On the Boltzmann equation, Quantitative studies and hydrodynamical limits

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Statement of Originality

I hereby declare that my dissertation entitled "On the Boltzmann equation, quantitative studies and hydrodynamical limits" is not substantially the same as any that I have submitted for a degree or diploma or other qualification at any other University. I further state that no part of my dissertation has already been or is concurrently submitted for any such degree of diploma or other qualification.

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except where specifically indicated in the text.

Chapter 1 introduces the work done in the present dissertation. It contains some elementary properties about the Boltzmann equation and mostly summarises the results presented later on.

Chapter 2 studies quantitatively the strict positivity of solutions to the Boltzmann equation. It is original work and has been submitted. Its arXiv link is: http://arxiv.org/abs/1302.1755.

Chapter 3 contains a detailed presentation of Hilbert's 6^{th} problem and a brief state of the art. It is mainly to motivate the studies made in the following two chapters.

Chapter 4 rigorously derives the incompressible Navier-Stokes limit of the Boltzmann equation with constructive arguments, in Sobolev spaces with exponential weight. It is original work and has been submitted. Its arXiv link is: http://arxiv.org/abs/1207.4379.

Chapter 5 extends the results obtained in the previous chapter to Sobolev spaces with polynomial weight. It is original work in collaboration with S. Merino-Aceituno and C. Mouhot.

Chapter 6 establishes a Cauchy theory on a quantic version of the Boltzmann equation and discusses the phenomenon of Bose-Einstein condensate. It is original work and its arXiv link is: http://arxiv.org/abs/1310.7220.

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Contents

Chapter 1

Introduction

The present thesis deals with the mathematical treatment of kinetic theory and focuses more precisely on the Boltzmann equation. The latter equation describes the evolution in position and velocity of rarefied gas particles with a statistical point of view. It plays a central role in mathematical physics as it builds a bridge between Newtonian systems of particles and fluid dynamics. In this chapter, we start with a brief overview of the Boltzmann equation and its main features (Section 1.1). We then present some mathematical problems such as the quantification of positivity of solutions (Section 1.2) and the Cauchy theory and the trend to equilibrium in a perturbative setting (Section 1.3); which is a short introduction to the hydrodynamical limits of the Boltzmann equation which will be studied more thoroughly in Part II. We conclude by a quantic version of the Boltzmann equation that is used to describe gases of bosons and fermions and also contains the mechanisms of the Bose-Einstein condensate (Section 1.4). We give, in each section, a brief description of our main contributions in those domains.

The reader will also find an index of the notations we use for functional spaces in Appendix 1.A.

Contents

1.1 General presentation of the Boltzmann equation

In this section we gather some general definitions, descriptions and properties of kinetic theory and the Boltzmann equation in rather informal statements. We refer to the standard books [28][32][30] or the review article [112] for deeper considerations and rigorous statements.

1.1.1 Models in kinetic theory

1.1.1.1 Kinetic theory and physics of particles and fluids

If one considers a system of N bodies, which can be particles, stars or galaxies for instance, moving in a domain $\Omega \subset \mathbb{R}^d$ with velocities in $\mathbb{R}^d$, the Newton's laws of motion describes the dynamics of the latter system. These laws are the foundations of classical mechanics and generate, in that case, an Hamiltonian system in the phase space $\left(\Omega \times \mathbb{R}^d\right)^N$. This system of evolution equations encodes the particular dynamics of each body, this is called the microscopic scale.

Unfortunately, the N-body problem is renowned to be of a tremendous mathematical difficulty (even when $N = 3$) and is therefore hardly useful to study or predict the behaviour of a large number of entities. In the case of a large system, it is often more interesting to look into the general, the average, behaviour rather than following each particle individually. Applying Newton's laws to infinitesimal volume of particles at equilibrium inside this volume leads to the equations of fluids mechanics, such as Euler or Navier-Stokes equations. This point of view is called the macroscopic scale.

The macroscopic scale proved itself useful to describe the average dynamics of systems (sea or car traffic for example). It however comes with the drawback that is the loss of the microscopic dynamics inside the system. Kinetic theory stands right between the microscopic and the macroscopic scale, it is called the mesoscopic scale. It adopts a probabilistic approach to the problem in order to decrease the degrees of freedom of the Hamiltonian system but still keeps track of the microscopic dynamics.

A deeper presentation of these physical points of view as well as their different connections is given in Chapter 3.

The aim of kinetic theory is to model a system constituted of a large number of particles by a distribution function, in the one particle phase space of position and velocity, that evolves with time. More precisely, the dynamics of the system is encoded in a density function

$$\begin{array}{rccc} f: & [0,T] \times \Omega \times \mathbb{R}^d & \longrightarrow & \mathbb{R}^+ \\ & (t,x,v) & \longmapsto & f(t,x,v), \end{array}$$

where $T > 0$ can be infinite and $\Omega \times \mathbb{R}^d$ is the particle phase space introduced earlier.

Physically speaking, for a given position x and velocity v, the quantity $f(t,x,v)dxdv$ is the probability of having a particle in the ball $B(x,dx)$ with a velocity in the ball $B(v,dv)$ at time t. One can understand f as an approximation, in the limit when N tends to infinity, of the first marginal of the empirical measure of the system

$$\frac{1}{N}\sum_{i=1}^{N}\delta_{x_i(t)}(x)\delta_{v_i(t)}(v),$$

where $(x_i(t),v_i(t))$ is the position and velocity of the i^{th} particle at time t. A more precise description is given in Chapter 3.

Moreover, for kinetic theory to have a physical meaning, one expects that the total mass of the system remains finite in bounded domains and therefore, the minimal assumption required for f is that

$$\forall t\in[0,T],\quad f(t,\cdot,\cdot)\in L^1_{loc}\left(\Omega, L^1_v\left(\mathbb{R}^d\right)\right).$$

In this point of view, physical observables can be expressed as averages in velocities. We therefore obtain the following local macroscopic quantities of the system of N particles.

- the local density:
$$\rho(t,x)=\int_{\mathbb{R}^d} f(t,x,v)\,dv,$$
- the local velocity:
$$u(t,x)=\frac{1}{\rho(t,x)}\int_{\mathbb{R}^d} vf(t,x,v)\,dv,$$
- the local temperature:
$$\theta(t,x)=\frac{1}{d\rho(t,x)}\int_{\mathbb{R}^d}|v-u|^2 f(t,x,v)\,dv,$$

  or, equivalently, the local energy:
$$E(t,x)=\int_{\mathbb{R}^d}\frac{|v|^2}{2}f(t,x,v)\,dv=\rho(t,x)\frac{|u|^2}{2}+d\frac{\rho(t,x)\theta(t,x)}{2}.$$

The mass, mean velocity and temperature of the system being the integral against the space variable x over the spatial domain Ω of the local observables. In order to be physically relevant, kinetic theory focuses on density functions f that have finite mass, mean velocity and energy at each time.

1.1.1.2 Evolution equations in kinetic theory

As described before, the kinetic theory point of view is to model the dynamics of a large number of particles thanks to an evolution equation satisfied by a density function $f = f(t, x, v)$. This equation has to take into account the free motion of a particle and the possible distortion it undergoes due to an external force or interactions with other particles. As we will see, the latter interactions play a major role in physical studies and contain most of the mathematical difficulties.

In the case of non-interacting particles and in the absence of external force, the motion remains straight lines travelled along with constant velocity. The corresponding equation is the free transport equation

$$\partial_t f + v \cdot \nabla_x f = 0. \tag{1.1.1}$$

When the system is influenced by an external force $F_{ext} = F_{ext}(x)$ acting on the particles, corrections have to be made to (1.1.1). The new equation is called the linear Vlasov equation and reads

$$\partial_t f + v \cdot \nabla_x f + F_{ext}(x) \cdot \nabla_v f = 0. \tag{1.1.2}$$

Even though these equations are deeper than they look, especially in bounded domains, they neglect the interactions which may exist between particles. These interactions could be attractive or repulsive, thinking of electromagnetism for instance, but also should model what happens when two, or more, particles collide with each other.

The modelling of one-to-one interaction between particles can be done in two different ways, and one can, of course, combine them. The idea of how to derive them from microscopic behaviours is given in Chapter 3, Section 3.2.

If the range of the interaction is macroscopic then the evolution equation is called a mean-field equation. This type of kinetic equation is non-linear and has the following form.

$$\partial_t f + v \cdot \nabla_x f + \nabla_x \Psi(t, x) \cdot \nabla_v f = 0, \tag{1.1.3}$$

with

$$\Psi(t, x) = -\psi *_x \int_{\mathbb{R}^d} f(t, x, v) dv.$$

A typical example of a mean-field equation is the Vlasov-Poisson equation used to described plasmas and for which ψ is the Coulomb interaction for electromagnetism:

$$\psi(z) = \frac{q^2}{4\pi\varepsilon_0 |z|},$$

where q is the electric charge of a particle and ε_0 is the vacuum permittivity.

Our work will, however, be about another way of modelling interactions between particles. In that case, the range of the interaction is assumed to be so small that it can be considered as a localised interaction. This happens when the trajectory of a particle is distorted when passing very close to another one or, in the simplest physical case, when the particles bounce again each other when colliding. The kinetic equations describing this type of interactions are called collision equations and read

$$\partial_t f + v \cdot \nabla_x f = Q(f), \tag{1.1.4}$$

where Q can be non-linear and encodes the physical properties of the collision process.

One of the most fundamental collision equation in kinetic theory gives the dynamics of rarefied gases. This equation is called the Boltzmann equation and will be the subject of this entire thesis.

1.1.2 The collisional model of the Boltzmann equation

As mentionned above, we will only be interested in the case of the collisional model (1.1.4) described by the Boltzmann equation (even if some of our results will apply to more models, see Chapter 4. We give below some elementary properties of the Boltzmann collisional operator Q.

1.1.2.1 The Boltzmann collision operator

The kinetic theory point of view begins with the microscopic modelling of the collisional interactions between particles. The Boltzmann equation rules a particular sort of many particles system. We restrain ourselves to the case of monoatomistic system with elastic collisions. The formal derivation of the Boltzmann equation relies on the following assumptions on the physical process. We refer to [28][30], first chapter, for a complete description.

1. We suppose that the interaction is a binary collision, which means that when two particles are close enough to each other their trajectories are deviated. The consequence of such a postulate is that one can neglect collisions involving more than two particles, which implies that the system is comparable to a dilute (rarefied) gas. Mathematically, if the system contains N particles of radius r, we suppose that we are in the Boltzmann-Grad limit: $Nr^3 << 1$ and $Nr^2 = O(1)$.

2. The collisions are considered to be localised both in space and time. This conveys the idea of the fact that the trajectories are deviated very quickly and it translates mathematically under the hypothesis that a collision takes place at a position x and a time t.

3. We also suppose that the collisions are elastic. In other terms, the momentum and the kinetic energy are preserved throughout the collision process. If two particles of respective velocities v' and v'_* collide, then their outcoming velocities v and v_* satisfies
$$\begin{cases} v' + v'_* = v + v_* \\ |v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2 . \end{cases}$$
We remark here that the mass is the same for all the particles in a monoatomistic gas and considering several species requires a different version of the preservation of kinetic energy and therefore different outcoming velocities.

4. The physics of the process is assumed to be microreversible, which means that the microscopic dynamics are reversible in time. In other terms, the probability that velocities (v', v'_*) are changed into (v, v_*) during a collision is equal to the probability of changing velocities from (v, v_*) into (v', v'_*).

5. We further suppose Boltzmann molecular choas inside the system. This states that before they collide, two particles evolve independently one from the other. This hypothesis implies an asymmetry in the arrow of time since after collision the velocities of the two particles are correlated (*via* the preservation of momentum and kinetic energy).

The formal derivation, from Newton's laws, of the kinetic model associated to the assumptions above (see Chapter 3.2 for a brief explaination or [28], chapter 3) yields the Boltzmann equation. Note that the rigorous mathematical derivation is still a very hard problem even if in 1974 Lanford [65], and recently ameliorated in [44][96], proved it for very short time (typically, shorter than the mean time of first collision).

The Boltzmann collision operator is therefore a bilinear operator encoding the probability for two particles with velocities v' and v'_* to undergo a collision resulting in velocities v and v_*. The laws of elasticity link (v', v'_*) to (v, v_*) in a bijective correspondance (easily deduced from Figure 1.1) we call the "σ-representation". If we denote

$$\sigma = \frac{v' - v'_*}{|v' - v'_*|},$$

then σ varies on $\mathbb{S}^{d-1}$ when (v', v'_*) varies in $\mathbb{R}^{2d}$ and we have the following relation

$$\begin{cases} v' = \dfrac{v + v_*}{2} + \dfrac{|v - v_*|}{2}\sigma \\[2mm] v'_* = \dfrac{v + v_*}{2} - \dfrac{|v - v_*|}{2}\sigma. \end{cases}$$

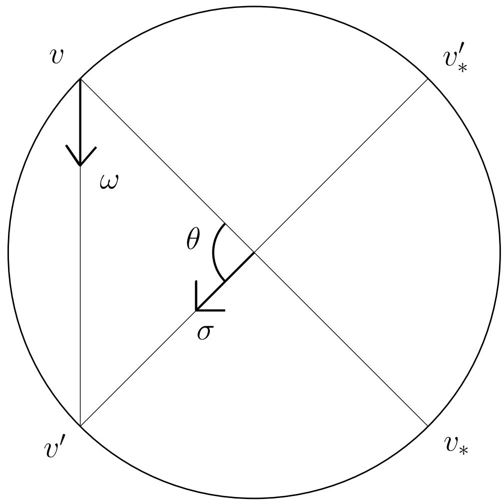

Figure 1.1: Correspondance between pre and post collision velocities

Under this representation we obtain an explicit form for the Boltzmann equation,

$$\partial_t f + v \cdot \nabla_x f = Q(f,f), \tag{1.1.5}$$

with Q being the Boltzmann collision operator given by

$$Q(f,f) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} B\left(|v - v_*|, \cos\theta\right) \left[f' f'_* - f f_*\right] dv_* d\sigma. \tag{1.1.6}$$

In the expression above, B is called the Boltzmann collision kernel and encodes the physics of the collision process, θ is the angle between $v - v_*$ and σ, and we use the standard notations $f = f(t,x,v)$, $f_* = f(t,x,v_*)$, $f' = f(t,x,v')$ and $f'_* = f(t,x,v'_*)$.

1.1.2.2 The different collision kernels of the Boltzmann operator

Alternative representations. The first thing one can notice about the Boltzmann collision kernel is that its form (1.1.6) depends on the choice we made to express (v', v'_*) in terms of (v, v_*). There exists other parametrisations and we refer to [112] Chapter 1 for advantages and inconveniences of each of them. We nonetheless present two alternative representations of the Boltzmann collision operator that will be use later in this work.

The most common alternative way of writing the Boltzmann operator Q is the so-called "ω-representation" (which can also be easily deduced from Figure 1.1). In this case, we consider the unit vector

$$\omega = \frac{v - v'}{|v - v'|}$$

to obtain a new bijective correspondance bewteen (v', v'_*) and (v, v_*), namely

$$\left\{\begin{array}{l} v' = v - \langle v - v_*, \omega\rangle\,\omega \\ v'_* = v_* - \langle v_* - v, \omega\rangle\,\omega. \end{array}\right.$$

In the "ω-representation", the collision operator reads

$$Q(f,f) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \tilde{B}\left(v - v_*, \omega\right)\left[f' f'_* - f f_*\right] dv_* d\omega,$$

where the correspondance with the "σ-representation" is given by

$$\tilde{B}\left(z,\omega\right) = 2^{d-1}\left|\left\langle \frac{z}{|z|}, \omega\right\rangle\right|^{d-2} B(|z|\,, \sigma).$$

The second representation we want to introduce has been proposed by Carleman [27] and takes v' and v'_* as integration parameters and defines $v_* = v' + v'_* - v$. With these choices, the Boltzmann collision operator reads

$$Q(f,f) = \int_{\mathbb{R}^d}\left(\int_{E_{v,v'}} \frac{1}{|v - v'|^{d-1}} \tilde{B}\left(2v - v' - v'_*, \frac{v' - v'_*}{|v' - v'_*|}\right)\left[f' f'_* - f f_*\right]\, dv'_*\right) dv',$$

where $E_{v,v'}$ is the hyperplane going through v and orthogonal to $v - v'$.

The Carleman representation will play an important role in Chapter 6.

Different collisional interactions. The physics of the collision process is encoded in the Boltzmann collision kernel B. For simplicity reasons, we will always assume that this kernel is of the form

$$B\left(|v - v_*|\,, \cos\theta\right) = \Phi(|v - v_*|)b(\cos\theta),$$

where Φ and b are positive and locally integrable functions except, eventually, at respectively 0 and 1. This assumption is made without loss of generality on the kernel but reduce the complexity of later computations. Moreover, it is satisfied in all the physically relevant cases that we describe below. We refer to [28] Chapter 2 or [112] Chapter 1 for a derivation of B from the interaction laws.

A very important case is the one of hard spheres which correspond to the case where particles are considered as billiard balls bouncing on each other. For this specific interaction one has

$$\exists\, C_\Phi > 0, \quad B\left(|v - v_*|\,, \cos\theta\right) = C_\Phi\,|v - v_*|\,.$$

In a more general setting, we will always assume that the kinetic collision kernel Φ

satisfies either

$$\forall z \in \mathbb{R}, \quad c_\Phi \, |z|^\gamma \leqslant \Phi(z) \leqslant C_\Phi \, |z|^\gamma$$

or a mollified version

$$\begin{cases} \forall \, |z| \geqslant 1 \in \mathbb{R}, & c_\Phi \, |z|^\gamma \leqslant \Phi(z) \leqslant C_\Phi \, |z|^\gamma \\ \forall \, |z| \leqslant 1 \in \mathbb{R}, & c_\Phi \leqslant \Phi(z) \leqslant C_\Phi, \end{cases}$$

c_Φ and C_Φ being strictly positive constants and γ belonging to $(-d, 1]$. The collision kernel is said to be "hard potential" in the case of $\gamma > 0$, "soft potential" if $\gamma < 0$ and "Maxwellian" if $\gamma = 0$.

The angular collision kernel b is seldom known explicitely. However, we will assume $(b \circ \cos)$ to be a continuous function on θ in $(0, \pi]$, strictly positive near $\theta \sim \pi/2$, which satisfies

$$b\,(\cos\theta) \sin^{d-2}\theta \underset{\theta \to 0^+}{\sim} b_0 \, \theta^{-(1+\nu)},$$

for $b_0 > 0$ and ν in $(-\infty, 2)$. The case when b is locally integrable, $\nu < 0$, is referred to by the Grad's cutoff assumption (first introduce in [48]) and therefore B will be said to be a cutoff collision kernel. This case is of tremendous importance since it allows to decompose the Boltzmann operator $Q = Q^+ - Q^-$. The case $\nu \geqslant 0$ will be designated by non-cutoff collision kernel.

We can mention here that in the physically important case of inverse-power laws in dimension $d = 3$,

$$\Phi(z) = C_\Phi \, |z|^\gamma$$

and γ and ν are not independent since there exists $s > 2$ such that

$$\begin{cases} \gamma = \dfrac{s-5}{s-1} \\ \nu = \dfrac{2}{s-1}. \end{cases}$$

Moreover, in the case of Coulomb interactions $s = 2$, we have an explicit formula for the angular kernel in dimension $d = 3$, which is

$$b\,(\cos\theta) = \frac{b_0}{\sin^4\theta}.$$

The mathematical treatment of these different collision kernels reveals different behaviours for solutions to the Boltzmann equation, depending where the singularities of both the kinetic and the angular collision kernels occur. In other terms, the decay at infinity or even the regularity properties, both in the velocity variable, of solutions to the

Boltzmann equation (1.1.5) are very sensitive to the way Φ behaves for small or large relative velocities and the way $(b \circ \cos)$ blows up at $\theta \sim 0^+$.

1.1.2.3 Initial data and boundary conditions

The Boltzmann equation has to describe, at a mesoscopic scale, the motion of particles evolving in time in the spatial domain Ω. We therefore need to prescribe an initial distribution $f_{in}(x,v)$ as well as a modelling of the interactions between a particle and the boundary of Ω, in the case it exists.

The problem of the initial data is quite obvious

$$\forall x \in \Omega,\ \forall v \in \mathbb{R}^d, \quad f(0,x,v) = f_{in}(x,v).$$

However, there exists density functions that are not physically relevant, as discussed in Section 1.1.1.1. In that respect, the physically relevant solutions to the Boltzmann equation must have finite mass and energy at least in bounded sets. The minimum requirements one should ask for f_{in} are thus

1. $f_{in}(x,v) \geqslant 0$ almost everywhere in $\Omega \times \mathbb{R}^d$,
2. for all K compact in Ω,
$$\int_K \int_{\mathbb{R}^d} \left(1 + |v|^2\right) f_{in}(x,v)\, dxdv < +\infty.$$

There exist several modellings of the interactions between a particle and the boundary of Ω.

In the case $\Omega = \mathbb{R}^d$, no boundary condition is needed. However, the relevant solutions need to satisfy an integrability condition at infinity.

In the case $\partial\Omega \neq \emptyset$, particles will interact with the frontier of the domain. The most common behaviours are the following.

- The bounce-back condition
$$\forall (t,x,v) \in \mathbb{R}^+ \times \partial\Omega \times \mathbb{R}^d, \quad f(t,x,v) = f(t,x,-v).$$
- If Ω is regular enough, then one can consider the specular reflection boundary condition
$$\forall (t,x,v) \in \mathbb{R}^+ \times \partial\Omega \times \mathbb{R}^d, \quad f(t,x,v) = f(t,x,\mathcal{R}_x(v)),$$

$\mathcal{R}_x$, for x on $\partial\Omega$, stands for the specular reflection at that point against the boundary. One can compute, denoting by n_x the outward normal at a point x on $\partial\Omega$,

$$\forall v \in \mathbb{R}^d, \quad \mathcal{R}_x(v) = v - 2\langle v, n_x\rangle n_x.$$

- If Ω is regular enough and $\partial\Omega$ has a temperature T_∂ then one can impose the Maxwellian diffusion boundary condition

$$\forall (t,x,v) \in \mathbb{R}^+ \times \partial\Omega \times \mathbb{R}^d, \quad f(t,x,v) = \left[\int_{v\cdot n_x > 0} f(t,x,v)\,(v\cdot n_x)\; dv\right] \frac{1}{(2\pi)^{\frac{d-1}{2}}\, T_\partial^{\frac{d+1}{2}}} e^{-\frac{|v|^2}{2T_\partial}}.$$

Note that the first two boundary conditions convey the idea of particles bouncing against the wall, in two different manners, whereas the third one expresses the fact that particles are absorbed by the wall and then emitted back into Ω according to the thermodynamical equilibrium distribution M_∂ between the wall and the gas,

$$M_\partial(v) = \frac{1}{(2\pi)^{\frac{d-1}{2}}\, T_\partial^{\frac{d+1}{2}}} e^{-\frac{|v|^2}{2T_\partial}}.$$

The last case we will consider is the periodic case when Ω is the d-dimensional torus $\mathbb{T}^d$. This will be of particular interest since it is a bounded domain without boundary conditions except for the periodicity condition. This case is also physically and mathematically interesting because it has been proven (see [30] Chapter 7) that it is equivalent to the case when Ω is a box with specular reflection boundary conditions.

1.1.2.4 Conservation laws and entropy dissipation

There are some few interesting facts that one can rapidly discover about the solutions to the Boltzmann equation, at least formally. Its collision operator

$$Q(f,f) = \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} B\left(|v-v_*|, \cos\theta\right) \left[f' f'_* - f f_*\right] dv_* d\sigma$$

encodes the microscopic behaviour of the gas in the case of elastic collisions. This particular case of interaction preserves the mass, the momentum and the energy and this reflects on the macroscopic observables. Indeed, the Boltzmann collision kernel is invariant, for instance, under the changes of variable

$$(v, v_*, \sigma) \to (v', v'_*, k) \quad \text{with} \quad k = \frac{v-v_*}{|v-v_*|}$$

and

$$(v, v_*, \sigma) \to (v_*, v, -\sigma).$$

These invariances formally give (see [46] Chapter 1 for a rigorous statement) that for a given test function $\phi(v)$,

$$\int_{\mathbb{R}^d} Q(f,f)(v)\phi(v)\, dv = -\frac{1}{4}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} B\left[f'f'_* - ff_*\right]\left(\phi'_* + \phi' - \phi_* - \phi\right)\, dvdv_*d\sigma. \tag{1.1.7}$$

The latter property has two major consequences that are related to macroscopic laws. In full generality (1.1.7) implies first that

$$\int_{\mathbb{R}^d} Q(f,f)\begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv = 0 \tag{1.1.8}$$

and second

$$D(f) := -\int_{\mathbb{R}^d} Q(f,f)\,\log f\, dv \geqslant 0. \tag{1.1.9}$$

In the case when f is solution to the Boltzmann equation

$$\partial_t f + v\cdot\nabla_x f = Q(f,f),$$

we have that, by integrating in v the differential equation, (1.1.8) comes out as

- the preservation of the total mass

$$\frac{d}{dt}\int_\Omega \rho(t,x)\, dx = 0,$$

- the preservation of total energy if Ω has no boundary or if boundary conditions are bounce-back or specular reflections

$$\frac{d}{dt}\int_\Omega E(t,x)\, dx = 0,$$

- the preservation of total momentum if Ω has no boundary

$$\frac{d}{dt}\int_\Omega \rho(t,x)u(t,x)\, dx = 0,$$

The consequence of (1.1.9) applied to solutions to the Boltzmann equation, at least at a formal level, is known as the Boltzmann H-theorem. The latter theorem states, if Ω has no boundary or if boundary conditions are bounce-back or specular reflections, that the entropy of a solution f, defined as

$$S(f) = \int_{\Omega\times\mathbb{R}^d} f \log f \, dxdv$$

which is the opposite of the thermodynamical entropy, can only decrease in time

$$\frac{d}{dt}S(f) = -\int_{\Omega} D(f)\, dx \leqslant 0.$$

Such a result is much deeper than it looks and was subject to a lot of discussions and rejections from the scientific community when Boltzmann discovered it. The entropy dissipation indeed implies time irreversibility of the Boltzmann equation which seems unnatural since the Newton dynamics of the microscopic processes it describes are time-reversible.

We conclude this brief introduction to the Boltzmann equation by describing its steady states. The entropy dissipation states that we are at a local thermodynamic equilibrium if

$$D(f)(t,x) = 0,$$

which is possible if and only if

$$\forall v \in \mathbb{R}^d, \quad f(t,x,v) = M_{(\rho(t,x),u(t,x),\theta(t,x))}(v),$$

where $M_{(\rho,u,\theta)}$ is called a Maxwellian distribution and is defined by

$$M_{(\rho,u,\theta)}(v) = \frac{\rho}{(2\pi\theta)^{d/2}} e^{-\frac{|v-u|^2}{2\theta}}.$$

Moreover, for all $\rho = \rho(t,x)$, $u = u(t,x)$ and $\theta = \theta(t,x)$ the following holds

$$Q(M_{(\rho,u,\theta)}, M_{(\rho,u,\theta)}) = 0,$$

and therefore a local thermodynamic equilibrium is global if and only if

$$\forall (x,v) \in \Omega\times\mathbb{R}^d, \quad v\cdot\nabla_x M_{(\rho,u,\theta)} = 0.$$

In the case of the torus, this condition yields a unique global equilibrium for the Boltzmann equation that is the Maxwellian independent of t and x that has the same total mass, momentum and energy as the initial configuration f_{in}. This is also the case if Ω is a non axis-symmetric bounded domain with bounce-back or specular reflection boundary conditions. We refer to [46] Chapter 1 and [112] Section 2.5 for more details and references.

In these cases, we can always consider, without loss of generality, that the equilibrium is a centred gaussian with mass and variance 1. We denote

$$\mu(v) = \frac{1}{(2\pi)^{d/2}} e^{-\frac{|v|^2}{2}}.$$

1.2 Quantitative study of the positivity of solutions (Part I)

Part I deals with some quantitative and qualitative aspects of solutions to the Boltzmann equation (1.1.5).

The aim of Chapter 2 is to prove that non-negative solutions to the Boltzmann equation are in fact strictly positive and bounded from below by an exponential lower bound. This is an *a priori* result where we will not tackle the issue of existence. We solely make the hypothesis that a solution exists and it has some uniform regularity properties we present below.

The framework of this study is quite wide since we consider all types of collision operator (hard and soft potentials with or without angular cutoff) and Ω is a C^2 convex bounded domain with specular reflection boundary conditions.

1.2.1 Motivations and state of the art

The issue of quantifying the positivity of solutions to the Boltzmann equation has been investigated for a long time. It is of great interest for physical purposes but, more recently, it has also proven itself of significant importance for the mathematical study of the Boltzmann equation. The development of entropy-entropy production methods (see [112] Chapter 3 and [113]) to study the convergence to equilibrium [35][36][37] requires this type of exponential lower bounds.

The first quantitative statement on positivity of the solutions to the Boltzmann equation goes back to Carleman [26] where he dealt with the spatially homogeneous equation. The radially symmetric solutions $f(t,v) = f(t,|v|)$ he constructed in dimension $d = 3$ for hard sphere collision operator, satisfy an almost Maxwellian lower bound

$$\forall t \geqslant t_0,\ \forall v \in \mathbb{R}^3, \quad f(t,v) \geqslant C_1 e^{-C_2|v|^{2+\varepsilon}},$$

$C_1, C_2 > 0$ for all $t_0 > 0$ and $\varepsilon > 0$. The constants C_1 and C_2 depends only on t_0, ε and *a priori* estimates on the solution f.

