilde{\mathfrak{h}}_2(u,u)) \\
&\geqslant \|\tilde{\nabla}u\|^2_{L^2} - |\Re(\tilde{\mathfrak{h}}_2(u,u))| \\
&\underset{(4.9)}{\geqslant} \|\tilde{\nabla}u\|^2_{L^2} - A_R(B_n\|\tilde{\nabla}u\|^2_{L^2} + A_n\|u\|^2_{L^2}) \\
&\geqslant -A_RA_n\|u\|^2_{L^2} + (1 - A_RB_n)\|\tilde{\nabla}u\|^2_{L^2}. \quad (4.14)
\end{aligned}$$

Hence, by choosing γ to be

$$\gamma = -2A_RA_n, \quad (4.15)$$

and noting that $-A_RA_n = \gamma + A_RA_n$, (4.14) becomes

$$\Re(\tilde{\mathfrak{h}}(u,u)) \geqslant (\gamma + A_RA_n)\|u\|^2_{L^2} + (1 - A_RB_n)\|\tilde{\nabla}u\|^2_{L^2},$$

which implies that

$$\Re(\tilde{\mathfrak{h}}(u,u)) - \gamma\|u\|^2_{L^2} \geqslant A_RA_n\|u\|^2_{L^2} + (1 - A_RB_n)\|\tilde{\nabla}u\|^2_{L^2}. \quad (4.16)$$

Hence, provided that the following condition is satisfied

$$\textbf{Cond1} \ : \ (1 - A_RB_n) \geqslant 0, \quad (4.17)$$

we are sure that (4.12) holds. Let us now try to find an appropriate ϑ that will make (4.13) hold. Note that from (4.16)we have

$$A_RA_n\|u\|^2_{L^2} \leqslant \Re(\tilde{\mathfrak{h}}(u,u)) - \gamma\|u\|^2 - (1 - A_RB_n)\|\tilde{\nabla}u\|^2_{L^2}\ . \quad (4.18)$$

Now, we shall assume that $A_R \neq 0$ and so we have

$$\begin{aligned}
|\Im(\tilde{\mathfrak{h}}(u,u))| &= |\Im(\tilde{\mathfrak{h}}_2(u,u))| \\
&\underset{(4.10)}{\leqslant} A_I(B_n\|\tilde{\nabla}u\|^2_{L^2} + A_n\|u\|^2_{L^2}) \\
&\leqslant \frac{A_I}{A_R}(A_RB_n\|\tilde{\nabla}u\|^2_{L^2} + A_RA_n\|u\|^2_{L^2}) \\
&\underset{(4.18)}{\leqslant} \frac{A_I}{A_R}(A_RB_n\|\tilde{\nabla}u\|^2_{L^2} + \Re(\tilde{\mathfrak{h}}(u,u)) - \gamma\|u\|^2_{L^2} - (1-A_RB_n)\|\tilde{\nabla}u\|^2_{L^2}) \\
&\leqslant \frac{A_I}{A_R}((2A_RB_n - 1)\|\tilde{\nabla}u\|^2_{L^2} + \Re(\tilde{\mathfrak{h}}(u,u)) - \gamma\|u\|^2_{L^2}) \ .
\end{aligned} \tag{4.19}$$

So by choosing ϑ such that

$$\tan(\vartheta) = \frac{A_I}{A_R}, \tag{4.20}$$

the inequality (4.13) holds, provided that the following condition

$$\textbf{Cond2}: \qquad (2A_RB_n - 1) \leqslant 0 \tag{4.21}$$

is satisfied. Note that if **Cond2** is satisfied, then **Cond1** is automatically satisfied. **Cond2** results in the following condition on the choice of n:

$$\textbf{Cond3}: \qquad n \geqslant \frac{1}{2}\left(\frac{4LA_R}{\cos(L)} - 3\right) \ . \tag{4.22}$$

It is always possible to find such $n > 0$. We therefore have a constructive way of finding γ and ϑ. For a given pair of impedances we can compute A_R and A_I using (4.11). Pick n such that **Cond3** is satisfied and choose γ and ϑ according to (4.15) and (4.20). It is then possible to look for the best possible γ, i.e γ as close to zero as possible (remember that with (4.15), γ is always negative). This can be done by optimising the choice of n and L in order to minimise the value of $A_n(L)$. In particular one may realise that $A_n(L)$ is strictly increasing with n for a given L, and that for a given n, $A_n(L)$ is always minimum when L is such that $\tan(L) = 1/L$. Note that we do not claim here to have the smallest sector possible, but this is not the focus of the present work.

5. Numerical method and results

5.1 *Numerical method*

In order to compute the eigenvalues we have used a numerical method based on surface finite element (see for example (**45**)). In particular, we used the open-source C^{++} library Deal.II ((**46**)). This particular library has the capability of dealing with computation on two dimensional surfaces embedded in a three dimensional space (Riemannian manifolds, typical characteristic of $\mathbb{S}$). For our purpose, we shall use a combination of step-38 and step-36 of the tutorial problems. Once the mass and stiffness matrices are constructed (see Appendix C), the eigenvalue problem is solved using the PETSc and SLEPc libraries. ((**47**), (**48**)).

One of the drawbacks of Deal.II is the fact that it can only deal with quadrilateral meshes for such cases. In order to mesh the sphere with the slit, we have made used of the open source meshing software Gmsh ((**49**)). Meshing the sphere with a slit is not a trivial issue, and the procedure used here was first to mesh the sphere with a diamond hole, and then to force the two extremities of the hole to be the same point. Hence, even if S^+ and S^- are at the same geometrical location, they do not have the same mesh points so that we can prescribe different boundary conditions on S^+ and S^-. We use the finite element space of continuous, piecewise polynomials of degree 2 in each coordinate direction. It is worth noting that other innovative methods such as (**50**), based on the properties of stereographic projections and fast multipole methods, have been developed to solve the Laplace-Beltrami equation on a sphere with "islands" subject to Dirichlet boundary conditions.

