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Stability of local quantum dissipative systems

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Abstract: Open quantum systems weakly coupled to the environment are modeled by completely positive, trace preserving semigroups of linear maps. The generators of such evolutions are called Lindbladians. In the setting of quantum many-body systems on a lattice it is natural to consider Lindbladians that decompose into a sum of local interactions with decreasing strength with respect to the size of their support. For both practical and theoretical reasons, it is crucial to estimate the impact that perturbations in the generating Lindbladian, arising as noise or errors, can have on the evolution. These local perturbations are potentially unbounded, but constrained to respect the underlying lattice structure. We show that even for polynomially decaying errors in the Lindbladian, local observables and correlation functions are stable if the unperturbed Lindbladian has a unique fixed point and a mixing time which scales logarithmically with the system size. The proof relies on Lieb-Robinson bounds, which describe a finite group velocity for propagation of information in local systems. As a main example, we prove that classical Glauber dynamics is stable under local perturbations, including perturbations in the transition rates which may not preserve *detailed balance.*

Contents

1. Background and previous work

The physical properties of a closed many-body quantum system are encoded in its Hamiltonian. Theoretical models of such systems typically assume some form of local structure, whereby the Hamiltonian decomposes into a sum over interactions between subsets of nearby particles. Similarly, the behavior of an open many-body quantum system is encoded in its Liouvillian. Again, this is typically assumed to have a local structure, decomposing into a sum over local Liouvillians acting on subsets of nearby particles.

Crucial to justifying such theoretical models is the question of whether their physical properties are stable under small perturbations to the local interactions. If the physical properties of a many-body Hamiltonian or Liouvillian depend sensitively on the precise mathematical form of those local terms, then it is difficult to conclude anything about physical systems, whose interactions will always deviate somewhat from theory.

Quantum information theory has motivated another perspective on many-body Hamiltonians. Rather than studying models of naturally occurring systems, it studies how many-body systems can be engineered to produce desirable behavior, such as long-term storage of information in quantum memories [12,19,20,48], processing of quantum information for quantum computing [34,51,13,11,14], or simulation of other quantum systems which are computationally intractable by classical means [8,7,3,28,30]. Again, stability of these systems under local perturbations is crucial, otherwise even tiny imperfections may destroy the desired properties. Stability in this context has been studied for self-correcting topological quantum memories, where one in addition requires robustness against local sources of *dissipative* noise, and the relevant quantity is the minimum time needed to introduce logical errors in the system. It has been known since [12,1] that a self-correcting quantum memory with local interactions is possible in four spatial dimensions. With the breakthrough of the Haah code [19], it seems it may be possible to engineer such self-correcting quantum memories in three dimensions.

Recently, and partially motivated by the dissipative nature of noise, this "engineering" approach has been extended to open quantum systems and many-body Liouvillians. First theoretical [38,61], and then experimental [39,5] work has shown that creating many-body quantum states as fixed points of engineered, dissipative Markovian evolutions can be more robust against undesirable errors and maintain coherence of quantum information for longer times. Intuitively, there is an inherent robustness in such models: the target state is independent of the initial state. If the dissipation is engineered perfectly, the system will always be driven back towards the desired state. This idea can be used to engineer dissipative systems both for carrying out computation via dissipative dynamics [61] and for storing quantum information [53]. However, it does not guarantee stability against errors *in the engineered Liouvillian itself.* Once again, stability against local perturbations – this time for many-body Liouvillians rather than Hamiltonians – is of crucial importance. Indeed, in [53] the authors consider depolarizing noise acting on top of a dissipative interaction enginereed for pro-

tecting a logical qubit encoded in the system, and provide positive numerical evidence for stability of a specific 4D model.

In the case of closed systems governed by Hamiltonians, recent breakthroughs have given rigorous mathematical justification to our intuition that the physical properties of many-body Hamiltonians are stable. Starting with [35, 10], it culminated in the work of [49] which showed that, under a set of mathematically well-defined and physically reasonable conditions, gapped many-body Hamiltonians are stable under perturbations to the local interactions.[1] More precisely, in the presence of *frustration-freeness*, *local topological quantum order*, and *local gap*, the spectral gap of a Hamiltonian with local (or quasi-local) interactions is stable against small (quasi-) local perturbations (see [49] for a formal definition of these conditions). The bound on the amount of imperfection tolerated by the system depends on the decay of the local gaps, the decay of the local topological order, and the strength (and decay rate) of the interactions. Furthermore, except for frustration freeness which is a technical condition required in the proof, these conditions are in a sense tight. There exist simple counterexamples to stability if any one of the conditions is lifted.

2. Stability of open quantum systems

In this work, we study stability of many-body Liouvillians. We consider dynamics generated by rapidly decaying interactions, where the notion of rapid decay is made precise in section 3. Moreover, in order to have a well-defined notion of scaling with system size, we restrict to Liouvillians whose local terms depend only on the subsystem on which they act, and thus are not redefined as we consider larger systems. We call such families of Liouvillians *uniform*.

Our main result shows that, under the above assumptions on the structure of the Liouvillian, logarithmic mixing time implies the desired stability in the dissipative setting.

However, although the result is analogous to [49], the proof and even the definition of stability in the case of Liouvillians necessarily differ substantially from the Hamiltonian case. For Hamiltonians, the relevant issue is stability of the spectral gap. Via the quasi-adiabatic technique [21, 25], this in turn implies a smooth transition between the initial and perturbed ground states, showing that both are within the same phase. Note that the existence of a smooth transition (no closing of the spectral gap in the thermodynamic limit) does not imply that both ground states are close in norm, as the simple example $H = \sum_{i=1}^{N} |0\rangle\langle 0|_i$ vs. $H(\varepsilon) = \sum_{i=1}^{N}(|0\rangle + \varepsilon |1\rangle)(\langle 0| + \varepsilon \langle 1|)_i/(1+\varepsilon^2)$ shows.[2] It does however imply a well-behaved perturbation in the expectation value of local observables – such as order parameters – and correlation functions, which in most experimental situations are the only measurable quantities.

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[1] Note that, in stark contrast to traditional perturbation theory, the perturbations considered here simultaneously change *all* the local interactions by a small amount. The strength of the total perturbation therefore scales with system size and standard perturbation theory does not apply. It is the structure of local ground states of the Hamiltonian that ensures stability.

[2] Note that each Hamiltonian is the sum of non-interacting projections for any $\varepsilon \in \mathbb{R}$. In particular, for each ε, there is a unitary $U(\varepsilon)$ acting on a single site, such that $H(\varepsilon) = U(\varepsilon)^{\otimes N} H U^\dagger(\varepsilon)^{\otimes N}$.

For Liouvillians, we are interested in a definition of stability more related to the evolution itself, which accounts at the same time for both the speed of convergence and the properties of the fixed point. Here, we consider the strongest definition of stability: we want our systems (initial and perturbed) to evolve similarly for all times and all possible initial states. Thus, not only should the speed of convergence to the fixed points be similar, the fixed points themselves should be close and so should the approach to the fixed points. Such "dynamical" definition of stability is radically different from other notions of stability previously considered in the literature, such as Local Thermodynamical Stability [2,56]. While there are cases in which both concepts are of interest, they are generally independent concepts.

Requiring stability of the evolution is significantly stronger than stability of the spectral gap alone,[3] and is more directly relevant to the applications discussed above. As in the Hamiltonian case, the analogous simple example shows that one cannot expect to attain such stability if we consider global measurements on the system. Therefore, in analogy with the Hamiltonian case, restrict our attention to local observables and few-body correlation functions. Since there are technical subtleties involved in extending this stronger definition of stability to dynamics with multiple fixed points, we defer consideration of multiple fixed points to a future paper, and restrict our attention here to dissipative dynamics with unique fixed points. It is important to note, however, that we do not make *any* assumption on the form of the unique fixed point. In particular, we do not assume that it is full-rank (primitivity); our results apply equally well to Liouvillians with pure fixed points. (Pure-state fixed points are particularly relevant to quantum information applications, such as dissipative state engineering and computation.)

A key technical ingredient in the stability proof for Hamiltonians is the quasi-adiabatic evolution technique [21,25], which directly uses the fact that Hamiltonian evolution is reversible. This is of course no longer true for Liouvillians, so we must use a different proof approach. We make use of the fact that evolution under a Liouvillian converges to a steady-state, together with dissipative generalizations [50] of the Lieb-Robinson bounds that are the other crucial ingredient in [49].

Among systems which satisfy our assumptions, one finds classical Glauber dynamics [46]. This immediately shows that Glauber dynamics is stable against errors. To the best of our knowledge, this is new even to the classical literature (related results, but with different assumptions, were given in [27]). Given the importance of Glauber dynamics to sampling from the thermal distributions of classical spin systems [41,46], we expect our results to have applications also to classical statistical mechanics.

The paper is structured as follows: After setting up notation and basic definitions in the next section, we state our main stability result in section 4 and discuss the assumptions it requires. In section 5 we prove various technical results used in the main proof, which is given in section 6. We apply these results in section 7 to the important example of classical Glauber dynamics, before concluding with a discussion of the results and related open questions in section 8.

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[3] Due to the recent work in [57], it is not clear whether the spectral gap in Liouvillians is the relevant quantity for convergence questions.

3. Setup and notation

We will consider a cubic lattice[4] $\Gamma = \mathbb{Z}^D$. The ball centered at $x \in \Lambda$ of radius r will be denoted by $b_x(r)$. At each site x of the lattice we will associate one elementary quantum system with a finite dimensional Hilbert space $\mathcal{H}_x$. We will use the Dirac notation for vectors: $|\phi\rangle$ will denote a vector in $\mathcal{H}_x$, $\langle\phi|$ its adjoint, and $\{|n\rangle\}_{n=0}^{\dim \mathcal{H}_x}$ the canonical basis for $\mathcal{H}_x$. Scalar product in $\mathcal{H}_x$ will be denoted by $\langle\phi|\psi\rangle$, and rank-one linear maps by $|\phi\rangle\langle\psi|$. For each finite subset $\Lambda \subseteq \Gamma$, the associated Hilbert space is given by

$$\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x, \tag{1}$$

and the algebra of observables supported on Λ is defined by

$$\mathcal{A}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{B}(\mathcal{H}_x).$$

If $\Lambda_1 \subset \Lambda_2$, there is a natural inclusion of $\mathcal{A}_{\Lambda_1}$ in $\mathcal{A}_{\Lambda_2}$ by identifying it with $\mathcal{A}_{\Lambda_1} \otimes \mathbb{1}$. The support of an observable $O \in \mathcal{A}_\Lambda$ is the minimal set Λ' such that $O = O' \otimes \mathbb{1}$, for some $O' \in \mathcal{A}_{\Lambda'}$, and will be denoted by $\operatorname{supp} O$. We will denote by $\|\cdot\|_p$ the Schatten p-norm over $\mathcal{A}_\Lambda$. Where there is no risk of ambiguity, $\|\cdot\|$ will denote the usual operator norm (i.e. the Schatten ∞-norm).

A linear map $\mathcal{T} : \mathcal{A}_\Lambda \to \mathcal{A}_\Lambda$ will be called a *superoperator* to distinguish it from operators acting on states. The support of a superoperator $\mathcal{T}$ is the minimal set $\Lambda' \subseteq \Lambda$ such that $\mathcal{T} = \mathcal{T}' \otimes \mathbb{1}$, where $\mathcal{T}' \in \mathcal{B}(\mathcal{A}_{\Lambda'})$. A superoperator is said to be Hermiticity preserving if it maps Hermitian operators to Hermitian operators. It is said to be positive if it maps positive operators (i.e. operators of the form O^*O) to positive operators. $\mathcal{T}$ is called *completely positive* if $\mathcal{T} \otimes \mathbb{1} : \mathcal{A}_\Lambda \otimes M_n \to \mathcal{A}_\Lambda \otimes M_n$ is positive for all $n \geqslant 1$. Finally, we say that $\mathcal{T}$ is trace preserving if $\operatorname{tr} \mathcal{T}(\rho) = \operatorname{tr} \rho$ for all $\rho \in A_\Lambda$. For a general review on superoperators, see [62].

The dynamics of the system is generated by a superoperator $\mathcal{L}$, which plays a similar role to the Hamiltonian in the non-dissipative case. The evolution will be given by the one parameter semigroup $T_t = e^{t\mathcal{L}}$. The natural assumptions to make about T_t are that it is a continuous semigroup of completely positive and trace preserving maps (CPTP, sometimes also called *quantum channels*). Such maps are always contractive, meaning that $\|T_t\|_{1\to 1,cb} \leqslant 1$, where the completely-bounded norm is defined as:

$$\|T\|_{1\to 1,cb} = \sup_n \|T \otimes \mathbb{1}_n\|_{1\to 1} = \sup_n \sup_{\substack{X \in \mathcal{A}_\Lambda \otimes M_n \\ X \neq 0}} \frac{\|T \otimes \mathbb{1}_n(X)\|_1}{\|X\|_1}. \tag{2}$$

We will also be interested in the $\|\cdot\|_{\infty\to\infty,cb}$ completely-bounded norm of super-operators, which is defined as follows:

$$\|T\|_{\infty\to\infty,cb} = \sup_n \|T \otimes \mathbb{1}_n\|_{\infty\to\infty} = \sup_n \sup_{\substack{X \in \mathcal{A}_\Lambda \otimes M_n \\ X \neq 0}} \frac{\|T \otimes \mathbb{1}_n(X)\|_\infty}{\|X\|_\infty}.$$

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[4] We restrict to cubic lattices for the sake of exposition. The results can be extended to more general settings, replacing the lattice $\mathbb{Z}^D$ with a graph with polynomial growth.

The relationship between $\|\cdot\|_{1\to 1,cb}$ and $\|\cdot\|_{\infty\to\infty,cb}$ is the following:

$$\|T\|_{1\to 1,cb} = \|T^*\|_{\infty\to\infty,cb},$$

where T^* is the dual of T, satisfying $\operatorname{tr} A\,T(B) = \operatorname{tr} T^*(A)\,B$. We will denote $\|\cdot\|_{\infty\to\infty,cb}$ simply by $\|\cdot\|_{cb}$ when there is no risk of confusing different completely-bounded norms.

Remark 1. As shown in [29], the supremum in equation (2) is reached when n is equal to the dimension of the space on which T is acting: if $T : \mathcal{M}_n \to \mathcal{M}_n$, then $\|T \otimes \mathbb{1}_n\|_{1\to 1} = \|T\|_{1\to 1,cb}$.

The generator $\mathcal{L}$ of the semigroup $T_t = e^{t\mathcal{L}}$, is called a *Liouvillian*. All such generators can be written in the following general form, often called the *Lindblad form* [15,42] (see [62]):

Proposition 1. $\mathcal{L}$ *generates a continuous semigroup of CPTP maps if and only if it can be written in the form:*

$$\mathcal{L}(\rho) = i[\rho, H] + \sum_j L_j \rho L_j^* - \frac{1}{2}\sum_j \{L_j^* L_j, \rho\}, \tag{3}$$

where H is a Hermitian matrix, $\{L_j\}_j$ a set of matrices called the Lindblad operators, $[\cdot, \cdot]$ denotes the commutator and $\{\cdot, \cdot\}$ the anticommutator.

We will use the term *Lindbladian* and Liouvillian interchangeably. Since we consider Lindbladians $\mathcal{L}$ corresponding to local dissipative dynamics, we assume that $\mathcal{L}$ is a *local Lindbladian* of the form:

$$\mathcal{L} = \sum_{u\in \Lambda}\sum_{r\geqslant 0} \mathcal{L}_{u,r}, \quad \operatorname{supp}\mathcal{L}_{u,r} = b_u(r), \tag{4}$$

where each term in the sum above can be written in the form given by equation (3).

Such a decomposition is obviously always trivially possible. We are interested in the cases in which the norms of $\mathcal{L}_{u,r}$ decay with r. Concretely, let us define the *strength of interaction* for a Lindbladian as the pair (J, f) given by:

$$J = \sup_{u,r}\|\mathcal{L}_{u,r}\|_{1\to 1,cb}, \quad f(r) = \sup_u \frac{\|\mathcal{L}_{u,r}\|_{1\to 1,cb}}{J}. \tag{5}$$

The behavior of $f(r)$ as r goes to infinity corresponds to various interaction regimes, listed in order of decreasing decay rate:

- finite range interaction: $f(r)$ is compactly supported;
- exponentially decaying: $f(r) \leqslant e^{-\mu r}$, for some $\mu > 0$;
- quasi-local interaction: $f(r)$ decays faster than any polynomial;
- power-law decay: $f(r) \leqslant (1+r)^{-\alpha}$, for some positive $\alpha > 0$.

