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Trading to Stops*

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Abstract. The use of trading stops is a common practice in financial markets for a variety of reasons: it reduces the frequency of trading and thereby transaction costs; it provides a simple way to control losses on a given trade, while also ensuring that profit-taking is not deferred indefinitely; and it allows opportunities to consider reallocating resources to other investments. In this paper, we try to explain why the use of stops may be desirable, by proposing a simple objective to be optimized. We investigate a number of commonly used rules for the placing and use of stops, either fixed or moving, with fixed costs, showing how to identify optimal levels at which to set stops, and compare the performance of different rules and strategies.

Key words. barriers, trailing stop, transaction costs, stopping time, Laplace transform

AMS subject classifications. 60G40, 91B60, 91G10, G11

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1. Introduction.

When an investor acquires fund shares, it is common to set stops at which he will come out of the position; for example, he may decide to come out of the position when the value has either risen by 0.1 or fallen by 0.03. Such a fixed-stop trading rule is the simplest to describe, but there are other possibilities, where perhaps the lower stop rises as the value of the position rises, thereby locking in any gain, while allowing the position to continue to rise in value. The latter kind of trading rule is called a trailing stop, and it is commonly used in practice.

In this paper, we shall study some simple explicit instances of trading to stops and try to answer four questions: Is it a good idea to trade to stops in some way? Is it sufficient to consider simple stopping rules? Given that we intend to trade to stops in some way, how would we go about placing them? And when we have reached one stop, how should we act then? To answer the third of these questions, we shall propose a simple objective which must be maximized over the parameters defining the stopping rule. The answer to the first question is more subtle. If we (just for now) restrict the discussion to rules which trade to fixed stops, what we find is that in most instances the best thing to do is to put the lower stop at $-\infty$, which is counterintuitive. It is counterintuitive because one of the reasons to use stops is to prevent the trade running up huge losses, and yet it seems from the theory that this is exactly what we should be doing. However, the theoretical predictions are based on very precise assumptions about the dynamics of the fund; if we relax these strong assumptions, we find a different picture emerging. Specifically, we shall assume that the value of the position evolves as a Brownian motion with constant drift and constant volatility; the volatility will always be

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assumed to be known, but we will relax the assumption that the drift is known with certainty to the more realistic assumption that we have some prior over the possible values of the drift. Given this, we find that there is good reason to place stops, either fixed or moving, as a means to protect against model uncertainty, and we compare various different ways of placing the stops. As a stop gives the opportunity of reallocating the investment capital to a different fund or to stick with the original one, the outcome of the trade may help to decide which action should be performed that gives an answer to the fourth question. To find out whether simple rules are sufficient, the results will be compared to an optimal stopping problem. There we see that we can get quite close to the optimal value by using a very simple stopping rule with fixed stops and a time-dependent slope.

2. Model setup.

We choose to study a specific situation where we will work with the assumptions given below. With these assumptions, we are able to derive *explicit expressions* for the solution to the problem, which we are then able to analyze and compare, leading to *quite concrete conclusions*.

- *The return process is an arithmetic Brownian motion.* We shall suppose that, at time 0, a wealthy individual has a sum Y_0 of money to invest in a fund. The value of his investment at time t will be supposed to be

$$Y_t = Y_0 + \mu t + \sigma W_t,$$

where W is a standard Brownian motion, and μ and σ are known constants until further notice.[1] We will mainly work with the gain process

$$\begin{aligned} X_t &= Y_t - Y_0 \\ &= \mu t + \sigma W_t. \end{aligned}$$

It might be more conventional to use geometric Brownian motion[2] to model the gain process, taking a CRRA utility to express the investor's preferences. However, this model leads to trivial solutions; the optimal strategy is either never to exit the position or to exit immediately. This is a sign that the geometric Brownian motion is an inappropriate model for our particular study. In practice, a fund might be a basket of stocks, with each stock symbolizing a company's value. Although it is common practice to model stock dynamics with the geometric Brownian motion, the weighted sum cannot be described by a geometric Brownian motion, although it might be a reasonable approximation. A reason for the inappropriateness of the geometric Brownian motion for a company's value is given in [13]. Using a geometric Brownian motion to model the company's value embodies an assumption of constant returns to scale which can only be realized if the retained earnings will be reinvested. But, in an infinite time horizon, the returns to scale have to decrease until the company reaches the optimal scale of operation. In contrast to that, the arithmetic Brownian motion describes a company which has already reached its optimal scale of operation. For

[1] The assumption that μ is known will be relaxed later on.

[2] The variation of using a geometric Brownian motion instead of an arithmetic Brownian motion is considered in [8].

sure, neither model perfectly describes reality, but either may be used as an adequate approximation. As we will deal with an infinite time horizon, the arithmetic Brownian motion is the right model to choose. The issue of the choice of a sensible model arises in the interesting paper [21], where the authors show for a similar objective that there is no nontrivial solution except in some situations where 0 is an entrance boundary point. The fact that the existence of an interesting stops trading rule (which in some sense should be a local phenomenon) is determined by *asymptotic* properties of the assumed diffusion model suggests that the choice of objective and model assumptions is a delicate issue.

- *The objective is the value of a repeated strategy with an infinite time horizon.* The investor's money is locked in the investment and cannot be spent until the position is closed out. Thus, the investor has some incentive to take profits from the investment, withdrawing the gain $X_T = Y_T - Y_0$ (which might be negative), at some stopping time $T = T_1$, for immediate consumption. Having closed out the position, we will suppose that the investor repeats the process, once again investing the remaining value Y_0 in the position and using the same stopping rule applied to the rebased process $(Y(T_1 + t) - Y(T_1))_{t\geq 0}$. Thus the stopping times T_n (which are the times at which the position gets closed and immediately reopened) form a renewal process. We suppose that withdrawal and reinvestment incur a cost c. The time-0 value of this repeated trading activity will be

$$\varphi \equiv E\left[\sum_{n\geq 0} e^{-\rho T_{n+1}}\, U(Y(T_{n+1}) - Y(T_n) - c\,)\right], \tag{2.1}$$

where $T_0 \equiv 0$ and ρ is the (constant) rate of discounting, and the utility U is some concave strictly increasing function. We do not consider a finite time horizon because finite horizon problems have time-dependent solutions which can only be solved numerically. With an infinite horizon objective, we obtain explicit solutions. Bankruptcy is not taken into account, as for the more sophisticated reallocating strategies we develop later on it is quite unlikely that the wealthy investor will go bankrupt.

- *The investor's preferences are constant absolute risk aversion.* If we cared only about the net present value of all the gains from trade over time, we would take $\rho = r$, the riskless rate of interest, and $U(x) = x$, and take expectations with respect to the pricing measure. However, this excludes any investor risk aversion. Indeed, we shall see that we must allow strict concavity of U to explain why an investor would wish to place stops; when it comes to studying this, we will always take the exponential utility function

$$U(x) = 1 - \exp(-\gamma x) \tag{2.2}$$

for some $\gamma > 0$, the coefficient of absolute risk aversion. Although we will concentrate on the exponential utility function only, the calculations can be used to get results for other utility functions. The (risk-neutral) case of linear U is regarded as a limiting case, using the limit as $\gamma \downarrow 0$ of $\gamma^{-1}(1 - e^{-\gamma x})$. Furthermore, all calculations can be carried out for a general utility function by replacing the boundary conditions by a general functional.[3] However, explicit solutions are not then guaranteed; some calculations may end in an integral which might only be solved numerically.

- *Only a small class of stopping rules is considered.* We focus on simple parametric stopping rules consisting of fixed and trailing stopping barriers as defined below. We should not expect this simple class of strategies to include the optimum, but the stopping rules considered are commonly used ones, and we are able to apply excursion theory to deduce the explicit solutions. Furthermore, when we compare our results to more general optimal stopping problems, which we have to solve numerically, it will be shown that these simple stopping rules do very well.

Related literature. In the literature, there exist several articles where the same quantities we are investigating are solved for many kinds of classes of dynamics. One can consult [2] for some of these computations. In [10] a double exponential jump diffusion model is considered, which is generalized to a hyperexponential jump diffusion model in [3]. The solutions derived there are, however, not that concrete. There are various other articles which deal with optimal investment decisions. For example, in [21] an investor solves an infinite horizon problem and the goal is to repeatedly sell the asset high and buy it back at a low price. Although it allows general diffusion dynamics, it assumes that the asset dynamics are known with certainty. The paper [5] covers drift uncertainty; it considers a regime switching model where the idea is to catch a bull market at its early stage, ride the trend, and liquidate the position at the first evidence of the subsequent bear market. However, the main focus of our study is not to fix some assumed stochastic control model and derive an optimal solution to it, but rather to compare various commonly used real-life stopping rules and to determine under what assumptions these might be appropriate or even good.

Coming back to our model, the following result reduces the calculation of φ to two simpler calculations.

Proposition 1. *The value φ of the trading strategy is*

$$\varphi = \frac{E[e^{-\rho T}\,U(X_T - c)]}{1 - E[e^{-\rho T}]}, \tag{2.3}$$

where $T \equiv T_1$.

Proof. By the strong Markov property and the stationary increments of Y, by decomposing the objective (2.1) at the first time $T = T_1$ that the position gets closed out we see that

$$\begin{aligned}
\varphi &= E[e^{-\rho T}U(X_T - c)] + E\Big[\sum_{n\geq 1} e^{-\rho T_{n+1}} U(Y(T_{n+1}) - Y(T_n) - c)\Big]\\
&= E[e^{-\rho T}U(X_T - c)] + E\Big[e^{-\rho T} E\Big[\sum_{n\geq 1} e^{-\rho(T_{n+1}-T)} U(Y(T_{n+1}) - Y(T_n) - c) | \mathcal{F}_T\Big]\Big]\\
&= E[e^{-\rho T}U(X_T - c)] + E[e^{-\rho T}]\varphi.
\end{aligned}$$

Rearrangement gives the result (2.3). ■

[3]The variation of using alternative utility functions is considered in [8].