5.2 *Results for boundary conditions of type I*

The first thing to do is to consider the mixed Dirichlet-Neumann problem, that is Dirichlet boundary condition on one face and Neumann boundary condition on the other. This is the simplest example that could not be dealt with by the simple method proposed in (**21**) to solve the pure Dirichlet and pure Neumann cases. These three cases, pure Dirichlet, pure Neumann and mixed Dirichlet-Neumann, do not depend on any impedance parameter and are represented by purple, green and red horizontal lines respectively in Figure 5.1. The next step is to apply Robin boundary conditions with $\alpha^{\pm}$ real. For these conditions, the LBO is also non–negative and self-adjoint, so we expect to obtain real positive eigenvalues. Figure 5.1 shows the results with Robin boundary conditions on one face and Neumann boundary conditions on the other in orange (λ_n^{NR}), Robin boundary condition on one face and Dirichlet on the other in pink (λ_n^{DR}) and Robin boundary conditions on both faces in blue (λ_n^{RR}). For plotting purposes, we have chosen $\alpha^{\pm} = \alpha \in [10^{-2}, 10^4]$, but cases when $\alpha^+ \neq \alpha^-$ can also be dealt with in a similar way. The eigenvalues are plotted against $\log(\alpha)$ and presented in Figure 5.1, while some of the normalised eigenfunctions are plotted in Figure 5.2. The results presented in Figure 5.1 illustrate the smooth transition from pure Neumann to pure Dirichlet via Robin conditions, the smooth transition from pure Neumann to mixed Dirichlet-Neumann via Robin-Neumann conditions and the smooth transition from Dirichlet-Neumann to pure Dirichlet via Dirichlet-Robin boundary conditions. As predicted in Subsection 3.3 and Remark 3.7, the eigenvalues are increasing functions of α and the inequalities (3.5) are satisfied.

It is interesting to note that contrary to the pure Dirichlet and the pure Neumann eigenfunctions, the mixed Neumann-Dirichlet eigenfunctions do not exhibit symmetry (or antisymmetry) across the equatorial plane (see Figure 5.2). This breakdown in symmetry can also be observed in the Robin case when different impedances are being used on each side of the slit. This is an important observation since the symmetry properties of the pure Neumann and pure Dirichlet eigenfunctions were essential for the method developed in (**21**) to work.

In order to have an idea of the precision with which our eigenvalues were computed in the Robin case, we performed a mesh sensitivity analysis by performing these computations for successive mesh refinements. The usual exponential convergence from above is obtained, as shown in Figure 5.3.

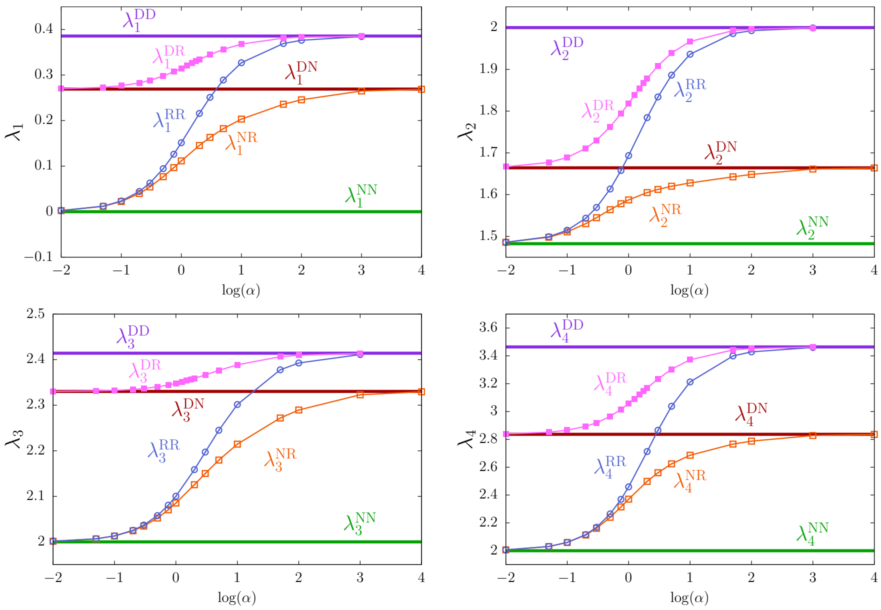

Fig. 5.1: The first four eigenvalues of the LBO for different boundary conditions of type I

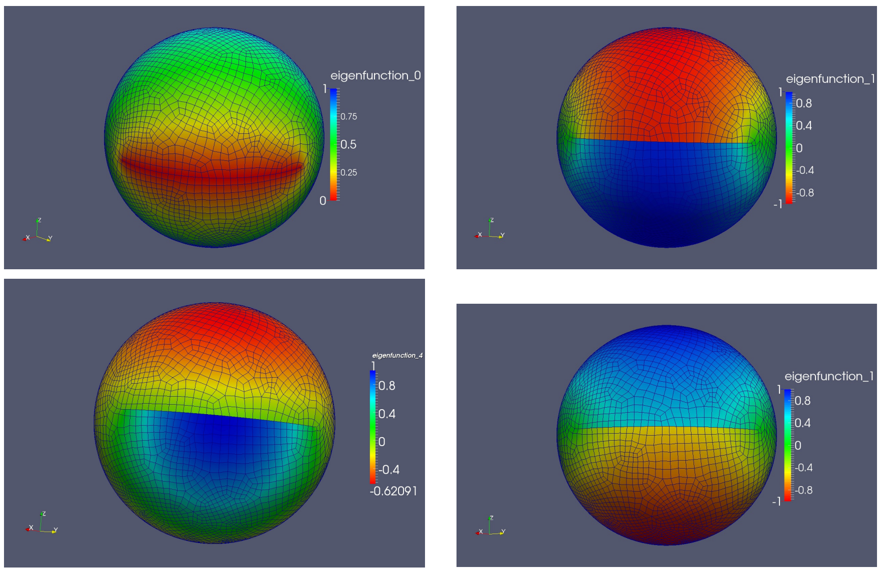

Fig. 5.2: Four typical normalised eigenfunctions for different boundary conditions of type I: 1st eigenfunction for pure Dirichlet (top left), 2nd eigenfunction for pure Neumann (top right), 5th eigenfunction for mixed Neumann-Dirichlet (bottom left) and 2nd eigenfunction for real Robin conditions on both faces with $\alpha^{\pm} = 1$.