As we will see later, our result will apply whenever $\mathcal{L}$ has finite range, exponentially decaying, or quasi-local interactions. It will also hold in the power-law decay regime, but we will require a lower bound on the decay exponent α, depending on the dimension of the underlying lattice. Not to overload the exposition, we will assume that $\mathcal{L}$ has finite range or exponentially decaying interactions, unless otherwise specified. The modifications needed to work with quasi-local interactions and power-law decay are presented in section 6.4. Also, we will say that functions we construct along the way are *fast-decaying*, if their decay rate is within the same decay class of $f(r)$ we are considering (or faster).

As shown in [63], from the spectral decomposition of $\mathcal{L}$ (and T_t) one can define two new CPTP maps which represent the infinite-time limit of the semigroup T_t. We will denote by T_∞ the projector onto the subspace of stationary states (fixed points), and by T_ϕ the projector onto the subspace of periodic states. They correspond, respectively, to the kernel of $\mathcal{L}$ and to the eigenspace of purely imaginary eigenvalues of $\mathcal{L}$, which we denote $\mathcal{F}_\mathcal{L}$ and $\mathcal{X}_\mathcal{L}$, respectively. Both subspaces are invariant under T_t: in particular, T_t acts as the identity over $\mathcal{F}_\mathcal{L}$, while it is a unitary operator over $\mathcal{X}_\mathcal{L}$. Note, also, that both subspaces are spanned by positive operators (i.e. density matrices) [62, Prop. 6.8, Prop. 6.12]. We will denote by $T_{\phi,t}$ the composition $T_t \circ T_\phi$.

Since we plan to exploit the local structure of $\mathcal{L}$, we will often make use of the restriction of $\mathcal{L}$ to a subset of the lattice. Given $A \subset \Lambda$, we define the *truncated*, or *localized*, generator:

$$\mathcal{L}_A = \sum_{b_u(r) \subseteq A} \mathcal{L}_{u,r}. \tag{6}$$

3.1. Uniform families. We are interested in how properties of dissipative dynamics scale with the size of the system. Hence, we are concerned with sequences of Lindbladians defined on lattices of increasing size, where all the Lindbladians in the sequence are from the same "family". To make this precise, we need to pin down how Lindbladians from the same family, but on different size lattices, are related to one-another. Our results will apply to very general sequences of Lindbladians, which we call *uniform families*. Before giving the precise definition, it is helpful to consider some special cases.

For local Hamiltonians on a lattice, one often considers translationally-invariant systems with various types of boundary conditions (e.g. open or periodic boundaries). There is then a natural definition of what it means to consider the same translationally-invariant Hamiltonian on different lattice sizes. Translationally-invariant Lindbladians are an important special case of a uniform family. In this special case, all the local terms in the Lindbladian that act in the "bulk" of the lattice are the same. Another way of thinking about this is to formally consider the translationally-invariant Lindbladian $\mathcal{M}$ defined on the infinite lattice $\Gamma = \mathbb{Z}^d$, and then consider each member of the family to be a restriction of this infinite Lindbladian to a finite sub-lattice $\Lambda \subset \Gamma$ of some particular size:

$$\mathcal{L} = \mathcal{M}_\Lambda.$$

This gives us translationally-invariant Lindbladians with *open boundary condition*. But of course, this is only one particular choice of boundary terms (in this case, no boundary terms at all). We are also interested in more general
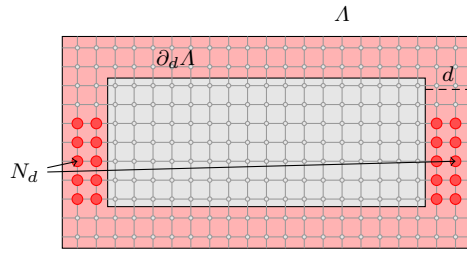
Fig. 1. Partition of the lattice Λ into the bulk and the boundary of thickness d, $\partial_d\Lambda$ (see Def. 2). The dark red regions on the boundary correspond to the interaction term N_d coupling distant regions in Λ.

boundary conditions, such as periodic boundaries. So, in addition to the "bulk" interactions coming from $\mathcal{M}$, we allow additional terms that play the role of *boundary conditions*:

$$\mathcal{L} = \mathcal{M}_\Lambda + \mathcal{L}^{\partial\Lambda}.$$

We allow greater freedom in the boundary terms $\mathcal{L}^{\partial\Lambda}$. For one thing, they are allowed to depend on the size of the lattice Λ. But more importantly, we allow strong interactions that *cross the boundary* of Λ, coupling sites that would otherwise be far apart. For example, the case of *periodic boundary conditions* corresponds to adding interaction terms that connect opposite boundaries of Λ, as if on a torus (see Fig. 1).

Now that we have given an intuition of what a uniform family is, it is time to present the formal definition. This includes all the special cases discussed so far, but also captures much more general families of Lindbladians that are not necessarily translationally-invariant, and many other types of boundary conditions (e.g. cylindrical boundaries, or boundary terms that give the sphere topology, or terms that force fixed states on the boundary[5]).

Definition 2. Given $\Lambda \subset \Gamma$, a *boundary condition* with strength (J, f) for Λ is a Lindbladian $\mathcal{L}^{\partial\Lambda} = \sum_{d\geqslant 1} N_d$, where

$$\|N_d\|_{1\rightarrow 1, c.b.} \leqslant J \, |\partial_d\Lambda| \, f(d)$$

with

$$\partial_d\Lambda := \{x \in \Lambda \,|\, \mathrm{dist}(x, \Lambda^c) \leqslant d\},$$
$$\mathrm{supp}\, N_d \subset \partial_d\Lambda.$$

Definition 3. A *uniform family* of Lindbladians $\mathcal{L}$ with strength (J, f) is given by the following:

  (i) *infinite Lindbladian*: a Lindbladian $\mathcal{M}$ on all of $\mathbb{Z}^D$ with strength (J, f);
 (ii) *boundary conditions*: a set of *boundary conditions* $\mathcal{L}^{\partial\Lambda}$, with strength (J, f) and $\Lambda = b_u(L)$, for each $u \in \mathbb{Z}^D$ and $L \geqslant 0$.

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[5] Or even Möbius strips, Klein bottles, and other exotic topologies.

We say that the family is *translationally invariant* if $\mathcal{M}$ is translationally invariant and $\mathcal{L}^{\partial b_u(L)}$ is independent of u.

Given a uniform family $\mathcal{L}$, we fix the following notation for evolutions defined on a subset Λ:

$$\mathcal{L}^\Lambda = \mathcal{M}_\Lambda \quad \text{``open boundary'' evolution;} \tag{7}$$

$$\mathcal{L}^{\overline{\Lambda}} = \mathcal{M}_\Lambda + \mathcal{L}^{\partial\Lambda} \quad \text{``closed boundary'' evolution,} \tag{8}$$

with the respective evolutions $T_t^\Lambda = \exp(t\mathcal{L}^\Lambda)$ and $T_t^{\overline{\Lambda}} = \exp(t\mathcal{L}^{\overline{\Lambda}})$.

Remark 2. In the rest of the paper, we will make use of the following notation:

$$A(s) = \{x \in \Lambda \,|\, \text{dist}(x, A) \leqslant s\}.$$

Since we are interested in observables whose support is not connected, we want to consider more general regions than balls: in particular, we are interested in disjoint unions of convex regions (for example, to calculate two-point correlation functions). Consider what happens to such a region $A = A_0 \sqcup A_1$ when we grow it by taking $A(s)$. When s becomes sufficiently large, $A_0(s)$ will merge with $A_1(s)$. At this point, $A(s)$ will not be a disjoint union of balls anymore. To avoid such complications, for s large enough that disjoint balls merge, we will replace $A(s)$ by the smallest ball containing it. This will not hurt us, as $|A(s)|$ will still grow asymptotically at the same rate, which will be sufficient for our purposes.
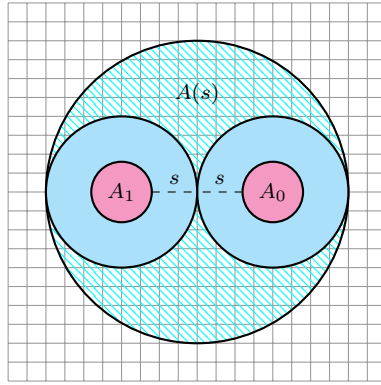
Fig. 2. The convention on how to grow a region $A = A_0 \sqcup A_1$.

Definition 4. We say that $\mathcal{L}$ has a unique fixed point if, for all $\Lambda = b_u(L)$, $\mathcal{X}_{\mathcal{L}^{\overline{\Lambda}}} = \mathcal{F}_{\mathcal{L}^{\overline{\Lambda}}} = \{\rho_\infty^{\overline{\Lambda}}\}$. In other words, $T_\phi^{\overline{\Lambda}}(\rho) = T_\infty^{\overline{\Lambda}}(\rho) = \rho_\infty^{\overline{\Lambda}}$, for all density matrices ρ.

Note that if for all pure states ρ, we have $T_t^{\overline{\Lambda}}(\rho) > 0$ (positive definite), for $t > 0$, then the evolution has a unique fixed point $\rho_\infty > 0$ (see [62, Thm. 6.7]).

We will drop the superscript from $T_t^{\overline{\Lambda}}$, and simply write T_t, when we consider some fixed $\Lambda \subset \Gamma$. In that case, we will refer to the number of lattice sites in Λ as the *system size*.

4. Main result

4.1. Assumptions for stability. In Hamiltonian systems, the spectral gap (the difference between the two lowest energy levels) plays a crucial role in a number of settings, from defining quantum phases and phase transitions [55] to understanding the entanglement and correlations present in the system [24,22,23] and analyzing its stability to perturbations [10,49]. On the other hand, it is known that for Lindbladians, the spectral gap (in this setting, the least negative real part of the non-zero eigenvalues) alone is not sufficient to fully characterize the convergence properties of the dissipative evolution [31,57]. Therefore, we will instead impose a more general requirement on the convergence of the dynamics. (The dependence of this requirement on spectral properties of $\mathcal{L}$, i.e. properties depending on the eigenvalues – like the gap – and eigenvectors – like the condition number, is an active area of research.)

Definition 5 (rapid mixing). Given a one-parameter semigroup of CPTP maps T_t, define the *contraction* of T_t as the following quantity:

$$\eta(T_t) = \frac{1}{2} \sup_{\substack{\rho \geqslant 0 \\ \operatorname{tr}\rho = 1}} \|T_t(\rho) - T_{\phi,t}(\rho)\|_1. \tag{9}$$

Given a family of such semigroups $\{T_t^\alpha\}_\alpha$, each of which is acting on $\mathcal{B}(H_\alpha)$ for some Hilbert space H_α of finite dimension d_α, we say that it satisfies *rapid mixing* if there exist constants $c, \gamma, \delta > 0$, such that for each α:

$$\eta(T_t^\alpha) \leqslant c \log^\delta(d_\alpha) \, e^{-t\gamma}. \tag{10}$$

We will write $\mathrm{RM}(\gamma, \delta)$ for short.

If each H_α has a tensor product structure of the type defined in equation (1), then the rapid mixing assumption can be restated as a logarithmic scaling with system size of the mixing time. Since the dimension of $\mathcal{H}_\Lambda$ is $(\dim \mathcal{H}_x)^{|\Lambda|}$, for uniform families condition (10) is equivalent to:

$$\eta(T_t^{\overline{\Lambda}}) \leqslant c \, |\Lambda|^\delta \, e^{-t\gamma} \quad \forall \Lambda. \tag{11}$$

Let us recall a result from [31].

Theorem 6 (Contraction for commuting Lindbladians). *Let $\{\mathcal{L}_j\}_{j=0}^n$ be a set of commuting Lindbladians. Define $\mathcal{L} = \sum_j \mathcal{L}_j$ and the corresponding evolutions $T_t^j = e^{t\mathcal{L}_j}$ and $T_t = e^{t\mathcal{L}}$. Then:*

$$\eta(T_t) \leqslant \sum_j \eta(T_t^j). \tag{12}$$

In particular, consider the definition of $T_t^{\overline{\Lambda}}$ given in remark 2 for $\Lambda \subset \Gamma$ being a disjoint union of balls. Then the previous theorem implies that, if $\mathcal{L}$ is translationally-invariant and it satisfies equation (10) for each of the connected components of Λ, then it also satisfies the same equation (up to constants) for Λ.

Finally, for translationally-invariant uniform families of Lindbladians, it is sufficient to satisfy equation (10) for lattices centered at the origin: $\Lambda = b_0(L)$, $L \geqslant 1$.

4.2. Stability. With the required assumptions laid out, we can now state our main result.

Theorem 7. *Let $\mathcal{L}$ be a uniform family of local Lindbladians with a unique fixed point, satisfying rapid mixing (equation* (11)*), and consider a perturbation of the form:*

$$E^{\overline{\Lambda}} = \sum_{u \in \Lambda} \sum_{r \geqslant 0} E_{u,r} + \sum_{d \geqslant 1} E_d,$$

where $E_{u,r}$ is supported on $b_u(r)$ and each E_d is supported on $\partial_d \Lambda$ (see definition 2*) and*

$$\|E_{u,r}\|_{1 \to 1, cb} \leqslant \varepsilon\, e(r), \quad \|E_d\|_{1 \to 1, cb} \leqslant \varepsilon\, |\partial_d \Lambda|\, e(d),$$

where $\varepsilon > 0$ is a constant (the strength of the perturbation) and $e(r)$ is a fast-decaying function. Consider the perturbed evolution

$$S_t = \exp t(\mathcal{L}^{\overline{\Lambda}} + E^{\overline{\Lambda}})$$

and suppose that the following assumptions hold:

  (i) *$E_{u,r}^*(\mathbb{1}) = E_d^*(\mathbb{1}) = 0$ (or, equivalently: $\operatorname{tr} E_{u,r}(O_A) = \operatorname{tr} E_d(O_A) = 0$, for all operators O_A).*
  (ii) *S_t is a contraction for each $t \geqslant 0$.*

For an observable O_A supported on $A \subset \Lambda$, we have for all $t \geqslant 0$:

$$\|T_t^*(O_A) - S_t^*(O_A)\| \leqslant c(|A|)\, \|O_A\| \left(\varepsilon + |\Lambda|\, \nu_\eta^{-1}(d_A)\right), \tag{13}$$

where $d_A = \operatorname{dist}(A, \Lambda^c)$; η is positive and independent of Λ; $\nu_\eta^{-1}(d) \leqslant (1 + d)^{-D-1}$; $c(|A|)$ is independent of Λ and t, and is bounded by a polynomial in $|A|$.

Remark 3. Note that, for a fixed A, if we let Λ grow then d_A will increase with the linear size of Λ and consequently $|\Lambda|\, \nu_\eta^{-1}(d_A)$ will vanish in the limit.

Remark 4. The assumptions (i)-(ii) on the perturbation E are satisfied whenever $\mathcal{M}_{u,r} + E_{u,r}$ and $N_d + E_d$ (as in definition 3) are Lindbladians, but the theorem also covers more general perturbations.

Remark 5. Since we are free to choose an O_A with support on two non connected regions, we can apply theorem 20 to two-point correlation functions (or more generally k-point correlation functions, for fixed k) and still obtain that the error introduced by the perturbation depends linearly on the strength of the perturbation (and not on its global norm).