Pulvirenti and Wennberg [95] then extended the latter inequality to solutions to the spatially homogeneous Boltzmann equation with hard potential and cutoff in dimension $d = 3$ with more general initial data. They proved that if the solution has finite mass, energy and entropy then it is bounded from below by a Maxwellian lower bound of the type

$$\forall t \geqslant t_0,\ \forall v \in \mathbb{R}^3, \quad f(t,v) \geqslant C_1 e^{-C_2|v|^2},$$

for all $t_0 > 0$.

Finally, Mouhot [78] dealt with the full Boltzmann equation on the torus. He not only proved the same result as Pulvirenti and Wennberg in the case of hard potential with angular cutoff but he also obtained the Maxwellian lower bound for soft potential with cutoff collision kernels. He also derived the same kind of results in the non-cutoff case in the torus, the immediate appearance of an exponential lower bound of the form

$$\forall t \geqslant t_0,\ \forall (x,v) \in \mathbb{T}^d \times \mathbb{R}^d, \quad f(t,v) \geqslant C_1(\varepsilon) e^{-C_2(\varepsilon)|v|^{K+\varepsilon}},$$

for all $t_0 > 0$, all $\varepsilon > 0$ and $K = K(\nu)$ with $K(0) = 2$ (thus recovering the cutoff case in the limit).

All these results deeply rely on a spreading property of the gain part Q^+ of the Boltzmann collision operator that arises as soon as the solution has a non-concentration property which means, roughly speaking, an initial lower bound. This "upheaval point" results from non-concentration properties of the gain operator ([95]) or continuity-compactness arguments (Chapter 2). The case of spatially inhomogeneous solutions [78] is based on these arguments and a method to make them uniform under the flow of characteristics.

The case where Ω is bounded implies a different behaviour for the characteristics and our main contribution is the derivation of a spreading method that remains invariant under the characteristics flow. For instance, boundaries imply the existence of grazing collisions where the strategy develop on the torus fails and we had to create a geometrical approach of those problematic trajectories.

1.2.2 The free transport equation in convex bounded domain (Chapter 2)

The first task is to establish a rigorous description of characteristic trajectories for the free transport equation

$$\partial_t f + v\nabla_x f = 0$$

with specular reflection boundary conditions, which can be seen has billiard balls trajectories. Although it has been studied in numerous works [93][33][104][105] and has been used in kinetic theory [52][59], a complete study in the case of mere specular reflections and convexity seemed to be missing. The cited works indeed contain assumptions on the boundary (electromagnetism or strict convexity for example) that lead to clear rebounds against the boundary whereas a general study should also consider rolling trajectories for instance.

Chapter 2 starts with an extensive and descriptive study of the characteristics of the free transport equation in a C^1 convex bounded domain. One of our most relevant contribution is the extension of a result of Tabachnikov [104] that states that the set of points (x, v) that comes from infinitely many rebounds in finite time is negligeable. More precisely, we proved the following.

Proposition 1.2.1 *Let Ω be a C^1 open, bounded domain in $\mathbb{R}^d$ and let (x, v) be in $\bar{\Omega} \times \mathbb{R}^d$. Then for all $t \geqslant 0$ the trajectory finishing at (x, v) after a time t has at most a countable number of rebounds.*

Moreover, this number is finite almost surely with respect to the Lebesgue measure on $\bar{\Omega} \times \mathbb{R}^d$.

The main idea was to generate a parametrisation of $\bar{\Omega}$ that links the trajectories to their footprints, where this result is known to hold thanks to the work of Tabachnikov.

1.2.3 An exponential lower bound (Chapter 2)

In what follows, we are going to need bounds on some physical observables of solution to the Boltzmann equation and we describe them below.

We consider a function $f(t, x, v) \geqslant 0$ defined on $[0, T) \times \Omega \times \mathbb{R}^d$ and we recall the definitions of its local physical quantities.

- its local energy
$$e_f(t, x) = \int_{\mathbb{R}^d} |v|^2 f(t, x, v)dv,$$
- its local weighted energy
$$e'_f(t, x) = \int_{\mathbb{R}^d} |v|^{\tilde{\gamma}} f(t, x, v)dv,$$
where $\tilde{\gamma} = (2 + \gamma)^+$,
- its local L^p norm ($p \in [1, +\infty)$)
$$l^p_f(t, x) = \|f(t, x, \cdot)\|_{L^p_v},$$
- its local $W^{2,\infty}$ norm
$$w_f(t, x) = \|f(t, x, \cdot)\|_{W^{2,\infty}_v}.$$

The solutions to the Boltzmann equation are assumed to satisfy some properties about their local hydrodynamical quantities. These properties will differ depending on which case of collision kernel we are considering and are given by what follows.

- In the case of hard or Maxwellian potentials with cutoff ($\gamma \geqslant 0$ and $\nu < 0$):

$$\sup_{(t,x)\in[0,T)\times\Omega} e_f(t,x) < +\infty. \tag{1.2.1}$$

- In the case of a singularity of the kinetic collision kernel ($\gamma \in (-d,0)$) we shall make the additional assumption

$$\sup_{(t,x)\in[0,T)\times\Omega} l_f^p(t,x) < +\infty, \tag{1.2.2}$$

  where $p_\gamma > d/(d+\gamma)$.

- In the case of a singularity of the angular collision kernel ($\nu \in [0,2)$) we shall make the additional assumption

$$\sup_{(t,x)\in[0,T)\times\Omega} w_f(t,x) \quad \text{and} \quad \sup_{(t,x)\in[0,T)\times\Omega} e'_f(t,x) < +\infty. \tag{1.2.3}$$

We now state the result of Chapter 2 in a rather informal way. For a more detailed and more rigorous statement, we refer to Section 2.2.

Theorem 1.2.2 *Let Ω be $\mathbb{T}^d$ or a C^2 open convex bounded domain in $\mathbb{R}^d$ and let f_{in} be a non-negative continuous function on $\bar{\Omega}\times\mathbb{R}^d$ with strictly positive mass and finite energy. Let $f(t,x,v)$ be a continuous non-negative solution of the Boltzmann equation in $\bar{\Omega}\times\mathbb{R}^d$ on some time interval $[0,T)$, $T\in(0,+\infty]$, which satisfies*

- *if the collision kernel is hard or Maxwellian potential with cutoff, then f satisfies (1.2.1);*

- *if the collision kernel is soft potential, then f satisfies (1.2.1) and (1.2.2);*

- *if the collision kernel is non-cutoff, then f satisfies (1.2.3).*

Then

(i) ***for cutoff collision kernels:*** *for all $\tau\in(0,T)$ there exists $\rho>0$ and $\theta>0$, depending on τ, E_f (and $L_f^{p_\gamma}$ if B is a soft potential kernel), such that for all $t\in[\tau,T)$*

the solution f is bounded from below, almost everywhere, by a global Maxwellian distribution with density ρ and temperature θ, i.e.

$$\forall t \in [\tau, T),\ \forall (x,v) \in \bar{\Omega} \times \mathbb{R}^d, \quad f(t,x,v) \geqslant \frac{\rho}{(2\pi\theta)^{d/2}} e^{-\frac{|v|^2}{2\theta}}.$$

(ii) ***for non-cutoff collision kernels:*** *for all $\tau \in (0,T)$ and for any exponent K such that*

$$K > 2\frac{\log\left(2 + \frac{2\nu}{2-\nu}\right)}{\log 2},$$

there exists $C_1, C_2 > 0$, depending on τ, K, E_f, E'_f, W_f (and $L_f^{p_\gamma}$ if B is a soft potential kernel), such that

$$\forall t \in [\tau, T),\ \forall (x,v) \in \bar{\Omega} \times \mathbb{R}^d, \quad f(t,x,v) \geqslant C_1 e^{-C_2|v|^K}.$$

As an important remark, let us emphasize that in the case of a C^3 bounded strictly convex domain with f having uniformly bounded local mass and entropy, our proofs are entirely constructive.

1.3 The incompressible Navier-Stokes limit of the Boltzmann equation (Part II)

1.3.1 Going from Boltzmann equation to incompressible Navier-Stokes equations (Chapter 3)

The Boltzmann equation rules the mesocopic evolution of a rarefied gas and is established on the microscopic dynamics of the particles. It therefore stands in between the microscopic scale and the macroscopic scale described by the acoustics and fluids evolution equations. A natural question thus arises: does there exist a link between the physical observables of solutions to the Boltzmann equation and solutions to fluid dynamics ?

It is physically relevant to derive a non-dimensional form of the Boltzmann equation [46][98] which reads

$$\partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon = \frac{1}{\varepsilon} Q(f_\varepsilon, f_\varepsilon), \tag{1.3.1}$$

where ε is called the Knudsen number of the gas. Physically, ε^{-1} represents the average number of collisions for each particle per unit of time. Therefore, as reviewed in [111], one can expect a convergence from the Boltzmann model towards the acoustics and the fluids dynamics as the Knudsen number ε tends to 0.

The study of the latter convergence is called the hydrodynamical limits of the Boltzmann equation and is of tremendous importance to mathematically prove the coherence of the different scales of description in physics. Chapter 3 is dedicated to the issue of hydrodynamical limits and gives a state of the art in the domain. We therefore just describe briefly here the incompressible Navier-Stokes framework that we thoroughly study in Chapters 4 and 5. The incompressible Navier-Stokes equations read

$$\begin{aligned} \partial_t u - \nu \Delta u + u \cdot \nabla u + \nabla p = 0, \\ \nabla \cdot u = 0, \\ \partial_t \theta - \kappa \Delta \theta + u \cdot \nabla \theta = 0, \end{aligned} \tag{1.3.2}$$

to which we can add the Boussineq relation

$$\nabla(\rho + \theta) = 0, \tag{1.3.3}$$

where p is the pressure, ν and κ are respectively the dynamic viscosity and the thermal conductivity of the fluid.

The Boltzmann equation and the incompressible Navier-Stokes equations describe physical phenomenon that do not evolve at the same timescale. As suggested in previous studies [46][111][98] we need to rescale (1.3.1) in time by a factor ε, to get rid of these time scale differences. Moreover, they also suggested that a perturbation of order ε around the global equilibrium

$$\mu(v) = \frac{1}{(2\pi)^{d/2}} e^{-\frac{|v|^2}{2}}$$

should approximate, as the Knudsen number tends to 0, the incompressible Navier-Stokes equations.

We hence study the following equation

$$\partial_t f_\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x f_\varepsilon = \frac{1}{\varepsilon^2} Q(f_\varepsilon, f_\varepsilon)\ , \text{ on } \mathbb{R}^+ \times \mathbb{T}^d \times \mathbb{R}^d, \tag{1.3.4}$$

under the linearization $f_\varepsilon(t,x,v) = \mu(v) + \varepsilon h_\varepsilon(t,x,v)$. This leads to the perturbed Boltzmann equation

$$\partial_t h_\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x h_\varepsilon = \frac{1}{\varepsilon^2} L(h_\varepsilon) + \frac{1}{\varepsilon} \Gamma(h_\varepsilon, h_\varepsilon). \tag{1.3.5}$$

that we will study thoroughly, and where we defined

$$\begin{cases} L(h) & = [Q(\mu,h) + Q(h,\mu)] \\ \Gamma(g,h) & = \dfrac{1}{2}\left[Q(g,h) + Q(h,g)\right]. \end{cases}$$

Roughly speaking, the dissipation of entropy discussed in Section 1.1.2.4 is expected, in the case of small initial perturbation $h_\varepsilon(0,x,v)$, to make $f_\varepsilon(t,x,v) = \mu(v) + \varepsilon h_\varepsilon(t,x,v)$ converge towards its global equilibrium $\mu(v)$ as time goes to infinity. This trend to equilibrium would give bounds on f_ε and if the latter bounds are uniform in ε one can study the hydrodynamical limit

$$\lim_{\varepsilon\to 0} \|f_\varepsilon\|,$$

where the norm of the convergence will be rigorously defined later. The main goal to study the limit for Boltzmann equation towards incompressible Navier-Stokes equation is therefore to develop a Cauchy theory and prove a trend to equilibrium for (1.3.5) that will be uniform in ε.

1.3.2 Hydrodynamical limit in $H^s_{x,v}\left(\mu^{-1/2}\right)$ (Chapter 4)

Chapter 4 rigorously justifies the discussion of previous subsection in the Sobolev space $H^s_{x,v}\left(\mu^{-1/2}\right)$ for s large. More precisely, it constructs a Cauchy theory for small initial data of the perturbed Boltzmann equation (1.3.5). This theory is uniform in the Knudsen number, that is to say the smallness assumption is independent of ε. Moreover, we show an exponential decay for h_ε, uniformly in ε. The latter decay allows us to rigorously prove the convergence of the observables of h_ε towards solutions to the incompressible Navier-Stokes equation (1.3.2), satisfying the Boussineq equation (1.3.3).

We emphasize here that all the results in Chapter 4 are obtained constructively, which is of great importance for physical purposes and seldom the case in Boltzmann perturbative theory. Our main contribution is the derivation of hypocoercive estimates independent on ε in new distorted norms catching the dependencies in the Knudsen number.

We refer to Sections 4.1.4 and 5.1.2 for a state of the art of the study of the semigroup and the Cauchy problem.

In this section we consider the Boltzmann equation with hard potential or Maxwellian potential ($\gamma = 0$), that is to say there is a constant $C_\Phi > 0$ such that

$$B\left(|v-v_*|,\theta\right) = \Phi\left(|v-v_*|\right) b\left(\cos\theta\right)$$

with

$$\Phi(z) = C_\Phi z^\gamma\,,\ \ \gamma\in[0,1],$$

and a strong form of Grad's angular cutoff, expressed here by the fact that we assume b to be C^1 with the controls from above

$$\forall z \in [-1,1], \quad b(z), b(z') \leqslant C_b.$$

For the sake of clearness, we study (1.3.4) with the linearization $f_\varepsilon(t,x,v) = \mu(v) + \varepsilon\mu^{1/2}h_\varepsilon(t,x,v)$ which amounts to working on h_ε in the space $H^s_{x,v}$ without any weight. The sole changes are in the linear and bilinear operators:

$$\left\{\begin{array}{ll} L(h) & = \left[Q(\mu,\mu^{\frac{1}{2}}h) + Q(\mu^{\frac{1}{2}}h,\mu)\right]\mu^{-\frac{1}{2}} \\ \Gamma(g,h) & = \dfrac{1}{2}\left[Q(\mu^{\frac{1}{2}}g,\mu^{\frac{1}{2}}h) + Q(\mu^{\frac{1}{2}}h,\mu^{\frac{1}{2}}g)\right]\mu^{-\frac{1}{2}}. \end{array}\right. \tag{1.3.6}$$

1.3.2.1 The linear Boltzmann operator

A common strategy in perturbative framework is to study the properties of the linear operator part and then consider the bilinear as a remainder term. We therefore focus first on the linear part of the perturbed Boltzmann equation

$$G_\varepsilon = \frac{1}{\varepsilon^2}L - \frac{1}{\varepsilon}v\cdot\nabla_x.$$

In the case of hard potential with angular cutoff, it is known that L is a negative self-adjoint operator in L^2_v. More importantly, L in hypocoercive. This translates into the following properties.

Properties in $H^1_{x,v}$:

(H1): Coercivity and general controls

$L : L^2_v \longrightarrow L^2_v$ is a closed and self-adjoint operator with $L = K - \Lambda$ such that:

- Λ is coercive:
  - it exists $\|.\|_{\Lambda_v}$ norm on L^2_v such that

$$\forall h\in L^2_v\,,\ \nu_0^\Lambda \|h\|^2_{L^2_v} \leqslant \nu_1^\Lambda \|h\|^2_{\Lambda_v} \leqslant \langle\Lambda(h),h\rangle_{L^2_v} \leqslant \nu_2^\Lambda \|h\|^2_{\Lambda_v}\,,$$

  - Λ has a defect of coercivity regarding its v derivatives:

$$\forall h\in H^1_v\,,\ \langle\nabla_v\Lambda(h),\nabla_v h\rangle_{L^2_v} \geqslant \nu_3^\Lambda \|\nabla_v h\|^2_{\Lambda_v} - \nu_4^\Lambda \|h\|^2_{\Lambda_v}\,.$$

- There exists $C^L > 0$ such that

$$\forall h\in L^2_v\,,\ \forall g\in L^2_v\,,\ \langle L(h),g\rangle_{L^2_v} \leqslant C^L \|h\|_{\Lambda_v}\|g\|_{\Lambda_v}\,,$$

where $(\nu_k^\Lambda)_{1\leqslant k\leqslant 4}$ are strictly positive constants depending on the operator and the dimension d.

We define a new norm on $L^2_{x,v}$:

$$\|\cdot\|_\Lambda = \left\| \|\cdot\|_{\Lambda_v} \right\|_{L^2_x} .$$

(H2): Mixing property in velocity

$$\forall \delta > 0\,,\ \exists C(\delta) > 0\,,\ \forall h \in H^1_v\,, \quad \langle \nabla_v K(h), \nabla_v h \rangle_{L^2_v} \leqslant C(\delta) \|h\|^2_{L^2_v} + \delta \|\nabla_v h\|^2_{L^2_v} .$$

(H3): Relaxation to equilibrium

The kernel of L is generated by $d+2$ functions which form an orthonormal basis for $\mathrm{Ker}(L)$:

$$\mathrm{Ker}(L) = \mathrm{Span}\{\phi_1(v), \dots, \phi_{d+2}(v)\}.$$

Moreover, the ϕ_i are of the form $P_i(v)e^{-|v|^2/4}$, where P_i is a polynomial.

Furthermore, denoting by π_L the orthogonal projector in L^2_v on $\mathrm{Ker}(L)$ we have the following local coercivity property:

$$\exists \lambda > 0\,,\ \forall h \in L^2_v\,, \quad \langle L(h), h \rangle_{L^2_v} \leqslant -\lambda \left\| h^\perp \right\|^2_{\Lambda_v} ,$$

where $h^\perp = h - \pi_L(h)$ denotes the microscopic part of h (the orthogonal to $\mathrm{Ker}(L)$ in L^2_v).

Assumptions in $H^s_{x,v}$, $s > 1$:

(H1'): Defect of coercivity for higher derivatives

L satisfies the following property: for all $s \geqslant 1$, for all $|j| + |l| = s$ such that $|j| \geqslant 1$,

$$\forall h \in H^s_{x,v}\,, \quad \langle \partial^j_l \Lambda(h), \partial^j_l h \rangle_{L^2_{x,v}} \geqslant \nu^\Lambda_5 \left\| \partial^j_l h \right\|^2_\Lambda - \nu^\Lambda_6 \|h\|_{H^{s-1}_{x,v}} ,$$

where ν^Λ_5 and ν^Λ_6 are strictly positive constants depending on L and d.
We also define a new norm on $H^s_{x,v}$:

$$\|\cdot\|_{H^s_\Lambda} = \left(\sum_{|j|+|l| \leqslant s} \left\| \partial^j_l \cdot \right\|^2_\Lambda \right)^{1/2} .$$

(H2'): Mixing properties

$(H2)$ extends to higher Sobolev's spaces: for all $s \geqslant 1$, for all $|j| + |l| = s$ such that $|j| \geqslant 1$,

$$\forall \delta > 0\,,\ \exists C(\delta) > 0\,,\ \forall h \in H^s_{x,v}\,, \quad \langle \partial^j_l K(h), \partial^j_l h \rangle_{L^2_{x,v}} \leqslant C(\delta) \|h\|^2_{H^{s-1}_{x,v}} + \delta \left\| \partial^j_l h \right\|^2_{L^2_{x,v}} .$$

All the constants are explicit thanks to the works of C. Baranger and C. Mouhot [4] and C. Mouhot [79].

The first important result derive in Chapter 4, is the fact that the linear part of the Boltzmann operator generates a contraction semigroup in $H^s_{x,v}$.

Theorem 1.3.1 *If L is a linear operator satisfying the conditions* (H1'), (H2') *and* (H3) *then there exists $0 < \varepsilon_d \leqslant 1$ such that for all s in $\mathbb{N}^*$,*

1. *for all $0 < \varepsilon \leqslant \varepsilon_d$, $G_\varepsilon = \varepsilon^{-2}L - \varepsilon^{-1}v \cdot \nabla_x$ generates a C^0-semigroup on $H^s_{x,v}$.*
2. *there exist $C_G^{(s)} > 0$ and a norm $\|\cdot\|_{\mathcal{H}^s_\varepsilon}$ such that for all $0 < \varepsilon \leqslant \varepsilon_d$:*

$$\|\cdot\|^2_{\mathcal{H}^s_\varepsilon} \sim \left(\|\cdot\|^2_{L^2_{x,v}} + \sum_{|l|\leqslant s} \left\|\partial^0_l \cdot\right\|^2_{L^2_{x,v}} + \varepsilon^2 \sum_{\substack{|l|+|j|\leqslant s \\ |j|\geqslant 1}} \left\|\partial^j_l \cdot\right\|^2_{L^2_{x,v}} \right),$$

and for all h in $H^s_{x,v}$,

$$\langle G_\varepsilon(h), h\rangle_{\mathcal{H}^s_\varepsilon} \leqslant -C_G^{(s)} \left\|h - \pi_{G_\varepsilon}(h)\right\|^2_{H^s_\Lambda}.$$

The modified norm $\|\cdot\|_{\mathcal{H}^s_\varepsilon}$ is dependent on ε. We can however make two remarks.

1. The dependence on ε only appears in front of v-derivatives which disappear in the process of the hydrodynamical limit since only integral against the v variable are of interest.
2. In the next subsection, another norm is constructed and do not involve any ε dependencies. With this norm, a same result than Theorem 1.3.1 can be obtained with similar arguments.

1.3.2.2 The perturbed Cauchy problem and trend to equilibrium

The hypocoercivity features of the linear Boltzmann operator and the generation of a strongly continuous semigroup in $H^s_{x,v}$ discussed in the previous subsection were used by C. Mouhot and L. Neumann [82] to obtain existence, uniqueness and exponential decay to equilibrium in the case $\varepsilon = 1$, with constructive methods. Such results were known to exist since the first rigorous studies by S. Ukai [107][108] but the methods of the proof were not constructive and thus did not give explicit statements.

The controls we have on the bilinear remainder term Γ are the following.

(H4): Control on the second order operator

$\Gamma : L^2_v \times L^2_v \longrightarrow L^2_v$ is a bilinear symmetric operator such that for all multi-indexes j and l such that $|j| + |l| \leqslant s$, $s \geqslant 0$,

$$\left|\langle \partial_l^j \Gamma(g,h), f\rangle_{L^2_{x,v}}\right| \leqslant \begin{cases} \mathcal{G}^s_{x,v}(g,h)\,\|f\|_\Lambda & , \text{ if } j \neq 0 \\ \mathcal{G}^s_{x}(g,h)\,\|f\|_\Lambda & , \text{ if } j = 0 \end{cases},$$

$\mathcal{G}^s_{x,v}$ and $\mathcal{G}^s_x$ being such that $\mathcal{G}^s_{x,v} \leqslant \mathcal{G}^{s+1}_{x,v}$, $\mathcal{G}^s_x \leqslant \mathcal{G}^{s+1}_x$ and satisfying the following property:

$$\exists s_0 \in \mathbb{N}\,,\ \forall s \geqslant s_0\,,\ \exists C_\Gamma > 0\,, \quad \begin{cases} \mathcal{G}^s_{x,v}(g,h) & \leqslant C_\Gamma \left(\|g\|_{H^s_{x,v}}\|h\|_{H^s_\Lambda} + \|h\|_{H^s_{x,v}}\|g\|_{H^s_\Lambda}\right) \\ \mathcal{G}^s_{x}(g,h) & \leqslant C_\Gamma \left(\|h\|_{H^s_x L^2_v}\|g\|_{H^s_\Lambda} + \|g\|_{H^s_x L^2_v}\|h\|_{H^s_\Lambda}\right). \end{cases}$$

The uniform Cauchy theory we present in Chapter 4 is an extension of the results derived in [82] to obtain estimates that are uniform in the Knudsen number. However, in the case $\varepsilon = 1$, the linear part G_1 and the bilinear remainder term Γ are of the same order and can be compared. The main difficulty for general Knudsen number is the fact that the linear part G_ε generates a contraction semigroup with a spectral gap of order 0(1) whereas the bilinear part is of order $O(\varepsilon^{-1})$. This makes impossible to consider $\varepsilon^{-1}\Gamma$ as a mere error term since it explodes as ε goes to 0. Our main contributions are

- A method mixing the hypocoercivity properties of the linear operator L with the *a priori* estimates on the bilinear operator, in particular thanks to an orthogonality property of the symmetrised operator Γ.
  (H5): Orthogonality to the Kernel of the linear operator

$$\forall h,\ g \in \mathrm{Dom}(\Gamma) \cap L^2_v\,, \quad \Gamma(g,h) \in \mathrm{Ker}(L)^\perp.$$

- The construction of a new norm in $H^s_{x,v}$ combining the idea of [82] and [56] to study both the microscopic and the fluid part of the solution.

The main result is the following theorem.

Theorem 1.3.2 *Let Q be a bilinear operator such that:*

- *the equation* (1.3.4) *admits an equilibrium* $0 \leqslant \mu \in L^1(\mathbb{T}^d \times \mathbb{R}^d)$*,*
- *the linearized operator $L = L(h)$ around μ with the scaling $f = \mu + \varepsilon\mu^{1/2}h$ satisfies* (H1'), (H2') *and* (H3)*,*
- *the bilinear remaining term $\Gamma = \Gamma(h,h)$ in the linearization satisfies* (H4) *and* (H5)*.*

Then

- *there exists $0 < \varepsilon_d \leqslant 1$ and a norm $\|\cdot\|_{\mathcal{H}^s_{\varepsilon\perp}}$ such that for any $s \geqslant s_0$ (defined in* (H4)*) and any $0 < \varepsilon \leqslant \varepsilon_d$, $\|\cdot\|_{\mathcal{H}^s_{\varepsilon\perp}} \sim \|\cdot\|_{H^s_{x,v}}$, independently of ε,*

- *there exist $\delta_s > 0$, $C_s > 0$ and $\tau_s > 0$ such that for all $0 < \varepsilon \leqslant \varepsilon_d$:*

For any distribution $0 \leqslant f_{in} \in L^1(\mathbb{T}^d \times \mathbb{R}^d)$ with $f_{in} = \mu + \varepsilon \mu^{1/2} h_{in} \geqslant 0$, h_{in} in $\mathrm{Ker}(G_\varepsilon)^\perp$ and

$$\|h_{in}\|_{\mathcal{H}^s_{\varepsilon\perp}} \leqslant \delta_s,$$

there exists a unique global smooth (in $H^s_{x,v}$, continuous in time) solution $f_\varepsilon = f_\varepsilon(t,x,v)$ to (1.3.4) which, moreover, satisfies $f_\varepsilon = \mu + \varepsilon \mu^{1/2} h_\varepsilon \geqslant 0$ with:

$$\|h_\varepsilon\|_{\mathcal{H}^s_{\varepsilon\perp}} \leqslant \delta_s e^{-\tau_s t},$$

We emphasize here that this Theorem is more general than just the case of the Boltzmann equation. It is indeed applicable to several other kinetic models such as the linear relaxation, the semi-classical relaxation, the linear Fokker-Planck equation and the Landau equation with hard and moderately soft potential.

1.3.2.3 The limit towards the incompressible Navier-Stokes equations

Theorem 1.3.2 implies that the sequence $(h_\varepsilon)_{0<\varepsilon\leqslant\varepsilon_d}$ is bounded in $L^\infty_t H^s_{x,v}$. Such a boundedness property is enough (see [8]) to obtain a weak convergence result $h_\varepsilon \rightharpoonup h$ in distributions as ε goes to 0 with

1. h is in $\mathrm{Ker}(L)$, so of the form
$$h(t,x,v) = \left[\rho(t,x) + v.u(t,x) + \frac{1}{2}(|v|^2 - d)\theta(t,x)\right]\mu(v)^{1/2},$$
2. $(\rho_\varepsilon, u_\varepsilon, \theta_\varepsilon)$ converges weakly-* in $L^\infty_t(H^s_x)$ towards (ρ, u, θ),
3. (ρ, u, θ) satisfies the incompressible Navier-Stokes equations (1.3.2) in the Leray sense [66] as well as the Boussineq equation (1.3.3).

In fact this convergence is strong and Chapter 4 gives explicit rates of convergence.

Theorem 1.3.3 *Consider $s \geqslant s_0$ and h_{in} in $H^s_{x,v}$ such that $\|h_{in}\|_{\mathcal{H}^s_\varepsilon} \leqslant \delta_s$.*

Then, $(h_\varepsilon)_{\varepsilon>0}$ exists for all $0 < \varepsilon \leqslant \varepsilon_d$ and converges weakly in $L^\infty_t(H^s_x L^2_v)$ towards h such that $h \in Ker(L)$, with $\nabla_x \cdot u = 0$ and $\rho + \theta = 0$.*

Furthermore, $\int_0^T hdt$ belongs to $H^s_x L^2_v$ and it exists $C > 0$ such that,

$$\left\| \int_0^{+\infty} hdt - \int_0^{+\infty} h_\varepsilon dt \right\|_{H^s_x L^2_v} \leqslant C\sqrt{\varepsilon\,|ln(\varepsilon)|}.$$

One can have a strong convergence in $L^2_{[0,T]}H^s_xL^2_v$ *only if* h_{in} *is in* $\mathrm{Ker}(L)$ *with* $\nabla_x \cdot u_{in} = 0$ *and* $\rho_{in} + \theta_{in} = 0$ *(initial layer conditions).*

Moreover, in that case we have

$$\|h - h_\varepsilon\|_{L^2_{[0,+\infty)}H^s_xL^2_v} \leqslant C\sqrt{\varepsilon\,|ln(\varepsilon)|},$$

and for all δ *in* $[0,1]$*, if* h_{in} *belongs to* $H^{s+\delta}_xL^2_v$*,*

$$\sup_{t\in[0,+\infty)} \|h - h_\varepsilon\|_{H^s_xL^2_v}(t) \leqslant C\varepsilon^{\min(\delta,1/2)}.$$

This theorem gives a strong convergence for $(\rho_\varepsilon, u_\varepsilon, \theta_\varepsilon)$ towards (ρ, u, θ) but above all it gives us that (ρ, u, θ) is the solution to the incompressible Navier-Stokes equations together with the Boussineq equation satisfying the initial conditions:

- $u(0,x) = Pu_{in}(x)$, where $Pu_{in}(x)$ is the divergence-free part of $u_{in}(x)$,
- $\rho(0,x) = -\theta(0,x) = \frac{1}{2}(\rho_{in}(x) - \theta_{in}(x))$.