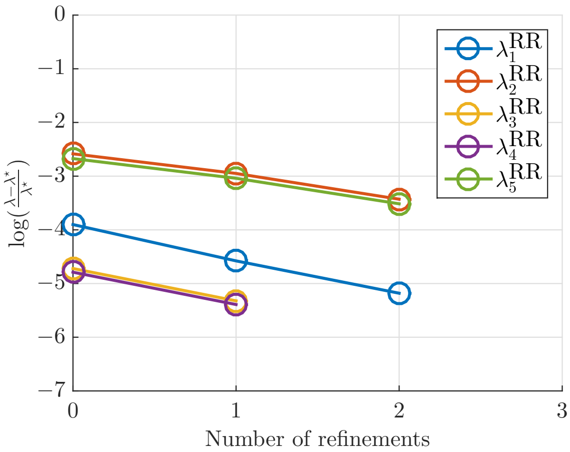

Fig. 5.3: Mesh sensitivity analysis for Robin boundary conditions on both faces, with $\alpha = 1$. The mesh has been refined three times and the logarithm (with base 10) of the error relative to the results of the third refinement, $\log\left(\dfrac{\lambda-\lambda^\star}{\lambda^\star}\right)$, is plotted against the number of refinements. The superscript * denotes the results of the third refinement. The results were recorded with 6 significant digits, so when the point of the second refinement is missing, this means that the exact same result (up to six digit accuracy) was obtained for the second and the third refinement.

5.3 *Results for boundary conditions of type II*

For these computations we use a very similar method to that presented in the previous subsection, that is a surface finite element method using the libraries Deal.II, PETSc and SLEPc and a similar mesh. However, at the time of performing the computations the use of complex numbers was not yet fully supported in Deal.II. Hence some work had to be carried out in order to solve the problem with boundary conditions of type II, leading to a partial implementation of complex numbers within the Deal.II library. For plotting purposes, and in order to match with Subsection 4.3, we are solving the problem for $\alpha^+ = \alpha^- = \alpha$ and $(\alpha^+ = 0, \alpha^- = \alpha)$, i.e. for Robin-Robin ($\mu_n^{\mathrm{RR}}$, blue line in Figures 5.4 and 5.5) and Neumann-Robin (μ_n^{NR}, orange line) boundary conditions of type II. We then rewrite α as $\alpha = c(1+i)$ and let c vary between 10^{-2} and 10^3. For completeness, we show that it is possible to deal with the case $\alpha^+ \neq \alpha^-$ and compute the Robin-Robin eigenvalues (μ_n^{RR}, pink line) when $\alpha^+ = c(1+i)$ and $\alpha^- = c\left(\dfrac{1}{2}+3i\right)$ and let c vary between 10^{-2} and 10^3. The results for the location of the eigenvalues in $\mathbb{C}$ are presented in Figure 5.4. The pure Neumann (λ_n^{NN}, green), Dirichlet-Neumann (λ_n^{DR}, red) and pure Dirichlet (λ_n^{DD}, purple) are represented as single points on these graphs. Once again, we observe a smooth transition from pure Neumann to pure Dirichlet in the case of Robin-Robin conditions, and a smooth transition from pure Neumann to Dirichlet-Neumann in the case of Neumann-

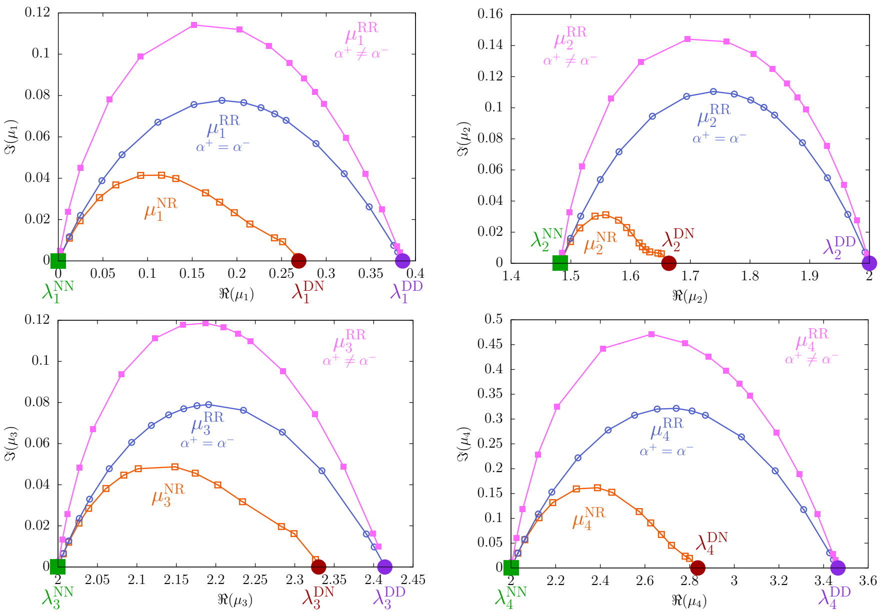

Fig. 5.4: The first four eigenvalues of the LBO for different boundary conditions of type II, when $\alpha = c(1+i)$ and $c \in [10^{-2}, 10^3]$. In the Robin-Robin case (blue) we have $\alpha^\pm = \alpha$, and in the Neumann-Robin case (orange), we have $\alpha^+ = 0$ and $\alpha^- = \alpha$. In the Robin-Robin case (pink), we have $\alpha^+ = c(1+i)$ and $\alpha^- = c(1/2+3i)$

Robin conditions. The main difference between these results and the results presented in Subsection 5.2 is that this time the smooth transition occurs through the upper half of the complex plane.

In particular, we observe that when c is small, the eigenvalues are very close to the pure Neumann case. This can actually be predicted using the perturbation theory of linear operators ((**36**)), as presented in Appendix B. In particular we can show that if λ is a pure Neumann eigenvalue of multiplicity m and is separated from the other pure Neumann eigenvalues by a distance d, and $B_{\frac{d}{2}}(\lambda)$ represents the ball of radius $\dfrac{d}{2}$ around λ, then if $|\alpha|$ is smaller than a certain value there are a finite number of α-Robin-Robin eigenvalues with total multiplicity m that lie within $B_{\frac{d}{2}}(\lambda)$. This is illustrated in Figure 5.5, where two cases are considered. The case of λ_2, which has a pure Neumann eigenvalue of multiplicity 1, and the case of λ_3 and λ_4, that have the same pure Neumann eigenvalue (equal to 2) and hence represent an eigenvalue of multiplicity 2. In both cases, we have $d = \lambda_{3,4}^{NN} - \lambda_2^{NN} \approx 0.518$.