For comparison, we also solve the optimal stopping problem for multiple stopping times $0 \equiv \tau_0 \le \tau \equiv \tau_1 \le \tau_2 \le \cdots$:

$$
\begin{aligned}
\bar{\varphi} &\equiv \sup_{0\le\tau_1\le\tau_2\le\cdots} E\Big[\sum_{n\ge 0} e^{-\rho\tau_{n+1}}\, U(Y(\tau_{n+1}) - Y(\tau_n) - c\,)\Big] \\
&= \sup_{0\le\tau_1\le\tau_2\le\cdots} E\Big[e^{-\rho\tau}\Big\{U(X_\tau - c) + E\Big[\sum_{n\ge 1} e^{-\rho(\tau_{n+1}-\tau)} U(Y(\tau_{n+1}) - Y(\tau_n) - c)\Big\}|\mathcal{F}_\tau\Big]\Big] \\
&= \sup_{\tau\ge 0} E\Big[e^{-\rho\tau}\Big\{U(X_\tau - c) + \sup_{\tau_2\le\tau_3\le\cdots} E\Big[\sum_{n\ge 1} e^{-\rho(\tau_{n+1}-\tau)} U(Y(\tau_{n+1}) - Y(\tau_n) - c)\Big\}|\mathcal{F}_\tau\Big]\Big] \\
&= \sup_{\tau\ge 0} E[e^{-\rho\tau}\{U(X_\tau - c) + \bar{\varphi}\}]. \qquad (2.4)
\end{aligned}
$$

It is clear that $\bar{\varphi} \ge \sup_{T\ge 0}\varphi$. On the other hand, μ is constant, and therefore X follows a *time-homogeneous* diffusion. See [18] or [12] to verify that in this case, the continuation region is time invariant and, thus, depends only on the state of the process and not explicitly on time. Hence, the optimal value is attained by the fixed-stops rule, which is defined below.[4]

To set the stage, we now offer a few natural examples which we will study in more detail later.

Example 1 (fixed stops). This is the easiest example of all. The investor trades if he made a gain of b or a loss of a. The upper stop can be seen as a take-profit stop, and the lower stop is used to limit the losses. So, we take $a > 0$, $b > 0$ and set

$$T \equiv \inf\{t : X_t = -a \text{ or } X_t = b\}. \qquad (2.5)$$

Example 2 (trailing stop and fixed stop). The trailing stop is not fixed, but it moves with the investment price. More precisely, the trailing stop occurs if the investment price drops by a fixed value from its maximum-to-date price. These kinds of stops are widely used by practitioners because once the investor has made a certain gain, the trailing stop preserves a part of this gain. Fix $a > 0$ and $b > 0$, and let $\bar{X}_t \equiv \sup_{0\le s\le t} X_s$. Then the stopping time is defined by a trailing stop at $-a + \bar{X}_t$ and a take-profit stop at $b > 0$:

$$T \equiv \inf\{t : \bar{X}_t - X_t = a \text{ or } X_t = b\}. \qquad (2.6)$$

Example 3 (trailing stop). As a special case of Example 2, we fix $a > 0$ and set

$$T \equiv \inf\{t : \bar{X}_t - X_t = a\}, \qquad (2.7)$$

which gives the trailing stop only. This has the stop-loss character of Example 2 but does not stop out at some high level, thereby stopping losses but running with profits.

Example 4 (converging stops). Fix $a > 0$ and $\varepsilon > 0$. Then we use the stopping time

$$T \equiv \inf\{t : (1+\varepsilon)\bar{X}_t - X_t = a\}. \qquad (2.8)$$

In this situation, it is easy to see that the trade stops out before X first hits a/ε; it has similarities to Example 2, and in the special case $\varepsilon = 0$ we recover Example 3.

[4]This idea is due to a referee's hint.

Since our main interest is in the case of CARA utility U (2.2), we see that the value of the problem can be expressed as

$$\begin{aligned}\varphi &= \frac{E[e^{-\rho T}] - e^{\gamma c}E[e^{-\rho T-\gamma X_T}]}{1-E[e^{-\rho T}]}\\ &= \frac{L(\rho,0)-e^{\gamma c}L(\rho,\gamma)}{1-L(\rho,0)},\end{aligned} \tag{2.9}$$

where for arbitrary $\rho,\ \gamma \geq 0$

$$L(\rho,\gamma) \equiv E[e^{-\rho T-\gamma X_T}] \tag{2.10}$$

is the joint Laplace transform of the time and place of stopping. Thus the first objective is to identify the joint Laplace transform L as explicitly as possible in each of the examples under investigation. As we shall see, this is not the end of the story, merely the start.

3. Analysis of the examples.

In this section, we shall analyze the examples presented in section 2 and derive explicit solutions for the joint Laplace transform L in each case. The first example is solved using differential equations techniques, which we can think of as an application of Itô calculus. Similar techniques may also be used to solve the other examples, but as the state variable is no longer one dimensional, the construction of the correct functions is not as simple or transparent. For this reason, we prefer to derive the answers using Itô excursion theory, introduced by Itô in [9]; see [14] or [16] for accessible accounts.

3.1. Example 1: Fixed stops.

We write

$$\mathcal{L} \equiv \frac{1}{2}\sigma^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mu\frac{\mathrm{d}}{\mathrm{d}x} - \rho \tag{3.1}$$

for the generator of the diffusion X with killing rate ρ. If $f : \mathbb{R} \mapsto \mathbb{R}$ is C^2 and satisfies $\mathcal{L}f = 0$, then by an application of Itô's formula we have that

$$M_t \equiv e^{-\rho t}f(X_t) \text{ is a local martingale}$$

which is bounded on the interval $[0, T]$, and therefore[5] $(M(t \wedge T))_{t\geq 0}$ is a martingale. By the optional sampling theorem, it follows[6] that

$$f(0) = E^0[e^{-\rho T}f(X_T)], \tag{3.2}$$

so in order to compute the numerator and denominator in (2.3) it is enough to solve the ODE $\mathcal{L}f = 0$ in $[-a, b]$ with the appropriate boundary conditions.

If we let $-\alpha < 0 < \beta$ be the roots of the quadratic

$$\frac{1}{2}\sigma^2z^2 + \mu z - \rho = 0, \tag{3.3}$$

[5]Here, of course, T is given by (2.5).

[6]The notation E^x denotes expectation under the initial condition $X_0 = x$.

then the solution to the ODE

$$\mathcal{L}f = 0, \quad f(-a) = A, \quad f(b) = B$$

is

$$f(x) = \frac{(Ae^{\beta b} - Be^{-\beta a})e^{-\alpha x} + (Be^{\alpha a} - Ae^{-\alpha b})e^{\beta x}}{e^{\alpha a+\beta b} - e^{-\alpha b-\beta a}}.$$

Evaluating at $x = 0$ simplifies to

$$f(0) = \frac{A(e^{\beta b} - e^{-\alpha b}) + B(e^{\alpha a} - e^{-\beta a})}{e^{\alpha a+\beta b} - e^{-\alpha b-\beta a}}. \tag{3.4}$$

If we now take $A = \exp(\gamma a)$ and $B = \exp(-\gamma b)$, we read off the joint Laplace transform L_1 for this first example:

$$L_1(\rho, \gamma) = \frac{e^{\gamma a}(e^{\beta b} - e^{-\alpha b}) + e^{-\gamma b}(e^{\alpha a} - e^{-\beta a})}{e^{\alpha a+\beta b} - e^{-\alpha b-\beta a}}. \tag{3.5}$$

Formula (3.5) can also be obtained by a variation of the calculations of section 2 in [7]. Substituting the form of L_1 into the expression (2.9) gives the value φ for this stopping rule. The dependence of the right-hand side on ρ is of course through the dependence of α, β on ρ as solutions to (3.3). The mean of the hitting time can be derived from the Laplace transform as

$$\begin{aligned} E[T] &= -\frac{\partial L_1}{\partial \rho}(0,0) \\ &= \frac{b(e^{ka} - 1) - a(1 - e^{-kb})}{\mu(e^{ka} - e^{-kb})} \end{aligned} \tag{3.6}$$

after some calculations, where $k \equiv 2\mu/\sigma^2$.

3.2. Example 2: Trailing stop and fixed stop.

We deal with this example first and read off the solution to Example 3 as the special case $b = \infty$. Recall that we take the stopping time

$$T \equiv \inf\{t : \bar{X}_t - X_t = a \text{ or } X_t = b\}, \tag{3.7}$$

where $\bar{X}_t \equiv \sup_{0\le s\le t} X_s$. The process[7] $Y \equiv X - \bar{X}$ is a continuous strong Markov process with values in $\mathcal{X} \equiv (-\infty, 0]$, and 0 is a recurrent point for this process. The Itô theory of excursions [9] applies to this process, and we will make use of it. Let V denote the space of all excursions of Y away from 0, that is, continuous functions $f : \mathbb{R}^+ \to \mathcal{X}$ with the property that for some $\zeta = \zeta(f) \in (0, \infty]$, the *lifetime* of the excursion, the set $f^{-1}((-\infty, 0))$ is of the form $(0, \zeta)$. Regarding V as a subset of $C(\mathbb{R}^+, \mathbb{R})$ induces the subset topology on V, and in fact V is a Polish space; see, for example, [16] for definitions and basic properties. The process $\bar{X}$ is a continuous homogeneous additive functional of Y, growing only when $Y = 0$, and acts as the local time at zero for Y. The open set $Y^{-1}((-\infty, 0))$ is the disjoint union of countably

[7] In what follows, Y is defined to be a new process.

many excursion intervals I_j, and the point process $\Pi \equiv \{(L_j, \xi^j) : j \in \mathbb{Z}\}$ is a Poisson point process in $(0,\infty) \times V$, where

$$L_j = \bar{X}(I_j),$$
$$\xi^j = Y|_{I_j} .$$

The mean measure of Π is Leb $\times\, n$, where n is the σ-finite *excursion measure*: see Itô [9]. The key to effective use of Itô excursion theory is an explicit characterization of the excursion measure n. Once the excursion has escaped from 0, it evolves like the diffusion $X - \bar{X}$ until it first hits zero, and it leaves 0 according to an entrance law.