A set of tools already applied in the setting of classical Markov chains [18,17, 16,46], and recently generalized to quantum dissipative systems [52,9,32], are the so-called *Logarithmic Sobolev inequalities* (in short, log-Sobolev inequalities). Introduced in a different setting to study hypercontractivity of semigroups [33], they provide the right asymptotic regime needed to satisfy the rapid mixing condition: in fact, the existence of a system size independent log-Sobolev constant implies a logarithmic scaling of the mixing time, which is exactly what is required in definition 5. Without going into the technical details of log-Sobolev inequalities (which can be found in [32]), we summarize this fact in the following corollary:

Corollary 8. *Let $\mathcal{L}$ belong to a uniform family of translationally-invariant Lind-bladians with a unique fixed point for each system size. If $\mathcal{L}$ satisfies the log-Sobolev inequality with a system-size independent constant, then the dissipative dynamics are stable, in the sense of theorem 7.*

In particular, in [60] it was shown that product evolutions, i.e. Lindbladians that can be decomposed as a sum of independent terms $\mathcal{L}_k$ acting on a single subsystem, satisfy a log-Sobolev inequality with a log-Sobolev constant lower bounded by the minimum of the spectral gaps of $\mathcal{L}_k$ (times a factor depending on the maximum dimension of the subsystems). Moreover, the authors of [60] were able to show that Davies maps associated to a graph state Hamiltonian [26] (which are not in a product form, but can be analyzed in a similar way) and the ones associated to free-fermionic Hamiltonians have a system-size independent log-Sobolev constant.

In all such cases, corollary 8 implies that the evolution of local observables is stable.

4.3. Local observables vs. global observables. The bound in equation (13) scales with the size of the support of the observable O_A. Although the dependence is polynomial, for observables with large support the result is not useful. Still, in most realistic experiments, we are interested in the behavior of observables with fixed support and low-degree correlation functions, making the above result widely applicable. Nonetheless, one might ask more generally for a system-size independent bound on:

$$\sup_{\rho} \|T_{\infty}(\rho) - S_{\infty}(\rho)\|_1, \tag{14}$$

where S_{∞} is the fixed-point projector for the evolution of the perturbed Lindbla-dian. However, this is not possible; the limitation to local observables is in some sense strict. There is no hope of finding such a bound for global observables, as the following simple example shows.[6]

Example 1. Consider N independent amplitude damping processes, with uniform rate γ (which we can suppose w.l.o.g. equal to 1). This Lindbladian can be written as

$$\mathcal{L}_N = \sum_{k=1}^{N} \mathbb{1}_{1\ldots k-1} \otimes \mathcal{L}_1 \otimes \mathbb{1}_{k+1\ldots N},$$

where

$$\mathcal{L}_1(\rho) = |0\rangle\langle 1| \, \rho \, |1\rangle\langle 0| - \frac{1}{2}\{\rho, |1\rangle\langle 1|\}$$

is an amplitude damping process on a single qubit, describing the decay of the state $|1\rangle$ into $|0\rangle$ at a constant rate $\gamma = 1$. This Lindabladian has gap $1/2$ and $e^{t\mathcal{L}_N} = (e^{t\mathcal{L}_1})^{\otimes N}$ has mixing time of order $O(\log N)$ [31, Sec. V. C.]. The fixed point is the pure state $|0\ldots0\rangle\langle0\ldots0|$.

---

[6] Indeed, all global stability results for quantum Linbladians we are aware of have a dependency on the total Hilbert space dimension [58].

Now consider $\mathcal{L}_1^\varepsilon$, a rotation of $\mathcal{L}_1$, which fixes $|\alpha_0\rangle = \sqrt{1-\varepsilon^2}\,|0\rangle + \varepsilon\,|1\rangle$. We have $\|\mathcal{L}_1 - \mathcal{L}_1^\varepsilon\|_{1\to1} = O(\varepsilon)$, but the new fixed point $|\alpha_0\rangle\langle\alpha_0|^{\otimes N}$ is almost orthogonal to the original one, since the overlap between the two is

$$\langle 0\ldots0|\alpha_0\ldots\alpha_0\rangle = \langle 0|\alpha_0\rangle^N = (1-\varepsilon^2)^{N/2} \sim e^{-N\varepsilon^2/2} \to 0 \text{ as } N \to \infty.$$

This shows that, in general, there is no good bound on (14) (note that we have $\|\,|0\ldots0\rangle\langle0\ldots0| - |a_0\ldots\alpha_0\rangle\langle a_0\ldots\alpha_0|\,\|_1 \geqslant 1 - |\langle0\ldots0|\alpha_0\ldots\alpha_0\rangle|^2$) and that the dependence on the support of the observable in equation (13) cannot be improved: to see this consider the observable $O_r = |0\ldots0\rangle\langle0\ldots0|_{1\ldots r}$ acting on $r \leqslant N$ spins. O_r has norm one, and

$$O_\infty := \lim_{t\to\infty} T_t^*(O_r) = \mathbb{1}, \quad O_\infty^\varepsilon := \lim_{t\to\infty} T_t^{\varepsilon\,*}(O_r) = \langle0|\alpha_0\rangle^{2r}\,\mathbb{1} = (1-\varepsilon^2)^r\,\mathbb{1}.$$

Consequently, we have:

$$\|O_\infty - O_\infty^\varepsilon\| = 1 - (1-\varepsilon^2)^r = r\varepsilon^2 + o(\varepsilon^2).$$

This implies that any upper bound to $\|O_\infty - O_\infty^\varepsilon\|$ has to be at least linear in r, which is the size of the support of O_r.

4.4. Do we need all the assumptions?. It is reasonable to ask if the assumptions of theorem 7 are all necessary. We have just shown that we must necessarily consider local observables if we are to have meaningful bounds, but what about the other conditions? We will now present three examples, each consisting of a family of Lindbladiands with periodic boundary conditions, such that, in order:

- The family is uniform and translationally invariant, satisfies rapid mixing, but does not have a unique fixed point;
- The family has a unique fixed point, but is not uniform and fails to satisfy rapid mixing;
- The family (presented in appendix A) has a unique fixed point, satisfies rapid mixing, but is not uniform.

All these systems will be shown to be unstable.

Example 2. Consider a 1D chain composed of N 4-level systems, with an independent Lindbladian acting on each site, having the following Lindblad operators

$$L_1 = |0\rangle\langle1|, \quad L_2 = |0\rangle\langle3|, \quad L_3 = |2\rangle\langle1|, \quad L_4 = |2\rangle\langle3|,$$

and denote by

$$\mathcal{L}_0(\rho) = \sum_{i=1}^4 L_i\rho L_i^* - \frac{1}{2}\{\rho, L_i^*L_i\}.$$

The global Lindbladian $\mathcal{L}_N$ is given by applying $\mathcal{L}_0$ independently on each site $k = 1\ldots N$:

$$\mathcal{L}_N = \sum_{k=1}^N \mathbb{1}_{1,\ldots,k-1} \otimes \mathcal{L}_0 \otimes \mathbb{1}_{k+1,\ldots,N}.$$

Then we have that

$$
\mathcal{L}_0(|i\rangle\langle j|) = \begin{cases} 0 & \text{if } i = j \in \{0, 2\} \\ |0\rangle\langle 0| + |2\rangle\langle 2| - 2\,|i\rangle\langle j| & \text{if } i = j \in \{1, 3\} \\ -[\chi_{\{1,3\}}(i) + \chi_{\{1,3\}}(j)]\,|i\rangle\langle j| & \text{if } i \neq j. \end{cases}
$$

Diagonal states of the form $|i\rangle\langle i|$ evolve according to the classical Markov process embedded in the Lindbladian, while off-diagonal elements $|i\rangle\langle j|$ evolve as

$$
T_t(|i\rangle\langle j|) = \exp(-t[\chi_{\{1,3\}}(i) + \chi_{\{1,3\}}(j)])\,|i\rangle\langle j|\,;
$$

where $\chi_{\{1,3\}}$ denotes the indicator function of the set $\{1, 3\}$. This implies that the space of fixed points $\mathcal{F}_{\mathcal{L}_0}$ is given by $\mathrm{span}\{|0\rangle\langle 0|, |2\rangle\langle 2|, |0\rangle\langle 2|, |2\rangle\langle 0|\}$. Since $\mathcal{L}_0$ has gap equal to 1, theorem 6 implies that $\mathcal{L}_N$ satisfies rapid mixing. $\mathcal{L}_N$ forms a uniform family, but it does not satisfy the unique fixed point condition.

Consider now the following additional Lindbladian

$$
\mathcal{E}_0(\rho) = \frac{2}{N}\left[|0\rangle\langle 2|\,\rho\,|2\rangle\langle 0| - \frac{1}{2}\{\rho, |2\rangle\langle 2|\} \right].
$$

Then, we have:

$$
(\mathcal{L}_0 + \mathcal{E}_0)(|i\rangle\langle j|) = \begin{cases} 0 & \text{if } i = j = 0 \\ |0\rangle\langle 0| + |2\rangle\langle 2| - 2\,|i\rangle\langle j| & \text{if } i = j = 1, 3 \\ \frac{2}{N}(|0\rangle\langle 0| - |i\rangle\langle j|) & \text{if } i = j = 2 \\ -\left(\chi_{\{1,3\}}(i) + \chi_{\{1,3\}}(j) + \frac{\chi_{\{i,j\}}(2)}{N}\right)|i\rangle\langle j| & \text{if } i \neq j. \end{cases}
$$

Again, this implies that $\mathcal{F}_{\mathcal{L}_0+\mathcal{E}_0} = \{|0\rangle\langle 0|\}$. Consequently $\mathcal{L}_N + \mathcal{E}_N$ has a unique fixed point. It is not a uniform family, and it does not satisfy rapid mixing, as it is not even globally gapped. To see this, note that for $\sigma = |200\ldots0\rangle\langle200\ldots0| - |020\ldots0\rangle\langle020\ldots0|$:

$$
(\mathcal{L}_N + \mathcal{E}_N)(\sigma) = -\frac{2}{N}\,\sigma.
$$

Analogously, $\mathcal{L}_N + \mathcal{E}_N^*$ satisfies the same conditions as $\mathcal{L}_N + \mathcal{E}_N$, but the unique fixed point is now $|2\ldots2\rangle\langle2\ldots2|$.

All three systems described above are unstable, since we can transform one into the other by applying a perturbation of order $O(1/N)$, yet the fixed points of $\mathcal{L}_N + \mathcal{E}_N$ and $\mathcal{L}_N + \mathcal{E}_N^*$ are locally orthogonal (while $\mathcal{L}_N$ has both of them as fixed points).

4.5. Relaxations of the rapid mixing condition. In the case of finite range or exponentially decaying interactions, the proof of theorem 7 still holds if we relax equation (11) by requiring only a polynomial decay in time, i.e. a bound of the form

$$
\eta(T_t^{\overline{\Lambda}}) \leqslant c\,|\Lambda|^\delta\,\gamma(t), \tag{15}
$$

if $\gamma(t)$ is a fast enough decaying function, where the threshold decay rate is determined by system-size independent constants (such as the Lieb-Robinson bound constants and the geometrical dimension of the underlying lattice structure).

Determining the precise value of such threshold requires an argument similar to the one given for the case of power-law decaying interactions in section 6.4, and is presented in section 6.5.

5. Toolbox for the proof

Before presenting the proof of theorem 7, we need to introduce some useful tools. We present them in full generality, including the case of power-law decay of interactions, without restricting here to exponentially decaying interactions.

5.1. Lieb-Robinson bounds for Lindbladian evolution. We first recall a generalization of Lieb-Robinson bounds to non-Hamiltonian evolution, due to [54] and [50], which we use to derive a number of useful tools that allow us to approximate the support of an evolving observable with a finite set which grows linearly in time. The following condition is sufficient for the bounds to hold.

Assumption 1 (Lieb-Robinson condition). Let $\mathcal{L} = \sum_{u,r} \mathcal{L}_{u,r}$ be a local Lindbladian. There exist positive constants μ and v, such that:

$$\sup_{x \in \Lambda} \sum_{u \in \Lambda} \sum_{r \geqslant \mathrm{dist}(u,x)} \|\mathcal{L}_{u,r}\|_{1 \to 1, cb} |b_u(r)| \, \nu_\mu(r) \leqslant \frac{v}{2} < \infty; \qquad (16)$$

where $\nu_\mu(\cdot)$ is one of the following:

$$\nu_\mu(r) = e^{\mu r}, \qquad\qquad\qquad (\text{LR-1})$$
$$\nu_\mu(r) = (1 + r)^\mu. \qquad\qquad (\text{LR-2})$$

Note that both functions are submultiplicative, in the sense that $\nu_\mu(r + s) \leqslant \nu_\mu(r)\nu_\mu(s)$. Moreover, $\nu_a(r)^b = \nu_{ab}(r)$.

The constant v is called the *Lieb-Robinson velocity* of $\mathcal{L}$, while the reciprocal function $\nu_\mu^{-1}(r) = 1/\nu_\mu(r)$ is called the *Lieb-Robinson decay* of $\mathcal{L}$.

Note that if $\mathcal{L}$ has interaction strength (J, f), then condition (16) can be replaced by:

$$J \sup_{x,y \in \Lambda} \sum_{n \geqslant 0} |b_x(r) \setminus b_x(r-1)| \sum_{r \geqslant n} f(r) \, |b_y(r)| \, \nu_\mu(r) \leqslant \frac{v}{2} < \infty. \qquad (17)$$

Since our systems are embedded in the lattice $\mathbb{Z}^D$, we have that $v < \infty$, as long as:

$$\sum_{n \geqslant 0} n^{D-1} F_\mu(n) < \infty, \quad F_\mu(n) := \sum_{r \geqslant n} r^D \, f(r) \, \nu_\mu(r). \qquad (18)$$

Remark 6. Condition (LR-1) is satisfied when $\mathcal{L}$ has finite-range or exponentially decaying interactions, while condition (LR-2) is satisfied when $\mathcal{L}$ has quasi-local interactions. If $\mathcal{L}$ has interactions decaying as a power-law with exponent α, then condition (LR-2) is satisfied whenever $\alpha > 2D+1$ (by choosing $\mu < \alpha-(2D+1)$).

Theorem 9 (Lieb-Robinson bound). *Suppose $\mathcal{L}$ is a local Lindbladian satisfying assumption 1. Let O_X be an observable supported on $X \subset \Lambda$, and denote by $O_X(t) = T_t^*(O_X)$ its evolution under $\mathcal{L}$. Let $K : \mathcal{A}_Y \to \mathcal{A}_Y$ be a super-operator supported on $Y \subset \Lambda$ which vanishes on $\mathbb{1}$. Then, the following bound holds [54, 50]:*

$$\|K(O(t))\| \leqslant \|K\|_{\infty \to \infty, cb} \, \|O_X\| \, C(X, Y) \frac{(e^{vt} - 1)}{\nu_\mu(\mathrm{dist}(X, Y))}, \qquad (19)$$

where $C(X, Y) = \min(|X|, |Y|)$.

From now on, we will only consider Lindbladians which satisfy equation (17) with either of the two possible assumptions on $\nu_\mu(\cdot)$.

Lemma 10 (Comparing different dynamics). *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two local Lindbladians, and suppose $\mathcal{L}_2$ has Lieb-Robinson speed and decay bounded by v and ν_μ^{-1}. Consider an operator O_X supported on $X \subset \Lambda$, and denote by $O_i(t)$ its evolution under $\mathcal{L}_i$, $i = 1, 2$. Suppose that $\mathcal{L}_1 - \mathcal{L}_2 = \sum_{r \geqslant 0} M_r$, where M_r is a superoperator supported on Y_r which vanishes on $\mathbb{1}$, and $\mathrm{dist}(X, Y_r) \geqslant r$. Then the following holds:*

$$\|O_1(t) - O_2(t)\| \leqslant \|O_X\| \, |X| \, \frac{e^{vt} - vt - 1}{v} \sum_{r=0}^{\infty} \|M_r\|_{1 \to 1, cb} \, \nu_\mu^{-1}(r). \qquad (20)$$

Proof. Let $h(t) = O_1(t) - O_2(t)$. Calculating its derivative, we obtain

$$h'(t) = \mathcal{L}_1^* O_1(t) - \mathcal{L}_2^* O_2(t) = \mathcal{L}_1^* h(t) + (\mathcal{L}_1^* - \mathcal{L}_2^*) O_2(t).$$

Since $h(0) = 0$, this differential equation for $h(t)$ has solution

$$
\begin{aligned}
h(t) = O_1(t) - O_2(t) &= \int_0^t e^{(t-s)\mathcal{L}_1^*}(\mathcal{L}_1^* - \mathcal{L}_2^*) O_2(s) \, \mathrm{d}s \\
&= \sum_{r \geqslant 0} \int_0^t e^{(t-s)\mathcal{L}_1^*} M_r^* O_2(s) \, \mathrm{d}s,
\end{aligned}
\qquad (21)
$$

giving us a useful integral representation for $O_1(t) - O_2(t)$. From this, we obtain the estimate

$$\|O_1(t) - O_2(t)\| \leqslant \sum_{r \geqslant 0} \int_0^t \|M_r^* O_2(s)\| \, \mathrm{d}s,$$

where we have used the fact that $e^{t\mathcal{L}_1^*}$ is a contraction with respect to $\|\cdot\|_\infty$ for each $t \geqslant 0$.