Moving towards the left on the blue and orange curves in Figure 5.5 corresponds to reducing the absolute value of α. Hence we can see from Figure 5.5a that in the case of pure Neumann eigenvalue of multiplicity 1, for $|\alpha|$ small enough, there is a single Robin-Robin eigenvalue inside $B_{d/2}$, while in the case of the eigenvalue of multiplicity 2 in Figure 5.5b, for $|\alpha|$ small enough, we always have two Robin-Robin eigenvalues inside $B_{d/2}$, meaning

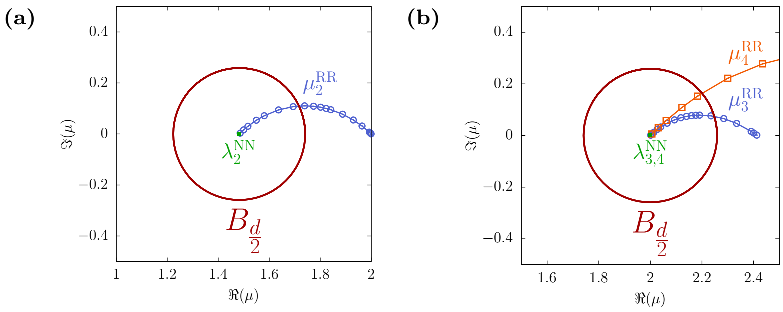

Fig. 5.5: (Colour online) Illustration of the proximity of the Robin and pure Neumann eigenvalues in the case of a pure Neumann eigenvalue of multiplicity 1 (a) and a pure Neumann eigenvalue of multiplicity 2 (b).

that the finite number of Robin-Robin eigenvalues inside $B_{d/2}$ has a total multiplicity of 2, as predicted by the theory in Appendix B.

6. Concluding remarks

The Laplace-Beltrami operator arising from the problem of diffraction by a quarter-plane has been carefully studied, numerically and theoretically, from a spectral point of view. Two types of boundary conditions have been considered, the boundary conditions of type I for which the operator is self-adjoint, and the boundary conditions of type II for which the operator ceases to be self-adjoint but remains m-sectorial. In the case of boundary conditions of type I, we have shown that the spectrum of the operator is an infinite set of isolated real positive eigenvalues with finite multiplicity. The relative position of these eigenvalues for different boundary conditions of type I has been studied. In the case of boundary conditions of type II, we have shown that the spectrum of the operator is an infinite set of isolated complex eigenvalues with finite multiplicity, and that these eigenvalues are contained in a sector of the complex plane. A constructive way of obtaining such a sector has been described. Note that although the theory in Section 4 has been developed for any value of the impedance parameters $\alpha^{\pm}$, most of the numerical results have been given for impedance parameters $\alpha^{\pm}$ with a positive real part. This is due to the fact that it becomes difficult to order the eigenvalues for large negative values of this real part. However, our sector estimate gives a good way of reducing the size of the region where we need to search.

Of course, the qualitative results obtained in this study are valid for any length of the cut between 0 and 2π, corresponding to the problem of diffraction by a plane sector of arbitrary angle. These qualitative results should also hold for arbitrary holes on the sphere with mixed boundary conditions, corresponding to the problem of a cone with arbitrary cross section. This knowledge will prove useful when trying to evaluate the diffraction coefficient of such scattering problems, when mixed boundary conditions are being used.

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APPENDIX A: GREEN'S IDENTITY FOR $

A.1 *Green's identity on a sphere with a hole*

Consider the surface of a sphere with a hole that represents a certain curved surface σ, such that the boundary of the surface, $\partial\sigma$ is oriented as described in Figure Ab. In order to apply Stokes' theorem on σ, it needs to be oriented, i.e., we need to make a choice of normal that is compatible with the right hand rule and the orientation of $\partial\sigma$. For this to work, we need to choose the normal $\tilde{\boldsymbol{n}}$ of σ as in Figure Aa, i.e. by choosing $\tilde{\boldsymbol{n}}=\boldsymbol{e}_r$. Let us now consider a vector field $\boldsymbol{F}$. We can apply Stokes' theorem to σ to get

$$\iint_\sigma \tilde{\boldsymbol{n}}\cdot(\nabla\times\boldsymbol{F})\,\mathrm{d}S \quad=\quad \int_{\partial\sigma}\boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}\ . \tag{A.1}$$

Let us now consider another vector field $\boldsymbol{G}$. In our case, it can be shown that the divergence of $\boldsymbol{G}$ restricted to σ, denoted $\tilde{\nabla}\cdot\boldsymbol{G}$, can be expressed as

$$\tilde{\nabla}\cdot\boldsymbol{G} \quad=\quad \tilde{\boldsymbol{n}}\cdot(\nabla\times(\tilde{\boldsymbol{n}}\times\boldsymbol{G}))\ . \tag{A.2}$$

Now, we can apply Stokes' theorem to the vector field $\boldsymbol{F}$ defined by $\boldsymbol{F}=\tilde{\boldsymbol{n}}\times\boldsymbol{G}$ and so (A.1) becomes

$$\iint_\sigma \tilde{\boldsymbol{n}}\cdot(\nabla\times(\tilde{\boldsymbol{n}}\times\boldsymbol{G}))\,\mathrm{d}S \quad=\quad \int_{\partial\sigma}(\tilde{\boldsymbol{n}}\times\boldsymbol{G})\cdot\mathrm{d}\boldsymbol{r}\ . \tag{A.3}$$

Now, let us parametrise $\partial\sigma$ by a parameter s say so that $\partial\sigma$ is defined by

$$\boldsymbol{r}(s) = (r(s), \theta(s), \varphi(s)) \quad \text{for} \quad s \in [s_0, s_1],$$

in the $(\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_\varphi)$ basis. Note that because we are on the sphere, we have $r(s) = 1$. The left hand side (LHS) of (A.3) can be rewritten as

$$\int_{\partial\sigma} (\tilde{\boldsymbol{n}} \times \boldsymbol{G}) \cdot \mathrm{d}\boldsymbol{r} \quad = \quad \int_{s_0}^{s_1} (\tilde{\boldsymbol{n}} \times \boldsymbol{G}) \cdot \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s}\, \mathrm{d}s\ .$$