We shall use excursion theory to calculate for any $\gamma \geq 0$ the expectation

$$L(\rho, \gamma) \equiv E[\, \exp(-\rho T - \gamma X_T)\,]; \tag{3.8}$$

evidently, once we have this, we can obtain the numerator and denominator in (2.3) by suitable substitutions and combinations. As explained in [14], we deal with expectations such as (3.8) by introducing an independent $\exp(\rho)$ time τ and writing

$$E[\, \exp(-\rho T - \gamma X_T)\,] = E[\, e^{-\gamma X_T} : T < \tau]. \tag{3.9}$$

The way this is handled by excursion theory is to think of τ as being the first event time τ_1 in a Poisson process on $\mathbb{R}^+$ of intensity ρ, with event times $\tau_1 < \tau_2 < \cdots$. This Poisson process of times can be dealt with by *marking* the excursions of Y, each independently of all others, according to a Poisson process of intensity ρ. The excursion point process Π gets modified to the marked excursion point process $\tilde{\Pi}$, where each excursion ξ^j gets augmented to $\tilde{\xi^j} \equiv (\xi^j, N^j)$, where N^j is an increasing $\mathbb{Z}^+$-valued path, representing the path of the marking process restricted to the excursion ξ^j. We observe the marked excursion process $\tilde{\Pi}$ until *either* local time $\bar{X}$ reaches b; *or* we see an excursion which gets to $-a$ before any mark; *or* we see an excursion which gets marked before it reaches $\{0, -a\}$. To set some notation, let

$$A \equiv \{\text{excursions which are marked before reaching 0 or } -a\}; \tag{3.10}$$

$$B \equiv \{\text{excursions which get to } -a \text{ with no mark before reaching } -a\}. \tag{3.11}$$

We shall calculate $n(A)$ and $n(B)$ quite simply, but for this we need to characterize the excursion measure effectively. Let $-\alpha < 0 < \beta$ be the roots of the quadratic $\frac{1}{2}\sigma^2 t^2 + \mu t - \rho$; then routine calculations lead to the conclusion that for any $-a < x < 0$

$$E^x[1 - e^{-\rho\, H_0 \wedge H_{-a}}] = \frac{1 - e^{-\beta a}}{e^{\alpha a} - e^{-\beta a}}(1 - e^{-\alpha x}) + \frac{e^{\alpha a} - 1}{e^{\alpha a} - e^{-\beta a}}(1 - e^{\beta x}), \tag{3.12}$$

$$E^x[e^{-\rho H_{-a}} : H_{-a} < H_0] = \frac{e^{-\alpha x} - e^{\beta x}}{e^{\alpha a} - e^{-\beta a}}, \tag{3.13}$$

where $H_z \equiv \inf\{t : X_t = z\}$ is the hitting time of z. Since the measure of excursions which reach $-\varepsilon$ is asymptotic to ε^{-1} as $\varepsilon \downarrow 0$ (see Williams' decomposition of the Brownian excursion law [19, II.67], we conclude that

$$n(A) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} E^{-\varepsilon}[1 - e^{-\rho\, H_0 \wedge H_{-a}}] = \frac{\beta e^{\alpha a} + \alpha e^{-\beta a} - (\alpha + \beta)}{e^{\alpha a} - e^{-\beta a}}, \tag{3.14}$$

$$n(B) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} E^{-\varepsilon}[e^{-\rho H_{-a}} : H_{-a} < H_0] = \frac{\alpha + \beta}{e^{\alpha a} - e^{-\beta a}}. \tag{3.15}$$

The first excursion in $A \cup B$ comes at local time rate

$$\nu \equiv n(A \cup B) = \frac{\beta e^{\alpha a} + \alpha e^{-\beta a}}{e^{\alpha a} - e^{-\beta a}}. \tag{3.16}$$

We shall stop the point process either at the first time we see an excursion in $A \cup B$, or when local time reaches b, whichever comes sooner.

Now we come back to the expectation (3.9) and consider how the event $T < \tau$ could happen: this could *either* be because $\bar{X}$ reaches b before the first excursion in $A \cup B$, *or* because the first excursion in $A \cup B$ happens before $\bar{X}$ reaches b, and is in fact an excursion in B. By simple properties of Poisson processes, we discover after a little thought that

$$\begin{aligned} L_2(\rho, \gamma) &\equiv E[\, \exp(-\rho T - \gamma X_T)\,] \\ &= E[e^{-\gamma X_T} : T < \tau] \\ &= e^{-\nu b - \gamma b} + \int_0^b \nu e^{-\nu y}\, \frac{n(B)}{\nu}\, e^{-\gamma(y-a)}\, \mathrm{d}y \\ &= e^{-(\nu+\gamma)b} + \frac{n(B) e^{\gamma a}}{\nu + \gamma}(1 - e^{-(\nu+\gamma)b}). \end{aligned} \tag{3.17}$$

To our knowledge, this explicit result is new. As before, the mean of T can be computed by differentiating the Laplace transform with respect to ρ at zero. We find[8] that

$$E[T] = \frac{\sigma^2}{2\mu^2}(e^{ka} - 1 - ka)(1 - e^{-mb}), \tag{3.18}$$

where $k = 2\mu/\sigma^2$ as before, and $m = k/(e^{ka} - 1)$.

3.3. Example 3: Trailing stop. When $b = \infty$, the results of subsection 3.2 reduce to simpler expressions

$$L_3(\rho, \gamma) = \frac{n(B) e^{\gamma a}}{\nu + \gamma}, \qquad E[T] = \frac{\sigma^2}{2\mu^2}(e^{ka} - 1 - ka). \tag{3.19}$$

The first of these agrees with the result of Taylor [17, equation (1.1)] and can easily be obtained from a result of Glynn and Iglehart [6]. Lehoczky determined this quantity in [11, equation 4] for the larger class of time homogeneous processes.

3.4. Example 4: Converging stops. In this example, the stopping time is given by (2.8):

$$T \equiv \inf\{t : (1 + \varepsilon)\bar{X}_t - X_t = a\}.$$

The analysis of this example is quite similar to Example 2, except that the excursion measure of the excursions which stop the process now depends on how much local time has elapsed. When local time $\bar{X}$ has reached ℓ, then any excursion which either contains a mark or reaches

[8] The calculations were carried out by a symbolic mathematics package and by traditional methods.

$-a+\varepsilon\ell$ will stop the Poisson point process. Exactly as in (3.14), (3.15), the intensity of excursions which are marked before reaching $-a+\varepsilon\ell$ or zero is

$$n_A(\ell) \equiv \frac{\beta e^{\alpha(a-\varepsilon\ell)} + \alpha e^{-\beta(a-\varepsilon\ell)} - (\alpha+\beta)}{e^{\alpha(a-\varepsilon\ell)} - e^{-\beta(a-\varepsilon\ell)}}\,, \tag{3.20}$$

and the intensity of excursions which get to $-a+\varepsilon\ell$ before getting marked is

$$n_B(\ell) \equiv \frac{\alpha+\beta}{e^{\alpha(a-\varepsilon\ell)} - e^{-\beta(a-\varepsilon\ell)}}\,. \tag{3.21}$$

So in total, the intensity of excursions which stop the Poisson point process is

$$n_{A\cup B}(\ell) = \frac{\beta e^{\alpha(a-\varepsilon\ell)} + \alpha e^{-\beta(a-\varepsilon\ell)}}{e^{\alpha(a-\varepsilon\ell)} - e^{-\beta(a-\varepsilon\ell)}}\,. \tag{3.22}$$

We can now calculate

$$\begin{aligned}
\bar{F}(t) &\equiv P(\bar{X} \text{ reaches } t \text{ before the stopping excursion}) \\
&= \exp\left[-\int_0^t n_{A\cup B}(s)\,\mathrm{d}s \right] \\
&= \exp\left\{ -\beta t - \varepsilon^{-1}\log\left(\frac{1-e^{-(\alpha+\beta)a}}{1-e^{-(\alpha+\beta)(a-\varepsilon t)}} \right) \right\} \\
&= e^{-\beta t}\left(\frac{1-e^{-(\alpha+\beta)(a-\varepsilon t)}}{1-e^{-(\alpha+\beta)a}} \right)^{1/\varepsilon},
\end{aligned}$$

which we notice is decreasing with t, and vanishes when $t=a/\varepsilon$, as it must. Using this, we deduce after some calculations that

$$\begin{aligned}
L_4(\rho,\gamma) &= \int_0^{a/\varepsilon} e^{-\gamma((1+\varepsilon)x-a)}\, n_B(x)\bar{F}(x)\,\mathrm{d}x \\
&= \int_0^{a/\varepsilon} e^{-\gamma((1+\varepsilon)x-a)}\, \frac{\alpha+\beta}{e^{\alpha(a-\varepsilon x)} - e^{-\beta(a-\varepsilon x)}}\,\bar{F}(x)\,\mathrm{d}x \\
&= \frac{1}{\varepsilon}\left(\frac{e^{-(\gamma+\beta)a}}{1-e^{-(\alpha+\beta)a}} \right)^{1/\varepsilon} \int_{\exp\{-(\alpha+\beta)a\}}^1 (1-t)^{(1-\varepsilon)/\varepsilon}\, t^{-\kappa}\,\mathrm{d}t,
\end{aligned} \tag{3.23}$$

where $\kappa = (\gamma+\beta)(1+\varepsilon)/\varepsilon(\alpha+\beta)$. The answer is therefore available in terms of incomplete beta functions.

4. Placing of the stops.

The identification of the joint Laplace transform of T and X_T in each of the previous examples now allows us to evaluate the objective φ (2.9), and by varying the parameters a and b we are able to optimize φ. However, numerical investigation shows that in many cases it is optimal to let $a\to\infty$. If this happens, then there would be no reason to place a lower stop, which is somewhat unexpected. We can analyze this phenomenon quite completely for the case of fixed stops, which coincides with the optimal stopping problem (2.4). In any case, since we observe that often the best thing is to use *no* lower stop, we are forced to reassess the modeling assumptions.

Accordingly, we will until further notice restrict our attention to the fixed-stops example, Example 1. The joint Laplace transform L_1 of T and X_T has been found (3.5), and so we are able to obtain an explicit expression for the value φ using (2.9). Since we are concerned with the behavior of this as $a \to \infty$ with all other parameters fixed, we shall use the (local) notation $\varphi(a)$, where we have explicitly

$$\begin{aligned}\varphi(a) &= \frac{L(\rho,0) - e^{\gamma c} L(\rho,\gamma)}{1 - L(\rho,0)} \\ &= -1 + \frac{1 - e^{\gamma c} L(\rho,\gamma)}{1 - L(\rho,0)} \\ &= -1 + \frac{e^{\alpha a} - B_1 e^{-\beta a} - (1 - B_1) e^{\gamma(a+c)} - B_2 e^{\gamma c}(e^{\alpha a} - e^{-\beta a})}{e^{\alpha a} - B_1 e^{-\beta a} - (1 - B_1) - B_3(e^{\alpha a} - e^{-\beta a})},\end{aligned} \tag{4.1}$$

where $B_1 = e^{-(\alpha+\beta)b}$, $B_2 = e^{-(\gamma+\beta)b}$, and $B_3 = e^{-\beta b}$, which are positive constants less than 1. The large-a behavior of this expression is determined in the following little result.

Proposition 2. *Consider the behavior of the objective* (2.9) *in the case of fixed stops* (2.5) *as $a \to \infty$, with b fixed.*

(i) *If $\gamma > \alpha$, then*

$$\lim_{a\to\infty} \varphi(a) = -\infty. \tag{4.2}$$

(ii) *If $\alpha > \gamma$ and $b > c$, then*

$$\varphi(a) < \varphi(\infty) \tag{4.3}$$

for all $a > 0$.