A similar convergence was known to exist (see [10]) in the case where the spatial domain was $\mathbb{R}^d$, but did not require any integration in time. Our main contribution was to adapt the arguments to the case of the torus where the integration in time is compulsory to control the Fourier transform of the semigroup generated by G_ε that was derived in [39].

1.3.3 Hydrodynamical limit in polynomial weighted spaces (Chapter 5)

This work has been done jointly with Sara Merino-Aceituno and Clément Mouhot, both from the University of Cambridge.

The aim of Chapter 5 is to extend the previous semigroup properties of the linear part G_ε and the Cauchy theory for the full perturbed equation to more general space. The ultimate goal is to derive those results, uniform in the Knudsen number, in $L^1_vL^\infty_x\left(1+|v|^2\right)$. This space is indeed optimal in the velocity variable, since it incorporates bounded mass and energy densities, in the Boltwmann framework whereas L^∞_x is problematic for the Navier-Stokes equations. We would therefore be able to construct solutions to the incompressible Navier-Stokes equations in L^∞_x *via* the Boltzmann equation and its hydrodynamical limit. Here again we hope to use constructive arguments and obtain explicit rates of convergence.

This aim has not been achieved yet but it is still a work in progress with Sara Merino-Aceituno and Clément Mouhot. We so far managed to drop the strong exponential weight for a polynomial one, almost optimal, and we can deal with spaces without any derivatives in v for the Cauchy problem. The semigroup properties are extended in all Lebesgue and

Sobolev spaces with a polynomial weight $(1+|v|)^k$, for k large enough and $k>2$ in the L^1_v case. Chapter 5 presents our joint work.

We recall that we are working on the dimensionless Boltzmann equation

$$\partial_t f_\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x f_\varepsilon = \frac{1}{\varepsilon^2} Q(f_\varepsilon, f_\varepsilon)\ , \text{ on } \mathbb{T}^N \times \mathbb{R}^N, \tag{1.3.7}$$

under the linearization $f_\varepsilon(t,x,v) = \mu(v) + \varepsilon h_\varepsilon(t,x,v)$, which leads to the perturbed Boltzmann equation

$$\partial_t h_\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x h_\varepsilon = \frac{1}{\varepsilon^2} \mathcal{L}(h_\varepsilon) + \frac{1}{\varepsilon} Q(h_\varepsilon, h_\varepsilon), \tag{1.3.8}$$

where we defined

$$\mathcal{L}(h) = 2Q(\mu, h).$$

Note that we will use curly letters for operators in that Section and standard ones to talk about the restrictions of these operators to $H^s_{x,v}\left(\mu^{-1/2}\right)$. For instance, we recover the operator of previous section

$$(\mathcal{G}_\varepsilon)|_{H^s_{x,v}\left(\mu^{-1/2}\right)} = G_\varepsilon.$$

We still consider the Boltzmann equation with hard potential or Maxwellian potential $(\gamma = 0)$, that is to say

$$B\left(|v-v_*|, \cos\theta\right) = \Phi\left(|v-v_*|\right) b\left(\cos\theta\right), \tag{1.3.9}$$

with Φ and b be positive functions. This hypothesis is satisfied for all physical model and is more convenient to work with but do not impede the generality of our results.

We also restrict ourselves to the case of *hard potential* or *Maxwellian potential* $(\gamma = 0)$, that is to say there is a constant $C_\Phi > 0$ such that

$$\Phi(z) = C_\Phi z^\gamma, \quad \gamma \in [0,1], \tag{1.3.10}$$

with a strong form of Grad's *angular cutoff* (see [48]), expressed here by the fact that we assume b to be C^1 with the controls from above

$$\forall z \in [-1,1], \quad b(z),\ b(z') \leqslant C_b. \tag{1.3.11}$$

1.3.3.1 Semigroup properties in Lebesgue and Sobolev spaces

In a recent article [51], an abstract extension theorem allows, under certain assumptions, to extend semigroup properties from a space E into a larger space $\mathcal{E}$. The latter theorem allowed to prove that $\mathcal{G}_1$ generates a strongly continuous semigroup in Lebesgue and Sobolev spaces with polynomial weight [51].

In the same spirit, we show that $\mathcal{G}_\varepsilon$ generates a strong continuous semigroup in Lebesgue and Sobolev spaces of the form $W^{\alpha,1}_v W^{\beta,p}_x\left(1+|v|)^k\right)$ for $\alpha \leqslant \beta$ and k large enough with explicit thresholds independent of ε. It is done by starting from existing results in $H^s_{x,v}\left(\mu^{-1/2}\right)$ and then decomposing the linear operator $\mathcal{G}_\varepsilon$ into a dissipative part and a regularising part that are then treated in larger and larger spaces up to the space where the semigroup properties have been derived in previous articles. We thus improve the existing result [23]. Our main contribution is an adapted version of the abstract extension theorem developed in [51] that takes into account the dependencies on the Knudsen number as well as a careful study of the dissipative and the regularising parts of the operator $\mathcal{G}_\varepsilon$.

Theorem 1.3.4 *Let B be a Boltzmann collision kernel satisfying (1.3.9)-(1.3.10)-(1.3.11). There exists $0 < \varepsilon_d \leqslant 1$ such that for all p, q in $[1,+\infty]$, all α, β in $\mathbb{N}$ with $\alpha \leqslant \beta$ and all $k > k^*_q$, where*

$$k^*_q = \frac{3+\sqrt{49-48/q}}{2} + \gamma\left(1-\frac{1}{q}\right), \tag{1.3.12}$$

1. *for all $0<\varepsilon\leqslant\varepsilon_d$, $\mathcal{G}_\varepsilon = \varepsilon^{-2}\mathcal{L} - \varepsilon^{-1} v\cdot\nabla_x$ generates a C^0-semigroup, $S_{\mathcal{G}_\varepsilon}(t)$, on $W^{\alpha,q}_v W^{\beta,p}_x\left(\langle v\rangle^k\right)$,*

2. *for all $\tau>0$, there exist $C_{\mathcal{G}}(\tau)$, $\lambda_0>0$, such that for all $0<\varepsilon\leqslant\varepsilon_d$ and for all h_{in} in $W^{\alpha,q}_v W^{\beta,p}_x\left(\langle v\rangle^k\right)$, for all $t\geqslant\tau$*

$$\left\|S_{\mathcal{G}_\varepsilon}(t)(h_{in}) - \Pi_{\mathcal{G}}(h_{in})\right\|_{W^{\alpha,q}_v W^{\beta,p}_x\left(\langle v\rangle^k\right)} \leqslant C_{\mathcal{G}}(\tau)e^{-\lambda_0 t}\left\|h_{in} - \Pi_{\mathcal{G}}(h_{in})\right\|_{W^{\alpha,q}_v W^{\beta,p}_x\left(\langle v\rangle^k\right)},$$

where $\Pi_{\mathcal{G}}$ is the spectral projector onto $\mathrm{Ker}\left(\mathcal{G}_\varepsilon\right)$ which is given, for all ε, by

$$\Pi_{\mathcal{G}}(g) = \sum_{i=0}^{d+1}\left(\int_{\mathbb{T}^d\times\mathbb{R}^d} g\phi_i\,dxdv\right)\phi_i\mu, \tag{1.3.13}$$

where $\phi_0(v)=1$, for $i=1,\dots,d$ we defined $\phi_i(v)=v_i$ and $\phi_{d+1} = \left(|v|^2-d\right)/\sqrt{2d}$. The constants ε_d, $C_{\mathcal{G}}(\tau)$ and λ_0 are constructive and only depends on d, p, q, k, α, β and the kernel of the Boltzmann operator.

The rate of decay λ_0 can be taken equal to the spectral gap of $\mathcal{L}|_{H^s_{x,v}\left(\mu^{-1/2}\right)}$ (see [23]), for s as large as wanted, when k is large enough (and we obtained a constructive threshold).

Finally, we emphasize that in the case $q=1$, the result holds for all $k>2$. This is almost the minimal regularity $L^2_v\left(1+|v|^2\right)$ for the Boltzmann equation, that is solutions with bounded mass and energy.

1.3.3.2 Cauchy problem and exponential decay in Sobolev spaces with polynomial weight

The second part of Chapter 5 deals with the uniform, in the Knudsen number, Cauchy problem and the exponential decay towards equilibrium in larger spaces than the exponential weight framework we dealt with in Chapter 4.

The spaces where we developed our theory are of the following forms

$$W_v^{\alpha,1} W_x^{\beta,1}\left(1+|v|^{2+0}\right) \quad \text{and} \quad W_v^{\alpha,1} H_x^{\beta}\left(1+|v|^{2+0}\right),$$

for s large enough and all $\alpha \leqslant \beta$. This improves the Cauchy theory developed in Chapter 4 by dropping the exponential weight and the v-derivatives. Again, the polynomial weight is almost the optimal one for the Boltzmann equation (conservation of mass and energy).

Such results have recently been obtained [51] for fixed ε, in which case the rate of decay of the semigroup generated by $\mathcal{G}_\varepsilon$ is of the same order than the remainder term $Q(h,h)$. However, in order to obtain uniform results we have to handle the remainder term $\varepsilon^{-1}Q$ and it cannot be treated as a mere perturbation that evolves under the flow of $S_{\mathcal{G}_\varepsilon}$, the semigroup generated by $\mathcal{G}_\varepsilon$, since the latter has an exponential decay of order $O(1)$.

Our main contribution is a new analytic point of view about the extension theorem in [51] and includes the bilinear term. More precisely, we decompose the perturbed equation (1.3.8) into a hierarchy of equations taking place in spaces that have more and more regularity up to $H^s_{x,v}\left(\mu^{-1/2}\right)$ where estimates had been derived in Chapter 4. At each step we use the dissipative part of the linear operator to control the remainder term $\varepsilon^{-1}Q$ whereas the regularising part is controlled in spaces with higher regularity.

We hence state the following theorem tackling the Cauchy problem and the exponential convergence towards the equilibrium μ.

Theorem 1.3.5 *Let B be a Boltzmann collision kernel satisfying* (1.3.9)-(1.3.10)-(1.3.11) *and let $p=1$ or $p=2$.*
There exists $0<\varepsilon_d \leqslant 1$ and β_0 in $\mathbb{N}$ such that

- *for all α, β in $\mathbb{N}$ such that $\beta \geqslant \beta_0$ and $\alpha \leqslant \beta$ and for all $k>2$ define*

$$\mathcal{E}^p = W_v^{\alpha,1} W_x^{\beta,p}\left(\langle v\rangle^k\right),$$

- *for any λ_0' in $(0,\lambda_0)$ (λ_0 defined in Theorem 1.3.4) there exist $C_{\alpha,\beta}$, $\eta_{\alpha,\beta}>0$ such that for any $0<\varepsilon\leqslant\varepsilon_d$, for any distribution $f_{in}=\mu+\varepsilon h_{in}\geqslant 0$:*

If

(i) h_{in} is in $\mathrm{Ker}(\mathcal{G}_\varepsilon)^\perp$ in $\mathcal{E}^p$,

(ii) $\|h_{in}\|_{\mathcal{E}^p} \leqslant \eta_{\alpha,\beta}$,

Then there exists a unique global solution $f_\varepsilon = f_\varepsilon(t,x,v)$ *to* (1.3.7) *in* $\mathcal{E}^p$ *which, moreover, satisfies* $f_\varepsilon = \mu + \varepsilon h_\varepsilon \geqslant 0$ *with:*

- h_ε *belongs to* $\mathrm{Ker}(\mathcal{G}_\varepsilon)^\perp$ *for all times,*
- $$\|h_\varepsilon\|_{\mathcal{E}^p} \leqslant C_{\alpha,\beta}\, \|h_{in}\|_{\mathcal{E}^p}\, e^{-\lambda_0' t}.$$

The constants $C_{\alpha,\beta}$ *and* $\eta_{\alpha,\beta}$ *are constructive and depends only on* α*,* β*,* k*,* d*,* λ_0' *and the kernel of the Boltzmann operator.*

1.4 A quantic version of Boltzmann equation (Part *III*)

1.4.1 The Boltzmann-Nordheim equation

As we mentionned before, the Boltzmann equation describes, at a mesoscopic level, the dynamics of a monoatomistic rarefied gas with elastic collisions. There exists different modifications of this kinetic model, for polyatomistic gases for instance (see [29]).

For all these models, the Boltzmann equation arises from microscopic behaviours ruled by classical physics, where the probability of two particles colliding depends only on the number of particles moving at the incoming velocities. The case of quantum mechanics is rather different since the probability of two particles colliding not only depends on the velocity of the particles undergoing the collision but also the outcoming velocity the collision yields. We refer to [32] Chapter 17 for more details.

Using quatum statistical physics instead of classical statistical physics, Nordheim [89] derived a quantic version of the Boltzmann equation for bosons and fermions.

The latter evolution equation is called the Boltzmann-Nordheim equation and reads as follow, with the usual shorthand notations.

$$\partial_t f + v\cdot\nabla_x f = Q_\alpha(f),$$

with

$$Q_\alpha(f) = \int_{\mathbb{R}^N\times\mathbb{S}^{d-1}} B\left(v,v_*,\theta\right)\left[f'(1+\alpha f)f'_*(1+\alpha f_*) - f(1+\alpha f')f_*(1+\alpha f'_*)\right] dv_* d\sigma.$$

The Boltzmann-Nordheim equation thus rules the dynamics of the distribution of particles for a dilute quantum gas of bosons ($\alpha = 1$) or fermions ($\alpha = -1$). Note that in the classical case $\alpha = 0$ one recovers the Boltzmann equation.

In Chapter 6, our study applies to the case where the collision kernel B is hard potential with angular cutoff. More precisely, the collision operator is supposed to satisfy the following properties.

1. $B(v, v_*, \theta) = \Phi\left(|v - v_*|\right) b\left(\cos\theta\right)$,
2. there exist $C_\Phi > 0$ and γ in $[0, 1]$ such that $\Phi(z) = C_\Phi z^\gamma$,
3. $(b \circ \cos)$ is continuous on $(0, \pi)$ and integrable on the sphere:

$$l_b = \int_{\mathbb{S}^{d-1}} b\left(\cos\theta\right) d\sigma = \left|\mathbb{S}^{d-2}\right| \int_0^\pi b\left(\cos\theta\right) \sin^{d-2}\theta \, d\theta < +\infty.$$

Moreover, we restrain ourselves to the spatially homogeneous case for a gas of bosons

$$\partial_t f + v \cdot \nabla_x f = Q(f), \tag{1.4.1}$$

with

$$Q(f) = \int_{\mathbb{R}^N \times \mathbb{S}^{d-1}} B\left(v, v_*, \theta\right) \left[f'(1+f)f'_*(1+f_*) - f(1+f')f_*(1+f'_*)\right] dv_* d\sigma. \tag{1.4.2}$$

1.4.2 Cauchy problem and the Bose-Einstein condensate (Chapter 6)

The Boltzmann-Nordheim collision operator (1.4.2) is in fact the addition of the classical Boltzmann collision operator with a trilinear operator. If some properties of the classical Boltzmann equation still hold true for the Boltzmann-Nordheim equation, such as the *a priori* preservation of mass, momemtum and energy

$$\int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} f(v)\, dv = \int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} f_0(v)\, dv,$$

the trilinear term implies rather different behaviours.

Indeed, physical observations and numerical simulations (see [40] for an overview of these results) in the isotropic setting $f(t, v) = f(t, |v|)$ showed that there exists a critical temperature $T_c(M_0)$, depending on the mass M_0 of the bosonic gas. If the temperature of the initial datum f_{in} is below $T_c(M_0)$ then the solution of the Boltzmann-Nordheim equation will develop a dirac mass at $|v| = 0$ in finite time. This blow-up phenomenon is known as the Bose-Einstein condensate.

From the mathematical point of view, the only rigorously known results focused on the isotropic framework. X. Lu [69][70][71] built solutions in L^1_2 and proved a Cauchy theory for measures. He also proved, with not entirely constructive methods, a concentration phenomenon for subcritical temperature in the limit t goes to infinity. We emphasize here that this asymptotic result does not imply the appearance of a Bose-Einstein condensate in finite time. Recently, other isotropic solutions have been constructed in $L^1(1+|v|^{3+0})$ by M. Escobedo and J. J. L. Velázquez [40]. Moreover, they made a major breakthrough by proving the appearance of a Bose-Einstein condensate in finite time under some assumptions on the solution [40] and for subcritical temperatures [41].

In Chapter 6 we develop a local in time Cauchy theory in the non-isotropic setting in $L^1_2 \cap L^\infty$. The latter space is the most general one can hope for a Cauchy theory that catches the Bose-Einstein condensate. Solutions are indeed physically expect to have finite mass and energy and the creation of a dirac mass creates a blow-up in L^∞ whereas it only leads to a loss of mass in L^1.

Our main contributions are a new version of Povzner inequality [94], which bounds the evolution of convex functions through a collision, and a new control on the operator Q^+ for high and small relative velocities $v - v_*$. We also control the higher moments of solutions to the Boltzmann-Nordheim equation and derive a precise estimate on the blow-up of the $(2+\gamma)^{th}$ moment of solutions at time $t = 0$, in the spirit of [77], to obtain uniqueness.

We denote, for all s and t in $\mathbb{R}^+$,

$$M_s(t) = \int_{\mathbb{R}^d} |v|^s \, f(t,v)\, dv,$$

the s^{th} moment of a function $f(t,v)$. The main result of Chapter 6 is the following Cauchy theorem for the Boltzmann-Nordheim equation for bosons.

Theorem 1.4.1 *Let $f_0(v)$ be in $L^1_{2,v} \cap L^\infty_v$.*
Then there exists $T_0 > 0$, depending only on C_Φ, l_b, γ, $\|f_0\|_{L^1_{2,v}}$ and $\|f_0\|_{L^\infty_v}$, such that there exists a unique f in $L^\infty_{loc}\left([0,T_0), L^1_{2,v} \cap L^\infty_v\right)$ solution of (6.1.3) on $[0,T_0) \times \mathbb{R}^d$ that preserves mass and energy.

Moreover, this solution satisfies

- $T_0 = +\infty$ *or* $\lim_{T \to T_0^-} \|f\|_{L^\infty_{[0,T]\times\mathbb{R}^d}} = +\infty$,
- *f preserves the momentum of f_0,*
- *for all $s > 0$ and for all $0 < T < T_0$,*

$$M_s(t) \in L^\infty_{loc}\left([T,T_0)\right).$$

- *for all $T < T_0$,*

$$\sup_{[0,T]\times\mathbb{R}^d} \left(f(t,v) + \int_0^t \left(1+|v|^\gamma\right) f(s,v)\, ds \right) < \infty.$$

Let us mention here that Theorem 6.2.1 implies a Bose-Einstein concentration phenomenon as time goes to infinity for subcritical initial data if they are globally defined thanks to the work of Lu ([71] Theorem 2).

The latter argument is however non explicit and does not prove any blow-up in finite time whereas [40] gives the apparition of the Bose-Einstein condensate in finite time in the isotropic setting. A work in progress is the proof of the creation of a condensate in finite time in our more general framework.

Appendices

1.A Notations

We will work in different function spaces. We gather in this appendix the different notations we will use throughout the sequel.

We first emphasize the fact that we consider that 0 belongs to the following sets: $\mathbb{N}$, $\mathbb{Z}^+$, $\mathbb{Q}^+$, $(\mathbb{R}-\mathbb{Q})^+$ and $\mathbb{R}^+$.
We then define the following shorthand notation,

$$\langle\cdot\rangle = \sqrt{1+|\cdot|^2}.$$

1.A.1 Function spaces for one variable

Here, the term “variable” has to be understood as being in a particular vector space of dimension N, namely $\mathbb{R}^N$. Basically, when there is not any combination of time, space and velocity spaces.

Let p be in $[1,+\infty)$, q in $\mathbb{R}$, s in $\mathbb{R}^+$ and $m : \mathbb{R}^N \longrightarrow \mathbb{R}^+$ a strictly positive measurable function.

Weighted Lebesgue spaces. We define the space $L^p(m)$ by the norm

$$\|f\|_{L^p(m)} = \left[\int_{\mathbb{R}^d} |f(y)|^p \, m(y)^p \, dy\right]^{\frac{1}{p}},$$

and the space $L^\infty(m)$ by the norm

$$\|f\|_{L^\infty(m)} = \sup_{y\in\mathbb{R}^N} \left(|f(y)| \, m(y)\right).$$

In the case when $m(y) = \langle y\rangle$ is a power of $\langle\cdot\rangle$ we use the shorthand notations

$$L^p_q = L^p(m^q) \quad \text{and} \quad L^\infty_q = L^\infty(m^q).$$

Weighted Sobolev spaces In the case where s is a natural number, for any multi-index $k = (k_1, \ldots, k_N)$ in $\mathbb{N}^N$ we denote

- the k^{th} partial derivative by

$$\partial^k = \frac{\partial^{k_1}}{\partial y_1^{k_1}} \cdots \frac{\partial^{k_N}}{\partial y_N^{k_N}},$$

- for i in $\{1, \ldots, N\}$ we denote by $c_i(k)$ the i^{th} coordinate of k,
- the length of k will be written $|k| = \sum_i c_i(k)$,
- the multi-index δ_{i_0} by : $c_i(\delta_{i_0}) = 1$ if $i = i_0$ and 0 elsewhere.

With these conventions, we define the space $W^{s,p}(m)$ by the norm

$$\|f\|_{W^{s,p}(m)} = \left[\sum_{|k| \leqslant s} \left\| \partial^k f \right\|_{L^p(m)}^p \right]^{\frac{1}{p}},$$

and the space $W^{\infty,p}(m)$ by the norm

$$\|f\|_{W^{s,\infty}(m)} = \sum_{|k| \leqslant s} \left\| \partial^k f \right\|_{L^\infty(m)}.$$

In the case $m(y) = \langle y \rangle$ we use the obvious shorthand notations $W_q^{s,p}$ and $W_q^{s,\infty}$.

These definitions can be extended by interpolation, or *via* the theory of Fourier transform, to the case s in $\mathbb{R}^+$.

In the particular case $p = 2$, we will write $H^s(m) = W^{s,2}(m)$ and $H_q^s = W_q^{s,2}$.

1.A.2 Function spaces for several varaiables

In the case where the functions we consider are functions of time, space and velocity we need distinctive notations. The convention we chose is to index the space by the name of the concerned variable. For instance, for a measurable function

$$f(t,x,v) : [0,T) \times \Omega \times \mathbb{R}^d \longrightarrow \mathbb{R}^+,$$

with $\Omega \subset \mathbb{R}^d$, we will denote for p in $[1,+\infty]$

$$L_t^p = L^p([0,T)), \quad L_x^p = L_x^p(\Omega) \quad \text{and} \quad L_v^p = L^p(\mathbb{R}^d).$$

We extend these notations verbatim to weighted Lebesgue and weighted Sobolev spaces.

In the case of norm involving all the different variables we need new definitions. We consider functions $f(x,v)$ defined in $\Omega \times \mathbb{R}^d$ with $\Omega \subset \mathbb{R}^d$.

Let p and q be in $[1, +\infty)$, α and s in $\mathbb{R}^+$ and $m : \mathbb{R}^d \longrightarrow \mathbb{R}^+$ a strictly positive measurable function.
In the case where s is a natural number, for any multi-indexes $j = (j_1, \dots, j_N)$ and $l = (l_1, \dots, l_N)$ in $\mathbb{N}^N$ we denote the $(j,l)^{th}$ partial derivative by

$$\partial^j_l = \partial^l_x \partial^j_v,$$

with multi-index partial derivatives being defined in previous subsection. We define the space $W^{\alpha,q}_v W^{\beta,p}_x(m)$ by the norm

$$\|f\|_{W^{\alpha,q}_v W^{\beta,p}_x(m)} = \sum_{\substack{|j|\leqslant\alpha,|l|\leqslant\beta \\ |l|+|j|\leqslant\max(\alpha,\beta)}} \left\|\partial^j_l f\right\|_{L^q_v L^p_x(m)}.$$

We emphasize here that in the case $\alpha = \beta$ and $p = q$ this definition is equivalent to the $W^{\beta,p}_{x,v}(m)$-norm on $\Omega \times \mathbb{R}^d$ we defined in the previous subsection. Again, in the particular case $p = 2$ or $q = 2$ we will use the notations, respectively, H^β_x and H^β_v.

Part I

QUALITATIVE AND QUANTITATIVE STUDY OF SOLUTIONS TO THE BOLTZMANN EQUATION

Chapter 2

Instantaneous filling of the vacuum for the full Boltzmann equation in bounded domains

We prove the immediate appearance of a lower bound for continuous mild solutions to the full Boltzmann equation in the torus or a C^2 convex domain with specular boundary conditions, under the sole assumption of regularity of the solution. We investigate a wide range of collision kernels, some satisfying Grad's cutoff assumption and others not. We show that this lower bound is exponential, independent of time and space with explicit constants depending only on the a priori bounds on the solution. In particular, this lower bound is Maxwellian in the case of cutoff collision kernels. A thorough study of characteristic trajectories, as well as a geometric approach of grazing collisions against the boundary are derived.

These results are entirely constructive if the domain is C^3 and strictly convex.

Contents

2.1 Introduction

This chapter deals with the Boltzmann equation, which rules the behaviour of rarefied gas particles moving in a domain Ω of $\mathbb{R}^d$ with velocities in $\mathbb{R}^d$ ($d \geqslant 2$) when the only interactions taken into account are binary collisions. More precisely, the Boltzmann equation describes the time evolution of $f(t,x,v)$, the distribution of particles in position and velocity, starting from an initial distribution $f_0(x,v)$.

We investigate the case where Ω is either the torus or a C^2 convex bounded domain. The Boltzmann equation reads

$$\begin{aligned} \forall t \geqslant 0 \quad , \quad \forall (x,v) \in \Omega \times \mathbb{R}^d, &\quad \partial_t f + v \cdot \nabla_x f = Q(f,f), \\ \forall (x,v) \in \bar{\Omega} \times \mathbb{R}^d, &\quad f(0,x,v) = f_0(x,v), \end{aligned} \tag{2.1.1}$$

with f being periodic in the case of $\Omega = \mathbb{T}^d$, the torus, or with f satisfying the specular reflections boundary condition if Ω is a C^2 convex bounded domain:

$$\forall (x,v) \in \partial\Omega \times \mathbb{R}^d, \quad f(t,x,v) = f(t,x,\mathcal{R}_x(v)). \tag{2.1.2}$$

$\mathcal{R}_x$, for x on the boundary of Ω, stands for the specular reflection at that point of the boundary. One can compute, denoting by $n(x)$ the outward normal at a point x on $\partial\Omega$,

$$\forall v \in \mathbb{R}^d, \quad \mathcal{R}_x(v) = v - 2(v \cdot n(x))n(x).$$

The quadratic operator $Q(f,f)$ is local in time and space and is given by

$$Q(f,f) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} B\left(|v - v_*|, \cos\theta\right) \left[f' f'_* - f f_*\right] dv_* d\sigma,$$

where f', f_*, f'_* and f are the values taken by f at v', v_*, v'_* and v respectively. Define:

$$\left\{ \begin{aligned} v' &= \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma \\ v'_* &= \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma \end{aligned} \right. , \text{ and } \quad \cos\theta = \langle \frac{v - v_*}{|v - v_*|}, \sigma \rangle.$$

The collision kernel $B \geqslant 0$ contains all the information about the interaction between two particles and is determined by physics (see [28] or [30] for a formal derivation for the hard sphere model of particles). In this chapter we shall only be interested in the case of

B satisfying the following product form

$$B(|v-v_*|,\cos\theta) = \Phi(|v-v_*|)\, b(\cos\theta), \tag{2.1.3}$$

which is a common assumption as it is more convenient and also covers a wide range of physical applications. Moreover, we shall assume that Φ satisfies either

$$\forall z \in \mathbb{R}, \quad c_\Phi |z|^\gamma \leqslant \Phi(z) \leqslant C_\Phi |z|^\gamma \tag{2.1.4}$$

or a mollified assumption

$$\begin{cases} \forall\, |z| \geqslant 1 \in \mathbb{R}, & c_\Phi |z|^\gamma \leqslant \Phi(z) \leqslant C_\Phi |z|^\gamma \\ \forall\, |z| \leqslant 1 \in \mathbb{R}, & c_\Phi \leqslant \Phi(z) \leqslant C_\Phi, \end{cases} \tag{2.1.5}$$

c_Φ and C_Φ being strictly positive constants and γ in $(-d,1]$. The collision kernel is said to be "hard potential" in the case of $\gamma > 0$, "soft potential" if $\gamma < 0$ and "Maxwellian" if $\gamma = 0$.

Finally, we shall consider b to be a continuous function on θ in $(0,\pi]$, strictly positive near $\theta \sim \pi/2$, which satisfies

$$b(\cos\theta)\sin^{d-2}\theta \underset{\theta\to 0^+}{\sim} b_0 \theta^{-(1+\nu)} \tag{2.1.6}$$

for $b_0 > 0$ and ν in $(-\infty, 2)$. The case when b is locally integrable, $\nu < 0$, is referred to by the Grad's cutoff assumption (first introduce in [48]) and therefore B will be said to be a cutoff collision kernel. The case $\nu \geqslant 0$ will be designated by non-cutoff collision kernel.