Upon writing $\boldsymbol{t} = \dfrac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s}$ (representing a vector field tangent to $\partial\sigma$), and using the vector calculus identity $(\tilde{\boldsymbol{n}} \times \boldsymbol{G}) \cdot \boldsymbol{t} = \boldsymbol{G} \cdot (\boldsymbol{t} \times \tilde{\boldsymbol{n}})$, we obtain

$$\int_{\partial\sigma} (\tilde{\boldsymbol{n}} \times \boldsymbol{G}) \cdot \mathrm{d}\boldsymbol{r} \quad = \quad \int_{s_0}^{s_1} \boldsymbol{G} \cdot (\boldsymbol{t} \times \tilde{\boldsymbol{n}})\, \mathrm{d}s.$$

Note that the quantity $(\boldsymbol{t} \times \tilde{\boldsymbol{n}})$ is tangent to σ and perpendicular to $\partial\sigma$, so it represents the normal to $\partial\sigma$ within σ, let us call this quantity $\boldsymbol{n}$, so that we have $\boldsymbol{n} = \boldsymbol{t} \times \tilde{\boldsymbol{n}}$ and the LHS of (A.3) becomes

$$\int_{\partial\sigma} (\tilde{\boldsymbol{n}} \times \boldsymbol{G}) \cdot \mathrm{d}\boldsymbol{r} \quad = \quad \int_{s_0}^{s_1} \boldsymbol{G} \cdot \boldsymbol{n}\ \mathrm{d}s\ .$$

Now using the definition of the divergence in (A.2), equation (A.3) becomes

$$\iint_{\sigma} \tilde{\nabla} \cdot \boldsymbol{G}\, \mathrm{d}S \quad = \quad \int_{s_0}^{s_1} \boldsymbol{G} \cdot \boldsymbol{n}\, \mathrm{d}s \underset{\text{def}}{=} \int_{\partial\sigma} \boldsymbol{G} \cdot \boldsymbol{n}\, \mathrm{d}s\ .$$

This is an equivalent of the divergence theorem for this curved surface σ. Note that in this case, $\boldsymbol{n}$ is oriented towards the centre of the hole as shown in the Figure Ab.

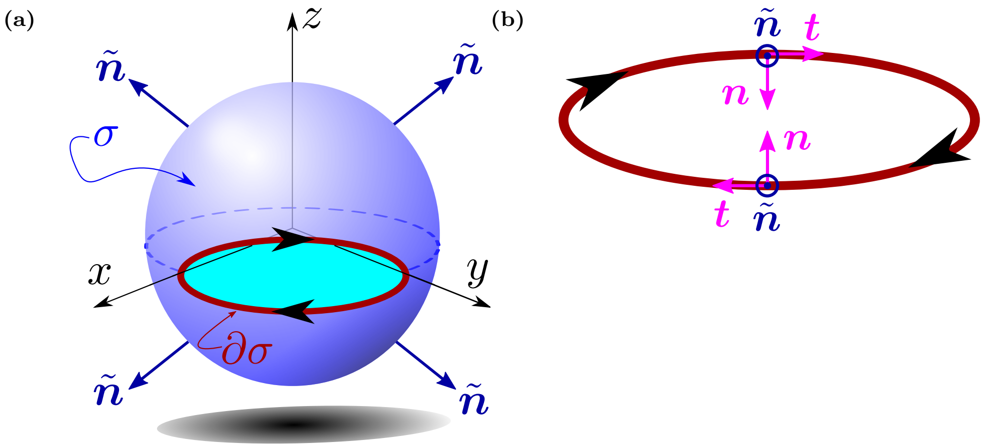

Fig. A: (Colour online) Choice of normal to σ so that it is compatible with the orientation of $\partial\sigma$ in view of the application of Stokes' theorem (a) and orientation of $\partial\sigma$ normal and tangent vectors $\boldsymbol{n}$ and $\boldsymbol{t}$ (b).

Note that even if $\boldsymbol{n}$ and $\mathbf{t}$ are three-dimensional vectors, they only depend on θ and φ and their

$\boldsymbol{e}_r$ component is always zero. Hence they can also be considered as two-dimensional vectors in the $(\boldsymbol{e}_\theta, \boldsymbol{e}_\varphi)$ basis, and so, for a function $u(\theta,\varphi)$ defined on σ, the quantity $\tilde{\nabla}u \cdot \boldsymbol{n}$ makes sense, and using the vector identity $\tilde{\nabla} \cdot \left(\bar{v}\tilde{\nabla}u\right) = \bar{v}\tilde{\Delta}u + \tilde{\nabla}u \cdot \tilde{\nabla}\bar{v}$ for any $u, v \in C^\infty(\bar{\sigma})$ and the divergence theorem derived above, we have the Green's identity

$$-\iint_\sigma \bar{v}\tilde{\Delta}u \,\mathrm{d}S \quad = \quad \iint_\sigma \tilde{\nabla}u \cdot \tilde{\nabla}\bar{v}\,\mathrm{d}S - \int_{s_0}^{s_1} \bar{v}\left(\tilde{\nabla}u \cdot \boldsymbol{n}\right)\,\mathrm{d}s. \qquad \text{(A.4)}$$