Proof. The proof is given in Appendix A. ∎

Remarks. It is easy to understand the content of Proposition 2. In the case where $\gamma > \alpha$, it is *not* advantageous to let $a \to \infty$ because although the expectation

$$E[e^{-\rho T} : X_T = -a] \sim e^{-\alpha a} \tag{4.4}$$

is getting exponentially small, the utility when this event happens is getting large negative exponentially, and *at a greater rate*. In contrast, if $\gamma < \alpha$, the exponential decay of the expectation (4.4) beats the growth of the penalty, and the investor can ignore the penalty for stopping at a low negative level. The condition $b > c$ is needed for the proof but has a natural interpretation; if $b < c$, we are certain to be losing money every time we review our portfolio, so we would never consider entering this trade.

For a reasonable solution, then, it seems that we require $\gamma > \alpha$. However, in typical examples, this can lead to coefficients γ of absolute risk aversion so high that the value φ is always negative, so we would never engage in this trade! The point is that $-\alpha$ solves the quadratic (3.3), and if $\mu > 0$, we will always have $\alpha > 2\mu/\sigma^2$, a lower bound which need not be small. So for a solution with realistic values of γ, and with a rationale for a lower stop at a finite position, it seems that we are forced to consider situations where μ is negative. But if the growth rate of the trade were negative, and we are paying transaction costs, we would certainly never want to enter into it!

So, we incorporate an uncertainty risk into our model, which means that when we pick a fund, *we are not certain of its true value of* μ. We merely have some prior distribution over possible μ values with a positive probability that μ is negative. Then, we will find that even for small values of γ the punishment for stopping at very low levels really hurts, and we will want to use a finite lower stop. On the other hand, if the probability of decently positive values of μ is quite high, we will be emboldened to take part in the trade.

Once we allow randomly picking a fund with an uncertain drift μ, there are several possible stories which one could tell. We might have the following:

(A) Each time we come out of a trade, we go back into the *same* fund.

(B) Each time we come out of a trade, we pick an *independent* fund with the same probabilistic structure and invest in that.

(C) Each time we come out of a trade on the down side, we pick an *independent* fund with the same probabilistic structure and invest in that.

(D) We perform an optimal stopping analysis for the situation where the drift parameter of the diffusion is being filtered from the observations; see, for example, [1] or [15]. Once we have picked a particular fund, we consider a learning process for the drift of that specific fund. Having observed the data up to some time τ_0, resulting in a prior μ_0, we assume the distribution of the drift to be $N(\mu_0, \sigma^2/\tau_0)$. Then, by Bayes' formula, the density of μ conditional on the observation filtration $\mathcal{X}_t$ yields

$$\begin{aligned} p_{\mu|X_t=x}(y) &= \frac{p_\mu(y)\, p_{X_t|\mu=y}(x)}{\int p_\mu(l)\, p_{X_t|\mu=l}(x)\mathrm{d}l} \\ &= \frac{1}{\sqrt{2\pi v_t}} e^{-\frac{(y-\hat{\mu}(t,x))^2}{2v_t}}, \end{aligned}$$

with

$$\hat{\mu}(t,x) = \frac{\tau_0\mu_0 + x}{\tau_0 + t} \quad \text{and} \quad v_t = \frac{\sigma^2}{\tau_0+t}. \tag{4.5}$$

Hence, μ conditional on $\mathcal{X}_t$ is normally distributed with mean $\hat{\mu}(t, X_t)$ and variance v_t. The gain process is

$$\begin{aligned} \mathrm{d}X_t &= \left(\int y\, p_{\mu|X_t}(y)\mathrm{d}y \right) \mathrm{d}t + \sigma \mathrm{d}\hat{W}_t \\ &= \hat{\mu}(t, X_t)\mathrm{d}t + \sigma\mathrm{d}\hat{W}_t, \end{aligned} \tag{4.6}$$

where $\hat{W}_t$ is a standard Brownian motion in the observation filtration $\mathcal{X}_t$.

Each time we reach the stopping region, we invest in an *independent* fund with the same probabilistic structure.

Let $\bar{\varphi}$ be the value function to be optimized; then $\bar{\varphi}$ depends on which of the stories we choose and has the subsequent forms.

Let m be the distribution of μ and E^μ the expectation when the drift is μ. For (A)–(C), every time we reallocate to a new investment, we pick a fund with constant drift according to the distribution m. Then, $\bar{\varphi} = \int \varphi(\mu) m(\mathrm{d}\mu)$ with a different kind of $\varphi(\mu)$.

(A) As in (2.3), we have value $\varphi(\mu)$ given by

$$\varphi(\mu) = E^\mu\big[\, e^{-\rho T} U(X_T - c) \big] + E^\mu\big[\, e^{-\rho T} \big]\varphi(\mu),$$

which is equivalent to

$$\varphi(\mu) = \frac{E^\mu\big[\, e^{-\rho T} U(X_T - c) \big]}{1 - E^\mu\big[\, e^{-\rho T} \big]},$$

so the overall value this time is given by

$$\begin{aligned} \bar\varphi &= \int \varphi(\mu)\ \ m(\mathrm{d}\mu) \\ &= \int \frac{E^\mu\big[\, e^{-\rho T} U(X_T - c) \big]}{1 - E^\mu\big[\, e^{-\rho T} \big]}\ \ m(\mathrm{d}\mu). \end{aligned} \tag{4.7}$$

(B) This time we have value $\varphi(\mu)$ given by

$$\varphi(\mu) = E^\mu\big[\, e^{-\rho T} U(X_T - c) \big] + E^\mu\big[\, e^{-\rho T} \big]\bar\varphi,$$

so the overall value will be

$$\begin{aligned} \bar\varphi &= \int \varphi(\mu)\, m(\mathrm{d}\mu) \\ &= \int E^\mu\big[\, e^{-\rho T} U(X_T - c) \big]\ m(\mathrm{d}\mu) + \int E^\mu\big[\, e^{-\rho T} \big]\ m(\mathrm{d}\mu)\ \bar\varphi, \end{aligned}$$

which can be rearranged to give

$$\bar\varphi = \int \varphi(\mu)\ \ m(\mathrm{d}\mu) = \frac{\int E^\mu\big[\, e^{-\rho T} U(X_T - c) \big]\ m(\mathrm{d}\mu)}{1 - \int E^\mu\big[\, e^{-\rho T} \big]\ m(\mathrm{d}\mu)}. \tag{4.8}$$

(C) We denote by H and H^c the events that the position closes out on the high side and down side, respectively. Then, $E^\mu[\cdot : H]$ and $E^\mu[\cdot : H^c]$ are the corresponding partial expectations. Then,

$$\varphi(\mu) = E^\mu\big[\, e^{-\rho T} U(X_T - c) \big] + E^\mu\big[\, e^{-\rho T} : H \big]\varphi(\mu) + E^\mu\big[\, e^{-\rho T} : H^c \big]\bar\varphi,$$

which is equivalent to

$$\varphi(\mu) = \frac{E^\mu\big[\, e^{-\rho T} U(X_T - c) \big] + E^\mu\big[\, e^{-\rho T} : H^c \big]\bar\varphi}{1 - E^\mu\big[\, e^{-\rho T} : H \big]},$$

so the overall value is

$$\begin{aligned} \bar\varphi &= \int \varphi(\mu)\, m(\mathrm{d}\mu) \\ &= \int \frac{E^\mu\big[\, e^{-\rho T} U(X_T - c) \big]}{1 - E^\mu\big[\, e^{-\rho T} : H \big]}\ m(\mathrm{d}\mu) + \int \frac{E^\mu\big[\, e^{-\rho T} : H^c \big]}{1 - E^\mu\big[\, e^{-\rho T} : H \big]}\ m(\mathrm{d}\mu)\ \bar\varphi. \end{aligned}$$

Rearranging leads to

$$\bar{\varphi} = \frac{\int \frac{E^\mu[\, e^{-\rho T} U(X_T - c)\,]}{1 - E^\mu[\, e^{-\rho T} : H\,]}\, m(\mathrm{d}\mu)}{1 - \int \frac{E^\mu[\, e^{-\rho T} : H^c\,]}{1 - E^\mu[\, e^{-\rho T} : H\,]}\, m(\mathrm{d}\mu)}. \tag{4.9}$$

For the above three stories, the main point is that once we are able to find an explicit expression for the Laplace transforms $L(\rho, \gamma|\mu) = E^\mu[\, \exp(-\rho T - \gamma X_T)\,]$ and $L_H(\rho, \gamma|\mu) = E^\mu\, [\exp(-\rho T - \gamma X_T) : H\,]$ for the different stopping rules, we are able to deduce the value $\bar{\varphi}$ just by doing at most two integrations. $L(\rho, \gamma|\mu)$ was already calculated in section 3 for all stopping rules. For Example 1, $L_H(\rho, \gamma|\mu)$ can be obtained in the same manner by simply changing the boundary conditions giving

$$L_H(\rho, \gamma|\mu) = \frac{(e^{\alpha a} - e^{-\beta a}) e^{-\gamma b}}{e^{\alpha a + \beta b} - e^{-\alpha b - \beta a}},$$

where μ is hidden in α and β, respectively. For Example 2, $L_H(\rho, \gamma|\mu)$ is a by-product of (3.17):

$$L_H(\rho, \gamma|\mu) = e^{-(\nu+\gamma)b}.$$

(D) As in (2.4), the value must satisfy

$$\bar{\varphi} = \sup_{\tau \geq 0} E\big[e^{-\rho\tau}\big\{U(X_\tau - c) + \bar{\varphi}\big\}\,\big], \tag{4.10}$$

but with gain process (4.6), which no longer follows a time-homogeneous diffusion. We propose solving this by recursively solving

$$\bar{\varphi}_{n+1} = \sup_{\tau \geq 0} E\big[e^{-\rho\tau}\big\{U(X_\tau - c) + \bar{\varphi}_n\big\}\,\big], \tag{4.11}$$

starting from $\bar{\varphi}_0 = 0$. We solve a Crank–Nicolson finite-difference scheme to obtain the answer. The calculations and the proof that (4.11) has a solution are given in Appendix B.

5. Numerical study.

We shall compare the stopping rules of section 3 in several examples. In all cases, we shall assume that $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, and $\rho = 0.1$. We have explored various other examples, and the behavior which we report in these examples appears to be quite typical. A further comparison we make is with a fixed-revision rule, where the investor chooses $T > 0$ fixed, and then revises his position at multiples of T, regardless of the performance of the fund. The objective is once again given by (2.3), though now, of course, T is constant. We find the investor's best choice of fixed T and compare the performance of this rule with the various rules determined by stops.