2.1.1 Motivations and comparison with previous results

The aim of this chapter is to show and to quantify the strict positivity of the solutions to the Boltzmann equation when the gas particles move in a bounded domain. This issue has been investigated for a long time since it not only presents a great physical interest but also appears to be of significant importance for the mathematical study of the Boltzmann equation.

Moreover, our results only require some regularity on the solution and no further assumption on its local density, which was assumed to be uniformly bounded from below in previous studies (which is equivalent of assuming *a priori* either that there is no vacuum or that the solution is strictly positive).

More precisely, we shall prove that continuous solutions to the Boltzmann equation with angular cutoff in a C^2 convex bounded domain or the torus which have uniformly bounded energy satisfy an immediate Maxwellian lower bound:

$$\forall t_0 > 0,\ \exists \rho, \theta > 0,\ \forall t \geqslant t_0,\ \forall (x,v) \in \Omega \times \mathbb{R}^d, \quad f(t,x,v) \geqslant \frac{\rho}{(2\pi\theta)^{d/2}} e^{-\frac{|v|^2}{2\theta}}.$$

The strict positivity of the solutions to the Boltzmann equation standing in the form of an exponential lower bound was already noticed by Carleman in [26] for the spatially homogeneous equation. In his article he proved that such a lower bound is created immediately in time in the case of hard potential kernels with cutoff in dimension 3. More precisely, the radially symmetric solutions he constructed in [26] satisfies an almost Maxwellian lower bound,

$$\forall t \geqslant t_0, \forall v \in \mathbb{R}^3, \quad f(t,v) \geqslant C_1 e^{-C_2 |v|^{2+\varepsilon}},$$

$C_1, C_2 > 0$ for all $t_0 > 0$ and $\varepsilon > 0$. His breakthrough was to notice that a part Q^+ of the Boltzmann operator Q satisfies a spreading property, roughly speaking

$$Q^+(\mathbf{1}_{B(\bar{v},r)}, \mathbf{1}_{B(\bar{v},r)}) \geqslant C_+ \mathbf{1}_{B\left(\bar{v},\sqrt{2}r\right)},$$

with $C_+ < 1$ (see Lemma 2.4.2 for an exact statement).

The spreading strategy was used by Pulvirenti and Wennberg in [95] to extend the latter inequality to solutions to the spatially homogeneous Boltzmann equation with hard potential and cutoff in dimension 3 with more general initial data. Their contribution was to get rid of the inital boundedness suggested in [26] by Carleman thanks to the use of an iterative regularity property of the Q^+ operator. This property allowed them to immediately create an "upheaval point" that they then spread with the method of Carleman. Moreover, they obtain an exact Maxwellian lower bound of the form by controlling the decay of C_+^n

$$\forall t \geqslant t_0, \forall v \in \mathbb{R}^3, \quad f(t,v) \geqslant C_1 e^{-C_2 |v|^2},$$

for all $t_0 > 0$.

Finally, Mouhot in [78] dealt with the full Boltzmann equation in the torus. He derived a spreading method that is invariant under the flow of the characteristics, obtaining lower bounds uniformly in space as long as the solution has uniformly bounded density, energy and entropy (for the hard potential case) together with uniform bounds on higher moments (for the soft and Maxwellian potentials case). However, he also implicitly assumed that the initial data had to be bounded from below uniformly in space. He also derived ([78]) the same kind of results in the non-cutoff case in the torus, the immediate appearance of an exponential lower bound of the form

$$\forall t \geqslant t_0, \forall (x,v) \in \mathbb{T}^d \times \mathbb{R}^d, \quad f(t,v) \geqslant C_1(\varepsilon) e^{-C_2(\varepsilon)|v|^{K+\varepsilon}},$$

for all $t_0 > 0$, all $\varepsilon > 0$ and $K = K(\nu)$ with $K(0) = 2$ (thus recovering the cutoff case in the limit). His idea was to split further the Q operator into a cutoff part and a non-cutoff part that is seen as a small perturbation of his original spreading method.

Our results extend those in [78] in the case of C^2 bounded convex domain. Our main contribution is the derivation of a spreading method that remains invariant under the characteristics flow that, unlike the torus case, changes the direction of velocities at the boundary. Moreover, we emphasize here that the existence of boundaries implies the existence of grazing collisions against the latter, where the strategies developped in [95] and [78] fail. We therefore to derive a geometrical approach to those problematic trajectories.

Furthermore, we do not assume any uniform boundedness on the initial data but we require the continuity of the solution to the Boltzmann equation. However, if we keep the assumptions made in [78] and further assume that the domain is C^3 and strictly convex then our proofs are constructive.

The quantification of the strict positivity, and above all the appearance of an exponential lower bound, has been seen to be of great mathematical interest thanks to the development of the entropy-entropy production method. This method (see [112], Chapter 3, and [113]) provides a useful way of investigating the long-time behaviour of solutions to kinetic equations. Indeed, it has been successfully used to prove convergence to the equilibrium in non-pertubative cases for the Fokker-Planck equation, [36], and the full Boltzmann equation in the torus or in C^1 bounded connected domains with specular reflections, [37]. This entropy-entropy production method requires (see Theorem 2 in [37]) uniform boundedness on moments and Sobolev norms for the solutions to the Boltzmann equation but also an *a priori* exponential lower bound of the form

$$f(t, x, v) \geqslant C_1 e^{-C_2|v|^q},$$

with $q \geqslant 2$.

Therefore, the present chapter allows us to prove that the latter *a priori* assumption is in fact satisfied for a lot of different cases (see [78], Section 5 for an overview). We also emphasize here that the assumption of continuity of the solution we have made does not reduce the range of applications since a lot more regularity is usually asked for the entropy-entropy production method. Moreover, our method, unlike the ones developed in [95] and [78], does not require a uniform bound on the local density of solutions, which is not a requirement for the entropy-entropy production method either (see [37], Theorem 2).

To conclude we note that our investigations require a deep and detailed understanding of the geometry and properties of characteristic trajectories for the free transport equation. In particular, a geometric approach of grazing collisions against the boundary is derived and is the key ingredient to study the strict positivity of solutions to the Boltzmann equation. The existing strategies as well as our improvements are discussed in the next section.

2.1.2 Our strategy

Our strategy to tackle this issue will follow the method introduced by Carleman [26] together with the idea of Mouhot [78] to find a spreading method that will be invariant along the characteristic trajectories. Roughly speaking we shall built characteristics in a C^2 bounded convex domain, create an "upheaval point" (as in [95] and [78]) that we spread and expand uniformly along the characteristics. Finally, once the lower bound can be compared to an exponential one we reach the expected result.

However, the existence of rebounds against the boundary leads to difficulties. We describe them below and point out how we shall overcome them.

Creating an "upheaval point" was achieved, in [95] and [78], by using an iterated Duhamel formula and a regularity property of the collision operator relying on a uniform lower bound of the local density of the function. But the use of this property requires a uniform control along the characteristics of the density, the energy and the entropy of the solutions to the Boltzmann equation which is natural in the homogeneous case but made Mouhot consider initial datum bounded from below uniformly in space. Our way of dealing with the appearance of the "upheaval point" is rather different but includes more general initial datum. We make the assumption of continuity of solutions to the Boltzmann equation and by compactness arguments we can construct a partition of our phase space where initial localised lower bounds exist, i.e., localised "upheaval points".

The case on the torus studied by Mouhot tells us that an exponential lower bound should arise immediately and therefore we expect the same to happen as long as the characteristic trajectory is a straight line. Unfortunately, the possibility for a trajectory to remain a line depends on the distance from the boundary of the starting point, which can be as short as one wants. This thought is the basis of our means for spreading the initial lower bound. We divided our trajectories into two categories, the ones which always stay close to the boundary (grazing collisions) and the others. For the latter we can spread our lower bound uniformly as noticed in [78]. The key contribution of our proof is a thorough investigation of the geometry of grazing collisions. We show that their velocity does not evolve a lot along time and mix it with the spreading property of the collision operator. Notice here that the convexity of Ω is needed for the study of grazing trajectories.

The last behaviour to notice is the fact that specular reflections completely change velocities but preserve their norm. Therefore, the existence of rebounds against the boundary prevents us from obtaining a uniform spreading method straight from the "upheaval point" unless it is depending only on the norm of the velocity. Our strategy is to spread the lower bound created at the "upheaval points" independently for grazing and non-grazing collisions up to the point when the lower bound we obtain depends only on the norm of the velocity. Roughly, our lower bounds will be balls in velocity that can be centred away from the origin and we shall grow them up finitely many times to balls containing the origin and finally be able to generate a uniform spreading method.

Collision kernels satisfying a cutoff property as well as collision kernels with a non-cutoff property will be treated following the strategy described above. The only difference is the decomposition of the Boltzmann bilinear operator Q we consider in each case. In the case of a non-cutoff collision kernel, we shall divide it into a cutoff collision kernel and a remainder. The cutoff part will already be dealt with and a careful control of the L^∞-norm of the remainder will give us the expected lower bound, smaller than a Maxwellian lower bound.

A preliminary to our study is to be able to construct the characteristic trajectories associated to the Boltzmann equation with specular reflections in a C^2 bounded convex domain. These trajectories are merely those of the free transport and so can be seen as the movement of a billiard ball inside the boundary of our domain.

Such a free transport in a convex domain has been studied in [33] (see also [93], [104] or [105] for geometrical properties) and has been used in kinetic theory by Guo, [52], or Hwang, [59], for instance. Yet, the common feature in [33], [52] and [59] is that their assumptions on the boundary always lead to clear rebounds of the characteristic trajectories. That is to say, the absoption phenomenon of [33], the electromagnetic field in [52] and [59] or the smooth strict convexity assumption used in [57], prevent the characteristics to roll on the boundary which is one of the possible behaviour we have to take into account in our general settings. As briefly mentionned in the introduction of [105], the behaviour at some specific boundary points is mathematically quite unexpected, even if that is of no physical relevance. We thus classify all the possible outcomes of a rebound against the boundary and study them carefully to analytically build the characteristics for the free transport equation in our domain Ω.

Finally, we need to control the number of rebounds that can happen in a finite time. In [104], Tabachnikov focuses on the footprints on the boundary of the trajectories of billiard balls and shows that the initial conditions leading to infinitely many rebounds on the boudary is a set of measure 0. We extend this to the whole trajectory (see Appendix 2.3.1, Proposition 2.3.4), not only its footprints on the boundary, allowing us to consider only finitely many rebounds in finite time and to have an analytic formula for the characteristics which we shall use throughout the chapter.

Notice that all this study of the free transport equation will be done in the case of a merely C^1 bounded domain.

2.1.3 Organisation of the chapter

Section 2.2 is dedicated to the statement and the description of the main results proved in this chapter. It contains four different parts

Section 2.2.1 defines all the notations which will be used throughout the chapter.

As mentioned above, we shall investigate in detail the characteristics and the free transport equation in a C^1 bounded domain. Section 2.2.2 mathematically formulates the intuitive ideas of trajectories.

The last subsections, 2.2.3 and 2.2.4, are dedicated to a mathematical formulation of the results related to the lower bound in, respectively, the cutoff case and the non-cutoff case, described in Section 2.1.2. It also defines the concept of mild solutions to the Boltzmann equation in each case.

Sections 2.4 to 2.7 focuse on the Maxwellian lower bound in the cutoff case. It is divided into the four main arguments of the proof.

Following our strategy, Section 2.4 creates the localised "upheaval points" whereas Section 2.5 and Section 2.6 spread them along non-grazing and grazing trajectories respectively.

Section 2.7 concludes by describing the immediate appearance of a lower bound depending only on the norm of the velocity (Proposition 2.2.4) as well as proving the immediate Maxwellian lower bound (proof of Theorem 2.2.3).

Finally, we deal with non-cutoff collision kernels in Section 2.8 where we prove the immediate appearance of an exponential lower bound (Theorem 2.2.6). The proof follows exactly the same steps as in the case of cutoff kernels and is thus divided into Section 2.8.1, where we construct a lower bound only depending on the norm of the velocity, and Section 2.8.2, where we derive the exponential lower bound.

As mentioned before, we need to study the free transport equation and the different important properties of the characteristics. Appendix 2.3 formulates these issues, investigates all the different behaviours of rebounds against the boundary (Section 2.3.1), builds the characteristics and derives their properties (Section 2.3.2) and solves the free transport equation (Section 2.3.3).

2.2 Main results

We begin with the notations we shall use all along the chapter.

2.2.1 Notations

We denote $\langle \cdot \rangle = \sqrt{1 + |\cdot|^2}$ and $y^+ = \max\{0, y\}$, the positive part of y.
This study will hold in specific functional spaces regarding the v variable that we describe

here and use throughout the sequel. Most of them are based on natural Lebesgue spaces $L^p_v = L^p\left(\mathbb{R}^d\right)$ with a weight:

- for $p \in [1, \infty]$ and $q \in \mathbb{R}$, $L^p_{q,v}$ is the Lebesgue space with the following norm

$$\|f\|_{L^p_{q,v}} = \|\langle v\rangle^q f\|_{L^p_v},$$

- for $p \in [1, \infty]$ and $k \in \mathbb{N}$ we use the Sobolev spaces $W^{k,p}_v$ by the norm

$$\|f\|_{W^{k,p}_v} = \left[\sum_{|s|\leqslant k} \|\partial^s f(v)\|^p_{L^p_v}\right]^{1/p},$$

  with the usual convention $H^k_v = W^{k,2}_v$.

In what follows, we are going to need bounds on some physical observables of solution to the Boltzmann equation (2.1.1).

We consider here a function $f(t, x, v) \geqslant 0$ defined on $[0, T) \times \Omega \times \mathbb{R}^d$ and we recall the definitions of its local hydrodynamical quantities.

- its local energy

$$e_f(t, x) = \int_{\mathbb{R}^d} |v|^2 f(t, x, v) dv,$$

- its local weighted energy

$$e'_f(t, x) = \int_{\mathbb{R}^d} |v|^{\tilde{\gamma}} f(t, x, v) dv,$$

  where $\tilde{\gamma} = (2 + \gamma)^+$,

- its local L^p norm ($p \in [1, +\infty)$)

$$l^p_f(t, x) = \|f(t, x, \cdot)\|_{L^p_v},$$

- its local $W^{2,\infty}$ norm

$$w_f(t, x) = \|f(t, x, \cdot)\|_{W^{2,\infty}_v}.$$

Our results depend on uniform bounds on those quantities and therefore, to shorten calculations we will use the following

$$E_f = \sup_{(t,x)\in[0,T)\times\Omega} e_f(t,x) \quad , \quad E'_f = \sup_{(t,x)\in[0,T)\times\Omega} e'_f(t,x),$$
$$L^p_f = \sup_{(t,x)\in[0,T)\times\Omega} l^p_f(t,x) \quad , \quad W_f = \sup_{(t,x)\in[0,T)\times\Omega} w_f(t,x).$$

In our theorems we are giving a priori lower bound results for solutions to (2.1.1) satisfying some properties about their local hydrodynamical quantities. Those properties will differ depending on which case of collision kernel we are considering. We will take them as assumptions in our proofs and they are the following.

- In the case of hard or Maxwellian potentials with cutoff ($\gamma \geqslant 0$ and $\nu < 0$):
$$E_f < +\infty. \tag{2.2.1}$$
- In the case of a singularity of the kinetic collision kernel ($\gamma \in (-d, 0)$) we shall make the additional assumption
$$L^{p_\gamma}_f < +\infty, \tag{2.2.2}$$
where $p_\gamma > d/(d+\gamma)$.
- In the case of a singularity of the angular collision kernel ($\nu \in [0,2)$) we shall make the additional assumption
$$W_f < +\infty, \;\; E'_f < +\infty. \tag{2.2.3}$$

As noticed in [78], in some cases several assumptions might be redundant.

Furthermore, in the case of the torus with periodic conditions or the case of bounded domain with specular boundary reflections, solutions to (2.1.1) also satisfy the following conservation laws (see [28], [30] or [112] for instance) for the total mass and the total energy:

$$\exists \mathrm{M},\ \mathrm{E} \geqslant 0,\ \forall t \in \mathbb{R}^+, \quad \left\{ \begin{array}{l} \displaystyle\int_\Omega \int_{\mathbb{R}^d} f(t,x,v)\,dxdv = M, \\ \displaystyle\int_\Omega \int_{\mathbb{R}^d} |v|^2 f(t,x,v)\,dxdv = E. \end{array} \right. \tag{2.2.4}$$

2.2.2 Results about the free transport equation

Our investigations start with the study of the characteristics of the free transport equation. We only focus on the case where Ω is not the torus (the characteristics in the torus being merely straight lines) but we will use the same notations in both cases. This is achieved by the following theorem.

Theorem 2.2.1 *Let Ω be an open, bounded and C^1 domain in $\mathbb{R}^d$.*
Let $u_0 : \bar{\Omega} \times \mathbb{R}^d \longrightarrow \mathbb{R}$ be C^1 in $x \in \Omega$ and in $L^2_{x,v}$.

The free transport equation with specular reflections reads

$$\forall t \geqslant 0 \quad , \quad \forall (x,v) \in \Omega \times \mathbb{R}^d, \quad \partial_t u(t,x,v) + D_x(v)(u)(t,x,v) = 0, \tag{2.2.5}$$

$$\forall (x,v) \in \bar{\Omega} \times \mathbb{R}^d, \quad u(0,x,v) = u_0(x,v), \tag{2.2.6}$$

$$\forall (x,v) \in \partial\Omega \times \mathbb{R}^d, \quad u(t,x,v) = u(t,x,\mathcal{R}_x(v)), \tag{2.2.7}$$

where $\mathcal{R}_x$ stands for the specular reflection at a point x and $D_x(v)$ is the directional derivative at x in the direction of v.
Then this equation has a unique solution $u : \mathbb{R}^+ \times \bar{\Omega} \times \mathbb{R}^d \longrightarrow \mathbb{R}$ which is C^1 in time, admits a directional derivative in space in the direction of v and is in $L^2_{x,v}$.

Moreover, for all (t,x,v) in $\mathbb{R}^+ \times \bar{\Omega} \times \mathbb{R}^d$, there exists $x_{fin}(t,x,v)$, $v_{fin}(t,x,v)$ and $t_{fin}(t,x,v)$ (see Definition 2.3.6) such that

$$u(t,x,v) = u_0\left(x_{fin} - (t - t_{fin})v_{fin}, v_{fin}\right).$$

2.2.3 Maxwellian lower bound for cutoff collision kernels

The final theorem we prove in the case of cutoff collision kernel is the immediate appearance of a uniform Maxwellian lower bound. We use, in that case, the Grad's splitting for the bilinear operator Q such that the Boltzmann equation reads

$$\begin{aligned} Q(g,h) &= \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \Phi\left(|v - v_*|\right) b\left(\cos\theta\right) \left[h' g'_* - h g_*\right] dv_* d\sigma \\ &= Q^+(g,h) - Q^-(g,h), \end{aligned}$$

where we used the following definitions

$$\begin{aligned} Q^+(g,h) &= \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \Phi\left(|v - v_*|\right) b\left(\cos\theta\right) h' g'_* \, dv_* d\sigma, \\ Q^-(g,h) &= n_b\left(\Phi * g(v)\right) h = L[g](v) h, \end{aligned} \tag{2.2.8}$$

where

$$n_b = \int_{\mathbb{S}^{d-1}} b\left(\cos\theta\right) d\sigma = \left|\mathbb{S}^{d-2}\right| \int_0^\pi b\left(\cos\theta\right) \sin^{d-2}\theta \, d\theta. \tag{2.2.9}$$

In Section 2.3 we prove that we are able to construct the characteristics $(X_t(x,v), V_t(x,v))$, for all (t,x,v) in $\mathbb{R}^+ \times \bar{\Omega} \times \mathbb{R}^d$, of the transport equation (Proposition (2.3.8)). Thanks to this Proposition we can define a mild solution of the Boltzmann equation in the cutoff case. This weaker form of solutions is actually the key point for our result and also gives a more general statement.

Definition 2.2.2 *Let f_0 be a measurable function, non-negative almost everywhere on $\bar{\Omega}\times\mathbb{R}^d$.*
A measurable function $f = f(t,x,v)$ on $[0,T)\times\bar{\Omega}\times\mathbb{R}^d$ is a mild solution of the Boltzmann equation associated to the initial datum $f_0(x,v)$ if

1. *f is non-negative on $\bar{\Omega}\times\mathbb{R}^d$,*
2. *for every (x,v) in $\Omega\times\mathbb{R}^d$:*
$$t\longmapsto L[f(t,X_t(x,v),\cdot)](V_t(x,v)),\ t\longmapsto Q^+[f(t,X_t(x,v),\cdot),f(t,X_t(x,v),\cdot)](V_t(x,v))$$
are in $L^1_{loc}([0,T))$,
3. *and for each $t\in[0,T)$, for all $x\in\Omega$ and $v\in\mathbb{R}^d$,*
$$\begin{aligned} f(t,X_t(x,v),V_t(x,v)) &= f_0(x,v)\mathrm{exp}\left[-\int_0^t L[f(s,X_s(x,v),\cdot)](V_s(x,v))\,ds\right] \\ &+\int_0^t \exp\left(-\int_s^t L[f(s',X_{s'}(x,v),\cdot)](V_{s'}(x,v))\,ds'\right) \\ &\qquad Q^+[f(s,X_s(x,v),\cdot),f(s,X_s(x,v),\cdot)](V_s(x,v))\,ds. \end{aligned} \tag{2.2.10}$$

Now we state our result.

Theorem 2.2.3 *Let Ω be $\mathbb{T}^d$ or a C^2 open convex bounded domain in $\mathbb{R}^d$ and let f_0 be a non-negative continuous function on $\bar{\Omega}\times\mathbb{R}^d$. Let $B=\Phi b$ be a collision kernel satisfying (2.1.3), with Φ satisfying (2.1.4) or (2.1.5) and b satisfying (2.1.6) with $\nu<0$. Let $f(t,x,v)$ be a mild solution of the Boltzmann equation in $\bar{\Omega}\times\mathbb{R}^d$ on some time interval $[0,T)$, $T\in(0,+\infty]$, which satisfies*

- *f is continuous on $[0,T)\times\bar{\Omega}\times\mathbb{R}^d$, $f(0,x,v)=f_0(x,v)$,$M>0$ and $E<\infty$ in (2.2.4);*
- *if Φ satisfies (2.1.4) with $\gamma\geqslant 0$ or if Φ satisfies (2.1.5), then f satisfies (2.2.1);*
- *if Φ satisfies (2.1.4) with $\gamma<0$, then f satisfies (2.2.1) and (2.2.2).*

Then for all $\tau\in(0,T)$ there exists $\rho>0$ and $\theta>0$, depending on τ, E_f (and $L^{p_\gamma}_f$ if Φ satisfies (2.1.4) with $\gamma<0$), such that for all $t\in[\tau,T)$ the solution f is bounded from below, almost everywhere, by a global Maxwellian distribution with density ρ and temperature θ, i.e.

$$\forall t\in[\tau,T),\ \forall (x,v)\in\bar{\Omega}\times\mathbb{R}^d,\quad f(t,x,v)\geqslant\frac{\rho}{(2\pi\theta)^{d/2}}e^{-\frac{|v|^2}{2\theta}}.$$

If we add the assumptions of uniform boundedness of f_0 and of the mass and entropy of the solution f we can use the arguments originated in [95] to construct explicitely the initial "upheaval point", without any compactness argument (see Section 2.4.2). Moreover, if we further suppose that Ω is C^3 and strictly convex, the use of tools developed by Guo [57] yields a constructive method to control grazing collisions (see Remark 2.6.3). We thus have the following corollary.

Corollary 1 *Suppose that conditions of Theorem 2.2.3 are satisfied (the continuity assumption on f_0 can be dropped) and further assume that Ω is C^3 and strictly convex, i.e. there exists $\xi : \mathbb{R}^d \longrightarrow \mathbb{R}$ to be C^3 such that*

$$\Omega = \{x \in \mathbb{R}^d, \quad \xi(x) < 0\}$$

and such that $\nabla\xi \neq 0$ on $\partial\Omega$ and there exists $C_\xi > 0$ such that

$$\partial_{ij}\xi(x)v_iv_j \geqslant C_\xi \left\|v\right\|^2$$

for all x in $\bar{\Omega}$ and all v in $\mathbb{R}^d$. Further assume that f_0 is uniformly bounded from below

$$\forall (x,v) \in \Omega \times \mathbb{R}^d, \quad f_0(x,v) \geqslant \varphi(v) > 0,$$

and that f has a bounded local mass and entropy

$$\begin{aligned} R_f &= \inf_{(t,x)\in[0,T)\times\Omega} \int_{\mathbb{R}^d} f(t,x,v)\,dv > 0 \\ H_f &= \sup_{(t,x)\in[0,T)\times\Omega} \left| \int_{\mathbb{R}^d} f(t,x,v)\log f(t,x,v)\,dv \right| < +\infty. \end{aligned}$$

Then conclusion of Theorem 2.2.3 holds true with the constants ρ and θ being explicitely constructed in terms of τ, E_f, H_f, $L^{p_\gamma}_f$ and upper and lower bounds on $|\nabla\xi|$ and $\left|\nabla^2\xi\right|$on $\partial\Omega$.

As stated in Subsection 2.1.2, the main result to reach Theorem 2.2.3 is the construction of an immediate lower bound only depending on the norm of the velocity:

Proposition 2.2.4 *Let f be the mild solution of the Boltzmann equation described in Theorem 2.2.3.*
For all $0 < \tau < T$ there exists r_V, $a_0(\tau) > 0$ such that

$$\forall t \in [\tau/2, \tau],\ \forall (x,v) \in \bar{\Omega} \times \mathbb{R}^d, \quad f(t,x,v) \geqslant a_0(\tau)\mathbf{1}_{B(0,r_V)}(v),$$

r_V and $a_0(\tau)$ only depending on τ, E_f (and $L^{p_\gamma}_f$ if Φ satisfies (2.1.4) with $\gamma < 0$).

2.2.4 Exponential lower bound for non-cutoff collision kernels

In the case of non-cutoff collision kernels ($0 \leqslant \nu < 2$ in (2.1.6)), Grad's splitting does not make sense anymore and so we have to find a new way to define mild solutions to the Boltzmann equation (2.1.1). The splitting we are going to use is a standard one and it reads

$$\begin{aligned} Q(g,h) &= \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \Phi\left(|v-v_*|\right) b\left(\cos\theta\right)\left[h'g'_* - hg_*\right] dv_* d\sigma \\ &= Q^1_b(g,h) - Q^2_b(g,h), \end{aligned}$$

where we used the following definitions

$$\begin{aligned} Q^1_b(g,h) &= \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \Phi\left(|v-v_*|\right) b\left(\cos\theta\right) g'_*\left(h'-h\right)\, dv_* d\sigma, \\ Q^2_b(g,h) &= -\left(\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \Phi\left(|v-v_*|\right) b\left(\cos\theta\right)\left[g'_* - g_*\right]\, dv_* d\sigma\right) h \\ &= S[g](v)h. \end{aligned} \tag{2.2.11}$$

We would like to use the properties we derived in the study of collision kernels with cutoff. Therefore we will consider additional splitting of Q.

For ε in $(0, \pi/4)$ we define a cutoff angular collision kernel

$$b^{CO}_\varepsilon\left(\cos\theta\right) = b\left(\cos\theta\right)\mathbf{1}_{|\theta|\geqslant\varepsilon}$$

and a non-cutoff one

$$b^{NCO}_\varepsilon\left(\cos\theta\right) = b\left(\cos\theta\right)\mathbf{1}_{|\theta|\leqslant\varepsilon}.$$

Considering the two collision kernels $B^{CO}_\varepsilon = \Phi b^{CO}_\varepsilon$ and $B^{NCO}_\varepsilon = \Phi b^{NCO}_\varepsilon$, we can combine Grad's splitting (2.2.8) applied to B^{CO}_ε with the non-cutoff splitting (2.2.11) applied to B^{NCO}_ε. This yields the splitting we shall use to deal with non-cutoff collision kernels,

$$Q = Q^+_\varepsilon - Q^-_\varepsilon + Q^1_\varepsilon - Q^2_\varepsilon, \tag{2.2.12}$$

where we use the shortened notations $Q^\pm_\varepsilon = Q^\pm_{b^{CO}_\varepsilon}$ and $Q^i_\varepsilon = Q^i_{b^{NCO}_\varepsilon}$, for $i = 1, 2$.

Thanks to the splitting (2.2.12) and the study of characteristics mentionned in Section 2.2.2, we are able to define mild solutions to the Boltzmann equation with non-cutoff collision kernels. This is obtained by considering the Duhamel formula associated to the splitting (2.2.12) along the characteristics (as in the cutoff case).