A.2 *Application to* $\mathbb{S}$

Let us now consider the cut $\partial\mathbb{S}$ as a degenerate hole, where the lower and upper boundaries of the hole have collapsed onto each other. The normal $\boldsymbol{n}$ remains well defined on $\partial\mathbb{S}$, apart from the end points. And in fact this normal is constant on S^+, where it is denoted $\boldsymbol{n}^+$ and on S^-, where it is denoted $\boldsymbol{n}^-$ as shown on Figure 3.1a. In fact, we clearly have $\boldsymbol{n}^+ = \boldsymbol{e}_\theta$ and $\boldsymbol{n}^- = -\boldsymbol{e}_\theta$. We can now parametrise S^+ and S^- with the position vectors $\boldsymbol{r}^+$ and $\boldsymbol{r}^-$ and the parameter s^+ and s^- by $\boldsymbol{r}^+(s^+) = \left(1, \frac{\pi}{2}^-, s^+\right)$ for $s^+ \in [0,\pi/2]$ and $\boldsymbol{r}^-(s^-) = \left(1, \frac{\pi}{2}^+, \frac{\pi}{2} - s^-\right)$ for $s^- \in [0,\pi/2]$. Hence for $\mathbb{S}$, the Green's identity becomes

$$-\iint_{\mathbb{S}} \bar{v}\tilde{\Delta}u \,\mathrm{d}\mathbb{S} \quad = \quad \iint_{\mathbb{S}} \tilde{\nabla}u \cdot \tilde{\nabla}\bar{v}\,\mathrm{d}\mathbb{S} - \int_{S^+} \bar{v}\left(\tilde{\nabla}u \cdot \boldsymbol{n}^+\right)\,\mathrm{d}s^+ - \int_{S^-} \bar{v}\left(\tilde{\nabla}u \cdot \boldsymbol{n}^-\right)\,\mathrm{d}s^-, \quad \text{(A.5)}$$

where the line integrals are defined by

$$\begin{aligned}\int_{S^+} \bar{v}\left(\tilde{\nabla}u \cdot \boldsymbol{n}^+\right)\,\mathrm{d}s^+ \quad &= \quad \int_{s^+=0}^{\pi/2} \bar{v}\left(\frac{\pi}{2}^-, s^+\right)\left(\tilde{\nabla}u\left(\frac{\pi}{2}^-, s^+\right)\cdot \boldsymbol{n}^+\right)\,\mathrm{d}s^+ \\ &\underset{\varphi\leftrightarrow s^+}{=} \int_{\varphi=0}^{\pi/2} \bar{v}\left(\frac{\pi}{2}^-, \varphi\right)\left(\tilde{\nabla}u\left(\frac{\pi}{2}^-, \varphi\right)\cdot \boldsymbol{n}^+\right)\,\mathrm{d}\varphi\end{aligned}$$

and

$$\begin{aligned}\int_{S^-} \bar{v}\left(\tilde{\nabla}u \cdot \boldsymbol{n}^-\right)\,\mathrm{d}s^- \quad &= \quad \int_{s^-=0}^{\pi/2} \bar{v}\left(\frac{\pi}{2}^+, \frac{\pi}{2} - s^-\right)\left(\tilde{\nabla}u\left(\frac{\pi}{2}^+, \frac{\pi}{2} - s^-\right)\cdot \boldsymbol{n}^-\right)\,\mathrm{d}s^- \\ &\underset{\varphi\leftrightarrow \frac{\pi}{2}-s^-}{=} \int_{\varphi=\pi/2}^{0} \bar{v}\left(\frac{\pi}{2}^+, \varphi\right)\left(\tilde{\nabla}u\left(\frac{\pi}{2}^+, \varphi\right)\cdot \boldsymbol{n}^-\right)(-\mathrm{d}\varphi) \\ &= \int_{\varphi=0}^{\pi/2} \bar{v}\left(\frac{\pi}{2}^+, \varphi\right)\left(\tilde{\nabla}u\left(\frac{\pi}{2}^+, \varphi\right)\cdot \boldsymbol{n}^-\right)\,\mathrm{d}\varphi\ .\end{aligned}$$

In particular, for any function u, here is what is meant by a line integral:

$$\int_{S^+} u\,\mathrm{d}\ell = \int_{\varphi=0}^{\pi/2} u\left(\frac{\pi}{2}^-, \varphi\right)\,\mathrm{d}\varphi \quad \text{and} \quad \int_{S^-} u\,\mathrm{d}\ell = \int_{\varphi=0}^{\pi/2} u\left(\frac{\pi}{2}^+, \varphi\right)\,\mathrm{d}\varphi\ . \qquad \text{(A.6)}$$

APPENDIX B: PERTURBATION THEORY APPLIED TO THE LBO

The aim of this appendix is to show that for impedances of small enough modulus, the eigenvalues of the LBO with boundary conditions of type II are close to the eigenvalues of a given LBO with boundary conditions of type I. In particular, we will show that for small enough impedances, $\mu_n^{\mathrm{RR}}, \mu_n^{\mathrm{RN}}$ and μ_n^{NR} are close to λ_n^{NN}, μ_n^{RD} is close to λ_n^{ND} and μ_n^{DR} is close to λ_n^{DN}. However, here we shall only focus on the most general case of μ_n^{RR}, the other cases being very similar. In order to do so, we shall apply the perturbation theory for linear operators. Let us start by defining a few important notions. Let consider $D_0 \subset \mathbb{C}$, and a family $\{\mathfrak{h}(x)\}_{x\in D_0}$ of sesquilinear forms acting on a Hilbert space $\boldsymbol{H}$. We say that the family $\{\mathfrak{h}(x)\}_{x\in D_0}$ is a holomorphic family of forms if (i) $\mathfrak{h}(x)$ is sectorial, densely defined and closed with domain independent of x, i.e. $D(\mathfrak{h}(x)) = D$ and (ii) for each $u \in D$, the function of x defined by $\mathfrak{h}(x)(u)$ is a holomorphic function for $x \in D_0$. Also, we need to define the operator norm $\|.\|_{\mathrm{op}}$ as follows. Let T be an operator acting on $\boldsymbol{H}$ with domain $\mathrm{Dom}(T)$, then the operator norm of T is given by

$$\|T\|_{\mathrm{op}} \quad = \sup_{u\in\mathrm{Dom}(T),\ u\neq 0} \{\|T(u)\|_{\boldsymbol{H}}/\|u\|_{\boldsymbol{H}}\} \ .$$

The main result of this section will be a consequence of the following theorem:

Theorem B.1. *[Thm. VII.4.8 and Thm. VII.4.9 in (**36**)]* Let $\{\mathfrak{h}^{(n)}\}_{n\in\mathbb{N}}$ be a sequence of sesquilinear forms defined on a Hilbert space $\boldsymbol{H}$. Let $\mathfrak{h} = \mathfrak{h}^{(0)}$ be densely defined with $D(\mathfrak{h}) = D$, sectorial and closable. Let $\mathfrak{h}^{(n)}$ with $n \geqslant 1$ be relatively bounded with respect to $\mathfrak{h}$ in the sense that $D \subset D(\mathfrak{h}^{(n)})$, and there exists $a, b, c \geqslant 0$ such that for $u \in D$ we have

$$|\mathfrak{h}^{(n)}(u,u)| \quad \leqslant \quad c^{n-1}(a\|u\|^2_{\boldsymbol{H}} + b\Re(\mathfrak{h}(u,u))).$$