In the first example, we assume that the investor knows $\mu = 0.15$ with certainty. There are four[9] stopping rules to be considered now, and the results are given in Table 1.

[9] The converging stops are left out in the following tables because we found that for this stopping rule it is optimal to mimic Example 2 by setting $a = b^*\epsilon$, where b^* is the optimal upper barrier for Example 2, and letting ϵ converge to 0.

Table 1
Known μ. Example with $\mu = 0.15$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, $\rho = 0.1$.

	Best a	Best b	Objective	$E[T]$
Fixed stops	∞	0.0184	4.1553	0.1224
Trailing stop and fixed stop	∞	0.0184	4.1553	0.1224
Trailing stop	0.0894		0.5314	0.0983
Fixed exit time			0.8724	0.3780

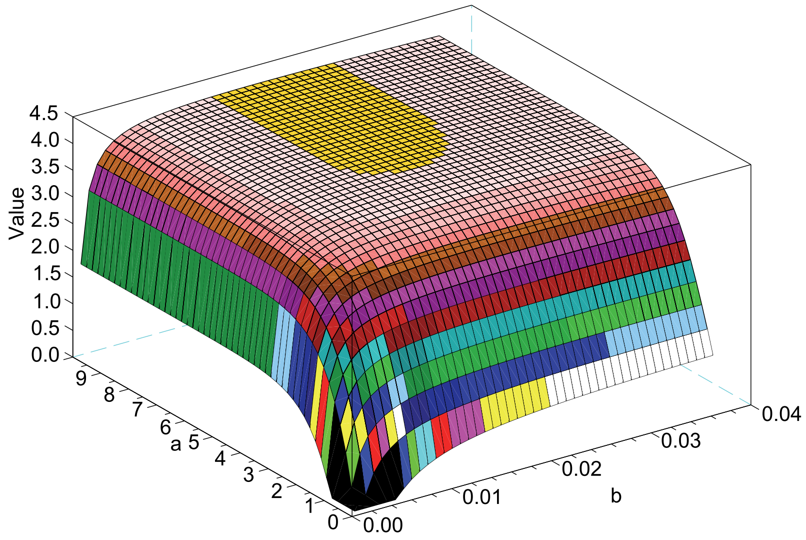

Figure 1. *Known μ. Two-stops rules (Examples 1 and 2). Example with $\mu = 0.15$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, $\rho = 0.1$.*

As the optimal solution to (2.4) is attained by fixed stops and the parameter are such that $\alpha > \gamma$, the optimal stopping rule is a one-sided trigger policy, where it is optimal to stop as soon as some threshold b is reached from below. Accordingly, with fixed stops or with a fixed upper stop and a trailing stop, the best choice of a is $a = \infty$; it always pays to push the lower stop all the way down. If this is done, then of course the two stopping rules amount to stopping at b, and so it is no surprise that the values, the optimal choices of b, and the mean time per trade all agree. The value φ as a function of a and b is displayed in Figure 1; for finite a, the pictures for Examples 1 and 2 are in principle different, but in this example they are not visibly different. Notice that the value for a fixed upper and trailing stop is substantially higher than for a trailing stop only; this is of course to be expected, as we have optimized over a larger set, but the magnitude of the improvement is noteworthy. The trailing stop example, Example 3, is quite different in character, with a much shorter mean time in trade. The fixed revision rule performs very poorly relative to the two-sided stops rules, Examples 1 and 2.

As we explained in section 4, it is uncertainty in the μ which vindicates trading to stops, and to illustrate this we study the stories where we do not suppose that μ is known. For (A)–(C), we assume that the drift of the fund we pick is a random variable with prior $N(\mu_0, \sigma_\mu^2)$ distribution. We suppose that $\mu_0 = 0.15$, and we take two different values for the standard deviation: first $\sigma_\mu = 0.3$ and second the more uncertain case $\sigma_\mu = 0.7$.

(A) The results obtained for the story where we go back into the same fund are reported in the following tables. Table 2 is for $\sigma_\mu = 0.3$ and Table 3 is for $\sigma_\mu = 0.7$.

Table 2

Story (A). *Example with* $\mu_0 = 0.15$, $\sigma_\mu = 0.3$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, $\rho = 0.1$.

	Best a	Best b	Objective	$E[T]$
Fixed stops	0.2159	0.0470	0.8416	0.0998
Trailing stop and fixed stop	0.2375	0.0464	0.8398	0.0984
Trailing stop	0.0603		0.2397	0.0439
Fixed exit time			0.5627	0.0670

Table 3

Story (A). *Example with* $\mu_0 = 0.15$, $\sigma_\mu = 0.7$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, $\rho = 0.1$.

	Best a	Best b	Objective	$E[T]$
Fixed stops	0.0837	0.0559	0.4410	0.0475
Trailing stop and fixed stop	0.1069	0.0523	0.4223	0.0452
Trailing stop	0.0411		-0.4264	0.0203
Fixed exit time			0.0698	0.0290

For $\sigma_\mu = 0.3$, the calculated values of the three stopping examples are displayed in Figures 2 and 3, respectively. Due to the risk aversion, the values of all the rules have dropped, particularly the stops trading rules. As with the certain growth rate, the two-stops rules do substantially better than either the trailing stop alone or the fixed time to revision. Mean times in trades have fallen in all cases. As before, there is no appreciable difference between Examples 1 and 2; the trailing stop has very little effect. Increasing the deviation of the drift to $\sigma_\mu = 0.7$ leads to even smaller objectives. In all cases, the parameter a has fallen to protect against huge losses.

(B) The next tables, Tables 4 and 5, show the results for $\sigma_\mu = 0.3$ and $\sigma_\mu = 0.7$, respectively, if we pick a new independent fund each time we come out of a trade.
The values of the three stopping examples which were calculated with respect to $\sigma_\mu = 0.3$ are displayed in Figures 4 and 5, respectively. Compared to story (A), the values of the stops trading rules have grown. It can be seen that the optimal lower barriers a have fallen to much lower values, while the values for the upper barriers b are much larger. The reason for this is that we do not want to stop a *good* investment having a large positive drift, but we get rid of those investments with negative drifts quickly. The difference in value of Examples 2 and 3 is comparatively small because the gain process in Example 2 will only occasionally get stopped at b; most will be caught by the trailing stop. We see, however, that the fixed-stops Example 1 performs

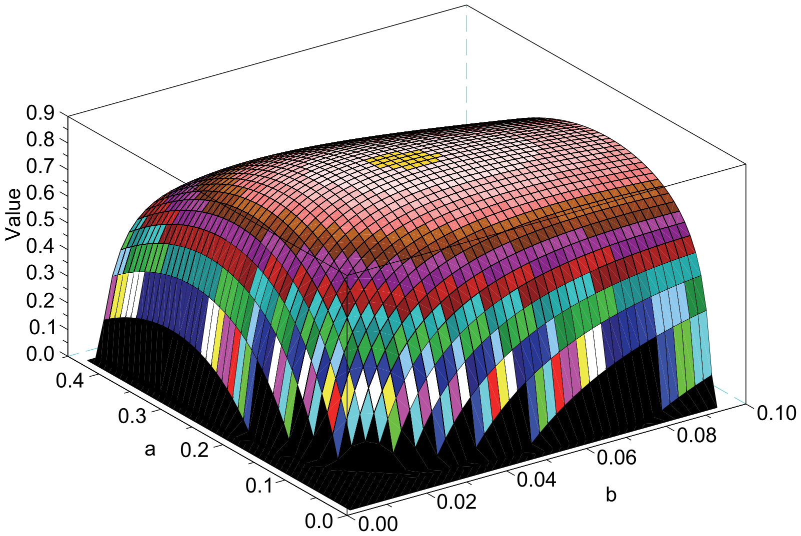

Figure 2. *Story* (A). *Fixed stops. Example with* $\mu_0 = 0.15$, $\sigma_\mu = 0.3$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, $\rho = 0.1$.

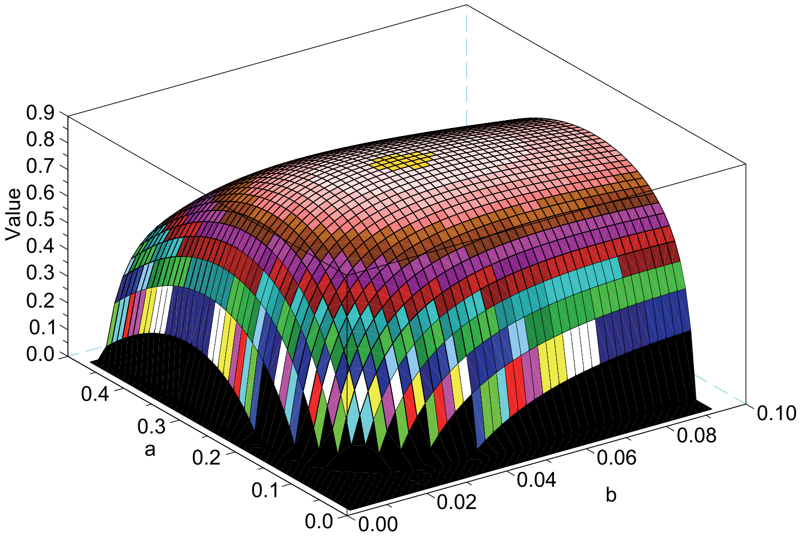

Figure 3. *Story* (A). *Trailing stop and fixed stop. Example with* $\mu_0 = 0.15$, $\sigma_\mu = 0.3$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, $\rho = 0.1$.

Table 4
Story (B). *Example with* $\mu_0 = 0.15$, $\sigma_\mu = 0.3$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, $\rho = 0.1$.

	Best a	Best b	Objective	$E[T]$
Fixed stops	0.0191	0.3409	1.4071	0.0804
Trailing stop and fixed stop	0.0985	0.6558	1.3431	0.1236
Trailing stop	0.0952		1.3354	0.1159
Fixed exit time			0.5596	0.0662

Table 5
Story (B). *Example with* $\mu_0 = 0.15$, $\sigma_\mu = 0.7$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, $\rho = 0.1$.