We can now apply the Lieb-Robinson bound (equation (19)) to each of the terms in the sum in the previous estimate, to obtain:

$$\|O_1(t) - O_2(t)\|$$

$$\leqslant \sum_{r \geqslant 0} \|M_r\|_{1 \to 1, cb} \, \|O_X\| \, C(X, Y_r) \nu_\mu^{-1}(\mathrm{dist}(X, Y_r)) \int_0^t (e^{vs} - 1) \, \mathrm{d}s,$$

which implies the claimed bound. $\qquad\square$

A particular application of the previous lemma is when $\mathcal{L}_2$ is a restriction of $\mathcal{L}_1$ onto a smaller region. Since this case occurs frequently, and is of particular interest, we state it as a separate lemma. Similar results were presented in [50, 6].

Lemma 11 (Localizing the evolution). *Let O_A be an observable supported on a finite $A \subset \Lambda$. Denote by $O_A(t) = T_t^*(O_A)$ its evolution under a local Lindbladian $\mathcal{L}$ with strength (J, f). Given $r > 0$, denote by $O_A(r; t)$ its evolution under the localized Lindbladian $\mathcal{L}_{A(r)}$.*

Then, the following bound holds:

$$\|O_A(t) - O_A(r; t)\| \leqslant \|O_A\| \, |A| \, J \frac{e^{vt} - 1 - vt}{v} \nu_\beta^{-1}(r), \qquad (22)$$

where $\nu_\beta^{-1}(r)$ decays exponentially if $\mathcal{L}$ satisfies condition (LR-1), while decays as $(1 + r)^{-\beta}$ if $\mathcal{L}$ satisfies condition (LR-2). In this case, if we denote by α the decay rate of $\mathcal{L}$, then β is given by:

$$\beta = \begin{cases} \alpha - 3D & \text{if } \alpha \geqslant 5D - 1; \\ \frac{1}{2}(\alpha - D - 1) & \text{if } \alpha \leqslant 5D - 1. \end{cases}$$

Proof. First, let us decompose $\mathcal{L} - \mathcal{L}_{A(r)}$ as a telescoping sum

$$\mathcal{L} - \mathcal{L}_{A(r)} = \sum_{l \geqslant r} \mathcal{L}_{A(l+1)} - \mathcal{L}_{A(l)}.$$

Since each element in the sum is the difference between restrictions on different subsets of the same global Lindbladian, it is easy to explicitly write their difference

$$\mathcal{L}_{A(l+1)} - \mathcal{L}_{A(l)} = \sum_{\delta=0}^{l+1} \sum_{\text{dist}(u,A)=\delta} \mathcal{L}_u(l + 1 - \delta).$$

We group the terms in the sum by their distance from A: Let

$$d = \text{dist}(A, b_u(l + 1 - \delta)) = \max\{0, 2\delta - l - 1\}$$

and

$$M_0 = \sum_{l \geqslant r} \sum_{\delta=0}^{\frac{l+1}{2}} \sum_{\text{dist}(u,A)=\delta} \mathcal{L}_u(l + 1 - \delta); \qquad (23)$$

$$M_d = \sum_{l \geqslant r} \sum_{\substack{\text{dist}(u,A)=\delta \\ \delta = \frac{l+1+d}{2}}} \mathcal{L}_u(l + 1 - \delta). \qquad (24)$$

Then, we can write:

$$\sum_{d \geqslant 0} M_d = \mathcal{L} - \mathcal{L}_{A(r)}; \qquad \text{dist}(A, \text{supp } M_d) = d.$$

Applying lemma 10, we obtain:

$$\|O_A(t) - O_A(r; t)\| \leqslant \|O_A\| \, |A| \, J \frac{e^{vt} - 1 - vt}{v} \zeta(r);$$

where, by denoting $q(l) = |A(l) \setminus A(l-1)|$, $\zeta(r)$ is the following:

$$\zeta(r) = \frac{1}{J} \sum_{d \geqslant 0} \|M_d\|_{1 \to 1, cb} \, \nu_\mu^{-1}(d) \leqslant$$

$$\sum_{l \geqslant r} \sum_{\delta=0}^{\frac{l+1}{2}} q(\delta) f(l+1-\delta) + \sum_{\delta=\frac{l+1}{2}}^{l+1} q(\delta) f(l+1-\delta) \nu_\mu^{-1}(2\delta - l - 1). \quad (25)$$

If $\delta \geqslant (l+1)/2$, since $\nu_\mu(\cdot)$ is submultiplicative, we have:

$$\nu_\mu(\delta) \leqslant \nu_\mu(l+1-\delta)\nu_\mu(2\delta - l - 1).$$

Otherwise, since $\nu_\mu(\cdot)$ is increasing, we have that $\nu_\mu(\delta) \leqslant \nu_\mu(l+1-\delta)$. Plugging these inequalities in the above sum, we get:

$$\zeta(r) \leqslant \sum_{l \geqslant r} \sum_{\delta=0}^{l+1} \left[q(\delta)\nu_\mu^{-1}(\delta) \right] \left[f(l+1-\delta)\nu_\mu(l+1-\delta) \right].$$

Since f satisfies equation (17), which in particular implies

$$\sum_{\delta \geqslant 0} f(\delta)\nu_\mu(\delta) \, |b_0(\delta)| < \infty,$$

then the sequence $f(\delta)\nu_\mu(\delta)$ is decreasing. We distinguish two cases: If ν_μ is of the type (LR-1), then the decay of $f(\delta)\nu_\mu(\delta)$ is exponential. Since $q(\delta)$ grows polynomially, $q(\delta)\nu_\mu^{-1}(\delta)$ is exponentially decaying. Then, the convolution of the two sequences, which is exactly:

$$\sum_{\delta=0}^{l+1} \left[q(\delta)\nu_\mu^{-1}(\delta) \right] \left[f(l+1-\delta)\nu_\mu(l+1-\delta) \right]$$

is exponentially decaying too, which implies an exponential decay rate for $\zeta(r)$. Thus, there exists some $\beta > 0$ such that $\zeta(r) \leqslant \nu_\beta^{-1}(r)$, and this concludes the proof for the case of exponential decay. Let us suppose now that ν_μ is of type (LR-2). Then, $f(\delta)\nu_\mu(\delta)$ decays as $(1+\delta)^{\mu-\alpha}$, while $q(\delta)\nu_\mu^{-1}(\delta)$ decays as $(1+\delta)^{D-1-\mu}$. This implies[7] that their convolution decays as $(1+l)^{-\min(\alpha-\mu, \mu-D+1)}$ and thus

$$\zeta(r) \leqslant c(1+r)^{-\min(\alpha-\mu-1, \mu-D)} = \nu_\beta^{-1}(r).$$

Recalling that condition (LR-2) requires $\mu < \alpha - (2D+1)$, a simple calculation shows that the above decay rate is maximized for

$$\mu < \min\left(\alpha - 2D - 1, \frac{\alpha + D - 1}{2} \right),$$

which gives the claimed formula for β. $\square$

---

[7] Consider two positive decreasing sequences (x_n) and (y_n). Since $0 < p < 1$ implies that $(x+y)^p \leqslant x^p + y^p$, it holds that $(x*y)_n^p \leqslant \sum_k x_k^p y_{n-k}^p = (x^p * y^p)_n$.

Another specialization of lemma 10, similar in spirit to the one just presented, is when we compare the evolution of local observables under $\mathcal{L}^{A(r)}$ and $\mathcal{L}^{\overline{A(r)}}$, as defined in definition 3.

Lemma 12. *Let O_A be an observable supported on $A \subset \Lambda$. Given $r > 0$, it holds that*

$$\left\| T_t^{*\overline{A(r)}}(O_A) - T_t^{*A(r)}(O_A) \right\| \leqslant \|O_A\| \, |A| \, J \, \frac{e^{vt} - 1 - vt}{v} \nu_\beta^{-1}(r). \qquad (26)$$

Proof. Without loss of generality, we consider the case of $A(r)$ being a convex set. By construction, $\mathcal{L}^{\overline{A(r)}} - \mathcal{L}^{A(r)} = \mathcal{L}^{\partial A(r)}$, and $\mathcal{L}^{\partial A(r)} = \sum_{d \geqslant 1} N_d$, where each N_d acts on sites that are closer than d to the border of $A(r)$. We group these terms by their distance from A. Let $k = \frac{1}{2} \operatorname{diam} A$ and set:

$$M_0 = \sum_{i=0}^{k} N_{r+1+i},$$

$$M_j = N_{r+1-j}, \quad j = 1 \ldots r.$$

It is easy to see that $\operatorname{dist}(A, \operatorname{supp} M_j) = j$. By applying lemma 10, we have that:

$$\left\| T_t^{*\overline{A(r)}}(O_A) - T_t^{*A(r)}(O_A) \right\| \leqslant \|O_A\| \, |A| \, \frac{e^{vt} - 1 - vt}{v} \sum_{j=0}^{r} \|M_j\|_{1 \to 1, c.b.} \, \nu_\mu^{-1}(j).$$

We are left to prove that the sum appearing on the r.h.s. is fast-decaying in r. From definition 3 it follows that for $j > 0$:

$$\|M_j\|_{1 \to 1, c.b.} \leqslant J \, |\partial_{r-j} A(r)| \, f(r+1-j) = J \, |A(r) \setminus A(j)| \, f(r+1-j),$$

while for $j = 0$:

$$\|M_0\|_{1 \to 1, c.b.} \leqslant \sum_{i=0}^{k} J \, |\partial_{r+i} A(r)| \, f(r+1+i).$$

Setting $h_{m,n} = |b_0(m) \setminus b_0(n)|$, we have that:

$$\sum_{j=0}^{r} \|M_j\|_{1 \to 1, c.b.} \, \nu_\mu^{-1}(j) \leqslant J\zeta(r), \qquad (27)$$

where

$$\zeta(r) := \sum_{i=0}^{k} h_{r+k, k-i} f(r+1+i) + \sum_{j=1}^{r} h_{r+k, k+j} f(r+1-j) \nu_\mu^{-1}(j).$$

An argument similar to the one in the proof of lemma 11 shows that $\zeta(r)$ is fast-decaying. Indeed, $h_{r+k,k-i} f(r+1+i)$ scales asymptotically as $r^D f(r)$, while $h_{r+k,k+j} f(r+1-j)$ scales as $(r-j)^D f(r+1-j)$. If $\mathcal{L}$ satisfies (LR-1), then $\zeta(r)$ will be exponentially decaying, with rate $\min(\alpha, \mu) - 1 = \mu - 1$.

If otherwise $\mathcal{L}$ satisfies (LR-2), then $\zeta(r)$ has a polynomial decay, with rate $\min(\alpha - D, \mu) - 1 = \mu - 1$. In both cases, then:

$$\zeta(r) \leqslant \nu_{\mu-1}^{-1}(r).$$

Notice that the constant β defined in lemma 11 is smaller than $\mu - 1$. $\qquad \square$

5.2. Local rapid mixing. The rapid mixing condition implies a local version of mixing that will be a useful tool for the proof of theorem 7. We state its definition here.

Definition 13 (Local rapid mixing). Take $A \subset \Lambda$, and define the *contraction of T_t relative to A* as

$$
\begin{aligned}
\eta^A(T_t) &:= \sup_{\substack{\rho \geqslant 0 \\ \operatorname{tr} \rho = 1}} \left\| \operatorname{tr}_{A^c} \left[T_t(\rho) - T_{\phi,t}(\rho) \right] \right\|_1 \\
&= \sup_{\substack{\rho \geqslant 0 \\ \operatorname{tr} \rho = 1}} \sup_{\substack{O_A \in \mathcal{A}_A \\ \|O_A\| = 1}} \operatorname{tr} \left(O_A \left[T_t(\rho) - T_{\phi,t}(\rho) \right] \right) \\
&= \sup_{\substack{\rho \geqslant 0 \\ \operatorname{tr} \rho = 1}} \sup_{\substack{O_A \in \mathcal{A}_A \\ \|O_A\| = 1}} \operatorname{tr} \left(\rho \left[T_t^*(O_A) - T_{\phi,t}^*(O_A) \right] \right).
\end{aligned}
\tag{28}
$$

We say that $\mathcal{L}$ satisfies *local rapid mixing* if, for each $A \subset \Lambda$, we have that

$$
\eta^A(T_t) \leqslant k(|A|)e^{-\gamma t}, \tag{29}
$$

where $k(r)$ grows polynomially in r, $\gamma > 0$ and all the constants appearing above are independent of the system size.

Remark 7. It follows from the definition that $\eta^A(T_t) \leqslant \eta^B(T_t)$ whenever $A \subset B$. In particular, $\eta^A(T_t) \leqslant \eta(T_t)$.

Note that, in contrast with definition 5, the quantity $\eta^A(T_t)$ depends on the evolution *on the whole system Λ*, and not just on the subset A. Thus local rapid mixing is a very strong condition: the term $k(r)$ appearing in equation (29) only depends on the support of A, so the local mixing time (i.e. the time it takes for the reduced density matrix on the subset A to converge) is required to be independent of system size.

Example 3. A simple dissipative system satisfying definition 13 is the tensor product of amplitude damping channels acting (with the same rate) on different qubits. Note that, though it might seem a trivial example, there are interesting dissipative systems of this form: among others, dissipative preparation of graph states [31] can be brought into this form by a non-local unitary rotation (which of course does not change the convergence rates).

6. Proof of main result

We are now ready to prove our main resul, theorem 7. The proof proceeds in three steps. First, we show that the assumptions of theorem 7 imply that the fixed points of $\mathcal{L}^{\overline{\Lambda}}$ for different Λ are locally indistinguishable. Then, we prove that rapid mixing implies local rapid mixing. Finally, we show how local rapid mixing and the uniqueness of the initial fixed point imply the desired stability result.

6.1. Step 1: closeness of the fixed points. Topological quantum order (TQO), namely the property of certain orthogonal quantum states to be locally indistinguishable from each other, is a widely studied property of ground state subspaces in the Hamiltonian setting. In the dissipative setting on the other hand, where the concept of ground states is no longer applicable, one may define the analogous concept for periodic states of Lindbladians. Below we describe the concept of Local Topological Quantum Order (LTQO) [49], which extends the concept of TQO to the invariant subspace (periodic states) of local restrictions of the global Lindbladian.

We note that, in contrast to the Hamiltonian case, in order to prove the desired stability result we do not require extra assumptions like LTQO, or frustration-freeness. Indeed, we show in this section that rapid mixing *implies* LTQO and a property similar to frustration-freeness. These properties will play a role in the proof of stability, via lemma 17.

Definition 14 (Local Topological Quantum Order (LTQO)). Consider a Lindbladian $\mathcal{L}$. Take a convex set $A \subset \Lambda$ and let $A(\ell) = \{x \in \Lambda | \operatorname{dist}(x, A) \leqslant \ell\}$. Given two states $\rho_i \in \mathcal{X}_{\mathcal{L}_{A(\ell)}}$, $i = 1, 2$, consider their reduced density matrices on A:

$$\rho_i^A = \operatorname{tr}_{A(\ell) \setminus A} \rho_i, \quad i = 1, 2.$$

We say that $\mathcal{L}$ has *local topological quantum order* (LTQO) if for each $\ell \geqslant 0$:

$$\left\| \rho_1^A - \rho_2^A \right\|_1 \leqslant p(|A|) \, \Delta_0(\ell), \tag{30}$$

where $\Delta_0(\ell)$ is a fast-decaying function, and $p(\cdot)$ is a polynomial.

As a first step in the proof, we will show that the conditions of theorem 7 imply that the fixed point of T_t, the fixed point of $T_t^{\overline{A}}$ and the periodic points of T_t^A are difficult to distinguish locally, in the same spirit as the LTQO condition.