Definition 2.2.5 *Let f_0 be a measurable function, non-negative almost everywhere on $\bar{\Omega} \times \mathbb{R}^d$.*
A measurable function $f = f(t, x, v)$ on $[0, T) \times \bar{\Omega} \times \mathbb{R}^d$ is a mild solution of the Boltzmann equation with non-cutoff angular collision kernel associated to the initial datum $f_0(x, v)$ if there exists $0 < \varepsilon_0 < \pi/4$ such that for all $0 < \varepsilon < \varepsilon_0$:

1. *f is non-negative on $\bar{\Omega} \times \mathbb{R}^d$,*
2. *for every (x, v) in $\Omega \times \mathbb{R}^d$:*

$$t \longmapsto L_\varepsilon[f(t, X_t(x,v), \cdot)](V_t(x,v)), t \longmapsto Q^+_\varepsilon[f(t, X_t(x,v), \cdot), f(t, X_t(x,v), \cdot)](V_t(x,v))$$

$$t \longmapsto S_\varepsilon[f(t, X_t(x,v), \cdot)](V_t(x,v)), t \longmapsto Q^1_\varepsilon[f(t, X_t(x,v), \cdot), f(t, X_t(x,v), \cdot)](V_t(x,v))$$

   are in $L^1_{loc}([0, T))$,

3. *and for each $t \in [0, T)$, for all $x \in \Omega$ and $v \in \mathbb{R}^d$,*

$$\begin{aligned} f(t, X_t(x,v), V_t(x,v)) = f_0(x,v) \exp\left[- \int_0^t (L_\varepsilon + S_\varepsilon)\left[f(s, X_s(x,v), \cdot)\right](V_s(x,v))\, ds \right] \\ + \int_0^t \exp\left(- \int_s^t (L_\varepsilon + S_\varepsilon)\left[f(s', X_{s'}(x,v), \cdot)\right](V_{s'}(x,v))\, ds' \right) \\ \left(Q^+_\varepsilon + Q^1_\varepsilon\right)\left[f(s, X_s(x,v), \cdot), f(s, X_s(x,v), \cdot)\right](V_s(x,v))\, ds. \end{aligned} \tag{2.2.13}$$

Now we state our result.

Theorem 2.2.6 *Let Ω be $\mathbb{T}^d$ or a C^2 open convex bounded domain in $\mathbb{R}^d$ and f_0 be a continuous function on $\bar{\Omega} \times \mathbb{R}^d$. Let $B = \Phi b$ be a collision kernel satisfying (2.1.3), with Φ satisfying (2.1.4) or (2.1.5) and b satisfying (2.1.6) with ν in $[0, 2)$. Let $f(t, x, v)$ be a mild solution of the Boltzmann equation in $\bar{\Omega} \times \mathbb{R}^d$ on some time interval $[0, T)$, $T \in (0, +\infty]$, which satisfies*

- *f is continuous on $[0, T) \times \bar{\Omega} \times \mathbb{R}^d$ and $f(0, x, v) = f_0(x, v)$,$M > 0$ and $E < \infty$ in (2.2.4);*
- *if Φ satisfies (2.1.4) with $\gamma \geqslant 0$ or if Φ satisfies (2.1.5), then f satisfies (2.2.1) and (2.2.3);*
- *if Φ satisfies (2.1.4) with $\gamma < 0$, then f satisfies (2.2.1), (2.2.2) and (2.2.3).*

Then for all $\tau \in (0,T)$ and for any exponent K such that

$$K > 2\frac{\log\left(2+\frac{2\nu}{2-\nu}\right)}{\log 2},$$

there exists $C_1, C_2 > 0$, depending on τ, K, E_f, E'_f, W_f (and $L^{p_\gamma}_f$ if Φ satisfies (2.1.4) with $\gamma < 0$), such that

$$\forall t \in [\tau, T),\ \forall (x,v) \in \bar{\Omega} \times \mathbb{R}^d, \quad f(t,x,v) \geqslant C_1 e^{-C_2|v|^K}.$$

Moreover, in the case $\nu = 0$, one can take $K = 2$ (Maxwellian lower bound).

We emphasize here that, in the same spirit as in the cutoff case, the main part of the proof will rely on the establishment of an equivalent to Proposition 2.2.4 for non-cutoff collision kernels.

Corollary 2 *As for Corollary 1, if if f_0 is bounded uniformly from below as well as the local mass of f, the local entropy of f is uniformly bounded from above and Ω is C^3 and strictly convex then the conclusion of Theorem 2.2.6 holds true with constants being explicitely constructed in terms of τ, K, E_f, E'_f, W_f, H_f, $L^{p_\gamma}_f$ and upper and lower bounds on $|\nabla\xi|$ and $\left|\nabla^2\xi\right|$ on $\partial\Omega$.*

Remark 2.2.7 *Throughout the chapter, we are going to deal with the case where Ω is a C^2 convex bounded domain since it is the case where the most important difficulties arise. However, if $\Omega = \mathbb{T}^d$, we can follow the same proofs by letting the first time of collision with the boundary to be $+\infty$ (see Section 2.3) and by making the definition that the distance to the boundary (which does not exist) is $+\infty$ (which rules out the case of grazing trajectories).*

2.3 The free transport equation: proof of Theorem 2.2.1

In this section, we study the transport equation with a given initial data and boundary condition in a bounded domain Ω. We will only consider the case of purely specular reflections on the boundary $\partial\Omega$. Those kind of interaction cannot occur for all velocities at the boundary. Indeed, for a particle to bounce back at the boundary, we need its velocity to come from inside the domain Ω. To express this fact mathematically, we define

$$\Lambda^+ = \left\{(x,v) \in \partial\Omega \times \mathbb{R}^d :\ v \cdot n(x) \geqslant 0\right\},$$

where we denote by $n(x)$ the exterior normal to $\partial\Omega$ at x.

Consider $u_0 : \bar{\Omega} \times \mathbb{R}^d \longrightarrow \mathbb{R}$ which is C^1 in $x \in \Omega$ and $L^2(\bar{\Omega} \times \mathbb{R}^d) = L^2_{x,v}$. We are interested in the problem stated in Theorem 2.2.1, (2.2.5) – (2.2.7).

If $D_x(v)(u)$ denotes the directional derivate of u in x in the direction of v we have, in the case of functions that are C^1 in x,

$$D_x(v)(u) = v \cdot \nabla_x u.$$

Therefore, instead of imposing that the solution to the transport equation should be C^1 in x, we reformulate the problem with directional derivatives.

Physically, the free transport equation means that a particle evolves freely in Ω at a velocity v until it reaches the boundary. Then it bounces back and moves straight until it reaches the boundary for the second time and so on so forth up to time t. The method of characteristics is therefore the best way to link $u(t,x,v)$ to u_0 by just following the path used by the particle, backwards from t to 0 (see Figure 2.1). This method has been used

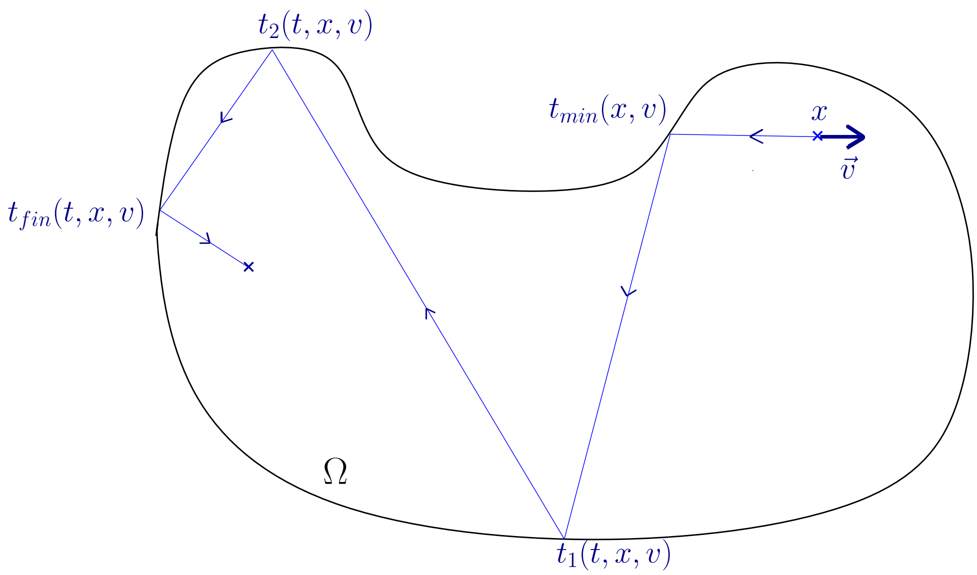

Figure 2.1: Backward trajectory with standard rebounds

in [52] on the half-line and in [33], [59], for instance, in the case of convex media. However, in both articles they only deal with finite, or countably many, numbers of rebounds in finite time. Indeed, the electrical field in [52] and [59] makes the particles always reach the boundary with $v \cdot n(x) > 0$ and [33] has a specular boundary problem with an absorption coefficient $\alpha \in [0,1)$: $u(t,x,v) = \alpha u(t,x,\mathcal{R}_x(v))$. Therefore, in the case the particle arrives tangentially to the boundary, i.e. $v \cdot n(x) = 0$, we have $\mathcal{R}_x(v) = v$ and so $u(t,x,v) = 0$.

This vanishing property allowed the authors to not care about the special cases where the particle starts to roll on the boundary.

Another way of looking at the characteristics method is to study the footprints of the trajectories on the boundary. This problem, as well as the possibility of having infinitely many rebounds in a finite time, has been tackled by Tabachnikov in [104]. Tabachnikov only focused on boundary points since the description of the trajectories by only considering their collisions with the boundary holds a symplectic property and a volume-preserving transformation. Such properties allowed him to show that the set of points on the boundary that lead to infinitely many rebounds in finite time is of measure 0 ([104], Lemma $1.7, 1$). Unfortunately, in our case we would like to follow the characteristics and the study of trajectories only via their footprints on the boundary is no longer a volume-preserving transformation.

In our case we need to follow the path of a particle along the characteristics of the equation to know the value of our function at each step. If the particle starts to roll on the boundary (see Figure 2.2) we require to know for how long it will do so. The major issue is the fact that $v \cdot n(x) = 0$ does not tell us much about the geometry of $\partial\Omega$ at x and the possibility, or lack of, for the particle to keep moving tangentially to the boundary. Moreover, some cases lead to non physical behaviour since the sole specular collision condition implies that some pairs $(x, v) \in \partial\Omega \times \mathbb{R}^d$ can only be starting points, they cannot be generated by any trajectories (see Figure 2.3). This case is mentioned quickly in the first chapter of [105] but not dealt with.

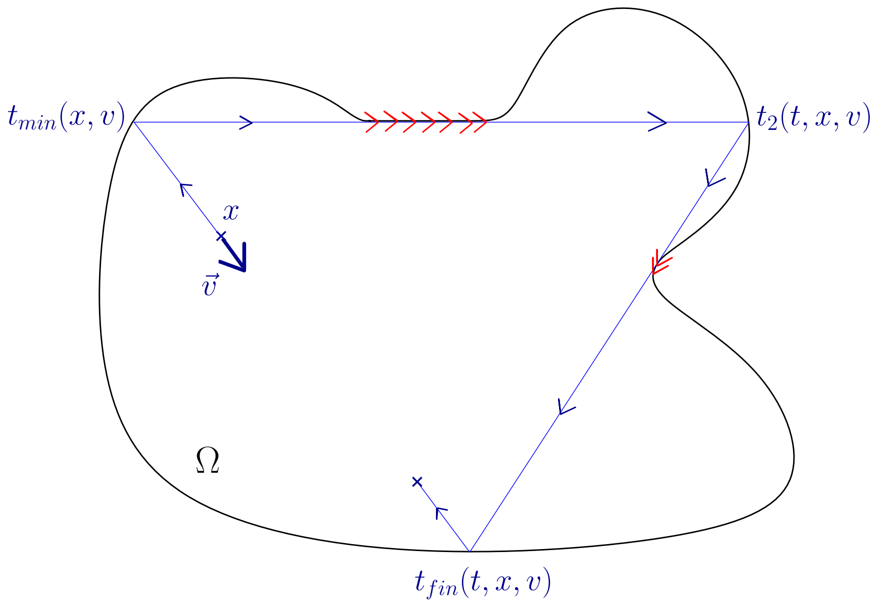

Figure 2.2: Backward trajectory rolling on the boundary

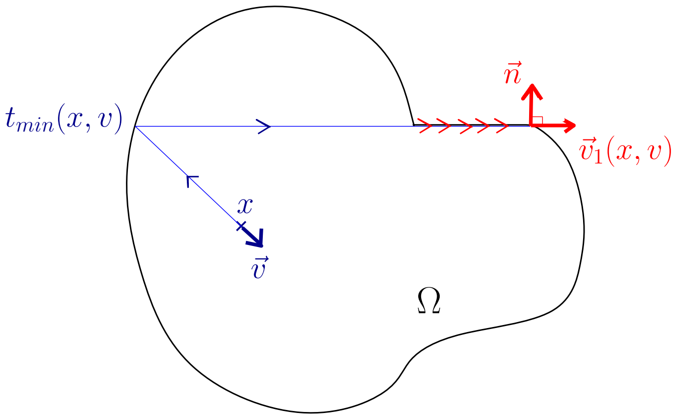

Figure 2.3: Backward trajectory that reaches an end

Therefore, in order to prove the well-posedness of the transport equation (2.2.5) − (2.2.7), we follow the ideas developed in [52] and [59], which consist of studying the backward trajectories that can lead to a point (t, x, v), combined with the idea of countably many collisions in finite time used in [33]. However, we have to deal with the issues described above and to do so we introduce a new classification of possible interactions with the boundary (see Definition 2.3.1). We also extend the result of [104], in terms of pair (x, v) leading to infinitely many rebounds in finite time, to the whole domain Ω (Proposition 2.3.4). To do so we link up the study on the boundary made in [104] with the Lebesgue measure on Ω by artificially creating volume on $\partial\Omega$ thanks to time and a foliation of the domain by parallel trajectories.

The section is divided as follows. First of all we shall describe and classify the collisions with the boundary in order to describe very accurately the backward trajectories of a point (x, v) in $\partial\Omega \times \mathbb{R}^d$. We will name trajectory or characteristic any solution $(X(t, x, v), V(t, x, v))$ satisfying the initial condition $(X(0, x, v), V(0, x, v)) = (x, v)$, the boundary condition (2.2.7) and satisfying, in Ω,

$$\begin{cases} \dfrac{dX}{dt} &= V \\ \dfrac{dV}{dt} &= 0. \end{cases}$$

This will give us an explicit form for the characteristics and allow us to link $u(t, x, v)$ with $u_0(x^*, v^*)$, for some x^* and v^*. Finally, we will show that the function we constructed is, indeed, a solution to the transport equation with initial data u_0 and specular boundary condition and that such a solution is unique.

2.3.1 Study of rebounds on the boundary

As mentionned in the introduction of this section, when a particle reaches a point at the boundary with a velocity v it can bounce back (Figure 2.1), keep moving straight (Figure 2.2) or stop moving because the specular reflection does not allow it to do anything else (Figure 2.3), which is physically unexpected. The next definition gives a partition of the points at the boundary which takes into account those properties.

Definition 2.3.1 *We define here a partition of $\partial\Omega \times \mathbb{R}^d$ that focuses on the outcome of a collision in each of the sets.*

- *The set coming from a rebound without rolling*

$$\Omega_{rebounds} = \left\{(x,v) \in \partial\Omega \times \mathbb{R}^d : \ v \cdot n(x) < 0\right\}.$$

- *The set coming from rolling on the boundary*

$$\Omega_{rolling} = \left\{(x,v) \in \partial\Omega \times \mathbb{R}^d : \ v \cdot n(x) = 0 \text{ and } \exists \delta > 0, \forall t \in [0,\delta], \ x - vt \in \bar{\Omega}\right\}.$$

- *The set of only starting points*

$$\Omega_{stop} = \left\{(x,v) \in \partial\Omega \times \mathbb{R}^d : \ v \cdot n(x) = 0 \text{ and } \forall \delta > 0, \exists t \in [0,\delta], \ x - vt \notin \bar{\Omega}\right\}.$$

- *The set coming from straight line*

$$\Omega_{line} = \left\{(x,v) \in \partial\Omega \times \mathbb{R}^d : \ v \cdot n(x) > 0\right\}.$$

One has to notice that any point of Ω_{line} indeed comes from a straight line arriving at x with direction v since Ω is open and is C^1 (so there is no cusp).
In order to understand the behaviour expected at Ω_{stop} we have the following proposition. The proof of it gives insight into the nature of specular reflections.

Proposition 2.3.2 *If we have (x,v) in Ω_{stop} then there is no trajectory with specular boundary reflections that leads to (x,v).*

Proof of Proposition 2.3.2 Let us assume the contrary, that is to say (x,v) is in Ω_{stop} comes from a trajectory with specular boundary reflection.

We have that (x,v) belongs to $\partial\Omega \times \mathbb{R}^d$ and so if (x,v) comes from a straight line it can only be (by definition of trajectories) a line containing x with direction v which means that (x, v) comes from $\{(x - vt, v),\ t \in [0, T]\}$, for some $T > 0$. But the trajectory is necessarily in $\bar{\Omega}$ and this is in contradiction with the definition of Ω_{stop}.

Therefore, (x, v) must come from a rebound after a straight line trajectory. But again we obtain a contradiction because the velocity before the rebound is $\mathcal{R}_x(v) = v$ and the backward trajectory is the one studied above. ■

Now we have our partition of points on the boundary of Ω, we are able to generate the backward trajectory associated to a starting point (x, v) in $\bar{\Omega} \times \mathbb{R}^d$. The first step towards its resolution is to find the first point of real collision (if it exists) that generates (x, v) (see Figure 2.1). The next proposition-definition proves mathematically what the figure shows.

Proposition 2.3.3 *Let Ω be an open, bounded and C^1 domain in $\mathbb{R}^d$. Let (x, v) be in $\bar{\Omega} \times \mathbb{R}^d$, then we can define*

$$t_{min}(x, v) = \max\left\{t \geqslant 0 : \ x - vs \in \bar{\Omega},\ \forall\, 0 \leqslant s \leqslant t\right\}.$$

Moreover we have the following properties:

1. *if there exists t in $(0, t_{min}(x, v))$ such that $x - vt$ hits $\partial\Omega$ then $(x - vt, v)$ belongs to $\Omega_{rolling}$.*

2. *$t_{min}(x, v) = 0$ if and only if (x, v) belongs to $\Omega_{stop} \cup \Omega_{rebounds}$.*

3. *$(x - vt_{min}(x, v), v)$ belongs to $\Omega_{stop} \cup \Omega_{rebounds}$.*

Property (1) emphasises the fact that if, on the straight line between x and $x - vt_{min}(x, v)$, the particle hits the boundary it will not be reflected and so just rolls on. Then property (2) tells us than $t_{min(x,v)}$ is always strictly positive except if (x, v) does not come from any trajectory of a particle or if it is the outcome of a rebound without rolling. Finally, property (3) finishes the study since at $x - vt_{min}(x, v)$ the particles either come from a reflection (case $\Omega_{rebounds}$), and we can keep tracking backwards, or started its trajectory at $x - vt_{min}(x, v)$ (case Ω_{stop}).

Proop of Proposition 2.3.3 First of all we have that Ω is bounded and so there exists R such that $\bar{\Omega} \subset B(0, R)$, the ball of radius R in $\mathbb{R}^d$.

Then we notice that 0 belongs to

$$A(x, v) = \left\{t \geqslant 0 : \ x - vs \in \bar{\Omega},\ \forall\, 0 \leqslant s \leqslant t\right\}.$$

Therefore $A(x, v)$ is not empty. Moreover, this set is bounded above by $2R/\|v\|$ since for all t in $A(x, v)$

$$R > \|x - vt\| \geqslant t\,\|v\| - \|x\|\,.$$

Therefore we can talk about the supremum $t_{min}(x, v)$ of $A(x, v)$. Let $(t_n)_{n\in\mathbb{N}}$ be increasing sequence in $A(x, v)$ that tends to $t_{min}(x, v)$. As $\bar{\Omega}$ is closed we have that $x - vt_{min}(x, v)$ belongs to $\bar{\Omega}$. Then, if $0 \leqslant s < t_{min}(x, v)$ there exists n such that $0 \leqslant s \leqslant t_n$ and so, by the property of t_n, $x - vs$ is in $\bar{\Omega}$. This conclude the fact that $t_{min}(x, v)$ belongs to $A(x, v)$ and so is a maximum.

We now turn to the proof of properties.

Let (x, v) be in $\bar{\Omega}$ and $0 < t < t_{min}(x, v)$ such that $x - vt$ belongs to $\partial\Omega$. Then for all $0 < t_1 < t < t_2 < t_{min}(x, v)$, $x - vt_1$ and $x - vt_2$ are in $\bar{\Omega}$ and so, by the definition of an exterior normal to a surface we have

$$[(x - vt) - (x - vt_1)] \cdot n(x - vt) \geqslant 0 \text{ and } [(x - vt) - (x - vt_2)] \cdot n(x - vt) \geqslant 0,$$

which gives $v \cdot n(x - vt) = 0$.
Moreover, since t_2 belongs to $A(x, v)$, for all s in $[0, t_2 - t]$, $(x - vt) - vs$ is in $\bar{\Omega}$, which means that $(x - vt, v)$ belongs to $\Omega_{rolling}$.

Property (2) is direct since if $t_{min}(x, v) = 0$ then for all $t > 0$, there exists $0 < s \leqslant t$ such that $x - vs$ does not belong to $\bar{\Omega}$ and then $v \cdot n(x) \leqslant 0$. So (x, v) belongs to $\Omega_{rebounds}$, if $v \cdot n(x) > 0$, or to Ω_{stop}.

Finally, property (3) is straightforward since $x - vt_{min}(x, v)$ is in $\partial\Omega$ (because Ω is open) and since for all $0 \leqslant t \leqslant t_{min}(x, v)$, $x - vt$ is in $\bar{\Omega}$. Thus $[(x - vt_{min}) - (x - vt)] \cdot n(x - vt_{min}(x, v)) \geqslant 0$, which yields $v \cdot n(x - vt_{min}(x, v)) \leqslant 0$.Then, by the definition of $A(x, v)$ and the fact that $t_{min}(x, v)$ is its maximum, we have that either $(x - vt_{min}(x, v), v)$ belongs to $\Omega_{rebounds}$ or belongs to Ω_{stop}. ■

Up to now we focused solely on the case of the first possible collision with the boundary. In order to conclude the study of rebounds for any given characteristics we have to, in some sense, count the number of rebounds without rolling that can happen in finite time. This is the purpose of the next proposition.

Proposition 2.3.4 *Let Ω be a C^1 open, bounded domain in $\mathbb{R}^d$ and let (x, v) be in $\bar{\Omega} \times \mathbb{R}^d$. Then for all $t \geqslant 0$ the trajectory finishing at (x, v) after a time t has at most a countable number of rebounds without rolling.*

Moreover, this number is finite almost surely with respect to the Lebesgue measure on $\bar{\Omega} \times \mathbb{R}^d$

Proof of Proposition 2.3.4 The fact that there is countably many rebounds without rolling comes directly from the fact that $t_{min}(x,v) > 0$ except if (x,v) is a starting/stopping point (and then did not move from 0 to t) or if (x,v) is the outcome of a rebound (and so comes from $(x, \mathcal{R}_x(v))$ which belongs to Ω_{line}, implying that $t_{min}(x, \mathcal{R}_x(v)) > 0$).

Now we shall prove that the set of points in $\bar{\Omega} \times \mathbb{R}^d$ which lead to an infinite number of rebounds in a finite time is of measure 0. To do so, we first need some definitions. The measure μ in $\bar{\Omega} \times \mathbb{R}^d$ is the one induced by the Lebesgue measure and we denote by λ the measure on $\partial\Omega \times \mathbb{R}^d$ (see Section 1.7 of [104]).
We will also denote

$$\begin{aligned} \Omega &= \left\{(x,v) \in \Omega \times \left(\mathbb{R}^d - \{0\}\right) \text{ coming from an infinite number of rebounds}\right\}, \\ \Omega_\partial &= \left\{(x,v) \in \partial\Omega \times \left(\mathbb{R}^d - \{0\}\right) \text{ coming from an infinite number of rebounds}\right\}. \end{aligned}$$

We know ([104] Lemma 1.7.1) that $\lambda(\Omega_\partial) = 0$ and we are going to establish a link between the measure of Ω and the one of Ω_∂. Those two sets do not live in the same topology nor same dimension and so we build a function that artificially recreates them via time.

Because Ω is bounded we can find time $T_M > 0$ such that for all x in $\bar{\Omega}$ and v in $\mathbb{R}^d - \{0\}$, $(x - T_M v/\left\|v\right\|)$ does not belong to $\bar{\Omega}$. Furthermore, in the same way as for $t_{min}(x,v)$, we can define, for (x,v) in $\bar{\Omega} \times \mathbb{R}^d$,

$$T(x,v) = \begin{cases} \min\{t > 0 : \; x + vt \in \partial\Omega\} \text{ if } (x,v) \in \Omega \cup \Omega_{rebounds} \\ 0 \text{ otherwise} \end{cases}.$$

We define the following function which is clearly C^1.

$$\begin{array}{rrcl} F: & [0,T_M] \times \mathbb{R}^d \times \left(\mathbb{R}^d - \{0\}\right) & \longrightarrow & \mathbb{R}^d \times \left(\mathbb{R}^d - \{0\}\right) \\ & (t,x,v) & \longmapsto & (x + \frac{v}{\|v\|}t, v). \end{array}$$

We also define the set

$$B = \left\{(t,x,v) : \; x \in \partial\Omega, \; v \in (\mathbb{R}^d - \{0\}), \; t \in [0, T(x,v))\right\}.$$

and claim that F is injective on the set B. Indeed, if (t,x,v) and (t^*,x^*,v^*) are in B such that $F(t,x,v) = F(t^*,x^*,v^*)$ then $v = v^*$ and $x + tv/\left\|v\right\| = x^* + t^*v/\left\|v\right\|$.
Let assume that $t^* > t$, therefore we have that

$$x = x^* + (t^* - t)\frac{v}{\|v\|} \in \partial\Omega$$

and thus $t^* - t \geqslant T(x^*, v)$. However, $t^* \leqslant T(x^*, v)$ so we reach a contradiction and $t^* \leqslant t$. By symmetry we have $t = t^*$ and then $x = x^*$. We also notice that $[0, T_M] \times \Omega_{stop}$ and $[0, T_M] \times \Omega_{rolling}$ do not intersect B.

Finally we have that $\Omega = F\left(B \cap ([0, T_M] \times \Omega_\partial)\right)$. Indeed, if (t, x, v) belongs to $B \cap ([0, T_M] \times \Omega_\partial)$ then $F(t, x, v) = (x + tv/\left\|v\right\|, v)$ and $x + tv/\left\|v\right\|$ is in Ω and its first rebound backward in time is (x, v) which lead to infinitely many rebounds in finite time. Therefore

$$x + t\frac{v}{\|v\|} \in \Omega.$$

The converse is direct, by considering the first collision with the boundary of the backward trajectory starting at (x, v) in Ω.

All those properties allow us to compute $\mu(\Omega)$ by a change of variable in $B \cap \Omega_\partial$.

$$\begin{aligned}
\mu(\Omega) &= \mu(F\left(B \cap ([0, T_M] \times \Omega_\partial)\right)) \\
&= \int_{\bar{\Omega} \times \mathbb{R}^d} \mathbf{1}_{F(B \cap ([0, T_M] \times \Omega_\partial))}(x, v)\,dxdvdt \\
&= \int_{B \cap ([0, T_M] \times \Omega_\partial)} \left|\mathrm{Jac}(F^{-1})\right| d\lambda(x, v)dt \\
&\leqslant T_M \sup_{[0, T_M] \times \bar{\Omega}} \left(\left|\mathrm{Jac}(F^{-1})\right|\right) \lambda(\Omega_\partial) = 0.
\end{aligned}$$

■

2.3.2 Description of characteristics

In the previous section we derived all the relevant properties of when, where and how a trajectory can bounce against the boundary of Ω. As was shown, the characteristic starting from a point (t, x, v) in $\mathbb{R}^+ \times \bar{\Omega} \times \mathbb{R}^d$ is the backward trajectory satisfying specular boundary reflections that leads to (x, v) in time t. Basically, it consists in a straight line as long as it stays inside Ω or it rolls on the boundary. Then it reaches a boundary point where it does not move any more (Ω_{stop}) or bounces back ($\Omega_{rebounds}$).

Thanks to Proposition 2.3.4 we can generate the countable (and almost surely finite) sequence of collisions with the boundary associated to the future point (x, v). We shall construct it by induction. We consider (x, v) in $\bar{\Omega} \times \mathbb{R}^d$.

- **Step 1: initialisation:** we define

$$\begin{cases} x_0(x,v) &= x, \\ v_0(x,v) &= v, \\ t_0(x,v) &= 0. \end{cases}$$

- **Step 2: induction:** if $(x_k(x,v), v_k(x,v)) \in \Omega_{stop}$ then we define

$$\begin{cases} x_{k+1}(x,v) &= x_k(x,v), \\ v_{k+1}(x,v) &= v_k(x,v), \\ t_{k+1}(x,v) &= +\infty, \end{cases}$$

  if $(x_k(x,v), v_k(x,v)) \notin \Omega_{stop}$ then we define

$$\begin{cases} x_{k+1}(x,v) &= x_k(x,v) - v_k(x,v) t_{min}(x_k(x,v), v_k(x,v)), \\ v_{k+1}(x,v) &= \mathcal{R}_{x_{k+1}(x,v)}(v_k(x,v)), \\ t_{k+1}(x,v) &= t_k(x,v) + t_{min}(x_k(x,v), v_k(x,v)). \end{cases}$$

Remark 2.3.5 *Let us make a few comments on the accuracy of the sequence we just built.*

1. *Looking at Proposition 2.3.3, we know that at each step (apart from* 0*) we necessary have that* $(x_k(x,v), v_k(x,v))$ *belongs to either* Ω_{stop} *or* $\Omega_{rebounds}$ *and so the characteristic stops for ever (case* 1 *in induction) or bounces without rolling and start another straight line (case* 2*). Thus the sequence of footprints defined above captures the trajectories as long as there are rebounds and then becomes constant once the trajectory reach a stopping point.*

2. *If* $t_{min}(x_k(x,v), v_k(x,v)) = 0$ *for some* $k > 0$ *then, by properties* $2.$ *and* $3.$ *of Proposition 2.3.3, we must have* $(x_k(x,v), v_k(x,v)) \in \Omega_{stop}$ *(since* $v_k(x,v)$ *is the specular reflection at* $x_k(x,v)$ *of* $v_{k-1}(x,v)$ *and* $(x_k(x,v), v_{k-1}(x,v))$ *is in* $\Omega_{rebounds} \cup \Omega_{stop}$*). Thus,* $(t_k(x,v))_{k\in\mathbb{N}}$ *is strictly increasing as long as it does not reach the value* $+\infty$*, where it remains constant.*

Finally, it remains to connect the time variable to those quantities. In fact, the time will determine how many rebounds can lead to (x,v) in a time t. The reader must remember that the backward trajectory can lead to a point in Ω_{stop} before time t.