	Best a	Best b	Objective	$E[T]$
Fixed stops	0.0083	0.3426	4.8955	0.0338
Trailing stop and fixed stop	0.0767	0.7572	4.1695	0.0792
Trailing stop	0.0728		4.1224	0.0720
Fixed exit time			0.0698	0.0290

better than Example 2 with a trailing stop, presumably because the trailing stop may prematurely close out a trade which might have turned out to be profitable. Interestingly, when we compare the values for Examples 1 and 2 in story (B) with the values for Examples 1 and 2 in the certain-drift case, we find that for the smaller value $\sigma_\mu = 0.3$ we do better if we know the drift, while for the larger value $\sigma_\mu = 0.7$ we do better if we have uncertainty in the drift. The reason is not hard to discern. For small σ_μ, risk aversion is the dominant effect, but for larger σ_μ we benefit from the wider spread of μ-values; the lower stop closes down the unprofitable trades, but we get more of an upside from the profitable trades.

(C) In story (B) we have seen that the two-stops examples have small a to shut down the unprofitable trades and large b to let the gains accumulate when we have found a profitable trade. In contrast, when we use story (C), which only changes funds if we come out at the lower stop, the results in Tables 6 and 7 look quite different.[10]
For $\sigma_\mu = 0.3$, the values are displayed in Figures 6 and 7, respectively. Notice first that the values of the objectives are substantially higher, because we are allowed to shop around for good funds, and once we have found one, we are allowed to play that fund until we get stopped out at a lower stop. Because of this, we have to be careful not to stop out a fund at the lower stop unless we are quite confident that it is a poor performer; the loss of profit from killing a good fund too early would be considerable. So this explains why we see larger a values than for story (B). We also see much smaller b values, which we understand as a desire to book profits quickly and avoid discounting them away; if we think we are playing a good fund, we will gladly do this, because we can just return to playing the same good fund immediately, in contrast to the situation of story (B) where we would have to pick a new independent fund and

[10]Of course, it does not make any sense to consider the single stopping rules, trailing stop, and fixed exit time because there we cannot distinguish between upper and lower outcomes.

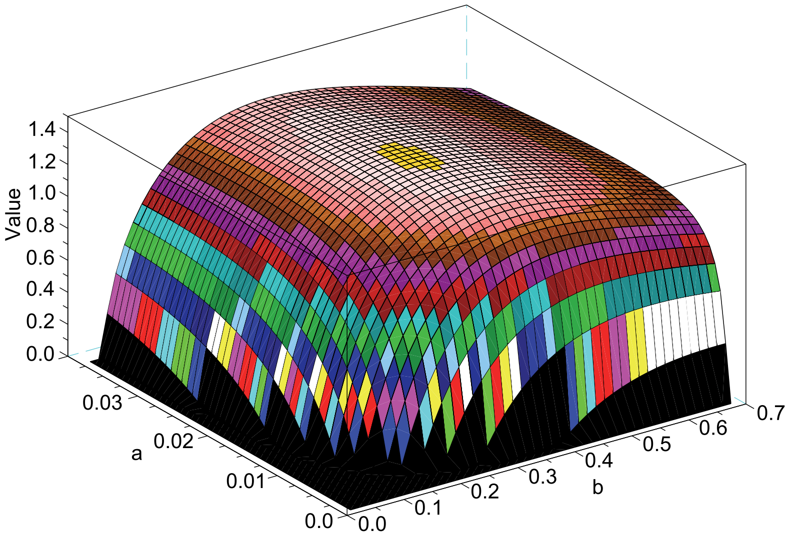

Figure 4. *Story* (B). *Fixed stops. Example with* $\mu_0 = 0.15$, $\sigma_\mu = 0.3$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, $\rho = 0.1$.

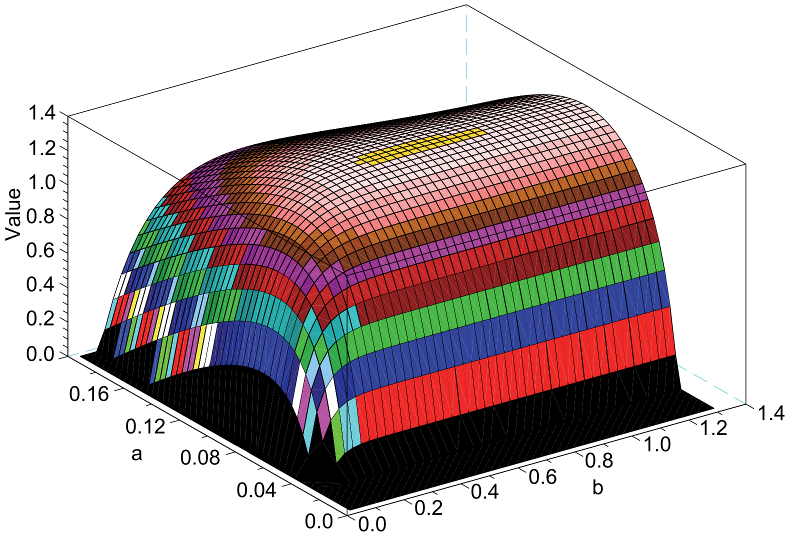

Figure 5. *Story* (B). *Trailing stop and fixed stop. Example with* $\mu_0 = 0.15$, $\sigma_\mu = 0.3$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, $\rho = 0.1$.

Table 6
Story (C). *Example with* $\mu_0 = 0.15$, $\sigma_\mu = 0.3$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, $\rho = 0.1$.

	Best a	Best b	Objective	$E[T]$
Fixed stops	0.3620	0.0239	6.8525	0.0370
Trailing stop and fixed stop	0.3735	0.0240	6.8539	0.0371

Table 7
Story (C). *Example with* $\mu_0 = 0.15$, $\sigma_\mu = 0.7$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, $\rho = 0.1$.

	Best a	Best b	Objective	$E[T]$
Fixed stops	0.2020	0.0213	20.216	0.0125
Trailing stop and fixed stop	0.2119	0.0219	20.231	0.0129

take our chances on its quality.

For the first time, we see Example 2 outperforming Example 1 (but only very slightly). This seems to be because the trailing stop will allow a slightly quicker closing out of bad trades, and since the lower stop is initially quite far from 0, this difference matters. Another way we could try to capture this advantage would be by adding a time-dependent slope to the barriers, and this is examined in section 6. For the same reasons as in story (B), a larger standard deviation $\sigma_\mu = 0.7$ yields a better objective.

(D) The Bayesian story has similarities to story (B); the stochastic nature of the funds is identical, but we allow *any* stopping rule. As was recorded at (4.6), we can model the gain process in the observation filtration as the solution of a stochastic differential equation, and the optimal stopping problem for this is found by solving the recursive scheme (4.11) by Crank–Nicolson. To compare with our results for story (B) using $\sigma_\mu = 0.3$, we propose taking a prior distribution for μ with mean 0.15 as before and precision $\tau_0 = \sigma^2/\sigma_\mu^2 = 1$. Table 8 compares the results from story (B) with the optimal solution obtained using story (D). Of course, we cannot report any fixed values for the optimal stopping solution, as the stopping boundary is a curve, which can be seen in Figure 8, that also covers a solution of a stopping rule which is considered later.

The shape of the stopping region can be interpreted as a time-dependent decreasing upper stop $\eta(t)$ and an increasing lower stop $\xi(t)$. The upper stop η begins at a high level to let *good* investments run and the lower stop ξ starts at a small negative value to immediately get rid of *bad* investments. As time goes by, the state of the gain process updates the drift estimator (4.5). The threshold $\eta(t)$ will decrease with t; if we decided to stop at time t when the gain value X_t was $y > 0$, then we would certainly want to stop at level y at any later time because our estimate of μ would then be smaller and we would be more confident in that estimate. A corresponding argument applies to the lower stop $\xi(t)$.

6. Time-dependent slope. In this section, we examine what happens if some time-dependent factors are added to the stops. Then, for a parameter q, the modified stopping times are as follows:

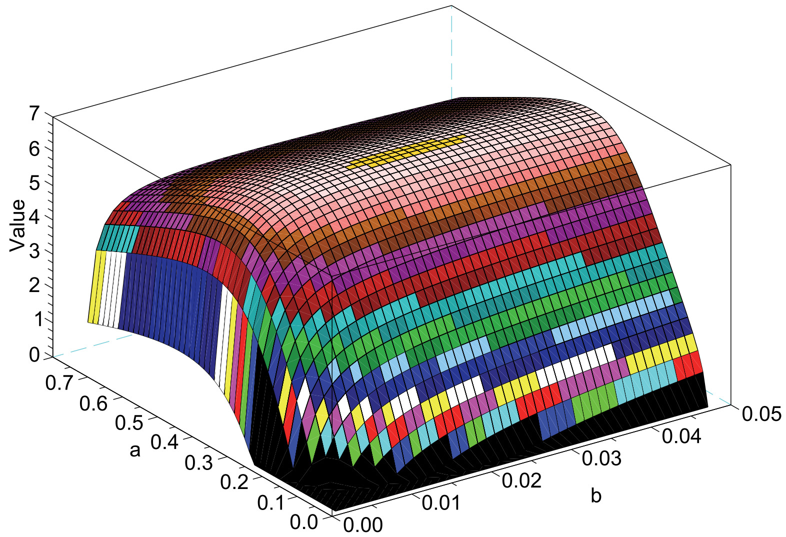

Figure 6. *Story* (C)*. Fixed stops. Example with* $\mu_0 = 0.15$*,* $\sigma_\mu = 0.3$*,* $\sigma = 0.3$*,* $\gamma = 2.5$*,* $c = 0.0005$*,* $\rho = 0.1$*.*

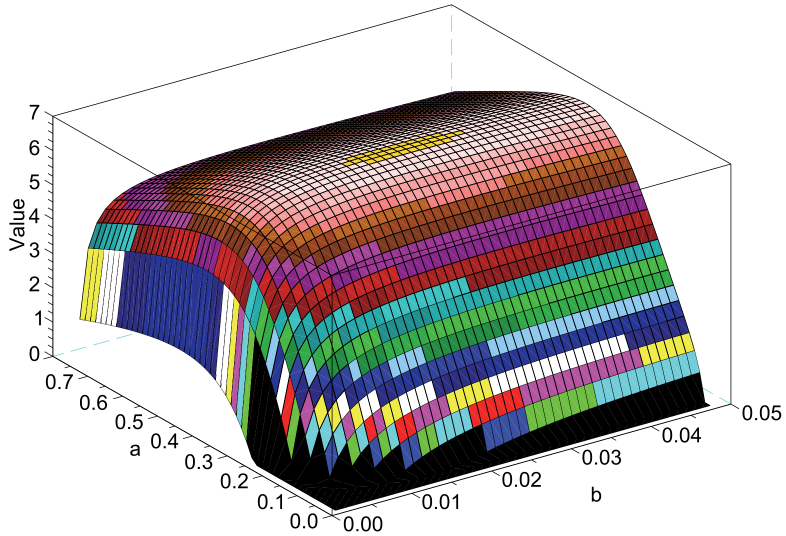

Figure 7. *Story* (C)*. Trailing stop and fixed stop. Example with* $\mu_0 = 0.15$*,* $\sigma_\mu = 0.3$*,* $\sigma = 0.3$*,* $\gamma = 2.5$*,* $c = 0.0005$*,* $\rho = 0.1$*.*

Table 8

Story (D) *and story* (B). *Example with* $\mu_0 = 0.15$, $\sigma_\mu = 0.3$, $\tau_0 = 1$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, $\rho = 0.1$.