Lemma 15. *Let $\mathcal{L}$ be a uniform family satisfying condition* (11)*, and suppose each $\mathcal{L}^{\overline{A}}$ has a unique fixed point and no other periodic points. Let O_A be an observable supported on $A \subset \Lambda$, ρ a periodic point of $T_t^{A(s)}$ and ρ_∞^s the unique fixed point of $T_t^{\overline{A(s)}}$. Then, we have*

$$|\operatorname{tr} O_A(\rho - \rho_\infty^s)| \leqslant \|O_A\| \left(\frac{J}{v} |A| + c |A|^\delta \right) \Delta_0(s), \tag{31}$$

where

$$\Delta_0(s) = (|A(s)| / |A|)^{\delta v/(v+\gamma)} \nu_{\beta'}^{-1}(s), \quad \beta' = \beta\gamma/(v+\gamma),$$

with c, γ, δ the constants defined in the rapid mixing condition $RM(\gamma, \delta)$, β the rate defined in lemma 12 and v the Lieb-Robinson velocity.

Proof. Fix a $t := t(s) \geqslant 0$, to be determined later. Since $T_t^{A(s)}$ acts on its space of periodic points as a unitary evolution, there exists a periodic point of $\mathcal{L}^{A(s)}$, ρ', such that $\rho = T_t^{A(s)}(\rho')$. Then, by the triangle inequality, we have:

$$|\operatorname{tr} O_A(\rho - \rho_\infty^s)| \leqslant \left| \operatorname{tr} O_A[T_t^{A(s)} - T_t^{\overline{A(s)}}](\rho')) \right| + \left| \operatorname{tr} O_A(T_t^{\overline{A(s)}}(\rho') - \rho_\infty^s) \right|. \tag{32}$$

The first term is bounded by lemma 12, since

$$\operatorname{tr} O_A(T_t^{A(s)}(\rho') - \overline{T_t^{A(s)}}(\rho')) = \operatorname{tr} \rho'(T_t^{*\,A(s)}(O_A) - \overline{T_t^{*\,A(s)}}(O_A))$$

and

$$\left| \operatorname{tr} \rho'(T_t^{*\,A(s)}(O_A) - \overline{T_t^{*\,A(s)}}(O_A)) \right| \leqslant \|\rho'\|_1 \|T_t^{*\,A(s)}(O_A) - \overline{T_t^{*\,A(s)}}(O_A)\|_\infty.$$

The second term is bounded using the rapid mixing condition on $\overline{T_t^{A(s)}}$. By putting the two bounds together, we obtain

$$|\operatorname{tr} O_A(\rho - \rho_\infty^s)| \leqslant \|O_A\| \, |A| \, \frac{J}{v} e^{vt} \nu_\beta^{-1}(s) + \|O_A\| \, c \, |A(s)|^\delta \, e^{-\gamma t}.$$

Setting $p(s) = (|A(s)| / |A|)^\delta$ and choosing $t(s)$ such that

$$e^{vt(s)} \nu_\beta^{-1}(s) = p(s) e^{-\gamma t(s)},$$

we have that $t(s) = \ln(\nu_\beta(s) \cdot p(s))^{1/(v+\gamma)}$. Under such choice, it holds that

$$e^{-\gamma t(s)} = (\nu_\beta(s) p(s))^{-\gamma/(v+\gamma)} = \nu_{\beta'}^{-1}(s) p(s)^{-\gamma/(v+\gamma)},$$

where $\beta' = \beta\gamma/(v+\gamma)$. Defining

$$\Delta_0(s) := (|A(s)| / |A|)^{\delta v/(v+\gamma)} \nu_{\beta'}^{-1}(s),$$

concludes the proof. $\square$

Corollary 16 (LTQO). *Under the assumptions of lemma 15, the Lindbladian $\mathcal{L}^{\overline{\Lambda}}$ satisfies LTQO (definition 14) for all Λ.*

Proof. Take $A \subset \Lambda$, and $s \geqslant 0$. Let ρ_1 and ρ_2 be two periodic points of $T_t^{A(s)}$. Then, by the triangle inequality, we have that:

$$|\operatorname{tr} O_A(\rho_1 - \rho_2)| \leqslant |\operatorname{tr} O_A(\rho_1 - \rho_\infty^s)| + |\operatorname{tr} O_A(\rho_\infty^s - \rho_2)| \leqslant$$
$$\leqslant 2 \|O_A\| \left(\frac{J}{v} |A| + c \, |A|^\delta \right) \Delta_0(s).$$

Since $\|\rho_1^A - \rho_2^A\|_1 = \sup_{\|O_A\|=1} |\operatorname{tr} O_A(\rho_1 - \rho_2)|$, the result follows immediately.
                                                                                $\square$

Lemma 17. *Under the same notation and assumptions of lemma 15, we have the following bound for ρ_∞ the unique fixed point of T_t:*

$$\sup_{\|O_A\|=1} |\operatorname{tr} O_A(\rho_\infty - \rho_\infty^s)| \leqslant \|O_A\| \left(\frac{J}{v} |A| + c \, |A|^\delta \right) \Delta_0(s). \qquad (33)$$

Proof. By the triangle inequality:

$$\left|\mathrm{tr}\, O_A(\rho_\infty - \rho_\infty^s))\right| \leqslant \left|\mathrm{tr}\, O_A(\rho_\infty - T_t^{\overline{A(s)}}(\rho_\infty)\right| + \left|\mathrm{tr}\, O_A(T_t^{\overline{A(s)}}(\rho_\infty) - \rho_\infty^s)\right|.$$

The first term on the right can be bounded using lemmas 11 and 12 along with $T_t(\rho_\infty) = \rho_\infty$:

$$\left|\mathrm{tr}\, O_A\big(T_t(\rho_\infty) - T_t^{\overline{A(s)}}(\rho_\infty)\big)\right| = \left|\mathrm{tr}\, \rho_\infty(T_t^*(O_A) - T_t^{*\overline{A(s)}}(O_A))\right|$$

$$\leqslant \|\rho_\infty\|_1 \left(\|T_t^*(O_A) - T_t^{*A(s)}(O_A)\|_\infty + \|T_t^{*A(s)}(O_A) - T_t^{*\overline{A(s)}}(O_A)\|_\infty \right)$$

$$\leqslant \|O_A\|\,|A|\,\tfrac{J}{v}e^{vt}\nu_\beta^{-1}(s).$$

The second term is bounded using the rapid mixing condition:

$$\left|\mathrm{tr}\, O_A\big(T_t^{\overline{A(s)}}(\rho_\infty) - \rho_\infty^s\big)\right| \leqslant \|O_A\|\,c\,|A|^\delta\,p(s)e^{-\gamma t}.$$

By making the same choice of $t = t(s)$ as in lemma 15, we get the desired bound. $\qquad\square$

Corollary 18 (Approximate frustration-freeness). *Under the same notation and assumptions of lemma 15, denote by ρ_∞ the unique fixed point of T_t, and by ρ a periodic point of $T_t^{A(s)}$. Then, we have the following bound:*

$$\sup_{\|O_A\|=1} |\mathrm{tr}\, O_A(\rho_\infty - \rho)| \leqslant 2\,\|O_A\| \left(\frac{J}{v}\,|A| + c\,|A|^\delta \right) \Delta_0(s). \qquad (34)$$

Proof. By the triangle inequality and lemmas 15 and 17, we have:

$$|\mathrm{tr}\, O_A(\rho_\infty - \rho))| \leqslant |\mathrm{tr}\, O_A(\rho_\infty - \rho_\infty^s)| + |\mathrm{tr}\, O_A(\rho_\infty^s - \rho)| \leqslant$$

$$\leqslant 2\,\|O_A\| \left(\frac{J}{v}\,|A| + c\,|A|^\delta \right) \Delta_0(s).$$

$\qquad\square$

6.2. Step 2: from global to local rapid mixing. As a second step in the proof, we show that the assumptions on $\mathcal{L}$ imply local rapid mixing.

Proposition 19 (From global to local rapid mixing). *Let $\mathcal{L}$ be a uniform family of Lindbladians with unique fixed point. Then, if condition (11) is satisfied, $\mathcal{L}$ satisfies local rapid mixing.*

Proof. Let O_A be an observable supported on A with $\|O_A\| = 1$. Denote by s_0 the minimum $s \geqslant 0$ such that $A(s) = \Lambda$. Fix $0 \leqslant s \leqslant s_0$, and let $B = A(s)$. Then, by the triangle inequality, we can bound the norm of $(T_t^* - T_\infty^*)$ as follows:

$$\|(T_t^* - T_\infty^*)O_A\| \leqslant \left\|(T_t^* - T_t^{\overline{B}*})O_A\right\| + \left\|(T_t^{\overline{B}*} - T_\infty^{\overline{B}*})O_A\right\| + \left\|(T_\infty^{\overline{B}*} - T_\infty^*)O_A\right\|. \qquad (35)$$

We bound the first term on the right using lemmas 11 and 12:

$$\left\| (T_t^* - T_t^{\overline{B}*})O_A \right\| \leqslant |A| \frac{J}{v}(e^{vt} - 1 - vt)e^{-\beta s}. \tag{36}$$

The second term is bounded by the rapid mixing condition (11), setting $p(s) = (|A(s)|\,/\,|A|)^{\delta}$:

$$\left\| (T_t^{\overline{B}*} - T_\infty^{\overline{B}*})O_A \right\| \leqslant \eta(T_t^{\overline{B}}) \leqslant c\,|A|^{\delta}\,p(s)e^{-\gamma t}. \tag{37}$$

Finally, the third term is bounded by using lemma 17:

$$\left\| (T_\infty^{\overline{B}*} - T_\infty^{*})O_A \right\| = |\mathrm{tr}\,O_A(\rho_\infty^s - \rho_\infty)| \leqslant \left(\frac{J}{v}|A| + c\,|A|^{\delta} \right) \Delta_0(s). \tag{38}$$

Substituting bounds (36), (37) and (38) into equation (35), we obtain, for $0 \leqslant s \leqslant s_0$ and for all $t \geqslant 0$:

$$\eta^A(T_t) \leqslant \frac{J}{v}|A|\,e^{vt}e^{-\beta s} + c\,|A|^{\delta}\,p(s)e^{-\gamma t} + \left(\frac{J}{v}|A| + c\,|A|^{\delta} \right)\Delta_0(s).$$

We want to show that we can choose $s = s(t) \in [0, s_0]$ in such a way that both $e^{vt}e^{-\beta s}$ and $e^{-t\gamma}p(s)$ are exponentially decaying in t. Choose $s := s(t) = t(v+\gamma)/\beta$. Since $\Delta_0(s) = (|A(s)|\,/\,|A|)^{\delta v/(v+\gamma)}\nu_{\beta'}^{-1}(s)$, denoting

$$\bar{p}(t) = p \circ s(t) = p(t(v+\gamma)/\beta),$$

we have that

$$\Delta_0(s(t)) = \bar{p}(t)^{v/(v+\gamma)}e^{-\gamma t}.$$

Therefore, since $\bar{p}(t) \geqslant 1$,

$$\eta^A(T_t) \leqslant \frac{J}{v}|A|\,e^{-\gamma t} + c\,|A|^{\delta}\,\bar{p}(t)e^{-\gamma t} + \left(\frac{J}{v}|A| + c\,|A|^{\delta} \right)\bar{p}(t)^{v/(v+\gamma)}e^{-\gamma t} \leqslant$$

$$\leqslant 2\left(\frac{J}{v}|A| + c\,|A|^{\delta} \right)\bar{p}(t)e^{-\gamma t}, \quad \forall t \leqslant \frac{\beta}{v+\gamma}s_0.$$

When $t \geqslant \beta/(v+\gamma)s_0$, we can simply bound $\eta^A(T_t)$ by $\eta(T_t)$ (see remark 7), obtaining:

$$\eta^A(T_t) \leqslant c\,|A|^{\delta}\,p(s_0)e^{-\gamma t} \leqslant c\,|A|^{\delta}\,\bar{p}(t)e^{-\gamma t}, \quad \forall t \geqslant \frac{\beta}{v+\gamma}s_0.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

6.3. Step 3: from local rapid mixing to stability. We now prove that local rapid mixing alone implies stability. This is the last step in the proof of theorem 7, as we already proved in the previous sections that the condition of theorem 7 imply local rapid mixing. However, the following result also stands independently: if a system can be shown to satisfy local rapid mixing by other means, it will also be stable. Moreover, the same proof holds if we relax the assumption on prefactor $k(|A|)$ in equation (29): a similar (but weaker) stability result will hold true as long as $|A|$ is independent of system size.

Theorem 20. *Let $\mathcal{L}$ be a local Lindbladian satisfying local rapid mixing, and having a unique fixed point ρ_∞ such that*

$$T_\phi^*(O_A) = T_\infty^*(O_A) = \mathrm{tr}(O_A \rho_\infty)\mathbb{1}.$$

Then, using the notation of theorem 7, for all observables O_A supported on $A \subset \Lambda$ we have that

$$\|T_t^*(O_A) - S_t^*(O_A)\| \leqslant c(|A|)\,\|O_A\|\left(\varepsilon + |\Lambda|\,\nu_\eta^{-1}(d_A)\right), \qquad (39)$$

where $d_A = \mathrm{dist}(A, \Lambda^c)$; η is positive and independent of Λ; $\nu_\eta^{-1}(d) \leqslant (1+d)^{-D-1}$; $c(|A|)$ is independent of Λ and t, and is bounded by a polynomial in $|A|$.

Proof. Let $O_0(t) = T_t^*(O_A)$ and $O_1(t) = S_t^*(O_A)$ and write the difference $O_0 - O_1$ using the integral representation from equation (21):

$$O_0(t) - O_1(t) = \int\limits_0^t S_{t-s}^* E^* T_s^*(O_A)\,\mathrm{d}s.$$

The triangle inequality implies:

$$\|O_0(t) - O_1(t)\| \leqslant \sum_u \sum_r \int\limits_0^t \left\|E_{u,r}^* O_0(s)\right\|\,\mathrm{d}s + \sum_d \int\limits_0^t \left\|E_d^* O_0(s)\right\|\,\mathrm{d}s,$$

where we used the fact that S_t is a contraction.

Fix a $K \in \{E_{u,r}\}_{u,r} \cup \{E_d\}_d$, and let $\delta = \mathrm{dist}(A, \mathrm{supp}\,K)$. We can split the integral at a time t_0 (to be fixed later, depending on δ). We bound the first part of the integral with Lieb-Robinson bounds:

$$\int\limits_0^{t_0} \|K^* O_0(s)\|\,\mathrm{d}s \leqslant \|K\|_{1\to 1,cb}\,\|O_A\|\,|A|\,\frac{e^{vt_0} - vt_0 - 1}{v\nu_\mu(\delta)}.$$

Now pick $t_0 = t_0(\delta)$ such that

$$\nu_\mu^{-1}(\delta)\frac{e^{vt_0} - vt_0 - 1}{v} \leqslant \nu_{\mu/2}^{-1}(\delta).$$

We can choose $t_0(\delta) = \frac{\mu}{2}\frac{\log v}{v}\delta = O(\delta)$, for exponentially decaying (or faster) $\nu_\mu^{-1}(\delta)$.

If $t \leqslant t_0(\delta)$, then we have bounded the entire integral, and we are done. Otherwise, we treat the second part of the integral as follows:

$$\int_{t_0(\delta)}^{t} \|K^*O_0(s)\| \, \mathrm{d}s = \int_{t_0(\delta)}^{t} \|K^*(O_0(s) - T_\infty^*(O_A))\| \, \mathrm{d}s$$

$$\leqslant \|K\|_{1\to 1, cb} \, \|O_A\| \int_{t_0(\delta)}^{\infty} \eta^A(T_s) \, \mathrm{d}s \leqslant \|K\|_{1\to 1, cb} \, \|O_A\| \, q(|A|) \int_{t_0(\delta)}^{\infty} e^{-\gamma s} \, \mathrm{d}s$$

$$= \|K\|_{1\to 1, cb} \, \|O_A\| \, k(|A|) \frac{1}{\gamma} e^{-\gamma t_0(\delta)}$$

where we used $K^* T_\infty^*(O_A) = K^*(\mathrm{tr}(\rho_\infty O_A) \mathbb{1}) = \mathrm{tr}(\rho_\infty O_A) K^*(\mathbb{1}) = 0$, together with the local rapid mixing condition.