Since the characteristics method helps us to find the value of the solution of the transport equation at a given point using its trajectory, the next definition links a triplet (t,x,v)

to the first rebound of the trajectory that leads to (x, v) in a time t.

Definition 2.3.6 *Let Ω be an open, bounded and C^1 domain in $\mathbb{R}^d$.*
Let (t, x, v) be in $\mathbb{R}^+ \times \bar{\Omega} \times \mathbb{R}^d$. Then we can define

$$n(t,x,v) = \begin{cases} \max\{k \in \mathbb{N} : \ t_k(x,v) \leqslant t\}, \ \textit{if it exists}, \\ +\infty, \ \textit{if } (t_k(x,v))_k \textit{ is bounded by } t. \end{cases}$$

The last rebound is then define by

- *if $n(t,x,v) < +\infty$ and $t_{n(t,x,v)+1} = +\infty$, then*

$$\begin{cases} x_{fin}(t,x,v) = x_{n(t,x,v)}(x,v), \\ v_{fin}(t,x,v) = v_{n(t,x,v)}(x,v), \\ t_{fin}(t,x,v) = t, \end{cases}$$

- *if $n(t,x,v) < +\infty$ and $t_{n(t,x,v)+1} < +\infty$, then*

$$\begin{cases} x_{fin}(t,x,v) = x_{n(t,x,v)}(x,v), \\ v_{fin}(t,x,v) = v_{n(t,x,v)}(x,v), \\ t_{fin}(t,x,v) = t_{n(t,x,v)}(x,v), \end{cases}$$

- *if $n(t,x,v) = +\infty$, then*

$$\begin{cases} x_{fin}(t,x,v) = \lim\limits_{k\to+\infty} x_k(x,v), \\ v_{fin}(t,x,v) = \lim\limits_{k\to+\infty} v_k(x,v), \\ t_{fin}(t,x,v) = \lim\limits_{k\to+\infty} t_k(x,v). \end{cases}$$

Remark 2.3.7 *Let us make a few comments on the definition above and the existence of limits.*

1. *After the last rebound, occuring at $t_{n(t,x,v)}$, the backward trajectory can only be a straight line during the time period $t - t_{n(t,x,v)}$ (see Figure 2.1). That is why we defined $t_{fin}(t,x,v) = t_{n(t,x,v)}$ if we reached a point on $\Omega_{rebounds}$ and $t_{fin}(t,x,v) = t$ if the last rebound reaches Ω_{stop} (the trajectory can only start from there).*

2. *In the last case of the definition, we remind the reader that $(t_k(x,v))_{k\in\mathbb{N}}$ is strictly increasing and so converges if bounded by t. But then, because $(\|v_k(x,v)\|)_{k\in\mathbb{N}}$ is constant and $x_k(x,v) = x_{k-1}(x,v) - t_{min}(x_k(x,v), v_k(x,v))v_k(x,v)$, we have that $(x_k(x,v))_{k\in\mathbb{N}}$ is a Cauchy sequence.*

3. *The last case in Definition 2.3.6 almost surely never happens, as proved in Proposition 2.3.4.*

To conclude this study of the characteristics we just have to make one more comment. We studied the characteristics that go backward in time because it simplifies the construction of a solution to the free transport equation. However, it is easy to prove (just requires the inductive construction of v_k and x_k) that the forward trajectory of (x,v) during a period t is the backward trajectory over a period t of $(x,-v)$. This gives the final proposition.

Proposition 2.3.8 *Let Ω be an open, bounded and C^1 domain in $\mathbb{R}^d$. Then for all (x,v) in $\bar{\Omega}\times\mathbb{R}^d$ we have existence and uniqueness of the characteristic $(X_t(x,v), V_t(x,v))$ given by, for all $t \geqslant 0$,*

$$\begin{aligned} X_t(x,v) &= x_{fin}(t,x,-v) + (t - t_{fin}(t,x,-v))v_{fin}(t,x,-v), \\ V_t(x,v) &= -v_{fin}(t,x,-v). \end{aligned}$$

Moreover, we have that $V_t(x,v) = O_{t,x,v}(v)$ with $O_{t,x,v}$ an orthogonal transformation, and that for almost every (x,v) in $\bar{\Omega}\times\mathbb{R}^d$ we have the following

$$\forall t \geqslant 0, \quad (x,v) = (X_t(X_t(x,-v), -V_t(x,-v)), V_t(X_t(x,-v), -V_t(x,-v))). \tag{2.3.1}$$

Proof of Proposition 2.3.8 By construction we have that

$$O_{t,x,v} = \mathcal{R}_{x_{fin}(t,x,v)} \circ \cdots \circ \mathcal{R}_{x_1(t,x,v)}.$$

It only remains to show the last equation (2.3.1), but it follows directly from the fact that the backward trajectory of (x,v) is the forward trajectory of $(x,-v)$.

We can reach a point on Ω_{stop} after a time t_1 and so the forward trajectory of that point during a time $t > t_1$ does not come back to the original point (since we stayed in Ω_{stop} for a period $t - t_1$).

However, the set of points that reach Ω_{stop} belongs to the set of points that bounce infinitely many times in a finite time and this set is of measure zero (see Proposition 2.3.4).
■

2.3.3 Existence and uniqueness of solution to (2.2.5) – (2.2.7)

2.3.3.1 Proof of uniqueness

The uniqueness of a solution with u_0 in $C^1_x \cap L^2_{x,v}$ comes directly from the fact that we have a preserved quantity through time, thanks to the specular reflection property. Indeed, let us assume that u is a solution to our free transport equation satisfying specular boundary condition and the initial value problem u_0. Then, a mere integration by part gives us

$$\forall t \geqslant 0, \quad \|u(t,\cdot,\cdot)\|^2_{L^2_{x,v}} = \|u_0\|^2_{L^2_{x,v}},$$

which directly implies the uniqueness of a solution, since the transport equation (2.2.5) is linear.

2.3.3.2 Construction of the solution

It remains to construct a function u that will be constant on the characteristic trajectories and check that we indeed obtain a function that is differentiable in t and x which satisfies the transport equation. The first point of Remark 2.3.7 gives us the answer as we expect the following behaviour

$$u(t,x,v) = u(t - t_1(x,v), x_1(x,v), v_1(x,v)) = \dots = u(t - t_k(x,v), x_k(x,v), v_k(x,v)),$$

up to the point where there are no more rebound in the time interval $[0,t]$. From there we continue in a straight line.

Thus, we define: $\forall (t,x,v) \in \mathbb{R}^+ \times \bar{\Omega} \times \mathbb{R}^d$,

$$u(t,x,v) = u_0\left(x_{fin}(t,x,v) - (t - t_{fin}(t,x,v))v_{fin}(t,x,v), v_{fin}(t,x,v)\right).$$

2.3.3.3 Boundary and initial conditions

First of all, u satisfies the initial condition (2.2.6) as $n(0,x,v) = 0$ (since $t_{min}(x,v) \geqslant 0$).

u also satifies the specular boundary condition (2.2.7). Indeed, if (x,v) is in Λ^+, then either $v \cdot n(x) = 0$ and the result is obvious since $\mathcal{R}_x(v) = v$, or $v \cdot n(x) > 0$ and thus $(x, \mathcal{R}_x(v))$ belongs to $\Omega_{rebounds}$ so $t_{min}(x, \mathcal{R}_x(v)) = 0$ (Proposition 2.3.3). An easy induction shows

$$x_k(x,v) = x_{k+1}(x, \mathcal{R}_x(v)),\ v_k(x,v) = v_{k+1}(x, \mathcal{R}_x(v)),\ t_k(x,v) = t_{k+1}(x, \mathcal{R}_x(v)),$$

for all k in $\mathbb{N}$.
The last equality gives us that $n(t,x,v) = n(t,x,\mathcal{R}_x(v)) - 1$ and therefore, combined with the two other equalities,

$$x_{fin}(t,x,v) = x_{fin}(t,x,\mathcal{R}_x(v)),\ v_{fin}(t,x,v) = v_{fin}(t,x,\mathcal{R}_x(v)),$$

$$t_{fin}(t,x,v) = t_{fin}(t,x,\mathcal{R}_x(v)),$$

which leads to the specular reflection boundary condition.

2.3.3.4 Time differentiability

Here we prove that u is differentiable in time on $\mathbb{R}^+$. Let us fix (x,v) in $\Omega \times \mathbb{R}^d$.
By construction, we know that $n(t,x,v)$ is piecewise constant. Since $(t_k(x,v))_{k\in\mathbb{N}}$ is strictly increasing up to the step where it takes the value $+\infty$, for $t_k(x,v) < t < t_{k+1}(x,v)$ we have that for all $s \in \mathbb{R}$ such that $t_k(x,v) < t+s < t_{k+1}(x,v)$,

$$x_{fin}(t,x,v) = x_{fin}(t+s,x,v),\ v_{fin}(t,x,v) = v_{fin}(t+s,x,v),$$

$$t_{fin}(t,x,v) = t_{fin}(t+s,x,v).$$

Therefore, we have that

$$\begin{aligned}
&\frac{u(t+s,x,v) - u(t,x,v)}{s} \\
&= \frac{u_0(x_{fin} - (t+s-t_{fin})v_{fin}, v_{fin}) - u_0(x_{fin} - (t-t_{fin})v_{fin}, v_{fin})}{s} \\
&\underset{s\to 0}{\to} -v_{fin} \cdot (\nabla_x u_0)\,(x_{fin} - (t-t_{fin})v_{fin}, v_{fin}),
\end{aligned}$$

because u_0 is C^1 in x. So u is differentiable at t if t in strictly between two times $t_k(x,v)$. We thus find that u is differentiable at t and that its derivative is continuous (since x_{fin}, v_{fin} and t_{fin} are continuous when x and v are fixed).

In the case $t = t_k(x,v)$ we can use what we just proved to show that we have the existence of right (except for $t=0$) and left limits of $\partial_t u(t,x,v)$ as t tends to $t_k(x,v)$. We use the specular reflection boundary condition of u_0 together with the fact that it is C^1 in x and that $t_k(x,v) = t_{k+1}(x,\mathcal{R}_x(v))$ to obtain the equality of the two limits.

2.3.3.5 Space differentiability and solvability of the transport equation

Here we prove that u is differentiable in x in Ω, which follows directly from the time differentiability. Let us fix t in $\mathbb{R}^+$ and v in $\mathbb{R}^d$, we shall study the differentiability of $u(t,\cdot,v)$ in the direction of v.

Ω is open and so

$$\forall x \in \Omega,\ \exists \delta > 0,\ \forall s \in [-\delta, \delta], \quad x + sv \in \Omega.$$

Thanks to the inductive construction, one find easily that

$$u(t, x + sv, v) = u(t - s, x, v).$$

Therefore, since u is time differentiable, we have that $u(t, \cdot, v)$ admits a directional derivative in the direction of v and that

$$\mathrm{D}_x(v)(u)(t, x, v) = -\partial_t u(t, x, v).$$

2.4 The cutoff case: localized "upheaval points"

In this section and the next three we are going to prove a Maxwellian lower bound for a solution to the Boltzmann equation (2.1.1) in the case where the collision kernel satisfies a cutoff property.

The strategy to tackle this result follows the main idea used in [95] and [78] which relies on finding an "upheaval point" (a first minoration uniform in time and space but localised in velocity) and spreading this bound, thanks to the spreading property of the Q^+ operator, in order to include larger and larger velocities.

We gather here two lemmas, proven in [78], that we will frequently use in this section. We remind the reader that we are using Grad's splitting (2.2.8). Let us first give an L^∞ bound on the loss term (Corollary 2.2 in [78]).

Lemma 2.4.1 *Let g be a measurable function on $\mathbb{R}^d$. Then*

$$\forall v \in \mathbb{R}^d, \quad |L[g](v)| \leqslant C_g^L \langle v \rangle^{\gamma^+},$$

where C_g^L is defined by:

1. *If Φ satisfies (2.1.4) with $\gamma \geqslant 0$ or if Φ satisfies (2.1.5), then*

$$C_g^L = \mathrm{cst}\, n_b C_\Phi e_g.$$

2. *If Φ satisfies (2.1.4) with $\gamma \in (-d, 0)$, then*

$$C_g^L = \mathrm{cst}\, n_b C_\Phi \left[e_g + l_g^p\right], \quad p > d/(d+\gamma).$$

The spreading property of Q^+ is given by the following lemma (Lemma 2.4 in [78]), where we define

$$l_b = \inf_{\pi/4 \leqslant \theta \leqslant 3\pi/4} b\,(\cos\theta)\,. \tag{2.4.1}$$

Lemma 2.4.2 *Let $B = \Phi b$ be a collision kernel satisfying (2.1.3), with Φ satisfying (2.1.4) or (2.1.5) and b satisfying (2.1.6) with $\nu \leqslant 0$. Then for any $\bar{v} \in \mathbb{R}^d$, $0 < r \leqslant R$, $\xi \in (0,1)$, we have*

$$Q^+(\mathbf{1}_{B(\bar{v},R)}, \mathbf{1}_{B(\bar{v},r)}) \geqslant \text{cst}\; l_b c_\Phi r^{d-3} R^{3+\gamma} \xi^{\frac{d}{2}-1} \mathbf{1}_{B\left(\bar{v},\sqrt{r^2+R^2}(1-\xi)\right)}.$$

As a consequence in the particular quadratic case $\delta = r = R$, we obtain

$$Q^+(\mathbf{1}_{B(\bar{v},\delta)}, \mathbf{1}_{B(\bar{v},\delta)}) \geqslant \text{cst}\; l_b c_\Phi \delta^{d+\gamma} \xi^{\frac{d}{2}-1} \mathbf{1}_{B\left(\bar{v},\delta\sqrt{2}(1-\xi)\right)},$$

for any $\bar{v} \in \mathbb{R}^d$ and $\xi \in (0,1)$.

The case of the torus, studied in [78], indicates that without rebounding the expected minoration is created after time $t = 0$ as quickly as one wants. Therefore we expect the same kind of bound to arise on each characteristic trajectory before its first rebound. However, in the case of a bounded domain, rebounds against the boundary can occur very close to the time $t = 0$ and a rebound preserves only the norm of the velocity. Therefore, we will fail finding a uniformly (in space) small time where a uniform bound arises. Nevertheless, the convexity and the smoothness of the domain implies that grazing collisions against the boundary do not change the velocity very much.

Thus our study will be split in three parts, which are the next three sections. The first step will be to partition the position and velocity spaces so that we have an immediate appearance of an "upheaval point" in each of those partitions. The second one is to obtain a uniform lower bound which will depend only on the norm of the velocity. Then the final part will use the standard spreading method used in [95] and [78] which will allow us to deal with large velocities and derive the exponential lower bound uniformly.

2.4.1 Partition of the phase space and first localised lower bounds

In this section we use the continuity of f together with the conservation laws (2.2.4) to obtain a point in the phase space where f is strictly positive. Then, thanks to the continuity of f, its Duhamel representation (2.2.10) and the spreading property of the Q^+ operator (Lemma 2.4.2) we extend this positivity to high velocities at that particular point (Lemma 2.4.3). Finally, the free transport part of the solution f will imply the immediate appearance of the localised lower bounds (Proposition 2.4.4).

Moreover we define constants that we will use in the next two subsections in order to have a uniform lower bound.

We define some shorthand notations. For x in $\bar{\Omega}$, v in $\mathbb{R}^d$ and $s,t \geqslant 0$ we denote the point at time s of the forward characteristic passing through (x,v) at time t by

$$\begin{aligned} X_{s,t}(x,v) &= X_s(X_t(x,-v),-V_t(x,-v)) \\ V_{s,t}(x,v) &= V_s(X_t(x,-v),-V_t(x,-v)), \end{aligned}$$

which has been derived from (2.3.1).

We start by the strict positivity of our function at one point for all velocities:

Lemma 2.4.3 *Let f be the mild solution of the Boltzmann equation described in Theorem 2.2.3.*
Then there exists (x_1,v_1) in $\Omega \times \mathbb{R}^d$ and $\Delta > 0$ such that for all $n \in \mathbb{N}$ and all t in $[0,\Delta]$, there exists $r_n > 0$, depending only on n, and $\alpha_n(t) > 0$ such that

$$\forall x \in B\left(x_1, \frac{\Delta}{2^n}\right), \; \forall v \in \mathbb{R}^d, \quad f(t,x,v) \geqslant \alpha_n(t)\mathbf{1}_{B(v_1,r_n)}(v),$$

with $\alpha_0 > 0$ independent of t and the induction formula

$$\alpha_{n+1}(t) = C_Q \frac{r_n^{d+\gamma}}{4^{d/2-1}} \int_0^{\min\left(t,\Delta/(2^{n+1}(2r_n+\|v_1\|)\right)} e^{-sC_L\langle 2r_n+\|v_1\|\rangle^{\gamma^+}} \alpha_n^2(s)\,ds$$

where $C_Q = cstl_b c_\Phi$ is defined in Lemma 2.4.2 and $C_L = cstn_b C_\Phi E_f$ (or $C_L = cstn_b C_\Phi (E_f + L^p_f)$) is defined in Lemma 2.4.1, and

$$r_0 = \Delta, \quad r_{n+1} = \frac{3\sqrt{2}}{4} r_n.$$

Proof of Lemma 2.4.3 The proof is an induction on n.

Step 1: Initialization. We recall the conservation laws satisfied by a solution to the Boltzmann equation, (2.2.4),

$$\forall t \in \mathbb{R}^+, \quad \int_\Omega \int_{\mathbb{R}^d} f(t,x,v)\,dxdv = M, \quad \int_\Omega \int_{\mathbb{R}^d} |v|^2 f(t,x,v)\,dxdv = E,$$

with $M > 0$ and $E < \infty$.
Since Ω is bounded, and so is included in, say, $B(0,R_X)$, we also have that

$$\forall t \in \mathbb{R}^+, \quad \int_\Omega \int_{\mathbb{R}^d} \left(|x|^2 + |v|^2\right) f(t,x,v)\,dxdv \leqslant \alpha = MR_X^2 + E < +\infty.$$

Therefore if we take $t=0$ and $R_{min}=\sqrt{2\alpha/M}$, we have the following

$$\int_{B(0,R_{min})}\int_{B(0,R_{min})} f_0(x,v)\,dxdv \geqslant \frac{M}{2} > 0.$$

Therefore we have that there exists x_1 in Ω and v_1 in $B(0,R_{min})$ such that

$$f_0(x_1,v_1) \geqslant \frac{\mathrm{M}}{4\mathrm{Vol}(B(0,R_{min}))^2} > 0.$$

The first step of the induction is then due to the continuity of f at $(0,x_1,v_1)$. Indeed, there exists $\delta_T,\delta_X,\delta_V>0$ such that

$$\forall t\in[0,\delta_T],\ \forall x\in B(x_1,\delta_X),\ \forall v\in B(v_1,\delta_V),\quad f(t,x,v)\geqslant \frac{\mathrm{M}}{8\mathrm{Vol}(B(0,R_{min}))^2}.$$

and we define $\Delta=\min(\delta_T,\delta_X,\delta_V)$.

Step 2: Proof of the induction. We assume the conjecture is valid for n. Let x be in $B(x_1,\Delta/2^{n+1})$, v in $B(0,\|v_1\|+2r_n)$ and t in $[0,\Delta]$.

We use the fact that f is a mild solution to write $f(t,x,v)$ under its Duhamel form (2.2.10). The control we have on the L operator, Lemma 2.4.1, allows us to bound from above the second integral term (the first term is positive). Moreover, this bound on L is independent on t, x and v since it only depends on an upper bound on the energy $e_{f(t,x,\cdot)}$ (and its local L^p norm $l^p_{f(t,x,\cdot)}$) which is uniformly bounded by E_f (and by L^p_f). This yields, for $\tau_n(t)=\min\left(t,\Delta/(2^{n+1}(2r_n+\|v_1\|))\right)$

$$f(t,x,v)\geqslant\int_0^{\tau_n(t)} e^{-sC_L\langle\|v_1\|+2r_n\rangle^{\gamma^+}}Q^+\left[f(s,X_{s,t}(x,v),\cdot),f(s,X_{s,t}(x,v),\cdot)\right](V_{s,t}(x,v))\,ds, \tag{2.4.2}$$

where $C_L=cst\,n_bC_\Phi E_f$ (or $C_L=cst\,n_bC_\Phi(E_f+L^p_f)$), see Lemma 2.4.1, and we used $\|V_{s,t}(x,v)\|=\|v\|\leqslant 2r_n+\|v_1\|$.

Besides, we have that $B(x_1,\Delta)\subset\Omega$ and also

$$\forall s\in\left[0,\frac{\Delta}{2^{n+1}(2r_n+\|v_1\|)}\right],\ \forall v_*\in B(0,\|v_1\|+2r_n),\quad \|x_1-(x+sv_*)\|\leqslant\frac{\Delta}{2^n}$$

which, by definition of the characteristics (see Section 2.3.2), yields

$$\forall s\in[0,\tau_n(t)],\ \forall v_*\in B(0,\|v_1\|+2r_n),\quad \begin{cases} X_{s,t}(x,v_*) &= x+sv_*\in B\left(x_1,\dfrac{\Delta}{2^n}\right)\\ V_{s,t}(x,v_*) &= v_*. \end{cases}$$

Therefore, by calling v_* the integration parametre in the operator Q^+ we can apply the induction property to $f(s, X_{s,t}(x,v), v_*)$ which implies, in (2.4.2),

$$f(t,x,v) \geqslant \int_0^{\tau_n(t)} e^{-sC_L\langle \|v_1\|+2r_n\rangle^{\gamma^+}} \alpha_n^2(s) Q^+\left[\mathbf{1}_{B(v_1,r_n)}, \mathbf{1}_{B(v_1,r_n)}\right] ds(v).$$

Applying the spreading property of Q^+, Lemma 2.4.2, with $\xi = 1/4$ gives us the expected result for the step $n+1$ since $B(v_1, r_{n+1}) \subset B(0, \|v_1\| + 2r_n)$. ■

We now have all the tools to prove the next proposition which is the immediate appearance of localised "upheaval points".

Proposition 2.4.4 *Let f be the mild solution of the Boltzmann equation described in Theorem 2.2.3.*
Then there exists $\Delta > 0$ such that for all $0 < \tau_0 \leqslant \Delta$, there exists $\delta_T(\tau_0)$, $\delta_X(\tau_0)$, $\delta_V(\tau_0)$, $R_{min}(\tau_0)$, $a_0(\tau_0) > 0$ such that for all N in $\mathbb{N}$ there exists N_X in $\mathbb{N}^*$ and $x_1, \dots, x_{N_X}$ in Ω and $v_1, \dots, v_{N_X}$ in $B(0, R_{min}(\tau_0))$ and

- $\bar{\Omega} \subset \bigcup_{1 \leqslant i \leqslant N_X} B\left(x_i, \delta_X(\tau_0)/2^N\right);$
- $\forall t \in [\tau_0, \delta_T(\tau_0)]$, $\forall x \in B(x_i, \delta_X(\tau_0)), \forall v \in \mathbb{R}^d$,

$$f(t,x,v) \geqslant a_0(\tau_0) \mathbf{1}_{B(v_i, \delta_V(\tau_0))}(v),$$

with $B(v_i, \delta_V(\tau_0)) \subset B(0, R_{min}(\tau_0))$.

Proof of Proposition 2.4.4 We are going to use the free transport part of the Duhamel form of f (2.2.10), to create localised lower bounds out of Lemma 2.4.3.

We take $0 < \tau_0 \leqslant \Delta$, where Δ is defined in Lemma 2.4.3.
Ω is bounded so let us denote its diameter by d_Ω. Let n be big enough such that $r_n \geqslant 2d_\Omega/\tau_0 + \|v_1\|$ and define $R_{min}(\tau_0) = 2d_\Omega/\tau_0$.
Thanks to Lemma 2.4.3 applied to this particular n we have that

$$\forall t \in \left[\frac{\tau_0}{2}, \Delta\right], \forall x \in B(x_1, \Delta/2^n), \quad f(t,x,v) \geqslant \alpha_n\left(\frac{\tau_0}{2}\right) \mathbf{1}_{B(v_1,r_n)}(v), \tag{2.4.3}$$

where we used the fact that $\alpha_n(t)$ is an increasing function.

Define

$$a_0(\tau_0) = \frac{1}{2}\alpha_n\left(\frac{\tau_0}{2}\right) e^{-\frac{\tau_0}{2}C_L\langle\frac{2d_\Omega}{\tau_0}\rangle^{\gamma^+}}.$$

We remark that f is continuous on the compact $[\tau_0, \Delta] \times \bar{\Omega} \times B(0, R_{min}(\tau_0))$ and hence uniformly continuous. Therefore it exists $\delta'_T(\tau_0)$, $\delta'_X(\tau_0)$, $\delta'_V(\tau_0) > 0$ such that

$$\forall |t - t'| \leqslant \delta'_T(\tau_0), \ \forall \left\|x - x'\right\| \leqslant \delta'_X(\tau_0), \ \forall \left\|v - v'\right\| \leqslant \delta'_V(\tau_0),$$
$$\left|f(t, x, v) - f(t', x', v')\right| \leqslant a_0(\tau_0). \tag{2.4.4}$$

We conclude our definition by taking

$$\begin{aligned} \delta_T(\tau_0) &= \min(\Delta, \tau_0 + \delta'_T(\tau_0)), \\ \delta_X(\tau_0) &= \min(\delta'_X(\tau_0), \Delta/2^n), \\ \delta_V(\tau_0) &= \min(\delta'_V(\tau_0), r_n). \end{aligned}$$

Finally, we take $N \in \mathbb{N}$ and notice that $\bar{\Omega}$ is compact so there exists $x_1, \dots, x_{N_X}$ in Ω such that $\bar{\Omega} \subset \bigcup_{1 \leqslant i \leqslant N_X} B\left(x_i, \delta_X(\tau_0)/2^N\right)$. Moreover, we construct them such that x_1 is the one defined in Lemma 2.4.3.

We then take v_1 to be the one defined in Lemma 2.4.3 and we define

$$\forall i \in \{2, \dots, N_X\}, \quad v_i = \frac{2}{\tau_0}(x_i - x_1).$$

Because Ω is convex we have that

$$\begin{aligned} X_{\tau_0/2, \tau_0}(x_i, v_i) &= x_1, \\ V_{\tau_0/2, \tau_0}(x_i, v_i) &= v_i, \end{aligned} \tag{2.4.5}$$

Using the fact that f is a mild solution of the Boltzmann equation, we write it under its Duhamel form (2.2.10) and we drop the last term which is positive. As in the proof of Lemma 2.4.3 we can control the L operator appearing in the first term in the right-hand side of (2.2.10) (corresponding to the free transport). Thus, we use the Duhamel form (2.2.10) between τ_0 and $\tau_0/2$ and we combine it with (2.4.5). This yields

$$\begin{aligned} f(\tau_0, x_i, v_i) &\geqslant f\left(\frac{\tau_0}{2}, x_1, v_i\right) e^{-\frac{\tau_0}{2} C_L \langle \frac{2}{\tau_0}(x_i - x_1)\rangle^{\gamma^+}} \\ &\geqslant \alpha_n\left(\frac{\tau_0}{2}\right) e^{-\frac{\tau_0}{2} C_L \langle \frac{2 d_\Omega}{\tau_0}\rangle^{\gamma^+}} \mathbf{1}_{B(v_1, r_n)}(v_i) \\ &\geqslant 2 a_0(\tau_0) \mathbf{1}_{B(v_1, r_n)}(v_i), \end{aligned}$$

where we used (2.4.3) for the second inequality. We see here that v_i belongs to $B(0, R_{min}(\tau_0))$ and that $B(0, R_{min}(\tau_0)) \subset B(v_1, r_n)$ and therefore

$$f(\tau_0, x_i, v_i) \geqslant 2a_0(\tau_0). \tag{2.4.6}$$

Finally, combining (2.4.6) with the uniform continuity of f, (2.4.4) we have that for all t in $[\tau_0, \delta_T(\tau_0)]$, x in $B(x_i, \delta_X(\tau_0))$ and v in $B(v_i, \delta_V(\tau_0))$,

$$f(t, x, v) \geqslant a_0(\tau_0).$$

■

Remark 2.4.5 *This last proposition tells us that localised lower bounds appear immediately, that is to say after any time $\tau_0 > 0$. The exponential lower bound we expect will appear immediately after those initial localised lower bounds, i.e. for all $\tau_1 > \tau_0$. Therefore, to shorten notation and lighten our presentation, we are going to study the case of solution to the Boltzmann equation which satisfies Proposition 2.4.4 at $\tau_0 = 0$. Then we will immediatly create the exponential lower bound after 0 and apply this result to $F(t, x, v) = f(t + \tau_0, x, v)$.*

2.4.2 A constructive approach to the initial lower bound, Corollary 1

The initial lower bounds we just derived relies on compactness arguments and their construction is therefore not explicit. However, as mentioned in Section 2.2.3, a few more assumptions on f_0 and f suffice to obatin a completely constructive approach for the "upheaval point". This method is based on a property of the iterated Q^+ operator discovered by Pulvirenty and Wennberg [95] and reformulated by Mouhot ([78] Lemma 2.3) as follows.