	Best a	Best b	Objective	$E[T]$
Optimal stopping			1.6770	
Fixed stops	0.0191	0.3409	1.4071	0.0804
Trailing stop and fixed stop	0.0985	0.6558	1.3431	0.1236
Trailing stop	0.0952		1.3354	0.1159
Fixed exit time			0.5596	0.0662

Example 1 (fixed stops). We take $a > 0$, $b > 0$ and set

$$T \equiv \inf\{t : X_t = -a + q\,t \text{ or } X_t = b + q\,t\}.$$

Example 2 (trailing stop and fixed stop). This time we fix $a > 0$ and $b > 0$ and define $\hat{X}_t = \sup_{0\le s\le t}\{X_s - q\,s\}$. Then we use the stopping time

$$T \equiv \inf\{t : X_t = \hat{X}_t - a + q\,t \text{ or } X_t = b + q\,t\}.$$

Example 3 (trailing stop). Fix $a > 0$ and set

$$T \equiv \inf\{t : X_t = \hat{X}_t - a + q\,t\}.$$

Regarding a process[11] $Y_t \equiv X_t - q\,t$, then for Y, the above stopping rules correspond to the time-independent ones defined in section 2. Thus, for process X and the time-dependent barriers, the joint Laplace transforms can be computed from the Laplace transforms of the previous sections with respect to process Y:

$$\begin{aligned} L_X(\rho,\gamma) &= E[e^{-\rho T - \gamma X_T}] \\ &= E[e^{-\rho T - \gamma(Y_T + qT)}] \\ &= E[e^{-(\rho+\gamma q)T - \gamma Y_T}] \\ &= L_Y(\rho + \gamma q, \gamma). \end{aligned}$$

Having added an additional parameter q which defines the slope of the barriers, we can optimize over a, b, and q to obtain the objectives for known drift μ and for the stories (A)–(C). As $q = 0$ is a feasible choice, the objectives we will find cannot be smaller than those found in section 5. For a better comparison, in the following tables, the optimal objectives for $q = 0$ are given as well. As we have seen that the upper barrier is of importance, we will concentrate on Examples 1 and 2 only. Furthermore, in case of uncertainty, only $\sigma_\mu = 0.3$ will be considered. All other parameters are as in section 5.

[11] In what follows, Y is defined to be a new process.

In case of certainty, the results given in Table 9 show that the optimal value for q is zero. This is due to the fact that for known μ, the fixed-stops rule solves the optimal stopping problem (2.4). Thus, a slope does not lead to an improvement and all results are as in Table 1.

Table 9

Known μ with slope q. Example with $\mu = 0.15$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, $\rho = 0.1$.

	Best a	Best b	Best q	Objective	$E[T]$	$q = 0$
Fixed stops	∞	0.0184	0.0000	4.1553	0.1224	4.1553
Trailing stop and fixed stop	∞	0.0184	0.0000	4.1553	0.1224	4.1553

Table 10

Story (A) with slope q. Example with $\mu_0 = 0.15$, $\sigma_\mu = 0.3$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, $\rho = 0.1$.

	Best a	Best b	Best q	Objective	$E[T]$	$q = 0$
Fixed stops	0.2244	0.0447	0.0347	0.8426	0.1005	0.8416
Trailing stop and fixed stop	0.2450	0.0441	0.0346	0.8407	0.0991	0.8398

Table 11

Story (B) with slope q. Example with $\mu_0 = 0.15$, $\sigma_\mu = 0.3$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, $\rho = 0.1$.

	Best a	Best b	Best q	Objective	$E[T]$	$q = 0$
Fixed stops	0.0345	0.2944	0.4949	1.5976	0.0782	1.4071
Trailing stop and fixed stop	0.1102	0.5445	0.2789	1.3519	0.1251	1.3431

(A) If we stick with the same fund forever, we will get the results given in Table 10.
The results are very close to those in section 5 because the optimal parameter q is close to 0. In other words, allowing the barriers to have a time-dependent drift does not yield a substantial improvement.

(B) Choosing a different investment from the marketplace on each side yields Table 11.
Just as in the case $q = 0$, we still get a large b to let *good* investments run and a small a to quickly stop *bad* investments. The improvement in the objective is much greater for Example 1 than for Example 2.

(C) If we pick an independent fund if we end up on the down side, this gives the results which are summarized in Table 12.
In this case there is a 1.7% improvement of the objective due to the slope q. As guessed above, the increasing lower barrier tackles below-average investments. The slope parameter q is considerably larger than 0, but it is not as high as in story (B), which reflects the risk of accidentally stopping a *good* investment.

(D) In section 5, we found that the objective for the best fixed-stops rule is quite far from the optimum. However, adding a time-dependent slope yields the situation given in Table 13.

Table 12

Story (C) *with slope q. Example with $\mu_0 = 0.15$, $\sigma_\mu = 0.3$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, $\rho = 0.1$.*

	Best a	Best b	Best q	Objective	$E[T]$	$q = 0$
Fixed stops	0.4181	0.0192	0.1212	6.9719	0.0330	6.8525
Trailing stop and fixed stop	0.4274	0.0192	0.1209	6.9721	0.0330	6.8539

Table 13

Story (D) *and story* (B) *with slope q. Example with $\mu_0 = 0.15$, $\sigma_\mu = 0.3$, $\tau_0 = 1$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, $\rho = 0.1$.*

	Best a	Best b	Best q	Objective	$E[T]$	$q = 0$
Optimal stopping				1.6770		
Fixed stops	0.0345	0.2944	0.4949	1.5976	0.0782	1.4071
Trailing stop and fixed stop	0.1102	0.5445	0.2789	1.3519	0.1251	1.3431

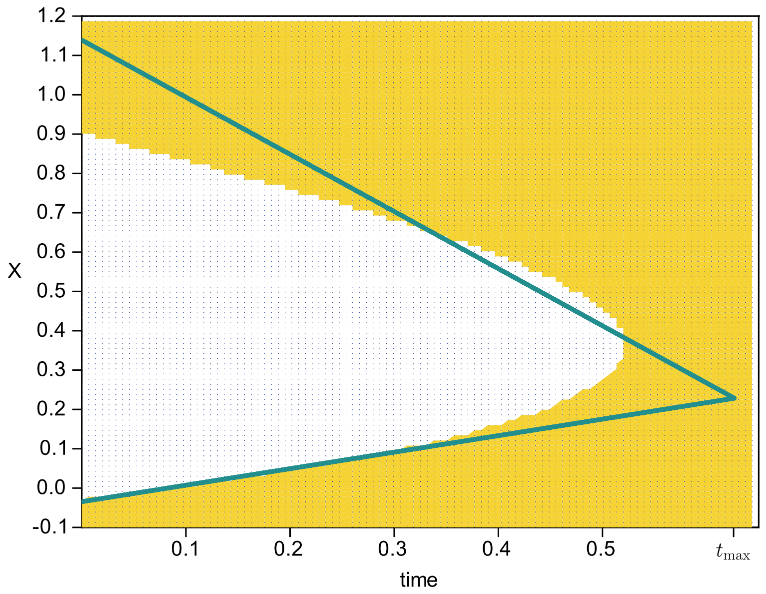

Figure 8. *Story* (D). *Boundary of the optimal stopping problem and the optimal fixed stops with two slopes. Example with $\mu_0 = 0.15$, $\tau_0 = 1$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, $\rho = 0.1$.*

The result of the optimal stopping problem cannot be improved, so we get the same objective as in section 5. But the time-dependent slope pushes the fixed stop's objective up by 13.5%, bringing the value much closer to the optimum, remarkably so given the very simple-minded nature of the stopping rule.

The shape of the stopping region in Figure 8 suggests using a falling upper stop with a rising lower stop. For fixed stops, this can be realized by using two different slope parameters

q_L and q_H. Then, for $a > 0$ and $b > 0$, the stopping time is defined by

$$T \equiv \inf\{t : X_t = -a + q_L t \text{ or } X_t = b + q_H t\}.$$

See [8] for a derivation of the joint Laplace transform for this stopping time. The representation of this Laplace transform contains two integrals which have to be computed numerically. This computation is very costly, and the numerical results in Table 14 show that the improvement of adding a second slope parameter is small. In Figure 8, the optimal fixed stops with two slopes are overlapping the figure of the optimal stopping problem.

Table 14

Story (D) *and story* (B) *with single slope q and two slopes q_L and q_H. Example with $\mu_0 = 0.15$, $\sigma_\mu = 0.3$, $\tau_0 = 1$, $\sigma = 0.3$, $\gamma = 2.5$, $c = 0.0005$, $\rho = 0.1$.*

	Best a	Best b	Best q_L	Best q_H	Objective	$q = 0$
Optimal stopping					1.6770	
Fixed stops (single slope)	0.0345	0.2944	0.4949	0.4949	1.5976	1.4071
Fixed stops (two slopes)	0.0324	1.1400	0.4198	-1.4532	1.6305	1.4071

7. Conclusions.

There are at least three reasons why we might in practice wish to trade to stops in some way. The first is to reduce transaction costs: trading strategies which rebalance infrequently are always preferred, and in some asset classes, such as EM currencies where costs might typically be 40bp, daily rebalancing will quickly eat into profits. The second reason for wishing to trade to stops in some way is that until a position has been closed out, the investor cannot make use of the gains. There is therefore an incentive not to let a position run indefinitely, but to take profits at some point. Following from this is a third reason; if our choice of stops rests on current estimates of fund dynamics, then it is important that we not sit in the trade long after the parameter estimates have wandered away; otherwise the expected performance may not materialize.

In this study, we have investigated several possible rules for placing fixed or moving stops and compared their performance. We have found that uncertainty over the growth rate of the fund is an essential feature of choosing stops; if we know that the fund is drifting up, we would never want to place a lower stop. The possibility that the drift might be negative is what makes us want to put in lower stops.