Since $t_0(\delta)$ is linear in δ, we have that:

$$h(\delta) := e^{-\frac{\mu\delta}{2}} + \frac{1}{\gamma} e^{-\gamma t_0(\delta)}$$

is exponentially decaying in δ.

Putting the different bounds together, we obtain:

$$\int_0^t \|K^*O_0(s)\| \, \mathrm{d}s \leqslant \|K\|_{1\to 1, cb} \, \|O_A\| \, k_1(|A|) h(\delta),$$

where $k_1(|A|) = \max(k(|A|), |A|)$.

Returning to the sum, we have proven that:

$$\|O_0(t) - O_1(t)\| \leqslant \varepsilon \, k_1(|A|) \, \|O_A\| \left[\underbrace{\sum_u \sum_r e(r) h(\mathrm{dist}(A, b_r(u)))}_{I_1(A; e, h)} \right.$$

$$\left. + \underbrace{\sum_d |\partial_d \Lambda| \, e(d) h(\mathrm{dist}(A, \partial_d \Lambda))}_{I_2(A; e, h)} \right]. \quad (40)$$

It suffices to show that I_1 and I_2 are finite (and independent of system size), and that I_2 decays exponentially in $\mathrm{dist}(A, \Lambda^c)$. Let us decompose the I_1 as follows

$$I_1(A; e, h) = \sum_u \sum_r e(r) h(\mathrm{dist}(A, b_r(u)))$$

$$= \sum_{\mathrm{dist}(u, A)=0} \sum_r e(r) h(0) + \sum_{d>0} \sum_{\mathrm{dist}(u, A)=d} \left(\sum_{r=0}^{d} e(r) h(d-r) + \sum_{r=d+1}^{\infty} e(r) h(0) \right)$$

$$= h(0) \, |A| \sum_r e(r) + \sum_{d>0} q(d) \left(\sum_{r=0}^{d} e(r) h(d-r) + h(0) \sum_{r=d+1}^{\infty} e(r) \right),$$

where $q(d) = |\{u : \text{dist}(u, A) = d\}|$ grows polynomially in d.

The first term is clearly bounded, since $e(r)$ is summable. Since e and h are both exponentially decaying functions, their discrete convolution $e \star h(d) = \sum_{r=0}^{d} e(r)h(d-r)$ is also exponentially decaying, and consequently summable against any polynomial. The same holds for $\sum_{r>d} e(r)$. This proves that the second term is also bounded.

On the other hand, we have that

$$I_2(A; e, h) = \sum_d |\partial_d \Lambda| \, e(d) h(\text{dist}(A, \partial_d \Lambda)) \leqslant |\Lambda| \left(e \star h(d_A) + \sum_{d \geqslant d_A} e(d) \right),$$

where $d_A = \text{dist}(A, \Lambda^c)$. We have just proven that $e \star h(d_A)$ and $\sum_{d \geqslant d_A} e(d)$ are exponentially decaying. This implies that there exists a positive η such that $\nu_\eta^{-1}(d_A)$ upper bounds both. Denoting $c(|A|) = k_1(|A|)I_1(A; e, h)$, we have the desired bound. $\qquad\square$

6.4. Power-law decay. As we stated before, the results and proofs presented above still hold when $\mathcal{L}$ has quasi-local or power-law interactions. In the latter case, this is only true when certain conditions are met on the decay of $\mathcal{L}$. In what follows, we highlight the changes one needs to make in the case of power-law decay, in order for the main stability results to hold.

Definition 21 (Compatibility condition). Let $\mathcal{L}$ be a local Lindbladian, and suppose it satisfies (LR-2) and rapid mixing RM(γ, δ). Let μ and v be the Lieb-Robinson constants for $\mathcal{L}$ defined in assumption 1 and β the constant defined in lemma 11. Then we say that $\mathcal{L}$ satisfies the weak compatibility condition for stability, if the following inequality is satisfied.

$$\beta\gamma - \delta D v > 0; \tag{CC-1}$$

we say that $\mathcal{L}$ satisfies the strong compatibility condition for stability if

$$\mu \frac{\beta\gamma - \delta D v}{\beta(\gamma + v)} > D + 2. \tag{CC-2}$$

Moreover, if the perturbation E, defined in theorem 7, is decaying polynomially and not exponentially, it must satisfy

$$\sum_n n^D \sum_{r>n} e(r) < \infty \tag{CC-e}$$

for the theorem to hold.

Remark 8. Clearly, the strong version of the compatibility condition implies the weak one. If $\mathcal{L}$ has quasi-local interactions, then the (polynomial) decay rate α of the interactions can be chosen to be larger than any fixed value. Consequently, since β and μ can be taken to be linear in α, quasi-local Lindbladians $\mathcal{L}$ satisfy the strong compatibility condition (CC-2).

Under the weak compatibility condition, all the results presented in sections 6.1 and 6.2 still hold true, while under the strong compatibility condition also the results presented in 6.3 are still valid, and in particular our main result, theorem 20.

We will now show this in the cases in which we made explicit use of condition (LR-1), and give the needed modifications to the proofs of lemma 15, proposition 19 and theorem 20 in order to make them valid for power-law decaying interactions.

From now on, we proceed under the working hypothesis that $\mathcal{L}$ satisfies (LR-2) and that the above compatibility conditions are satisfied.

Proof (Modifications in the proof of lemma 15). The argument below follows closely the proof of the original lemma, but now one must check that $\Delta_0(s)$ is still decaying. Recall the definition of $\Delta_0(s)$ from the original proof of lemma 15:

$$\Delta_0(s) = (|A(s)| / |A|)^{\delta v/(v+\gamma)} \nu_{\beta'}^{-1}(s), \quad \beta' = \beta\gamma/(v+\gamma).$$

Since $(|A(s)| / |A|)^{\delta v/(v+\gamma)}$ grows as $(1+s)^{\delta Dv/(v+\gamma)}$, we have:

$$\Delta_0(s) \sim (1+s)^{-\gamma'},$$

where $\gamma' = \frac{\beta\gamma - \delta Dv}{v+\gamma}$ is positive because of (CC-1).   □

Proof (Modifications in the proof of proposition 19). Keeping the notation introduced in the original proof of this proposition, we have already shown that, for each $0 \leqslant s \leqslant s_0$:

$$\eta^A(T_t) \leqslant \frac{J}{v}|A|\, e^{vt}\nu_\beta^{-1}(s) + c\,|A|^\delta\, p(s)e^{-\gamma t} + \left(\frac{J}{v}|A| + c\,|A|^\delta\right)\Delta_0(s).$$

At this point, we can no longer choose $s = s(t)$ to scale linearly in t, since the decay $\nu_\beta^{-1}(s)$ is polynomial in s and the prefactor e^{vt} would render the bound trivial. Still, we may choose $s = s(t) \in [0, s_0]$ in such a way that the r.h.s. above is exponentially decaying in t.

Fix $k > 0$ (to be determined later), and consider:

$$s(t) = e^{kt} - 1,$$

in such a way that for $t \leqslant \log(1 + s_0)/k$, we have:

$$\bar{p}(t) = p \circ s(t) = \left(\left|A(e^{kt} - 1)\right| / |A|\right)^\delta \sim e^{kD\delta t}.$$

Then, the r.h.s. of the desired bound for $\eta^A(T_t)$ contains the following exponentials:

$$e^{vt}\nu_\beta^{-1}(s) = e^{-(\beta k - v)t}; \quad p(s)e^{-\gamma t} \sim e^{-(\gamma - kD\delta)t},$$

and

$$\Delta_0(s) \sim (1+s)^{-\gamma'} = e^{-k\gamma't},$$

where

$$\gamma' = \frac{\beta\gamma - \delta Dv}{v + \gamma}$$

is defined in the modified proof of lemma 15. We want to show that we can choose k in such a way that all the exponential functions appearing above are decaying, i.e. each exponent is negative for $t > 0$. (CC-1) implies that $\Delta_0(s)$ is decaying for all $k > 0$. Let

$$k' = \frac{v + \gamma}{\beta + \delta D},$$

such that $\beta k' - v = \gamma - k' D\delta = k'\gamma'$, making all of the above exponents equal to $-(\beta\gamma - \delta Dv)/(\beta + \delta D)$ and negative (due to (CC-1)), as desired.

When $t \geqslant \log(1 + s_0)/k'$, as in the proof for exponentially decaying interactions, we bound $\eta^A(T_t)$ by $\eta(T_t)$ (see remark 7), thus obtaining:

$$\eta^A(T_t) \leqslant c\,|A|^\delta\,p(s_0)e^{-\gamma t} \leqslant c\,|A|^\delta\,\bar{p}(t)e^{-\gamma t} \sim c\,|A|^\delta\,e^{-k'\gamma' t}.$$

$\square$

Proof (Modifications in the proof of theorem 20). Following the same steps as in the original proof, but now using the constants for the local rapid mixing obtained in the modified proof of proposition 19, we have that, for each $0 \leqslant t_0 \leqslant t$:

$$\int_0^t \|K^*O_0(s)\|\,\mathrm{d}s \leqslant \|K\|_{1\to 1,cb}\,\|O_A\| \left(|A|\,\frac{1}{v}e^{vt_0}\nu_\mu^{-1}(d) + k(|A|)e^{-t_0\frac{\beta\gamma-\delta Dv}{\beta+\delta D}}\right),$$

where $d = \mathrm{dist}(A, \mathrm{supp}\,K)$.

Let us define $t_0(d) = k\log(1+d)$ for some positive k (to be determined later), and denote $h(d) = \nu_{vk-\mu}(d) + \nu_{-k\frac{\beta\gamma-\delta Dv}{\beta+\delta D}}(d)$, such that

$$\int_0^t \|KO_0(s)\|\,\mathrm{d}s \leqslant \|K\|_{1\to 1,cb}\,\|O_A\|\,k_1(|A|)h(d),$$

where $k_1(|A|) = \max(k(|A|), |A|/v)$. Then we have that h has a maximum decay rate of

$$\mu' = \sup_{k\geqslant 0}\min\left(\mu - vk, k\frac{\beta\gamma - \delta Dv}{\beta + \delta D}\right).$$

The optimal choice of k is $k = \frac{\mu}{\beta}\frac{\beta+\delta D}{v+\gamma}$, in such a way that $\mu' = \frac{\mu}{\beta}\frac{\beta\gamma-\delta Dv}{v+\gamma}$. μ' is positive because of condition (CC-1).

Recalling the following definitions from the original proof of theorem 20:

$$q(d) = |\{u : \mathrm{dist}(u, A) = d\}|, \qquad\qquad l(d) = |\partial_d\Lambda|\,e(d),$$

$$x \star y(d) = \sum_{r=0}^d x(r)y(d-r), \qquad\qquad d_A = \mathrm{dist}(A, \Lambda^c),$$

we need to show that

$$I_1(A; e, h) = h(0)\,|A|\sum_r e(r) + \sum_{d>0} q(d)\left(e \star h(d) + h(0)\sum_{r>d} e(r)\right)e \star h(d)$$

is finite, and that

$$I_2(A;\, e, h) = l \star h(d_A) + \sum_{d \geqslant d_A} l(d) \leqslant \nu_\eta^{-1}(d_A)$$

for some positive η. Notice that

$$I_1(A;\, e, h) \leqslant (1 + |A|)h(0) \sum_d q(d) \sum_{r>d} e(r) + \sum_d q(d)\, e \star h(d).$$

Since $q(d)$ grows as $(1+d)^D$, $\sum_d q(d) \sum_{r \geqslant d} e(r)$ is finite if $\sum_n n^D \sum_{r>n} e(r) < \infty$, which is condition (CC-e).

On the other hand, $e \star h$ decays as the slowest of the two functions. Since we have already assumed that $\sum_d q(d) \sum_{r>d} e(r)$ is finite, we only need to satisfy that $\sum_d q(d) \sum_{r>d} h(r)$ is finite. For this to happen, it suffices that $\mu' > D + 2$, which is condition (CC-2).

In order to bound $I_2(A; e, h)$, note that $l(d) \leqslant |A|\, e(d)$, and therefore

$$I_2(A; e, h) \leqslant |A|\, (e \star h(d_A) + \sum_{d \geqslant d_A} e(d)).$$

We have already proven that conditions (CC-2) and (CC-e) imply that the r.h.s. of the latter bound is decaying polynomially in d_A at least as fast as $\nu_{D+1}^{-1}(d_A)$.

This concludes the proof.   □

6.5. Relaxing rapid mixing. In this section, we will show that, in the case of exponentially decaying interactions (LR-1), the proof of theorem 7 still holds if

$$\sum_n n^D \sum_{d>n} \int_d^\infty \left(1 + \frac{v}{\beta}s - \frac{1}{\beta}\log \gamma(s)\right)^{\delta D} \gamma(s)\, \mathrm{d}s < \infty. \tag{41}$$

We will directly prove proposition 19, without the intermediate step of a lemma like 15. Nonetheless, results of that kind can be proven using exactly the same arguments that we will use in the following proof.

Proof (Proof of proposition 19). Using the same notation as in the original proof, we have that for $0 \leqslant s \leqslant s_0$:

$$\|(T_t^* - T_\infty^*)O_A\| \leqslant \left\|(T_t^* - T_t^{\overline{B}*})O_A\right\| + \left\|(T_t^{\overline{B}*} - T_\infty^{\overline{B}*})O_A\right\| + \left\|(T_\infty^{\overline{B}*} - T_\infty^*)O_A\right\|.$$

We will bound the first two terms as in the original proof (using lemma 11, lemma 12 and equation (15)) while we rewrite the third term as in the proof of lemma 15:

$$\left\|(T_\infty^{\overline{B}*} - T_\infty^*)O_A\right\| = |\mathrm{tr}\, O_A(\rho_\infty^s - \rho_\infty)| \leqslant$$

$$\left|\mathrm{tr}\, O_A(\rho_\infty - T_t^{\overline{A(s)}}(\rho_\infty))\right| + \left|\mathrm{tr}\, O_A(T_t^{\overline{A(s)}}(\rho_\infty) - \rho_\infty^s)\right| =$$

$$\left|\mathrm{tr}\, O_A(T_t(\rho_\infty) - T_t^{\overline{A(s)}}(\rho_\infty))\right| + \|O_A\| \left\|T_t^{\overline{A(s)}}(\rho_\infty) - \rho_\infty^s\right\|_1 \leqslant$$

$$\left\|(T_t^* - T_t^{\overline{B}*})O_A\right\| + \eta(T_t^{\overline{B}}).$$

Thus we have that

$$\|(T_t^* - T_\infty^*)O_A\| \leqslant 2\frac{J}{v}\,|A|\,e^{vt}\nu_\beta^{-1}(s) + 2c\,|A|^\delta\,p(s)\gamma(t),$$

where $p(s) = (|(A(s)|\,/\,|A|)^\delta \sim (1+s)^{\delta D}$.

We have claimed that the result only holds in the case of exponentially decaying or faster decay of interaction. Suppose $\nu_\beta(s) = (1+s)^\beta$ (i.e., if $\mathcal{L}$ satisfies (LR-2)). Defining $s = s(t)$ as

$$s(t) = e^{\frac{v}{\beta+\delta D}t}\gamma(t)^{-\frac{1}{\beta+\delta D}} - 1,$$

then it holds that

$$\frac{e^{vt}}{\gamma(t)} = p(s(t))\nu_\beta(s(t)) \quad \forall t \leqslant t_0,$$

where t_0 is such that $s(t_0) = s_0$. Thus

$$\delta_0(t) := e^{vt}\nu_\beta^{-1}(s) = e^{\frac{\delta D}{\beta+\delta D}vt}\gamma(t)^{\frac{\beta}{\beta+\delta D}}.$$

We have that this last function is decaying in t if

$$\gamma(t) < e^{-(v\delta D/\beta)t}.$$

This forces $\gamma(t)$ to be exponentially decaying, and thus there is no possible relaxation of the rapid mixing condition.