Lemma 2.4.6 *Let $B = \Phi b$ be a collision kernel satisfying (2.1.3), with Φ satisfying (2.1.4) or (2.1.5) and b satisfying (2.1.6) with $\nu \leqslant 0$. Let $g(v)$ be a nonnegative function on $\mathbb{R}^d$ with bounded energy e_g and entropy h_g and a mass ρ_g such that $0 < \rho_g < +\infty$. Then there exist R_0, δ_0, $\eta_0 > 0$ and $\bar{v} \in B(0, R_0)$ such that*

$$Q^+\left(Q^+\left(g\mathbf{1}_{B(0,R_0)}, g\mathbf{1}_{B(0,R_0)}\right), g\mathbf{1}_{B(0,R_0)}\right) \geqslant \eta_0 \mathbf{1}_{B(\bar{v},\delta_0)},$$

with R_0, δ_0, η_0 being constructive in terms on ρ_g, e_g and h_g.

We now suppose that $0 < \rho_{f_0} < +\infty$, $h_{f_0} < +\infty$ and that

$$\forall (x, v) \in \Omega \times \mathbb{R}^d, \quad f_0(x, v) \geqslant \varphi(v) > 0$$

and we follow the argument used in[78].

By the Duhamel definition (2.2.10) of f being a mild solution and Lemma 2.4.1 we have

$$f(t, X_t(x,v), V_t(x,v)) \geqslant f_0(x,v) e^{-tC_L \langle v \rangle^{\gamma^+}} \tag{2.4.7}$$

and

$$f(t,x,v) \geqslant \int_0^t e^{-(t-s)C_L\langle v \rangle^{\gamma^+}} Q^+ \left[f(s, X_{s,t}(x,v), \cdot), f(s, X_{s,t}(x,v), \cdot) \right] (V_{s,t}(x,v)) \; ds.$$

Define $t(x,v) > 0$ the time of first contact with $\partial\Omega$ of the trajectory $x + sv$ (see rigorous definition in Proposition 2.3.3). For all t in $[0, t(x,v)]$ we have

$$\begin{aligned} X_{0,t}(x,v) &= x + tv, \\ V_{0,t}(x,v) &= v. \end{aligned}$$

Thus, for all $0 \leqslant t \leqslant t(x,v)$,

$$f(t,x,v) \geqslant \int_0^t e^{-(t-s)C_L\langle v \rangle^{\gamma^+}} Q^+ \left[f(s, x+sv, \cdot), f(s, x+sv, \cdot) \right] (v) \; ds,$$

and we can iterate the latter inequality

$$\begin{aligned} &f(t,x,v) \geqslant \int_0^t e^{-(t-s)C_L\langle v \rangle^{\gamma^+}} \\ &Q^+ \left[\int_0^s e^{-(s-s')C_L\langle v \rangle^{\gamma^+}} Q^+ \left(f(s, x+s'v, \cdot), f(s, x+s'v, \cdot) \right)(\cdot) ds', f(t, x+sv, \cdot) \right] (v) \, ds. \end{aligned} \tag{2.4.8}$$

(2.4.7) and (2.4.8) are exactly the same bounds than the ones obtained in [78], Step 1 of proof of Proposition 3.2, and we can therefore conclude the same way with Lemma 2.4.6

$$f(t,x,v) \geqslant a_0(\tau_0) \mathbf{1}_{B(\bar{v},\delta_0)},$$

as long as v is in $B(0, R_0)$ and $0 \leqslant t \leqslant \tau_0$.

The only difference with [78] is the fact that we need τ_0 to be in $[0, t(x,v)]$, giving local lower bounds instead of a global one.

2.4.3 A lower bound depending only on the norm of the velocity: strategy of the proof of Proposition 2.2.4

As stated in the introduction, the spreading property of the bilinear operator Q^+ cannot be used (at least uniformly in time and space) when we are really close to the boundary due to the lack of control over the rebounds. However, if we have a lower bound depending only on the norm of the velocity then the latter bound will not take into account rebounds as they preserve the norm, allowing us to spread this minoration up to an exponential one.

The next two sections are dedicated to the creation of such a uniform lower bound depending solely on the norm of the velocity. In order to do so we restrain the problem without taking into account large velocities and divide the study to two cases: if the trajectory stays close to the boundary or if it does not. In both cases we will start from the localised "upheaval points" constructed in Section 2.4.1 and spread them to the point where one gets a lower bound depending only on the norm of the velocity.

The next sections tackle each of these points. We first study the case when a characteristic reaches a point far from the boundary and finally we focus on the case of grazing characteristics. We fix δ_T, δ_X, δ_V, R_{min} and a_0 to be the ones described in Proposition 2.4.4 at time $\tau_0 = 0$.

The result we will derive out of those studies is Proposition 2.2.4 and from now on, dependencies on physical observables of f (E_f and $L^{p_\gamma}_f$) will be mentionned but will not be explicitly written everytime.

2.5 The cutoff case: characteristics passing by a point far from the boundary

In this section we manage to spread the lower bounds created in Proposition 2.4.4 up to a ball in velocity centred at zero as long as the trajectory we look at reaches a point far enough from the boundary.

First, we pick N in $\mathbb{N}^*$ and cover $\bar{\Omega}$ with $\bigcup_{1\leqslant i\leqslant N_X} B(x_i, \delta_X/2^N)$ as in Proposition 2.4.4. Then for $l \geqslant 0$ we define

$$\Omega_l = \{x \in \Omega : \; d(x, \partial\Omega) \geqslant l\}, \tag{2.5.1}$$

where $d(x, \partial\Omega)$ is the distance from x to the boundary of Ω.

For any $R > 0$ we define two sequences in $\mathbb{R}^+$ by induction, for all $\tau \geqslant 0$ and $l \geqslant 0$,

$$\begin{cases} r_0 &= \delta_V \\ r_{n+1} &= \dfrac{3\sqrt{2}}{4} r_n \end{cases} \tag{2.5.2}$$

and

$$\begin{cases} a_0(l,\tau) &= a_0 \\ a_{n+1}(l,\tau) &= C_Q \dfrac{r_n^{d+\gamma}}{4^{d/2-1}} \dfrac{l}{2^{n+3}R} e^{-\tau C_L \langle R \rangle^{\gamma^+}} a_n^2 \left(\dfrac{l}{8}, \tau\right), \end{cases} \tag{2.5.3}$$

where C_Q and C_L were defined in Lemma 2.4.3.

We express the spreading of the lower bound in the following proposition.

Proposition 2.5.1 *Let f be the mild solution of the Boltzmann equation described in Theorem 2.2.3 and suppose that f satisfies Proposition 2.4.4 with $\tau_0 = 0$.*
Consider $0 < \tau \leqslant \delta_T$ and N in $\mathbb{N}$. Let $(x_i)_{i\in\{1,\dots,N_X\}}$ and $(v_i)_{i\in\{1,\dots,N_X\}}$ be given as in Proposition 2.4.4 with $\tau_0 = 0$.
Then for all n in $\{0,\dots,N\}$ we have that the following holds: for all $0 < l \leqslant \delta_X$, and $R > 0$ such that $l/R < \tau$, for all t in $[l/(2^n R), \tau]$, and for all $x \in \bar{\Omega}$ and $v \in B(0,R)$, if there exists $t_1 \in [0, t - l/(2^n R)]$ such that $X_{t_1,t}(x,v)$ belongs to $\Omega_l \cap B(x_i, \delta_X/2^n)$ then

$$f(t,x,v) \geqslant a_n(l,\tau) \mathbf{1}_{B(v_i,r_n)}(V_{t_1,t}(x,v)),$$

where (r_n) and (a_n) are defined by (2.5.2)-(2.5.3).

Proof of Proposition 2.5.1 This Proposition will be proved by induction on n.

Step 1: Initialization. The initialisation is simply Proposition 2.4.4 and the first term in the Duhamel formula (2.2.10) starting at τ.

Indeed, we use the definition of f being a mild solution to write $f(t,x,v)$ under its Duhamel form (2.2.10) starting at t_1 where both parts are positive. The control we have on the L operator, Lemma 2.4.1, allows us to bound from above the first term. Moreover, this bound on L is independent on x and v (see proof of Lemma 2.4.3). This gives

$$f(t,x,v) \geqslant e^{-(t-t_1)C_L \langle R \rangle^{\gamma^+}} f\left(t_1, X_{t_1,t}(x,v), V_{t_1,t}(x,v)\right). \tag{2.5.4}$$

Finally, Proposition 2.4.4 applied to $f(t_1, X_{t_1,t}(x,v), V_{t_1,t}(x,v))$ gives us the property for $n = 0$.

Step 2: Proof of the induction. We consider the case where the proposition is true for n.

Given $l \in (0, \delta_X]$, $t \in [l/(2^{n+1}R), \tau]$, $x \in \bar{\Omega}$ and $v \in B(0,R)$.

We suppose now that there exists $t_1 \in [0, t - l/(2^{n+1}R)]$ such that $X_{t_1,t}(x,v) \in \Omega_l \cap B(x_i, \delta_X/2^{n+1})$.

Similar to what we did in the first step of the induction, but concentrating on the second part of the Duhamel formula (2.2.10) we conclude that

$$
\begin{aligned}
&f(t,x,v) \geqslant \\
&e^{-C_L \tau \langle R \rangle^{\gamma^+}} \left(\int_{t_1 + \frac{l}{2^{n+3}R}}^{t_1 + \frac{l}{2^{n+2}R}} Q^+ \left[f(s, X_{s,t}(x,v), \cdot), f(s, X_{s,t}(x,v), \cdot) \right] \, ds \right) (V_{t_1,t}(x,v)) .
\end{aligned} \tag{2.5.5}
$$

The goal is now to apply the induction to the triplet $(s, X_{s,t}(x,v), v_*)$, where v_* is the integration parametre inside the Q^+ operator, with $\|v_*\| \leqslant R$.

One easily shows that $X_{s,t}(x,v) = X_{t_1,t}(x,v) + (s - t_1)V_{t_1,t}(x,v)$, for s in $[t_1 + \frac{l}{2^{n+3}R}, t_1 + \frac{l}{2^{n+2}R}]$, and therefore we have that

$$
\|X_{t_1,t}(x,v) - X_{s,t}(x,v)\| \leqslant \frac{l}{2^{n+2}}, \tag{2.5.6}
$$

and so that $X_{s,t}(x,v)$ belongs to $\Omega_{l - l/2^{n+2}}$.

Finally, we have to find a point on the characteristic trajectory of $(s, X_{s,t}(x,v), v_*)$ that is in $\Omega_{l'}$ for some l'. This is achieved at the time t_1 (see Figure 2.4).

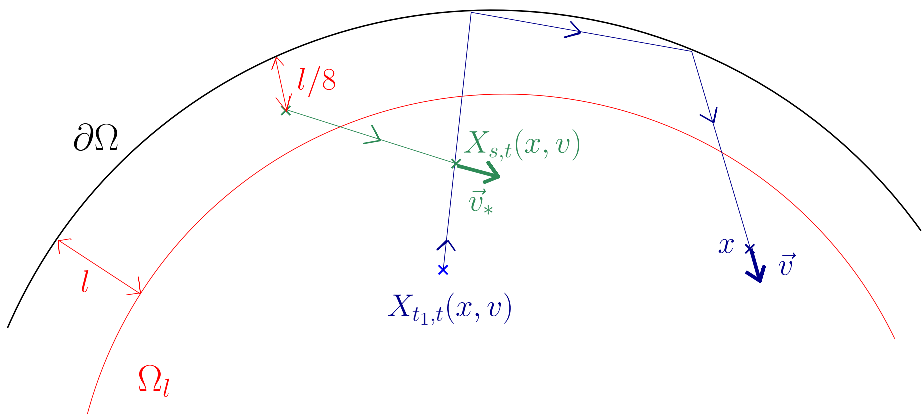

Figure 2.4: Study of $(s, X_{s,t}(x,v), v_*)$ far from the boundary

Indeed, we have s in $[t_1 + l/(2^{n+3}R), t_1 + l/(2^{n+2}R)]$ so, for $\|v_*\| \leqslant R$

$$\forall s' \in [t_1, s], \quad \left\|X_{s,t}(x,v) - \left(X_{s,t}(x,v) - (s-s')v_*\right)\right\| \leqslant \frac{l}{2^{n+2}}. \tag{2.5.7}$$

This gives us the characteristics trajectory backward starting from s, since $X_{s,t}(x,v) - (s-s')v_*$ remains in Ω, and therefore

$$\forall s' \in [t_1, s], \quad \begin{cases} X_{s',s}\left(X_{s,t}(x,v), v_*\right) & = X_{s,t}(x,v) - (s-s')v_* \\ V_{s',s}\left(X_{s,t}(x,v), v_*\right) & = v_*. \end{cases}$$

To conclude we just need to gather the upper bounds we found about the trajectories reaching $(X_{s,t}(x,v), v_*)$ in a time s in $[t_1 + l/(2^{n+3}R), t_1 + l/(2^{n+2}R)]$, equations (2.5.6) and (2.5.7)

$$\|X_{t_1,t}(x,v) - X_{t_1,s}\left(X_{s,t}(x,v), v_*\right)\| \leqslant \frac{l}{2^{n+1}}.$$

We have that $X_{t_1,t}(x,v)$ belongs to $\Omega_l \cap B(x_i, \delta_X/(2^{n+1}))$ and therefore we have that for all s in $[t_1+l/(2^{n+3}R), t_1+l/(2^{n+2}R)]$, $X_{t_1,s}\left(X_{s,t}(x,v), v_*\right)$ belongs to $\Omega_{l/2} \cap B(x_i, \delta_X/2^n)$.

Finally, if s belongs to $[t_1+l/(2^{n+3}R), t_1+l/(2^{n+2}R)]$ we have that $(l/8)/(2^n R) \leqslant s \leqslant \tau$ and t_1 is in $[0, s - (l/8)/(2^n R)]$.

We can therefore apply the induction assumption for $l' = l/8$ inside the Q^+ operator in (2.5.5), recalling that $V_{t_1,s}(X_{s,t}(x,v), v_*) = v_*$.

$$f(t,x,v) \geqslant a_n\left(\frac{l}{8}, \tau\right)^2 e^{-C_L \tau \langle R \rangle^{\gamma^+}} \left(\int_{t_1 + \frac{l}{2^{n+3}R}}^{t_1 + \frac{l}{2^{n+2}R}} Q^+\left[\mathbf{1}_{B(v_i, r_n)}, \mathbf{1}_{B(v_i, r_n)}\right] \, ds\right)(V_{t_1,t}(x,v)).$$

Applying the spreading property of Q^+, Lemma 2.4.2, with $\xi = 1/4$ gives us the expected result for the step $n+1$. $\blacksquare$

One easily notices that $(r_n)_{n \in \mathbb{N}}$ is a strictly increasing sequence. Moreover, for all N in $\mathbb{N}$ we have that for all $1 \leqslant i \leqslant N_X$, v_i belongs to $B(0, R_{min})$. Therefore, by taking N large enough (greater than N_1 say) we have that

$$\forall i \in \{1, \dots, N_X\}, \quad B(0, 2R_{min}) \subset B(v_i, r_N).$$

This remark leads directly to the following corollary which stands for Proposition 2.2.4 in the case when a point on the trajectory is far from the boundary of Ω.

Corollary 3 *Let f be the mild solution of the Boltzmann equation described in Theorem 2.2.3 and suppose that f satisfies Proposition 2.4.4 with $\tau_0 = 0$.*
Let Δ_T be in $(0, \delta_T]$ and take τ_1 in $(0, \Delta_T]$.
Then for all $0 < l \leqslant \delta_X$, there exists $a(l, \tau_1, \Delta_T) > 0$ and $0 < \tilde{t}(l, \tau_1, \Delta_T) < \tau_1$ such that for all t in $[\tau_1, \Delta_T]$, and every (x, v) in $\bar{\Omega} \times \mathbb{R}^d$: if there exists $t_1 \in [0, t - \tilde{t}(l, \tau_1, \Delta_T)]$ such that $X_{t_1,t}(x, v)$ belongs to Ω_l then

$$f(t, x, v) \geqslant a(l, \tau_1, \Delta_T) \mathbf{1}_{B(0,2R_{min})}(v).$$

Proop of Corollary 3 This is a direct consequence of Proposition 2.5.1.

Indeed, take $0 < l \leqslant \delta_X$, $0 < \tau_1 \leqslant \Delta_T$ and $R = R(\Delta_T) > 0$ such that $R \geqslant 3R_{min}$ and $l/R \leqslant \Delta_T$. Then take $N_2 \geqslant N_1$ large enough such that $l/(2^{N_2}R) < \tau_1$. We emphasize here that N_2 depends on to τ_1 so we write $N_2(\tau_1)$.

Now apply Proposition 2.5.1 with $N = N_2(\tau_1)$ and for t in $[\tau_1, \Delta_T]$. We obtain exactly Corollary 3 (since $B(0, 2R_{min}) \subset B(v_i, r_N)$ for all i and $R \geqslant 3R_{min}$) with

$$a(l, \tau_1, \Delta_T) = a_{N_2(\tau_1)}(l, \Delta_T) \quad \text{and} \quad \tilde{t}(l, \tau_1, \Delta_T) = \frac{l}{2^{N_2} R(\Delta_T)},$$

and the fact that $\bigcup_{1 \leqslant i \leqslant N_X} B\left(x_i, \delta_x/2^N\right)$ covers $\bar{\Omega}$. ■

2.6 The cutoff case: geometry and grazing trajectories

We now turn to the case when the characteristic trajectory never escapes a small distance from the boundary of our convex domain Ω.

Intuitively, by considering the case where Ω is a circle, one can see that such a behaviour is possible only when the angles of collisions with the boundary remain small (which corresponds in high dimension to the scalar product of the velocity with the outside normal being close to zero), or the angle is important but the norm of the velocity or the time of motion is small. Thus, by using the spreading property of the Q^+ operator we may be able to create larger balls in between two rebounds against the boundary because the latters should not change the velocity too much.

The study of grazing collisions will follow this intuition. First of all Section 2.6.1 proves a geometric lemma dealing with the fact that if the velocities are bounded from below and above, then for short times, the possibility for a trajectory to stay very close to the boundary implies that the velocity do not change a lot over time. Then Section 2.6.2 spreads a lower bound, in the same spirit as the last subsection, up to the point when this lower bound covers a centred ball in velocity. Notice that the geometric property forces us to work with velocities whose norm is bounded from below and so we shall have to take into account the speed of the spreading.

2.6.1 Geometric study of grazing trajectories

The key point of the study of grazing collisions is the following geometric lemma. We emphasize here that this is the only part of the chapter where we need the fact that Ω is C^2.

Proposition 2.6.1 *Let Ω be an open convex bounded C^2 domain in $\mathbb{R}^d$ and let $0 < v_m < v_M$.*
Then, for all $\varepsilon > 0$ there exists $t_\varepsilon(v_M)$ such that for all $0 < \tau_2 \leqslant t_\varepsilon(v_M)$ there exists $l_\varepsilon(v_m, \tau_2) > 0$ such that for all x in $\bar{\Omega}$ and all v in $\mathbb{R}^d$ with $v_m \leqslant \|v\| \leqslant v_M$,

$$\left(\forall s \in [0, \tau_2],\ X_s(x,v) \notin \Omega_{l_\varepsilon(v_m,\tau_2)}\right) \Longrightarrow \left(\forall s \in [0, t_\varepsilon(v_M)],\ \|V_s(x,v) - v\| \leqslant \varepsilon\right).$$

Furthermore, $l_\varepsilon(v_m, \cdot)$ is an increasing function.

The following is dedicated to the proof of Proposition 2.6.1.

We recall that for x in $\bar{\Omega}$ and v in $\mathbb{R}^d$ we define, see Section 2.3, $t_{min}(x,v)$ to be the time of the first proper rebound when we start from x with a velocity $-v$. This means that $t_{min}(x,v)$ does not take into account the case where a ball rolls on the boundary. This implies that one cannot hope to get continuity of the function t_{min} because changing the velocity slightly may lead to a proper rebound instead of a rolling movement.

This being said, we define a time of collision against the boundary which will not take into account the possibility of rolling along the boundary of Ω. This will not be too restrictive as we are considering a C^2 convex domain and therefore a trajectory that stays on the boundary will only reach a stopping point which happens only on a set of measure zero in the phase space (see Section 2.3). Therefore we define for x in $\bar{\Omega}$ and v in $\mathbb{R}^d$, the first forward contact with the boundary, $t(x,v)$. It exists by the same arguments as for t_{min}. Notice that if x is on $\partial\Omega$ then for all $v \neq 0$ we have that $t(x,v) = 0$ if and only if $n(x) \cdot v \geqslant 0$, with $n(x)$ being the outward normal to $\partial\Omega$ at the point x.

We have the following Lemma dealing with the continuity of the outward normal to $\partial\Omega$ at the first forward contact point which will be of great interest for proving the crucial Proposition 2.6.1.

Lemma 2.6.2 *Let Ω be an open convex bounded C^1 domain in $\mathbb{R}^d$.*
Then $t : (x,v) \longrightarrow t(x,v)$ is continuous from $\bar{\Omega} \times \left(\mathbb{R}^d - \{0\}\right)$ to $\mathbb{R}^+$.

Proof of Lemma 2.6.2 Let suppose that t is not continuous at (x_0, v_0) in $\Omega \times \left(\mathbb{R}^d - \{0\}\right)$. Then

$$\exists \varepsilon > 0,\ \forall N \geqslant 1,\ \exists (x_N, v_N), \quad \begin{cases} \|x_0 - x_N\| \leqslant 1/N \\ \|v_0 - v_N\| \leqslant 1/N \end{cases} \text{and} \quad |t(x_0, v_0) - t(x_N, v_N)| > \varepsilon.$$

If we still denote by d_Ω the diameter of Ω, we obviously have that for all N, $0 \leqslant t(x_N, v_N) \leqslant d_\Omega / \|v_N\|$. Thus, $(t(x_N, v_N))_{N\in\mathbb{N}}$ is a bounded sequence of $\mathbb{R}$ and we can extract a converging subsequence $\left(t(x_{\phi(N)}, v_{\phi(N)})\right)$ such that $T = \lim\limits_{N\to+\infty} t(x_{\phi(N)}, v_{\phi(N)})$.

By construction (see Section 2.3) we have that for all N in $\mathbb{N}$, $x_{\phi(N)} + t(x_{\phi(N)}, v_{\phi(N)}) v_{\phi(N)}$ belongs to $\partial\Omega$ which is closed. Moreover, this sequence converges to $x_0 + T v_0$ which therefore is on $\partial\Omega$.

Finally we have that $|t(x_0, v_0) - T| \geqslant \varepsilon$. Since $\bar{\Omega}$ is convex, the segment $[x_0, x_0 + \max(t(x_0, v_0), T) v_0]$ stays in $\bar{\Omega}$ and intersect the boundary at least at two distinct points. By convexity of the domain, this implies that the extreme points of the latter segment have to be on the boundary which means that x_0 belongs to $\partial\Omega$ which is a contradiction.

Therefore, t is continuous in $\Omega \times \left(\mathbb{R}^d - \{0\}\right)$. By the definition of $t(x, v)$ we have its continuity at the boundary. Indeed, $n(x) \cdot v \geqslant 0$ means we came from inside the domain to reach that point and we have

$$\left\| t(x', v) - t(x, v) \right\| \leqslant \frac{x - x'}{\|v\|}.$$

■

We are now ready to prove the geometric Proposition 2.6.1.

Proof of Proposition 2.6.1 Consider $\varepsilon > 0$ and $0 < v_m < v_M$.

Step 1: the case of segments. The first step is to understand that if a whole trajectory stays close to the boundary, then the angle made by the velocity with respect to the normal at the point of collision is close to $\pi/2$ for dimension $d = 2$. The same behaviour in higher dimensions is described by the scalar product of the direction of the trajectory and the normal being close to zero. One has to remember that controlling $\|V_s(x, v) - v\|$ is the same as controlling the scalar products of the trajectory and the normal on the boundary at each collision point (see definition of $V_s(x, v)$ in Section 2.3).

Let x be on $\partial\Omega$ and p in $\mathbb{N}^*$. We define

$$\Gamma_p(x) = \left\{ |n(x) \cdot v| :\ v \in \mathbb{S}^{d-1} \text{ s.t. } n(x) \cdot v < 0 \text{ and } \forall s \in [0, t(x, v)],\ x + sv \notin \Omega_{1/p} \right\},$$

with $\Omega_{1/p}$ being defined by (2.5.1).

$\Gamma_p(x)$ gives us the values of scalar products between a normal on the boundary and all the directions that create a characteristic trajectory which stays at a distance less than $1/p$ from the boundary in between two distinct rebounds (see Figure 2.5). This is exactly what we would like to control uniformly on the boundary.

We remark that $\Gamma_p(x)$ is not empty because Ω and, thus, $\Omega_{1/p}$ are convex and by the geometric theorem of Hahn-Banach we can separate $\Omega_{1/p}$ and a disjoint convex ball containing x. It is also straightforward, a mere Cauchy-Schwartz inequality, that $\Gamma_p(x)$ is bounded from above by 1. Therefore we can define, for all p in $\mathbb{N}^*$,

$$\begin{array}{rccl} h_p: & \partial\Omega & \longrightarrow & \mathbb{R}^+ \\ & x & \longmapsto & \sup \Gamma_p(x). \end{array}$$

We are going to prove that $(h_p)_{p\in\mathbb{N}^*}$ satisfies the following properties: it is a decreasing sequence of functions, h_p is continuous in x for each $p \geqslant 1$ and for all x in $\partial\Omega$ $(h_p(x))_{p\in\mathbb{N}^*}$ converges to 0.

The fact that (h_p) is decreasing is obvious.

In order to prove the continuity of h_p we take an x on the boundary and v in $\mathbb{S}^{d-1}$ such that $|n(x)\cdot v|$ is in $\Gamma_p(x)$. We have that for all s in $[0, t(x,v)]$

$$d(x+sv, \partial\Omega) < 1/p.$$

The distance to the boundary is a continuous function and $[0, t(x,v)]$ is compact so there exists $s(x,v)$ in the latter interval such that $d(x+s(x,v)v, \partial\Omega)$ is maximum. Because Ω is convex we have that $\Omega_{1/p}$ is convex and therefore

$$\forall s \in [0, t(x,v)], \quad B\left(x+sv, \frac{d(x+s(x,v)v, \Omega_{1/p})}{2}\right) \cap \Omega_{1/p} = \emptyset.$$

Then for all x' on the boundary such that $\|x-x'\| \leqslant d(x+s(x,v)v, \Omega_{1/p})/2$ we have that for all s in $[0, t(x',v)]$, $x'+sv$ is not in $\Omega_{1/p}$. Lemma 2.6.2 gives us that if x' is close to x then $t(x',v) > 0$ and thus v is not tangential at x' either. Moreover Ω is C^2 so the outward normal to the boundary is continuous and therefore for x' even closer to x we have that v is such that $|n(x')\cdot v|$ is also in $\Gamma_p(x')$. To conclude, we notice that the scalar product is continuous and therefore for all $\eta > 0$ we obtain

$$-\eta \leqslant \Big| |n(x')\cdot v| - |n(x)\cdot v| \Big| \leqslant \eta,$$

when x' is close enough to x.

The same arguments with the same constants (since our continuous functions act on compact sets and therefore are uniformly continuous) if x' is close to x then taking $|n(x') \cdot v|$ in $\Gamma_{1/p}(x')$ we have $|n(x) \cdot v|$ in $\Gamma_{1/p}(x)$ and the same inequality as above. This gives us the continuity of h_p at x. Indeed, we showed that for all x' close to x and for all element u in $\Gamma_{1/p}(x)$ we can find an element u' in $\Gamma_{1/p}(x')$ that is close to u.

Finally, it remains to show that for x on the boundary we have that $h_p(x)$ tends to 0 as p tends to $+\infty$.
One can notice that the vector $-n(x)$ is the maximum possible in $\Gamma_p(x)$ and is exactly the direction of the diametre in Ω passing by x. Hence, simple convexity arguments lead to the fact that if all the segments of the form $[x, x - t(x, -n(x))n(x)]$ intersect $\Omega_{1/p}$ then we have that for all x on the boundary, there exists $v_p(x)$ in $\mathbb{S}^{d-1}$ such that $n(x) \cdot v_p(x) = -h_p(x)$. Moreover, the segment $[x, x + t(x, v_p(x))v_p(x)]$ is tangent to $\Omega_{1/p}$ and we denote by x_p its first contact point (see Figure 2.5). The convexity of Ω and $\Omega_{1/p}$ shows that, as p increases, x_p gets closer to x and to the boundary (Ω is convex). Therefore $v_p(x)$ tends to a tangent vector of the boundary at x. This shows that

$$\lim_{p \to +\infty} h_p(x) = 0$$

in the case where all the segments of the form $[x, x - t(x, -n(x))n(x)]$ intersect $\Omega_{1/p}$.

We now come to the case where the segments of the form $[x, x - t(x, -n(x))n(x)]$ do not all intersect $\Omega_{1/p}$. If for all p, this segment does not intersect $\Omega_{1/p}$ this implies by convexity of Ω that $[x, x - t(x, -n(x))n(x)]$ is included in $\partial\Omega$. But then $-n(x)$ is not only a normal vector to the boundary at x but also a tangential one at x. Geometrically this means that x is a corner of $\partial\Omega$ and $n(x)$ is ill-defined. This is impossible for Ω being C^2. Hence, for all x on the boundary, it exists $p(x)$ such that the segment at x intersect $\Omega_{p(x)}$. However, Ω is C^2 and we also have Lemma 2.6.2. Those two facts implies that $p(x)$ is continuous on $\partial\Omega$ which is compact and therefore $p(x)$ reaches a maximum. Let us call this maximum P. For all $p \geqslant P$, all the segments of the form $[x, x - t(x, -n(x))n(x)]$, x in $\partial\Omega$, intersect Ω_P and we conclude thanks to the previous case.