By hypothesizing the existence of many alternative investment opportunities, we investigated a number of different responses which could be taken when our existing position stops out. Of these, story (B), which supposes that we start with a new investment when we have stopped out, leads to stopping rules closest to the market wisdom of "run large gains and stop small losses"; however, the performance of the contrasting story (C), where we stick with a trade until it stops out at a lower stop, was much better than story (B). The difference between the performance of the fixed-stops Example 1 and the mixed-stops Example 2 was very small in story (C), whereas in story (B) the difference was more substantial. This could lead us to choose to use fixed stops, though the trailing stop does guard against the possibility that the drift might deteriorate while we are in the trade; this is an effect which we have not considered but which can be found in [8].

Our final study simply puts a constant drift onto the stopping examples studied earlier. Since we optimize over a larger class, the performance improves when we do this, often by an appreciable margin. Two points are noteworthy here. The first is that if we pick an independent fund each time we come out of the market, then the optimal solution for the full Bayesian learning analysis is only a little better than the best fixed-stops solution (objective 1.6770 against 1.5976 in the numerical example). This means that the much simpler fixed-stops rule applied to the fund with suitably chosen drift is close to optimal and can be recommended.

The second noteworthy point is that by far the best strategy coming out of our study is to use fixed stops with a constant drift, but only to change the fund on those occasions when we come out at the lower stop. Of course, this conclusion depends on the particular parameter values chosen, but in the example we report that the value of the objective is approximately quadrupled.

Appendix A. Proof of Proposition 2.

Proof of Proposition 2. The case $\gamma > \alpha$ is easy; the dominant term in (4.1) is the term $k_1 e^{\gamma(a+c)}$ in the numerator, and this makes it obvious[12] that $\varphi(a) \to -\infty$ as $a \to \infty$.

The second case is more delicate. The limit of $\varphi(a)$ is easily seen to be

$$\varphi(\infty) = -1 + \frac{1 - B_2 e^{\gamma c}}{1 - B_3}.$$

If we now consider $\varphi(\infty) - \varphi(a)$, we find a rational expression whose denominator is positive, and whose numerator is (a positive multiple of)

$$H \equiv (1 - B_3)z - (B_2 e^{\gamma c} - B_3)y - (1 - B_2 e^{\gamma c}), \tag{A.1}$$

where we set $z \equiv e^{\gamma(a+c)}$, $y \equiv e^{-\beta a}$ for brevity. Thus it will be sufficient to prove that the expression H is nonnegative.

Since $b > c$, we may write $\varepsilon = b - c > 0$, and then H becomes

$$\begin{aligned} H &= (1 - B_3)z + B_3(1 - e^{-\gamma-\varepsilon})y - (1 - B_3 e^{-\gamma\varepsilon}) \\ &= (1 - B_3)(z - 1) + B_3(1 - e^{-\gamma\varepsilon})y - B_3(1 - e^{-\gamma\varepsilon}) \\ &= (1 - B_3)(z - 1) - B_3(1 - e^{-\gamma\varepsilon})(1 - y). \end{aligned} \tag{A.2}$$

It is clear from the final equation that if we now hold $a > 0$ fixed and consider H as a function of γ, then H is convex and vanishes as $\gamma \downarrow 0$. To prove the nonnegativity of H, we now investigate the gradient of H with respect to γ, which is

$$\begin{aligned} \frac{\partial H}{\partial \gamma} &= (1 - B_3)(a + c)e^{\gamma(a+c)} - \varepsilon B_3(1 - y)e^{-\gamma\varepsilon} \\ &= e^{-\gamma\varepsilon}\big[(1 - B_3)(a + c)e^{\gamma(a+b)} - (1 - y)B_3(b - c)\big]. \end{aligned}$$

[12]The denominator is asymptotic to $e^{\alpha a}(1 - B_3)$, which is certainly positive.

As $\gamma \downarrow 0$, we obtain the limit

$$\begin{aligned}\frac{\partial H}{\partial \gamma}(0) &= (1-B_3)(a+c) - (1-y)B_3(b-c)\\ &= (1-e^{-\beta b})(a+c) - e^{-\beta b}(b-c)(1-e^{-\beta a})\\ &= e^{-\beta b}\left[\,(a+c)e^{\beta b} + (b-c)e^{-\beta a} - (a+b)\,\right]\\ &= (a+b)e^{-\beta b}\left[\,\frac{a+c}{a+b}\,e^{\beta b} + \frac{b-c}{a+b}\,e^{-\beta a} - 1\right]\\ &\geq (a+b)e^{-\beta b}\left[e^{\beta c} - 1\right]\\ &> 0,\end{aligned}$$

where we have used convexity of the exponential function for the first inequality. Since H is convex and its derivative at zero is positive, it follows that H is increasing, and therefore is everywhere nonnegative, since it is zero at $\gamma = 0$. ∎

Appendix B. Crank–Nicolson finite-difference scheme. In what follows, it will be shown how to calculate the value $\bar{\varphi}_{n+1}$ with $\bar{\varphi}_n$ given to solve (4.11). The gain process satisfies the diffusion equation (4.6). The stopping reward process is of the form

$$\begin{aligned}Z(t, X_t) &= e^{-\rho t}(U(X_t - c) + \bar{\varphi}_n)\\ &\equiv e^{-\rho t} g(X_t),\end{aligned}$$

where $\rho \geq 0$. We fix a final time $\bar{T}$ which should be large enough to be outside the continuation region and define the value function

$$V(t,x) \equiv \sup_{t \leq \tau \leq \bar{T}} E[e^{-\rho(\tau - t)} g(X_\tau) | X_t = x]; \tag{B.1}$$

then we have that $V \geq g$ everywhere, and that

$$\mathcal{L}V + V_t - \rho V \leq 0, \tag{B.2}$$

which holds with equality when $V > g$, where $\mathcal{L}$ is the generator of the diffusion,

$$\mathcal{L} \equiv \frac{1}{2}\sigma^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mu(t, X_t)\frac{\mathrm{d}}{\mathrm{d}x}.$$

This problem can be solved numerically by using the Crank–Nicolson finite-difference scheme. See [4] for the original paper or [20] for a more general description. We set down a grid of x-values and a grid $0 = t_0 < t_1 < \cdots < t_N = \bar{T}$ of time values, and we let $L^{(n)}$ be a discrete approximation of the diffusion generator at $t = t_n$. If $v^{(n)}$ denotes the approximation of the value function at time t_n, then the Crank–Nicolson method approximates (B.2) by

$$\frac{1}{2}\{L^{(n)}v^{(n)} + L^{(n+1)}v^{(n+1)}\} - \frac{1}{2}\rho(v^{(n)} + v^{(n+1)}) + \Delta t_n^{-1}(v^{(n+1)} - v^{(n)}) \leq 0, \tag{B.3}$$

with equality where it is optimal to continue. Here, we write $\Delta t_n = t_{n+1} - t_n$. The unknown in this equation is $v^{(n)}$; we start with $v^{(N)}(x) = g(\bar{T}, x)$ since, by assumption, the final time point $\bar{T}$ is outside the continuation region, and we work recursively back through the grid in the usual dynamic-programming fashion. Rewriting (B.3) to make the unknown the subject, we have, say,

$$\begin{aligned}(L^{(n)} - \rho - 2\Delta t_n^{-1})v^{(n)} &\leq -(L^{(n+1)} - \rho + 2\Delta t_n^{-1})v^{(n+1)} \\ &\equiv -\alpha^{(n)},\end{aligned} \tag{B.4}$$

with equality at all places where it is optimal to continue and with $v^{(n)}(x) = g(t_n, x)$ in places where it is optimal to stop.

However, the problem (B.4) is an optimal stopping problem for the Markov chain with generator[13] $L^{(n)}$, with discount rate $\rho + 2\Delta t_n^{-1}$, and with running reward $\alpha^{(n)}$. It is quite straightforward (and very fast) to solve this by policy improvement. Probably the simplest thing to do at the boundaries is to insist that the process gets absorbed there, so in the original stopping problem, we have to stop when we reach one end or the other end of the x-grid.

Proof that (4.11) *has a solution.* If $\bar{\varphi}_1 > 0$, then from (4.11) we see that $\bar{\varphi}_2 > \bar{\varphi}_1$ and hence that the sequence $\bar{\varphi}_n$ is increasing. Similarly, if $\bar{\varphi}_1 < 0$, we deduce that the sequence $\bar{\varphi}_n$ is decreasing. Therefore a limit for the $\bar{\varphi}_n$ exists, and what remains is to prove that this limit must be finite. To establish a lower bound, we consider the fixed stopping time $\tau = 1$. Then from (4.11) we see, say,

$$\begin{aligned}\bar{\varphi}_{n+1} &\geq e^{-\rho} E\left[U(X_1 - c) + \bar{\varphi}_n\right] \\ &\equiv a + \beta\bar{\varphi}_n.\end{aligned}$$

Iterating this inequality gives the lower bound $\bar{\varphi}_n \geq a/(1-\beta)$ for all n.

For the upper bound, we note that $U(x) \leq \gamma x$, so

$$\bar{\varphi}_{n+1} \leq \sup_\tau E[e^{-\rho\tau}(\gamma X_\tau + \bar{\varphi}_n - \gamma c)]. \tag{B.5}$$

This leads us to consider the problem

$$V(x) = \sup_\tau E^x[e^{-\rho\tau}(\gamma X_\tau + \bar{\varphi}_n - \gamma c)]. \tag{B.6}$$

By considering fixed times τ tending to infinity, it is clear that $V \geq 0$. Similarly, $V(\cdot)$ must be increasing and bounded below by $x \mapsto \gamma(x - c) + \bar{\varphi}_n$. Where we do not optimally choose to stop, V solves

$$-\rho V + \frac{1}{2}\sigma^2 V'' + \mu V' = 0. \tag{B.7}$$

If $-\alpha_- < 0 < \alpha_+$ are the roots of the characteristic quadratic $\frac{1}{2}\sigma^2 t^2 + \mu t - \rho$, the optimal solution V is

$$V(x) = \frac{\gamma}{\alpha_+} \exp[\,\alpha_+(x - c - \alpha_+^{-1} + \bar{\varphi}_n/\gamma)]. \tag{B.8}$$

[13] With a three-point finite difference scheme, the matrix $L^{(n)}$ will usually be a Q-matrix; the calculations need to check this.

Evaluating at $x = 0$ and writing $\xi_n \equiv \alpha_+\bar{\varphi}_n/\gamma$, we learn that

$$\xi_{n+1} \le \exp(\xi_n - 1 - c\alpha_+), \tag{B.9}$$

with the initial condition $\xi_0 = 0$. The most extreme case is when this inequality holds with equality. In that case, it is easy to see that the ξ_n increase to the unique fixed point less than 1 of $x \mapsto \exp(x - 1 - c\alpha_+)$. Hence we deduce that the $\bar{\varphi}_n$ are bounded above. ■

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