On the other hand, if $\nu_\beta(s) = e^{\beta s}$ (i.e., if $\mathcal{L}$ satisfies (LR-1)), we define

$$s(t) = \frac{v}{\beta}t - \frac{1}{\beta}\log\gamma(t),$$

such that $e^{vt}\nu_\beta^{-1}(s) = \gamma(t)$ and

$$\bar{p}(t) = p \circ s(t) \sim \left(1 + \frac{v}{\beta}t - \frac{1}{\beta}\log\gamma(t)\right)^{\delta D}$$

grows polynomially.

In this case, we have proved that

$$\|(T_t^* - T_\infty^*)O_A\| \leqslant 2\left(\frac{J}{v}\,|A| + c\,|A|^\delta\right)\bar{p}(t)\gamma(t),$$

and this concludes the proof since equation (41) implies that $\bar{p}(t)\gamma(t)$ is decaying in t. □

Proof (Proof of theorem 20). Following the same steps as in the original proof, we have that for any $K \in \{E_{u,r}\}_{u,r} \cup \{E_d\}_d$, and let $\delta = \mathrm{dist}(A, \mathrm{supp}\,K)$

$$\int_0^t \|K^*O_0(s)\|\,\mathrm{d}s \leqslant \|K\|_{1\to 1,cb}\,\|O_A\|\,k_1(|A|)h(\delta).$$

where $h(\delta)$ is now

$$h(\delta) = e^{-\mu\delta/2} + \int\limits_{\frac{\mu}{2}\frac{\log v}{v}\delta}^{\infty} \bar{p}(s)\gamma(s)\,\mathrm{d}s.$$

We want to show that $h(\delta)$ is decaying fast enough for the r.h.s. of equation (40) to be summable. This is the case (see the proof of theorem 20 in section 6.4) if equation (41) holds. □

7. Glauber dynamics

7.1. Quantum embedding of Glauber dynamics. As an example of a non-trivial dynamics for which we can now prove stability using our results, we turn to one of the most studied dynamics in classical statistical mechanics: Glauber dynamics, a Markov process that samples thermal states of local (classical) Hamiltonians on lattices. Apart from being an interesting model in itself, it has important applications in Monte-Carlo Markov chain algorithms for numerical many-body physics [41]. Determining whether Glauber dynamics is stable against noise or errors is therefore an important question and, as far as we are aware, still open (with partial results obtained under the assumption of attractiveness [27]).

In this section, we present a natural embedding of Glauber dynamics into the Linbdlabian setting, showing how this embedded dynamics inherits properties from the classical Markov chain[8]. We will then apply the results of section 4 to prove, in the appropriate regime, stability of Glauber dynamics.

We will consider a lattice spin system over $\Gamma = \mathbb{Z}^D$ or $\Gamma = (\mathbb{Z}/L\mathbb{Z})^D$, with (classical) configuration space of a single spin a finite set S. For simplicity, we will consider the case $S = \{+1, -1\}$. For each $\Lambda \subset \Gamma$, we will denote by Ω_Λ the space of configurations over Λ, namely S^Λ. Λ^c will denote the complementary of Λ in Γ, namely $\Gamma \setminus \Lambda$.

Definition 22. A finite range, translationally-invariant potential $\{J_A\}_{A \subset \Gamma}$ is a family of real functions indexed by the non empty finite subsets of Γ satisfying the following properties:

1. $J_A : \Omega_A \to \mathbb{R}$.
2. For all $A \subset \Gamma$ and all $x \in \Gamma$:

$$J_A(\sigma) = J_{A+x}(\eta) \quad \text{if} \quad \sigma(y+x) = \eta(y) \quad \forall y \in A.$$

3. There exists a positive $r > 0$ such that $J_A = 0$ if $\operatorname{diam} A > r$, called the range of interaction.

Given a finite-range, translationally-invariant potential, we can define a Hamiltonian for each finite lattice $\Lambda \subset \Gamma$ and each *boundary condition* $\tau \in \Omega_{\Lambda^c}$ by

$$H_\Lambda^\tau(\sigma) = - \sum_{A \cap \Lambda \neq 0} J_A(\sigma \times \tau) \quad \forall \sigma \in \Omega_\Lambda,$$

---

[8]   A similar construction was proposed in [4].

where $\sigma \times \tau$ is the configuration that agrees with σ over Λ and with τ over Λ^c. For each such Hamiltonian, we define the Gibbs state state as

$$\mu_\Lambda^\tau(\sigma) = (Z_\Lambda^\tau)^{-1} \exp(-H_\Lambda^\tau(\sigma)),$$

where Z_Λ^τ is a normalizing constant.[9] The convex hull of the set of Gibbs states over Λ will be denoted by $\mathcal{G}(\Lambda)$:

$$\mathcal{G}(\Lambda) = \operatorname{conv}\{\mu_\Lambda^\tau \mid \tau \in \Omega_{\Lambda^c}\}.$$

Definition 23. The Glauber dynamics for a potential J is the Markov process on Ω_Λ with the following generator:

$$(Q_\Lambda f)(\sigma) = \sum_{x \in \Lambda} c_J(x, \sigma) \nabla_x f(\sigma),$$

where $\nabla_x f(\sigma)$ if defined as $f(\sigma^x) - f(\sigma)$, and σ^x is the configuration obtained by flipping the spin at position x:

$$\sigma^x(y) = \begin{cases} \sigma(y) & \text{if } x \neq y \\ -\sigma(x) & \text{if } x = y. \end{cases}$$

The numbers $c_J(x, \sigma)$ are called transition rates and must satisfy the following assumptions:

1. Positivity and boundedness: There exist positive constants c_m and c_M such that:
$$0 < c_m \leqslant c_J(x, \sigma) \leqslant c_M < \infty \quad \forall x, \sigma.$$

2. Finite range: $c_J(x, \cdot)$ depends only on spin values in $b_r(x)$.

3. Translational invariance: for all $k \in \Gamma$,
$$c_J(x, \sigma') = c_J(x + k, \sigma) \quad \text{if} \quad \sigma'(y) = \sigma(y + k) \quad \forall y.$$

4. Detailed balance: for all $x \in \Gamma$ and all σ
$$\exp\left(-\sum_{A \ni x} J_A(\sigma)\right) c_J(x, \sigma) = c_J(x, \sigma^x) \exp\left(-\sum_{A \ni x} J_A(\sigma^x)\right).$$

These assumptions are sufficient to ensure that Q_Λ generates a Markov process which has the Gibbs states over Λ as stationary points.

Definition 24. A quantum embedding of the classical Glauber dynamics for a potential J is generated by the following Lindblad operators

$$L_{x,\eta} = \sqrt{c_J(x, \eta)} \, |\eta^x\rangle\langle\eta| \otimes \mathbb{1}, \quad \forall x \in \Lambda, \forall \eta \in \Omega_{b_x(r)}; \tag{42}$$

$$\mathcal{L}_{x,\eta}(\rho) = L_{x,\eta} \rho L_{x,\eta}^* - \frac{1}{2}\left\{\rho, c_J(x, \eta) \, |\eta\rangle\langle\eta|\right\};$$

$$\mathcal{L}_\Lambda(\rho) = \sum_{x \in \Lambda} \sum_\eta L_{x,\eta} \rho L_{x,\eta}^* - \frac{1}{2}\{\rho, K\}, \quad K = \sum_\sigma \left(\sum_x c_J(x, \sigma)\right) |\sigma\rangle\langle\sigma|; \tag{43}$$

---

[9] Following [46], in our notation we have incorporated the usual inverse temperature parameter β directly into the potential J.

plus a dephasing channel acting independently and uniformly on all sites $x \in \Lambda$:

$$D_{x,0} = \sqrt{\gamma}\,|0\rangle\langle 0|\,, \quad D_{x,1} = \sqrt{\gamma}\,|1\rangle\langle 1|\,, \quad \mathcal{D}(\rho) = \sum_{x\in\Lambda}\sum_{i=0,1} D_{x,i}\rho D_{x,i}^* - |\Lambda|\,\gamma\rho.$$

(44)

$\mathcal{L}_\Lambda$ satisfies translational invariance because the transition rates c_J do, and it easy to see that this family of Lindbladians is uniform.

Remark 9. Take $|\alpha\rangle\langle\beta|$ an element of the computational basis, and let $d(\alpha,\beta)$ be the Hamming distance between α and β. Then it holds that

$$\mathcal{D}(|\alpha\rangle\langle\beta|) = -\gamma d(\alpha,\beta)\,|\alpha\rangle\langle\beta|\,.$$

In other words, $\mathcal{D}$ is a Schur multiplier in the computational basis, represented by $(-\gamma d(\alpha,\beta))_{\alpha,\beta}$.

On the other hand, we have that for all x:

$$\sum_{\eta\in\Omega_{b_x(r)}} \mathcal{L}_{x,\eta}(|\alpha\rangle\langle\beta|) = \begin{cases} c_J(x,\alpha)\,(|\alpha^x\rangle\langle\beta^x| - |\alpha\rangle\langle\beta|) & \text{if } \alpha|_{b_x(r)} = \beta|_{b_x(r)}, \\[2mm] -\frac{1}{2}\,(c_J(x,\alpha) + c_J(x,\beta))\,|\alpha\rangle\langle\beta| & \text{otherwise.} \end{cases}$$

(45)

Since $d(\alpha^x,\beta^x) = d(\alpha,\beta)$, $[\mathcal{D},\sum_\eta \mathcal{L}_{x,\eta}] = 0$ for all $x \in \Lambda$, and in particular $\mathcal{D}$ and $\mathcal{L}_\Lambda$ commute.

This quantum dissipative system inherits various properties from its classical counterpart.

Definition 25. Let μ be a full-rank positive state. Denote by

$$\Gamma_\mu(\rho) = \mu^{\frac{1}{2}}\rho\mu^{\frac{1}{2}}\,.$$

We say that $\mathcal{L}$ is in detailed balance [59,37,44,45] with respect to μ if $\Gamma_\mu \circ \mathcal{L} = \mathcal{L}^* \circ \Gamma_\mu$.

Proposition 26. *Let μ_Λ^τ be a Gibbs state over Λ. Then $\mathcal{L}_\Lambda$ and $\mathcal{D}$ are in detailed balance with respect to μ_Λ^τ.*

Proof. Note that $\Gamma_{\mu_\Lambda^\tau}$ is a Schur multiplier in the computational basis:

$$\Gamma_{\mu_\Lambda^\tau}(|\eta_1\rangle\langle\eta_2|) = \mu_\Lambda^\tau(\eta_1)^{\frac{1}{2}}\mu_\Lambda^\tau(\eta_2)^{\frac{1}{2}}\,|\eta_1\rangle\langle\eta_2|\,.$$

From the detailed balance condition for the transition rates $c_J(x,\sigma)$, it follows that for all $x \in \Lambda$, denoting $\mathcal{L}_x = \sum_{\eta\in\Omega_{b_x(r)}} \mathcal{L}_{x,\eta}$,

$$\Gamma_{\mu_\Lambda^\tau} \circ \mathcal{L}_x \circ \Gamma_{\mu_\Lambda^\tau}^{-1}(|\eta_1\rangle\langle\eta_2|)$$

$$= \delta_{\eta_1,\eta_2}^x\left(c_J(x,\eta_1)\frac{\mu_\Lambda^\tau(\eta_1^x)}{\mu_\Lambda^\tau(\eta_1)}\right)|\eta_1^x\rangle\langle\eta_2^x| - \frac{c_J(x,\eta_1) + c_J(x,\eta_2)}{2}\,|\eta_1\rangle\langle\eta_2|$$

$$= \delta_{\eta_1,\eta_2}^x c_J(x,\eta_1^x)\,|\eta_1^x\rangle\langle\eta_2^x| - \frac{c_J(x,\eta_1) + c_J(x,\eta_2)}{2}\,|\eta_1\rangle\langle\eta_2|$$

$$= \mathcal{L}_x^*(|\eta_1\rangle\langle\eta_2|)),$$

where

$$\delta^x_{\eta_1,\eta_2} = \begin{cases} 1 & \text{if } \eta_1|_{b_x(r)} = \eta_2|_{b_x(r)} \\ 0 & \text{otherwise.} \end{cases}$$

To prove detailed balance for $\mathcal{D}$, note that Schur multipliers commute, thus $[\mathcal{D}, \Gamma_\mu] = 0$. This, together with the fact that $\mathcal{D}^* = \mathcal{D}$, implies that $\mathcal{D}$ is in detailed balance w.r.t. μ^τ_Λ. □

The above proposition implies that Gibbs states are stationary states for the quantum Glauber dynamics. Let us prove that there are no other fixed points apart from the classical ones (i.e. states that are diagonal in the computational basis). Clearly, $\mathcal{D}$ has all classical states as stationary points. We just have to check $\mathcal{L}_\Lambda$.

Proposition 27. *The set of fixed points of $\mathcal{L}_\Lambda$ is equal to $\mathcal{G}(\Lambda)$, the set of Gibbs states over Λ.*

Proof. Let ρ be a fixed point of $\mathcal{L}_\Lambda$. We want to prove that ρ is diagonal, i.e. that it is of the form

$$\rho = \sum_\sigma p_\sigma \, |\sigma\rangle\langle\sigma| \, .$$

Consider a non-diagonal element $|\alpha\rangle\langle\beta|$, and suppose $\alpha(x) \neq \beta(x)$ for some $x \in \Lambda$. Then, from equation (45), we have that for all $y \in b_x(r)$,

$$\mathcal{L}_y(|\alpha\rangle\langle\beta|) = -\frac{1}{2}(c_J(y,\alpha) + c_J(y,\beta)) \, |\alpha\rangle\langle\beta| \, .$$

For $y \notin b_x(r)$, $\mathcal{L}_y$ is not supported on x, and thus cannot change the configuration there. This implies that the evolution cannot change the configurations over the set $\Delta(r)$, where $\Delta = \{x \in \Lambda \, | \, \alpha(x) \neq \beta(x)\}$. In turn, this implies that $\mathcal{L}_\Delta$ commutes with $\mathcal{L} - \mathcal{L}_\Delta$ (since it acts as a Schur multiplier whose entries depend only on the sites in $\Delta(r)$). Finally, this means that

$$\left\| e^{t\mathcal{L}_\Lambda}(|\alpha\rangle\langle\beta|) \right\|_1 \leqslant \left\| e^{t\mathcal{L}_\Delta}(|\alpha\rangle\langle\beta|) \right\|_1 = \exp\left(-t\frac{1}{2}\left(\sum_{x\in\Delta} c_J(x,\alpha) + c_J(x,\beta) \right) \right)$$

$$\leqslant \exp\left(-t\frac{1}{2}c_m d(\alpha,\beta) \right) \to 0.$$

Since the off-diagonal elements are killed, ρ must be of the form $\sum_\sigma p_\sigma \, |\sigma\rangle\langle\sigma|$. Writing the equation $\mathcal{L}_\Lambda(\rho) = 0$ we obtain

$$\sum_\sigma \sum_x c_J(x,\sigma)p_\sigma \, |\sigma^x\rangle\langle\sigma^x| - \sum_\sigma \sum_x c_J(x,\sigma)p_\sigma \, |\sigma\rangle\langle\sigma| = 0,$$

which implies

$$\sum_x c_J(x,\sigma^x)p_{\sigma^x} = \sum_x p_\sigma c_J(x,\sigma).$$

The last equation is simply a rewriting of the fact that (p_σ) is a stationary distribution for Q_Λ, that is, it is exactly a Gibbs state on Λ. □

Since $\mathcal{L}_\Lambda$ and $\mathcal{L}_\Lambda + \mathcal{D}$ have the same stationary distributions, even locally, all properties that depend just on the structure of the fixed-point sets will be shared by both: this is the case, for example, of frustration freeness (which we will prove next) and LTQO (which will be proved later).