Then the forms $\mathfrak{h}(x)$ with $D(\mathfrak{h}(x)) = D$, defined for $u, v \in D$ by

$$\mathfrak{h}(x)(u,v) \quad = \quad \sum_{n=0}^{\infty} x^n \mathfrak{h}^{(n)}(u,v) \ ,$$

are well defined for $|x| < 1/c$ and are sectorial and closable for $|x| < 1/(b+c)$. Let $\tilde{\mathfrak{h}}(x)$ denotes the family of their closures. Then $\{\tilde{\mathfrak{h}}(x)\}$ is a holomorphic family of forms and as such, this family is associated to a family of m-sectorial operators $\{\tilde{H}(x)\}$. If we further assume that $\mathfrak{h}$ is symmetric, and hence associated to a self-adjoint operator $\tilde{H}$, then for any $\zeta \in \rho(\tilde{H})$, the resolvent $R(\zeta, \tilde{H}(x))$ exists and is a convergent power series of x for

$$|x| \quad < \quad (2\|(a + bH)R(\zeta, H)\|_{\mathrm{op}} + c)^{-1} \ .$$

Let us consider the LBO with Robin boundary conditions of type II on both sides of the cut. Let the two closed sesquilinear forms $\tilde{\mathfrak{h}}_1$ and $\tilde{\mathfrak{h}}_2$ with $D(\tilde{\mathfrak{h}}_1) = D(\tilde{\mathfrak{h}}_2) = H^1(\mathbb{S})$ be defined as in Section 4.3 for $u, v \in H^1(\mathbb{S})$ by

$$\tilde{\mathfrak{h}}_1(u,v) = \langle \tilde{\nabla} u, \tilde{\nabla} v\rangle_{L^2} \quad \text{and} \quad \tilde{\mathfrak{h}}_2(u,v) = \alpha^+ \int_{S^+} u\bar{v}\,\mathrm{d}\ell + \alpha^- \int_{S^-} u\bar{v}\,\mathrm{d}\ell \ .$$

Recall that $\tilde{\mathfrak{h}}_1$ is also symmetric and non-negative and associated to a self-adjoint operator $\tilde{H}_1 = \tilde{T}_{\mathrm{NN}}$. Let us now define the family of forms $\{\mathfrak{h}(\varepsilon)\}$ for $\varepsilon \in \mathbb{C}$ by $D(\mathfrak{h}(\varepsilon)) = H^1(\mathbb{S})$ and

$$\mathfrak{h}(\varepsilon) \quad = \quad \tilde{\mathfrak{h}}_1 + \varepsilon\tilde{\mathfrak{h}}_2 \ .$$

Note that for $\varepsilon > 0$, $\mathfrak{h}(\varepsilon)$ is associated with a LBO with boundary conditions of type II, $\tilde{H}(\varepsilon)$ with impedances $\varepsilon\alpha^+$ and $\varepsilon\alpha^-$. Recall that we have proven in Subsection 4.3 that $\tilde{\mathfrak{h}}_2$ is relatively bounded with respect to $\tilde{\mathfrak{h}}_1$ with

$$|\tilde{\mathfrak{h}}_2(u,u)| \quad \leqslant \quad A\|u\|^2_{L^2} + B\Re(\tilde{\mathfrak{h}}_1(u,u)),$$

for A and B given in (4.8). Hence we can apply theorem B.1, with $a = A$, $b = B$, and $c = 0$, $\mathfrak{h} = \mathfrak{h}^{(0)} = \tilde{\mathfrak{h}}_1$, $\mathfrak{h}^{(1)} = \tilde{\mathfrak{h}}_2$ and $\mathfrak{h}^{(n)} = 0$ for $n \geqslant 2$ to conclude that the family $\tilde{\mathfrak{h}}(\varepsilon)$ forms a holomorphic family of forms and is associated to a family of m-sectorial operators $\tilde{H}(\varepsilon)$ for $|\varepsilon| < 1/B$ (note from (4.5) that B can be made as big as possible, so this is in fact true for all ε). Moreover, since $\tilde{\mathfrak{h}}_1$

is symmetric, we know that for any $\zeta \in \rho(\widetilde{H}_1)$, the resolvent $R(\zeta, \tilde{H}(\varepsilon))$ exists and is a convergent power series for

$$|\varepsilon| \quad < \quad (2\|(A + B\tilde{H}_1)R(\zeta, \widetilde{H}_1)\|_{\mathrm{op}})^{-1} \; . \tag{B.1}$$

Now, let $\lambda \in \sigma(\tilde{H}_1)$ be an eigenvalue of $\tilde{H}_1$. As has been shown in Subsection 3.3, we know that λ is real, positive, isolated and has a finite multiplicity, say m. Let $d > 0$ be such that $|\lambda - \lambda'| \geqslant d$ for all $\lambda' \in \sigma(\tilde{H}_1)\backslash\{\lambda\}$. Let γ be the circle of centre λ and radius $d/2$ so that $\gamma \subset \rho(\widetilde{H}_1)$ and let $\zeta \in \gamma$, we shall note $B_{d/2}(\lambda)$ the resulting ball. According to ((**36**), V. Eqn. (4.9)), because $\tilde{H}_1$ is self-adjoint, we know that

$$\begin{aligned}\|(A + B\tilde{H}_1)R(\zeta, \widetilde{H}_1)\|_{\mathrm{op}} \quad &= \quad A \sup_{\lambda' \in \sigma(\tilde{H}_1)} |\lambda' - \zeta|^{-1} + B \sup_{\lambda' \in \sigma(\tilde{H}_1)} |\lambda'||\lambda' - \zeta|^{-1} \\ &\leqslant \quad A(d/2) + B(2 + 2\lambda/d),\end{aligned}$$

where we have used the fact that for $\lambda' \in \sigma(\tilde{H}_1)$,

$$|\lambda'||\lambda' - \zeta|^{-1} \quad = \quad 1 + |\zeta||\lambda' - \zeta|^{-1} \leqslant 1 + |\lambda - \zeta||\lambda' - \zeta|^{-1} + \lambda|\lambda' - \zeta|^{-1} \leqslant 2 + 2\lambda/d.$$