Proposition 28. $\mathcal{L}_\Lambda$ *(and consequently $\mathcal{L}_\Lambda + \mathcal{D}$) is frustration free.*

Proof. By the previous proposition, we have that $\mathcal{X}_{\mathcal{L}_\Lambda} = \mathcal{G}(\Lambda)$. We know [41] that for Gibbs states it holds that

$$\Delta \subset \Lambda \Rightarrow \mathcal{G}(\Lambda) \subset \mathcal{G}(\Delta),$$

but this is exactly the frustration-freeness condition for $\mathcal{L}_\Lambda$.   □

7.2. Stability of Glauber dynamics. We want to show that the contraction of the semigroup generated by $\mathcal{L}_\Lambda + \mathcal{D}$ can be controlled by the contraction of the classical Glauber dynamics. To fix notation, denote by $\mathcal{C} : \mathcal{A}_\Lambda \to \mathcal{A}_\Lambda$ the projector on the diagonal subspace with respect to the computational basis. $\mathcal{C}$ is a completely positive, trace preserving map, and it also satisfies $\mathcal{C} = \lim_{t\to\infty} \exp(t\mathcal{D})$. Since $\mathcal{L}_\Lambda$ commutes with $\mathcal{D}$, it also commutes with $\mathcal{C}$. Then we can prove the following:

Lemma 29. *If $T_t = \exp\left(t(\mathcal{L}_\Lambda + \mathcal{D})\right)$, then*

$$\eta(T_t) \leqslant \eta(T_t \circ \mathcal{C}) + \eta(\exp(t\mathcal{D})). \tag{46}$$

Proof. Fix an initial state ρ. Then we can write

$$
\begin{aligned}
\|T_t(\rho) - T_\infty(\rho)\|_1 &\leqslant \|T_t \circ \mathcal{C}(\rho) - T_\infty(\rho)\|_1 + \|T_t \circ (1 - \mathcal{C})(\rho)\|_1 \\
&\leqslant \|T_t \circ \mathcal{C}(\rho) - T_\infty \circ \mathcal{C}(\rho)\|_1 + \|\exp(t\mathcal{D}) \circ (1 - \mathcal{C})(\rho)\|_1 \\
&\leqslant \eta(T_t \circ \mathcal{C}) + \eta(\exp(t\mathcal{D})),
\end{aligned}
$$

where we have used the fact that $\mathcal{L}_\Lambda$ and $\mathcal{D}$ commute, and that the fixed points of $\mathcal{L}_\Lambda$ are invariant under $\mathcal{C}$.   □

We know, because of theorem 6, that

$$\eta(\exp(t\mathcal{D})) \leqslant |\Lambda|\, e^{-\frac{\gamma}{2}t}, \tag{47}$$

and this implies the following result.

Corollary 30. *If the classical Glauber dynamics satisfies rapid mixing, then also the quantum embedded Glauber dynamics generated by $\mathcal{L}_\Lambda + \mathcal{D}$ does.*

Remark 10. Convergence rates of classical Glauber dynamics are a well studied subject. It is known that, in some regimes, classical Glauber dynamics satisfies a Log Sobolev inequality with system-size independent Log Sobolev constant (for a review on the subject see [46]). In such situations the classical chain has a logarithmic mixing time, and thus satisfies rapid mixing.

For this class of classical dynamical systems it is possible to apply our main result 7. In particular, we can arbitrary perturb the transition rates $c_J(x, \sigma)$ by some $e(x, \sigma)$, not necessary preserving detailed balance. If we denote by $\mathcal{E}$ the maximum of $|e(x, \sigma)|$, the difference between the perturbed and the original evolution of local observables can be bounded by $\mathcal{E}$ times a factor depending on the size of the support of the observables taken into account.

Theorem 31. *Let Q_Λ the generator of a classical Glauber dynamics, having a unique fixed point and satisfying a Log Sobolev inequality with constant independent of system size. Let E be the generator of another classical Markov process of the form*

$$(Ef)(\sigma) = \sum_{x \in \Lambda} e(x, \sigma) \nabla_x f(\sigma).$$

Suppose that $\mathcal{E} = \sup_{x, \sigma} |e(x, \sigma)| < \infty$ and that $e(x, \cdot)$ has support bounded uniformly in x. Denote by T_t the evolution generated by Q_Λ and by S_t the evolution generated by $Q_\Lambda + E$. Then, for each function f supported on $A \subset \Lambda$, it holds that

$$\|T_t(f) - S_t(f)\|_\infty \leqslant c(|A|) \|f\|_\infty \mathcal{E},$$

for some $c(\cdot)$ independent of system size and polynomially growing.

Remark 11. It is known [47, 43] that the Ising model on $\mathbb{Z}^2$ or $(\mathbb{Z}/n\mathbb{Z})^2$ has a system size independent Log Sobolev constant for high temperatures (when the inverse temperature β is lower than the critical value β_c), or at any temperature in presence of an external magnetic field. In this regime the Glauber dynamics sampling the Ising model is stable (in the sense of theorem 7).

7.3. Weak mixing and LTQO. As a nice observation, though not necessary to prove theorem 31, we want show that *weak mixing*, a condition on Gibbs states defined in [46], is equivalent to the LTQO condition given in section 6. The weak mixing conditions for two-dimensional systems has been shown [47] to imply L_2 convergence of the corresponding Glauber dynamics.

Definition 32. We say that the Gibbs measures in $\mathcal{G}(\Lambda)$ satisfy the *weak mixing* condition in $V \subset \Lambda$ if there exist constants C and m such that, for every subset $\Delta \subset V$, the following holds:

$$\sup_{\tau, \tau' \in \Omega_{V^c}} \left\| \mu^\tau_{V, \Delta} - \mu^{\tau'}_{V, \Delta} \right\|_1 \leqslant C \sum_{\substack{x \in \Delta, \\ y \in \partial_r^+ V}} e^{-m \operatorname{dist}(x, y)}, \tag{48}$$

where $\partial_r^+ V = \{x \in V^c \mid \operatorname{dist}(x, V) \leqslant r\}$ and $\mu^\tau_{V, \Delta} = \operatorname{tr}_{V \setminus \Delta} \mu^\tau_V$.

Proposition 33. *If $\mathcal{G}(\Lambda)$ satisfies the weak mixing condition for each $V \subset \Lambda$, then $\mathcal{L}_\Lambda$ (and consequently $\mathcal{L}_\Lambda + \mathcal{D}$) satisfies LTQO.*

Proof. Take $A \subset \Lambda$, $\ell \geqslant 0$, and let V be $A(\ell)$. The weak mixing condition for V implies that there exist constants C and m such that

$$\sup_{\tau, \tau' \in \Omega_{V^c}} \left\| \mu^\tau_{V, A} - \mu^{\tau'}_{V, A} \right\|_1 \leqslant C \sum_{\substack{x \in A, \\ y \in \partial_r^+ V}} e^{-m \operatorname{dist}(x, y)} \leqslant C e^{-m\ell} |A| \left| \partial_r^+ A(\ell) \right|.$$

This is the LTQO condition with $\Delta_0(\ell) = Ce^{-m\ell} |A| |\partial_r^+ A(\ell)|$. The bound, proven for states of the form μ_V^τ, can be extended by convexity to all $\mathcal{G}(V)$. Let $\eta_0, \eta_1 \in \mathcal{G}(V)$. By definition, η_0 and η_1 are convex combination of states of the form μ_V^τ, thus we can write

$$\eta_0 = \sum_i p_i \mu_V^{\tau_i}, \quad \eta_1 = \sum_j q_j \mu_V^{\sigma_j}, \quad \sum_i p_i = \sum_j q_j = 1; \quad p_i, q_j \geqslant 0.$$

Then we have

$$
\begin{aligned}
\|\eta_{0,A} - \eta_{1,A}\|_1 &= \left\| \sum_i p_i \mu_{V,A}^{\tau_i} - \sum_j q_j \mu_{V,A}^{\sigma_j} \right\|_1 \\
&= \left\| \sum_i p_i \Big(\sum_j q_j \mu_{V,A}^{\tau_i} \Big) - \sum_j q_j \Big(\sum_i p_i \mu_{V,A}^{\sigma_j} \Big) \right\|_1 \\
&\leqslant \sum_{i,j} p_i q_j \left\| \mu_{V,A}^{\tau_i} - \mu_{V,A}^{\sigma_j} \right\|_1 \leqslant \sup_{\tau,\sigma} \left\| \mu_{V,A}^{\tau} - \mu_{V,A}^{\sigma} \right\|_1.
\end{aligned}
$$

$\qquad\square$

8. Conclusions and open questions

In the context of local perturbations of local Hamiltonians, changes in the ground state can be detected by the lack of smoothness of the expectation value of local observables. Via the quasi-adiabatic technique [25], the regularity of such expectation values can be related to the study of the effect that the perturbation has on the spectral gap of the Hamiltonian. In [49], the stability of the spectral gap was shown under the assumptions of frustration-freeness and local indistinguishability between ground states of local patches of the original Hamiltonian.

In this paper we have studied a class of open quantum systems described by local Lindbladian evolutions with unique fixed points, focusing on the problem of the smoothness of evolution of local observables in the presence of local perturbations. Given any initial configuration, the system will converge toward the fixed point with a certain rate. The slowest rate over all possible initial configurations defines a mixing property of the Lindbladian, and we consider how this scales with the system size. In the case of power-law decay of interactions, we show that a logarithmic scaling is sufficient for the stability of the evolution of local observables, while for exponentially decaying and finite range interactions a scaling at least as fast as a certain polynomial, determined by equation (41), is also sufficient. Moreover, the same assumptions imply certain properties of the fixed point, such as local topological quantum order. It should be emphasized that Log Sobolev inequalities provide strong enough convergence-time estimates to satisfy our assumptions, but that our results also apply more generally.

The most important open question involves state engineering of degenerate topologically ordered states, such as topologically protected quantum codes. For such states, all known preparation maps have a convergence time that is slower than required for our result to apply [36]. It is an interesting question whether it is possible to exploit the very weak requirements in terms of locality of the

boundary condition in our definition of uniform families (see definition 2) to construct faster mixing maps for which one could prove stability, since logical observables partially supported on such boundaries are not necessarily localizable in the sense of [36].

Appendix A   The non-stable example

The following example will satisfy all the conditions of theorem 7, except forming an uniform family, and will be shown to be unstable. Interestingly, the system *is* rapid mixing, showing that without the correct structure with respect to system size scaling, rapid mixing alone is not sufficient to imply stability of local observables. This example is the generalization to dissipative systems of the globally gapped but not locally gapped example in [49]. We will show that the characteristics of the dynamics are essentially determined by a classical Markov chain embedded into the Lindbladian. For a general review on convergence of Markov chains, see [40].

Example 4. Consider a chain of $2N$ classical spins, with values in $\{0,1\}$. Let us define a generator Q^{2N} of a classical Markov chain over the configuration space $\{0,1\}^{2N}$. We will define Q^{2N} in a translationally-invariant way as follows:

$$
Q_c = \begin{array}{c} \\ |10\rangle \\ |00\rangle \\ |11\rangle \\ |01\rangle \end{array}
\begin{array}{cccc} |10\rangle & |00\rangle & |11\rangle & |01\rangle \end{array}
\left(\begin{array}{cccc} -\frac{2}{3N} & 0 & 0 & \frac{2}{3N} \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right), \quad
Q_r = \begin{array}{c} |10\rangle \\ |00\rangle \\ |11\rangle \\ |01\rangle \end{array}
\begin{array}{cccc} |10\rangle & |00\rangle & |11\rangle & |01\rangle \end{array}
\left(\begin{array}{cccc} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),
$$

$$
Q_l = \begin{array}{c} |10\rangle \\ |00\rangle \\ |11\rangle \\ |01\rangle \end{array}
\begin{array}{cccc} |10\rangle & |00\rangle & |11\rangle & |01\rangle \end{array}
\left(\begin{array}{cccc} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right), \quad
\delta_0 = |0\rangle\langle 0|, \quad \delta_1 = |1\rangle\langle 1|.
$$

We then define for each $i = 1 \ldots N$, a generator matrix Q_i acting on spins $(2i-2, \ldots, 2i+1)$ by

$$
Q_i = \mathbb{1} \otimes Q_c \otimes \mathbb{1} + \mathbb{1} \otimes Q_r \otimes \delta_0 + \delta_1 \otimes Q_l \otimes \mathbb{1};
$$

and $Q^{2N} = \sum_{i=1}^{N} Q_i$.

The matrix Q_i can only change spins $(2i-1, 2i)$: its transition graph restricted to such spins is presented in figure 3.
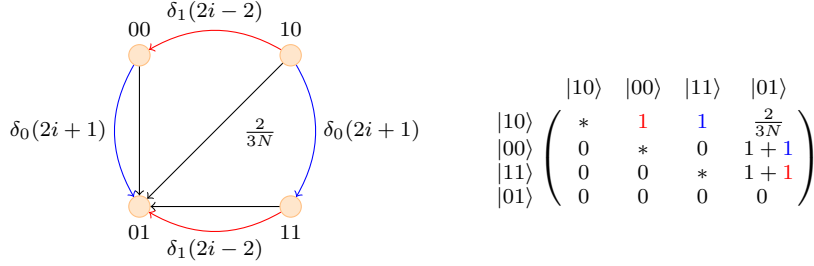
Fig. 3. The transition matrix for Q_i on the spins $(2i-1, 2i)$. The blue and the red transitions are present depending on the nearby sites: the blue ones if there is a 0 on the right, the red ones if there is a 1 on the left. Asterisks in the diagonal are such that the sum of each row is zero.

By construction, Q^{2N} is upper triangular. Thus the elements on the diagonal are the eigenvalues. The unique steady state is then $|0101\ldots01\rangle$, and the smallest non-zero eigenvalue, corresponding to the state $|1010\ldots10\rangle$, is $\frac{2}{3}$. Furthermore, it is easy to see that the diameter of the graph of the transitions of Q^{2N} is N, and in turn this implies that the mixing time for Q^{2N} is of order $O(\log N)$[10].

Let us now embed this classical Markov chain into a Lindbad operator, in a similar fashion as we have done in section 7 with Glauber dynamics. We will consider then a chain of $2N$ qubits, and define the following Lindblad operators: if k is odd, then

$$L_{k,1} = \sigma_x^{k+1} |0\rangle\langle0|_k \otimes |0\rangle\langle0|_{k+1},$$
$$L_{k,2} = \sigma_x^{k} \quad |1\rangle\langle1|_k \otimes |1\rangle\langle1|_{k+1},$$
$$L_{k,3} = \sqrt{\frac{2}{3N}}\sigma_x^k \otimes \sigma_x^{k+1} |1\rangle\langle1|_k \otimes |0\rangle\langle0|_{k+1};$$

if k is even, then

$$L_{k,1} = \sigma_x^{k} \quad |0\rangle\langle0|_k \otimes |0\rangle\langle0|_{k+1},$$
$$L_{k,2} = \sigma_x^{k+1} |1\rangle\langle1|_k \otimes |1\rangle\langle1|_{k+1},$$
$$L_{k,3} = 0.$$

The Lindbladian is then defined translationally-invariantly as

$$\mathcal{L}^{2N} = \sum_{k=1}^{2N}\sum_{i=1}^{3}\mathcal{L}_{k,i} + \mathcal{D}_k;$$

where $\mathcal{D}_k$ is a dephasing channel acting on site k, as in equation (44). Since $L_{k,3}$ depends on N, the family we have defined is not a uniform family.

---

[10] This can be seen from the upper triangular form of Q^{2N}, noticing that the polynomials appearing in $e^{tQ^{2N}}$ have degree of at most the diameter of the transition graph.

It is easy to see that the action of $\mathcal{L}^{2N}$ on diagonal states of the form $|\alpha\rangle\langle\alpha|$, with $\alpha \in \{0,1\}^{2N}$, is equal to that of Q^{2N} acting on α: this is indeed an embedding of Q^{2N}.

Then, by a similar argument as in section 7, we can prove that the fixed points of $\mathcal{L}^{2N}$ are exactly the same as those of Q^{2N} (namely, the unique state $|0101\ldots01\rangle\langle0101\ldots01|$), and that the mixing time of $\mathcal{L}^{2N}$ is bounded by the sum of the mixing times of Q^{2N} and of $\mathcal{D}$. Since both of them are mixing in time $O(\log N)$, we see that $\mathcal{L}^{2N}$ satisfies rapid mixing.

But the system is unstable: if we perturb $\mathcal{L}^{2N}$ by removing the terms generated by $L_{k,3}$ (which is a perturbation of order $O(\frac{1}{N})$), the diagonal state $|1010\ldots10\rangle\langle1010\ldots10|$ becomes a stationary state, and it is clearly locally ortogonal from the original one $|0101\ldots01\rangle\langle0101\ldots01|$.

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