Hence, provided we have

$$|\varepsilon| \quad < \quad \frac{d}{4A + 4B(d + \lambda)},$$

the bound (B.1) is satisfied, and $R(\zeta, \tilde{H}(\varepsilon))$ is well defined for all $\zeta \in \gamma$, i.e. $\gamma \subset \rho(\tilde{H}(\varepsilon))$. We can then define the well studied spectral projection operator $P(\varepsilon)$ by

$$P(\varepsilon) \quad = \quad \frac{1}{2i\pi} \int_\gamma R(z, \tilde{H}(\varepsilon))\, \mathrm{d}z \; .$$

Note that since $\gamma \subset \rho(\tilde{H}(\varepsilon))$, γ separates its spectrum, and we can write $\sigma(\tilde{H}(\varepsilon)) = \boldsymbol{S}(\varepsilon) \cup \boldsymbol{T}(\varepsilon)$, with $\boldsymbol{S}(\varepsilon) \cap \boldsymbol{T}(\varepsilon) = \emptyset$ where $\boldsymbol{S}(\varepsilon) = \sigma(\tilde{H}(\varepsilon)) \cap B_{d/2}(\lambda)$ and $\boldsymbol{T}(\varepsilon) = \sigma(\tilde{H}(\varepsilon))\backslash\boldsymbol{S}(\varepsilon)$. Moreover (see (**37**), Thm. 11.1.5), we know that $P(\varepsilon)$ is idempotent and that the restriction of $\tilde{H}(\varepsilon)$ to the range of $P(\varepsilon)$ (denoted $P(\varepsilon)(\boldsymbol{H})$) has spectrum $\boldsymbol{S}(\varepsilon)$. And hence, if $\mathrm{rank}(P(\varepsilon)) = m$, then $\boldsymbol{S}(\varepsilon)$ is made of finitely many eigenvalues with total multiplicity m. But because the projections $P(\varepsilon)$ depend analytically on β, we have $\mathrm{rank}(P(\varepsilon)) = \mathrm{rank}(P(0)) = m$ by ((**37**), Lemma 1.5.5), and so we have proved the following theorem:

Theorem B.2. Suppose that $\lambda \in \rho(\widetilde{H}_1)$ has multiplicity m and that $d > 0$ is such that $|\lambda - \lambda'| \geqslant d$ for all $\lambda' \in \sigma(\tilde{H}_1)\backslash\{\lambda\}$. Denote the ball of radius $d/2$ around λ by $B_{d/2}(\lambda)$. Then given β such that $\beta < d/(4A + 4B(d + \lambda))$, $\tilde{H}(\beta)$ has finitely many eigenvalues in $B_{d/2}(\lambda)$ with total multiplicity m.

APPENDIX C: FINITE ELEMENT IMPLEMENTATION

In order to obtain the finite element formulation of our eigenvalue problem, we want to express our solution u as a finite linear combination of shape functions $\psi_j(\theta, \varphi)$, $j = 0 \ldots N-1$, i.e., we write

$$u \quad = \quad \sum_{j=0}^{N-1} u_j \psi_j, \tag{C.1}$$

where u_j are constant. Let us now consider one particular shape function ψ_i and take the inner product of it with equation (2.4), i.e

$$\langle -\tilde{\Delta} u, \psi_i \rangle_{L^2} \quad = \quad \lambda \langle u, \psi_i \rangle_{L^2} \; .$$

Using the Green's identity (A.5) for $\mathbb{S}$, this becomes

$$\left\langle \tilde{\nabla} u, \tilde{\nabla} \psi_i \right\rangle_{L^2} + \alpha^+ \langle u, \psi_i \rangle_+ + \alpha^- \langle u, \psi_i \rangle_- \quad = \quad \lambda \langle u, \psi_i \rangle_{L^2},$$

where

$$\langle u, v \rangle_\pm \quad = \quad \int_{S^\pm} u \bar{v} \, \mathrm{d}\ell \ .$$

Now, using the sum decomposition (C.1) and the linearity of the inner products in their first argument, we get

$$\sum_{j=0}^{N-1} u_j \left\langle \tilde{\nabla} \psi_j, \tilde{\nabla} \psi_i \right\rangle_{L^2} + \alpha^+ u_j \langle \psi_j, \psi_i \rangle_+ + u_j \alpha^- \langle \psi_j, \psi_i \rangle \quad = \quad \lambda \sum_{j=0}^{N-1} u_j \langle \psi_j, \psi_i \rangle_{L^2} \ , \qquad \text{(C.2)}$$

So if we define the $N \times N$ matrices $\boldsymbol{A} = (A_{ij})$ and $\boldsymbol{M} = (M_{ij})$ by

$$\begin{aligned} A_{ij} &= \left\langle \tilde{\nabla} \psi_j, \tilde{\nabla} \psi_i \right\rangle_{L^2} + \alpha^+ \langle \psi_j, \psi_i \rangle_+ + \alpha^- \langle \psi_j, \psi_i \rangle_- \\ M_{ij} &= \langle \psi_j, \psi_i \rangle_{L^2} \ , \end{aligned}$$

the expansion (C.2) can be rewritten as

$$\sum_{j=0}^{N-1} A_{ij} u_j \quad = \quad \lambda \sum_{j=0}^{N-1} M_{ij} u_j,$$

which, upon defining the vector $\boldsymbol{u} = (u_0, u_1, \ldots, u_{N-1})$ becomes

$$\boldsymbol{A} \boldsymbol{u} \quad = \quad \lambda \boldsymbol{M} \boldsymbol{u},$$

which is a typical generalised eigenvalue problem. The matrix $\boldsymbol{A}$ is called the *stiffness matrix*, while the matrix $\boldsymbol{M}$ is called the *mass matrix.* Note that α^+ and α^- being complex in general, the matrix $\mathbf{A}$ and hence the vector $\boldsymbol{u}$ and eigenvalues λ can be complex.