Thanks to these three properties and the fact that $\partial\Omega$ is compact, we are able to use Dini's theorem. We therefore find that $(h_p)_{p \in \mathbb{N}^*}$ converges uniformly to 0. By taking p_ε large enough we have that for a segment of a characteristic trajectory joining two points on the boundary to be outside Ω_{p_ε} we must have $\Gamma_{p_\varepsilon} \leqslant \varepsilon$ for any x on the boundary (see Figure 2.5).

Step 2: more general trajectories. We take x in $\partial\Omega$ and v such that $v_m \leqslant \|v\| \leqslant v_M$ and we suppose that for a given $t > 0$

$$\forall s \in [0, t], \quad X_s(x, v) \notin \Omega_{1/p_{(\varepsilon/2N_{max})}},$$

N_{max} to be define later.

We are about to find a uniformly small time such that trajectories having at least two collisions against the boundary do not undergo an important evolution of velocity. This will be achieved thanks to the facts that $\|v\| \leqslant v_M$ and that the maximum of the scalar product is attained at a critical vector and which is the only one that needs to be controlled.

Thanks to Proposition 2.3.4, $(X_s(x,v))_s$ has countably many rebounds against the boundary (almost surely a finite number in fact). We denote by $(t_i)_{(i\in\mathbb{N})}$ the sequence of times between consecutive collisions and by $(l_i)_{i\in\mathbb{N}}$ the distance travelled during these respective times. We have that

$$\forall i \in \mathbb{N}, \quad l_i = |v|\, t_i \quad \text{and} \quad v_m t \leqslant \sum_{i\in\mathbb{N}} l_i \leqslant v_M t.$$

Therefore, for all $\eta > 0$, there exists $N_\eta(x,v)$ in $\mathbb{N}$ such that

$$\sum_{i>N_\eta(x,v)} t_i \leqslant \eta. \tag{2.6.1}$$

By continuity of $t(x,v)$, see Lemma 2.6.2, and the fact that $t(x,v) = 0$ if and only if $n(x)\cdot v \geqslant 0$, we have that for η small enough (2.6.1) yields

$$\sum_{i>N_\eta(x,v)} |n(x_i)\cdot v_i| \leqslant \varepsilon/4, \tag{2.6.2}$$

where v_i is the velocity after the i^{th} rebound and x_i is the i^{th} footprint.

$t(x,v)$ is uniformly continuous on the compact $\partial\Omega\times\{|v| = v_M\}$ (see Lemma 2.6.2) therefore the footprints of $(X_s(x,v))_{s\in[0,t]}$ are uniformly continuous and therefore there exists $\alpha_X^{(1)} > 0$ and N_{max} in $\mathbb{N}$ such that

$$\forall x, x' \in \partial\Omega \;\text{ s.t. }\; \left\|x - x'\right\| \leqslant \alpha_X,\ \forall v_m \leqslant |v| \leqslant v_M, \quad N_\eta(x,v) \leqslant N_{max} - 1. \tag{2.6.3}$$

We have now defined N_{max}.

The first property to notice is that if $(X_s(x,v))_{s\in[0,t]}$ has at least two rebounds against the boundary, then at each of them the scalar product between the incoming velocity and the outward normal is less than $\varepsilon/2N_{max}$.

Secondly, Ω is C^2 and therefore $n(x)$ is uniformly continuous on the boundary. Thus, the specular reflection operator $\mathcal{R}_x$ is uniformly continuous on $\partial\Omega\times B(0,v_M)$:

$$\exists \alpha_X^{(2)} > 0,\ \forall x, x' \in \partial\Omega \;\text{ s.t. }\; \left\|x - x'\right\| \leqslant \alpha_X, \quad \|\mathcal{R}_x - \mathcal{R}_{x'}\| \leqslant \varepsilon/4N_{max}. \tag{2.6.4}$$

We want to be sure that straight trajectories stay in our domain of uniformity so we consider

$$t \leqslant t_\varepsilon(v_M) = \max\left(\frac{\alpha_X}{v_M}, \frac{1}{p_{\varepsilon/2N_{max}} v_M}\right),$$

where $\alpha_X = \min(\alpha_X^{(1)}, \alpha_X^{(2)})$ defined in (2.6.3) and (2.6.4). To conclude, thanks to (2.6.3) and (2.6.2), if $(X_s(x,v))_{s\in[0,t]}$ collides at least twice with the boundary then

$$\forall s \in [0,t], \quad \|v - V_s(x,v)\| \leqslant 2\sum_{i\in\mathbb{N}} |n(x_i)\cdot v_i| \leqslant 2 \sum_{i\leqslant N_{max}-1} \frac{\varepsilon}{4N_{max}} + 2\frac{\varepsilon}{4} = \varepsilon.$$

Roughly speaking we do not allow the velocities near the critical direction to bounce against the wall and for the grazing ones we run them for a short time, preventing them from escaping a small neighbourhood where the collisions behave almost the same everywhere (see Figure 2.5).

To conclude our proof, it only remains to find $l \leqslant 1/p_{\varepsilon/2N_{max}}$ that prevents trajectories staying in Ω_l but go through only one rebound with a scalar product greater than $\varepsilon/2$ from happening. This is easily achieved by taking l small enough such that not a single trajectory with a scalar product greater than $\varepsilon/2N_{max}$ can stay inside Ω_l during a time τ. Indeed, one part of these trajectories will overcome a straight line of lenght at least $v_m\tau/2$ and making a scalar product greater than $\varepsilon/2N_{max}$. The distance from the boundary of the extremal point of these straight lines is therefore, by convexity, uniformly bounded from below (e.g. in dimension 2 it is bounded by $v_m\tau\varepsilon/4N_{max}$. Taking $l_\varepsilon(v_m,\tau)$ being the minimum between this lower bound and $1/p_{\varepsilon/2N_{max}}$ gives us the required distance from the boundary.

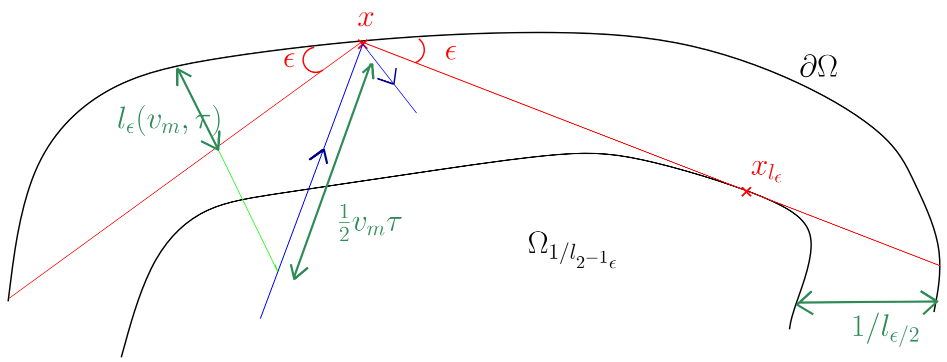

Figure 2.5: Control on grazing trajectories

■

Remark 2.6.3 *In the case of Ω is a strictly convex C^3 domain, the proof of Proposition 2.6.1 can be easily made constructive thanks to the tools developed by Guo [57].*

In that case we have the existence of $\xi : \mathbb{R}^d \longrightarrow \mathbb{R}$ to be C^3 such that

$$\Omega = \{x \in \mathbb{R}^d, \quad \xi(x) < 0\}$$

and such that $\nabla\xi \neq 0$ on $\partial\Omega$ and there exists $C_\xi > 0$ such that

$$\partial_{ij}\xi(x)v_iv_j \geqslant C_\xi \|v\|^2$$

for all x in $\bar{\Omega}$ and all v in $\mathbb{R}^d$. It allows us to define the following bounded functional along a characteristic trajectories (X_s, V_s),

$$\alpha(s) = \xi^2(X_s) + [V_s \cdot \nabla\xi(X_s)]^2 - 2\left[V_s \cdot \nabla^2\xi(X_s) \cdot V_s\right]\xi(X_s) \geqslant 0.$$

The latter functional satisfies that if X_{s_0} is on $\partial\Omega$ then

$$\alpha(s_0) = [V_{s_0} \cdot \nabla\xi(X_{s_0})]^2 = [V_{s_0} \cdot n(X_{s_0})]^2 |\nabla\xi(X_{s_0})|^2 .$$

α thus encodes the evolution of the scalar product between the velocity of the trajectory and the normal to Ω at the footprints of the characteristic. If the characteristic trajectory starts with a velocity v such that $v_m \leqslant \|v\| \leqslant v_M$, as in Proposition 2.6.1, Lemma 1 and Lemma 2 of [57] shows that in between two consecutive collision with the boundary at time s_1 and s_2 we have the existence of $C_\xi > 0$ such that

$$\begin{aligned} |s_1 - s_2| &\geqslant C_\xi \frac{\sqrt{\alpha(s_1)}}{v_M^2}, && (2.6.5)\\ e^{C_\xi(v_m+1)s_1}\alpha(s_1) &\leqslant e^{C_\xi(v_M+1)s_2}\alpha(s_2), && (2.6.6)\\ e^{-C_\xi(v_M+1)s_1}\alpha(s_1) &\geqslant e^{-C_\xi(v_m+1)s_2}\alpha(s_2). && (2.6.7) \end{aligned}$$

With (2.6.5) we can control the minimum time between two consecutive collisions with the boundary and therefore the minimum lenght of a segment between two consecutive collisions, uniformly in x and v (since $\nabla\xi$ is bounded from below on $\partial\Omega$ and non-vanishing). We therefore obtain a uniform maximum number of collisions during the given time T. Finally, (2.6.6) and (2.6.7) bounds uniformly the evolution of the scalar product between two consecutive collision and therefore the maximum evolution of $V_s(x,v)$ on the whole trajectory for a given time T. Plugging those constructive constants into the study we just made gives explicit constants in Proposition 2.6.1.

Now that we understand how grazing trajectories behave geometrically we can turn our attention to their effects combined with the spreading property of the Boltzmann Q^+

operator.

2.6.2 Spreading effect along grazing trajectories

In order to use the geometrical behaviour of grazing characteristic trajectories, one needs to consider velocities that are bounded from below. However, we would like to spread a lower bound up to ball centred at 0 where a lower bound on the norm of velocities is impossible. We shall overcome this problem using the flexibility of the spreading property of the Q^+ operator, Lemma 2.4.2, which allows us to extend the radius of the ball from 0 up to $\sqrt{2}$ times the initial radius.

The idea is to spread the initial lower bound by induction as long as the origin is strictly outside, where we are allowed to use the geometrical property of grazing characteristics. Finally, a last iteration of the spreading property, not requiring any *a priori* knowledge on characteristics, will include 0 in the lower bound.

In Corollary 3 we can fix a special time τ_1 of crossing the frontier of some Ω_l allowing us to derive a lower bound for our function in this special case. The second case of grazing trajectories is dealt with Proposition 2.6.1 where we can find an l for Ω_l to control the evolution of the velocity. Our goal now will be to find all the constants that are still free and to finally find a time of collision small enough that it will remain the same during all the iteration scheme.

We now fix all the constants that remain to be fixed in Corollary 3 thanks to Proposition 2.6.1.

Let

$$\Delta_T = \min\left(\delta_T, t_{\delta_V/4}(3R_{min})\right). \tag{2.6.8}$$

Next we define, for ξ in $(0,1)$,

$$\left\{\begin{array}{ll} r_0(\xi) & = \delta_V \\ r_{n+1}(\xi) & = \sqrt{2}(1-\xi)r_n(\xi) - \dfrac{\delta_V}{4}. \end{array}\right. \tag{2.6.9}$$

We have that $\left(r_n(1/2 - 5/(8\sqrt{2}))\right)_{n\in\mathbb{N}}$ is a strictly increasing sequence. Therefore, it exists N_{max} such that

$$r_{N_{max}}\left(\frac{1}{2} - \frac{5}{8\sqrt{2}}\right) \geqslant 2R_{min}.$$

Now we fix N in $\mathbb{N}^*$ greater than N_{max}. With this N and Proposition 2.4.4 at $\tau_0 = 0$, we construct $v_1, \dots, v_{N_X}$.

For i in $\{1,\dots,N\}$ we take $\xi^{(i)}$ in $(0, 1/4 - 5/(8\sqrt{2})]$ and we define $N_{max}(i)$ to be such that $0 \notin B\left(v_i, r_n(\xi^{(i)})\right)$ for all $n < N_{max}(i)$ and $0 \in B\left(v_i, r_{N_{max}(i)}(\xi^{(i)})\right)$. We can in fact take $\xi^{(i)}$ such that $0 \in \mathrm{Int}\left(B\left(v_i, r_{N_{max}(i)}(\xi^{(i)})\right)\right)$.

Therefore we have that for all i in $\{1, \dots, N_X\}$,

$$\delta_i = \|v_i\| - r_{N_{max}(i)-1}(\xi^{(i)}) \geqslant 0,$$

which is strictly positive if and only if $N_{max}(i) > 0$. We consider

$$v_m = \min_{i \in \{1,\dots,N_X\}} \{\delta_i;\ \delta_i > 0\}. \tag{2.6.10}$$

We can now define:

$$\forall\, 0 < \tau \leqslant \Delta_T, \qquad R(\tau) = \max\left(3R_{min}, \frac{2\delta_X}{\tau} + 1\right), \tag{2.6.11}$$

$$\tau_1(\tau) = \tau - \frac{2\delta_X}{R(\tau)} > 0, \tag{2.6.12}$$

$$\tilde{t}(\tau) = \tilde{t}(l(\tau), \tau_1(\tau), \Delta_T). \tag{2.6.13}$$

Finally, we define $l(\tau)$

$$\forall\, 0 < \tau \leqslant \Delta_T, \qquad \tau_2(\tau) = \min\left(\Delta_T, \frac{\delta_X}{R(\tau)}\right), \tag{2.6.14}$$

$$l(\tau) = \min\left(\delta_X, l_{\delta_V/4}\left(v_m, \tau_2(\tau)\right)\right). \tag{2.6.15}$$

We also build up the following sequence, where R, l and τ_1 depend on τ,

$$\begin{cases} b_0^{(i)}(\tau, \Delta_T) &= a_0 e^{-(\Delta_T - \tau) C_L \langle R \rangle^{\gamma^+}} \\ b_{n+1}^{(i)}(\tau, \Delta_T) &= \min\left(C_Q r_n^{d+\gamma} (\xi^{(i)})^{d/2-1} \dfrac{\delta_X}{2^{n+2} R} e^{-\tau C_L \langle R \rangle^{\gamma^+}} b_n^{(i)}(\tau, \Delta_T)^2; a(l, \tau_1, \Delta_T)\right) \end{cases} \tag{2.6.16}$$

$\xi^{(i)}$ was defined above and $a(l, \tau, \Delta_T)$ was defined in Corollary 3.

We are now ready to state the next Proposition which is the complement of Proposition 2.5.1 in the case when the trajectory stays close to the boundary. We remind the reader that $0 < \tilde{t}(\tau) < \tau_1(\tau)$.

Proposition 2.6.4 *Let f be the mild solution of the Boltzmann equation described in Theorem 2.2.3 and suppose that f satisfies Proposition 2.4.4 with $\tau_0 = 0$.*

Consider $0 < \tau \leqslant \Delta_T$ *and take* i *in* $\{1, \dots, N_X\}$ *such that* $N_{max}(i) > 1$.
For all n *in* $\{0, \dots, N_{max}(i) - 1\}$ *we have that for all* t *in* $[\tau - \delta_X/(2^n R(\tau)), \Delta_T]$*, all* x *in* $B(x_i, \delta_X/2^n)$ *and all* v *in* $B(0, R(\tau))$*, if*

$$\forall s \in [0, t - \tilde{t}(\tau)], \quad X_{s,t}(x,v) \notin \Omega_{l(\tau)}$$

then

$$f(t,x,v) \geqslant b_n^{(i)}(\tau, \Delta_T) \mathbf{1}_{B\left(v_i, r_n(\xi^{(i)})\right)}(v),$$

all the constants being defined in (2.6.8), (2.6.9), (2.6.15), (2.6.11), (2.6.12), (2.6.13) *and* (2.6.16).

Proof of Proposition 2.6.4 We are going to use the same kind of induction we used to prove Proposition 2.5.1. So we start by fixing i such that $N_{max}(i) > 1$.

Step 1**: Initialization**. The initialisation is simply Proposition 2.4.4 and the first term in the Duhamel formula (2.2.10) starting at τ, with the control from above on L thanks to Lemma 2.4.1.

Stef 2**: Proof of the induction**. We consider the case where the Proposition is true at $n \leqslant N_{max}(i) - 2$.
We take t in $[\tau - \delta_X/(2^{n+1}R(\tau)), \Delta_T]$, x in $B(x_i, \delta_X/2^{n+1})$ and all v in $B(0, R(\tau))$.

We suppose now that for all $s \in [0, t - \tilde{t}(\tau)]$ we have that $X_{s,t}(x,v)$ does not belongs to $\Omega_{l(\tau)}$.

To shorten notation we will skip the dependence in τ of the constant.

We use the definition of f being a mild solution to write $f(t,x,v)$ under its Duhamel form (2.2.10) where both parts are positive. As in the proof of Proposition 2.5.1, we control, uniformly on t, x and v, the L operator from above. This yields

$$f(t,x,v) \geqslant e^{-C_L \tau \langle R \rangle^{\gamma^+}} \int_{t - \frac{\delta_X}{2^{n+1}R}}^{t - \frac{\delta_X}{2^{n+2}R}} Q^+ \left[f(s, X_{s,t}(x,v), \cdot), f(s, X_{s,t}(x,v), \cdot) \right] (V_{s,t}(x,v)) \, ds, \tag{2.6.17}$$

where we used $\|V_{s,t}(x,v)\| = \|v\| \leqslant R$. We also emphasize here that this inequality holds true thanks to the definition of (2.6.11):

$$t - \frac{\delta_X}{2^{n+1}R} \geqslant \tau - \frac{\delta_X}{R} > 0.$$

The goal is now to apply the induction to the triplet $(s, X_{s,t}(x,v), v_*)$, where v^* is the integration parameter inside the Q^+ operator, with $\|v_*\| \leqslant R$.

We notice first that for all s in $[t-\delta_X/(2^{n+1}R), t-\delta_X/(2^{n+2}R)]$

$$\begin{aligned}\|x_i - X_{s,t}(x,v)\| &\leqslant \frac{\delta_X}{2^{n+1}} + \|x - X_{s,t}(x,v)\| \\ &\leqslant \frac{\delta_X}{2^{n+1}} + (t-s)R \leqslant \frac{\delta_X}{2^n},\end{aligned}$$

so that for all s in $[t-\delta_X/(2^{n+1}R), t-\delta_X/(2^{n+2}R)]$, $X_{s,t}(x,v)$ belongs to $B(x_i, \delta_X/2^n)$.

We also note that

$$\left[t - \frac{\delta_X}{2^{n+1}R}, t - \frac{\delta_X}{2^{n+2}R}\right] \subset \left[\tau - \frac{\delta_X}{2^n R}, \Delta_T\right].$$

We have two different cases to consider for $(X_{s',s}(X_{s,t}(x,v),v_*))_{s'\in[0,s-\tilde{t}]}$.
Either for some s' in $[0,s-\tilde{t}]$, $X_{s',s}(X_{s,t}(x,v),v_*)$ belongs to Ω_l and then we can apply Corollary 3:

$$\begin{aligned}f(s, X_{s,t}(x,v), v_*) &\geqslant a(l,\tau_1,\Delta_T)\mathbf{1}_{B(0,2R_{min})}(v_*) \\ &\geqslant b_n^{(i)}(\tau,\Delta_T)\mathbf{1}_{B\left(v_i, r_n(\xi^{(i)})\right)}(v),\end{aligned} \tag{2.6.18}$$

since v_i is in $B(0,R_{min})$.

Or for all s' in $[0,s-\tilde{t}] \subset [0,\tau_2]$, $X_{s',s}(X_{s,t}(x,v),v_*)$ does not belong to Ω_l and then we can apply our induction property at rank n and we reach the same lower bound (2.6.18).

Plugging (2.6.18) into (2.6.17) implies, thanks to the spreading property of Q^+, Lemma 2.4.2 with $\xi = \xi^{(i)}$,

$$\begin{aligned}&f(t,x,v) \geqslant \\ &C_Q r_n^{d+\gamma}(\xi^{(i)})^{d/2-1} e^{-\tau C_L\langle R\rangle^{\gamma^+}} (b_n^{(i)})^2 \int_{t-\frac{\delta_X}{2^{n+1}R}}^{t-\frac{\delta_X}{2^{n+2}R}} \mathbf{1}_{B(v_i,\sqrt{2}(1-\xi^{(i)})r_n(\xi^{(i)}))}(V_{s,t}(x,v))\, ds.\end{aligned} \tag{2.6.19}$$

To conclude we use the fact that for all s in $[0,t-\tilde{t}]$ we have that $X_{s,t}(x,v)$ does not belong to Ω_l and that $t-\tilde{t} > \tau_2$. Moreover, $n+1 \leqslant N_{max}(i)-1$ and so if v belongs to $B\left(v_i, r_n(\xi^{(i)})\right)$ we have that $v_m \leqslant \|v\|$. We apply Proposition 2.6.1, raising

$$\forall s \in \left[t - \frac{\delta_X}{2^{n+1}R}, t - \frac{\delta_X}{2^{n+2}R}\right], \quad \|v - V_{s,t}(x,v)\| \leqslant \frac{\delta_V}{4}.$$

Therefore, if v belongs to $B\left(v_i, r_{n+1}(\xi^{(i)})\right)$ we have that $V_{s,t}(x,v)$ belongs to $B(v_i, \sqrt{2}(1-\xi^{(i)})r_n(\xi^{(i)}))$ for all s in $[t-\delta_X/(2^{n+1}R), t-\delta_X/(2^{n+2}R)]$.

Therefore if v belongs to $B\left(v_i, r_{n+1}(\xi^{(i)})\right)$ we can compute explicitly (2.6.19) and obtain the expected induction. ■

Thanks to Proposition 2.6.4, we can build, for all x and all v, a lower bound that will contain 0 in its interior after another use of the spreading property of the Q^+ operator. The next Corollary is the complement of Corollary 3.

Corollary 4 *Let f be the mild solution of the Boltzmann equation described in Theorem 2.2.3 and suppose that f satisfies Proposition 2.4.4 with $\tau_0 = 0$.*
Let Δ_T be defined by (2.6.8).
There exists $r_V > 0$ such that for all $\tau \in (0, \Delta_T]$ there exists $b(\tau) > 0$ such that for all t in $[\tau, \Delta_T]$

If, for $\tilde{t}(\tau)$ and $l(\tau)$ being defined by (2.6.13) − (2.6.15),

$$\forall s \in [0, t-\tilde{t}(\tau)], \quad X_{s,t}(x,v) \notin \Omega_{l(\tau)}.$$

Then

$$f(t,x,v) \geqslant b(\tau)\mathbf{1}_{B(0,r_V)}(v).$$

Proof of Corollary (4) We are going to use the spreading property of Q^+ one more time. We recall that we chose $N \geqslant N_{max} \geqslant N_{max}(i)$ for all i. By definition of $N_{max}(i)$,

$$\forall i \in \{1,\dots,N_X\}, \quad 0 \in \mathrm{Int}\left(B\left(v_i, r_{N_{max}(i)}(\xi^{(i)})\right)\right).$$

We define

$$r_V = \min\left\{r_{N_{max}(i)}(\xi^{(i)}) - \|v_i\|\,;\, i \in \{1,\dots,N_X\}\right\},$$

which only depends on δ_V and $(v_i)_{i\in\{1,\dots,N_X\}}$. By construction we see that

$$\forall i \in \{1,\dots,N_X\}, B(0,r_V) \subset B\left(v_i, r_{N_{max}(i)}(\xi^{(i)})\right). \tag{2.6.20}$$

Now we take τ in $(0,\Delta_T]$ and we take t in $[\tau,\Delta_T]$, x in $B(x_i, \delta_X/2^N)$ and v in $B(0, R(\tau))$ such that

$$\forall s \in [0, t-\tilde{t}(\tau)], \quad X_{s,t}(x,v) \notin \Omega_{l(\tau)},$$

We have that t is in $[\tau - \delta_X/(2^{N_{max}(i)-1}R(\tau)), \Delta_T]$ and x in $B(x_i, \delta_X/2^{N_{max}(i)-1})$ $(N \geqslant N_{max}(i))$. By the same methods we reached (2.6.19), we obtain for $n = N_{max}(i)$

$$f(t,x,v) \geqslant \tag{2.6.21}$$
$$C_Q r_n^{d+\gamma} (\xi^{(i)})^{d/2-1} e^{-\tau C_L \langle R \rangle^{\gamma^+}} (b_n^{(i)})^2 \int_{t-\frac{\delta_X}{2^{n+1}R}}^{t-\frac{\delta_X}{2^{n+2}R}} \mathbf{1}_{B(v_i, \sqrt{2}(1-\xi^{(i)})r_n(\xi^{(i)}))} \left(V_{s,t}(x,v)\right)\, ds.$$

This time the conclusion is different because we cannot bound the velocity from below since our lower bound contains 0. However, (2.6.20) allows us to bound from below the integrand in (2.6.21) by a function depending only on the norm. Moreover, $\|v\| = \|V_{s,t}(x,v)\|$ along characteristic trajectories (see Proposition (2.3.8)). Thus we obtain the expected result by taking

$$b(\tau) = \min\left\{ b^{(i)}_{N_{max}(i)};\ i \in \{1, \dots, N_X\} \right\}.$$

■

2.7 Maxwellian lower bound in the cutoff case: proof of Theorem 2.2.3

This section gathers all the results we proved above and proves the main Theorem in the case of a cut-off collision kernel.

2.7.0.1 Proof of Proposition (2.2.4)

By combining Corollary 3 and Corollary 4 we can deal with any kind of characteristic trajectory. This is expressed by the following lemma.

Lemma 2.7.1 *Let f be the mild solution of the Boltzmann equation described in Theorem 2.2.3 and suppose that f satisfies Proposition 2.4.4 with $\tau_0 = 0$.*
There exists $\Delta_T > 0$ and $r_V > 0$ such that for all $0 < \tau \leqslant \Delta_T$ there exists $a(\tau)$ and

$$\forall t \in [\tau, \Delta_T],\ a.e.\ (x,v) \in \bar{\Omega} \times \mathbb{R}^d, \quad f(t,x,v) \geqslant a(\tau) \mathbf{1}_{B(0,r_V)}(v).$$

Proof of Lemma 2.7.1 In Corollary 4 we constructed Δ_T and r_V.

We now take τ in $(0, \Delta_T]$ and consider t in $[\tau, \Delta_T]$, (x,v) in $\bar{\Omega} \times \mathbb{R}^d$ where f is a mild solution of the Boltzmann equation.

We remind the reader that $l(\tau)$ and $\tilde{t}(\tau)$ have been introduced in (2.6.15) and (2.6.13). Either $(X_{s,t}(x,v))_{s\in[0,t-\tilde{t}(\tau)]}$ meets $\Omega_{l(\tau)}$ and then we use Corollary 3 to get

$$f(t,x,v) \geqslant a(l(\tau),\tau_1(\tau),\Delta_T)\mathbf{1}_{B(0,r_V)}(v).$$

Or $(X_{s,t}(x,v))_{s\in[0,t-\tilde{t}(\tau)]}$ stays out of $\Omega_{l(\tau)}$ and then we use Corollary 4 to get

$$f(t,x,v) \geqslant b(\tau)\mathbf{1}_{B(0,r_V)}(v).$$

We obtain Lemma 2.7.1 with $a(\tau) = \min\left(a(l(\tau),\tau_1(\tau),\Delta_T), b(\tau)\right)$. ∎

We now have all the tools to prove Proposition 2.2.4.

Proof of Proposition 2.2.4 Let τ be strictly positive and consider t in $[\tau/2,\tau]$.

First case. We suppose that f satisfies Proposition 2.4.4 with $\tau_0 = 0$.

We can compare t with Δ_T constructed in Lemma 2.7.1.

If $t \leqslant \Delta_T$ then we can apply the latter lemma and obtain for almost every (x,v) in $\bar{\Omega}\times\mathbb{R}^d$

$$f(t,x,v) \geqslant a\left(\frac{\tau}{2}\right)\mathbf{1}_{B(0,r_V)}(v). \tag{2.7.1}$$

If $t \geqslant \Delta_T$ then we can use Duhamel formula (2.2.10) and bound $f(t,x,v)$ by its value at time Δ_T (as we did in the first step of the induction in the proof of Proposition 2.5.1) and use Lemma 2.7.1 at Δ_T. This gives, for $\|v\| \leqslant r_V$,

$$\begin{aligned}
f(t,x,v) &\geqslant f(\Delta_T, X_{\Delta_T,t}(x,v), V_{\Delta_T,t}(x,v))e^{-(t-\Delta_T)C_L\langle r_V\rangle^{\gamma^+}} \\
&\geqslant a(\Delta_T)e^{-(\tau-\Delta_T)C_L\langle r_V\rangle^{\gamma^+}}\mathbf{1}_{B(0,r_V)}(V_{\Delta_T,t}(x,v)) \\
&= a(\Delta_T)e^{-(\tau-\Delta_T)C_L\langle r_V\rangle^{\gamma^+}}\mathbf{1}_{B(0,r_V)}(v).
\end{aligned} \tag{2.7.2}$$

We just have to take the minimum of the two lower bounds (2.7.1) and (2.7.2) to obtain Proposition 2.2.4.

Second case. We do not assume anymore that f satisfies Proposition 2.4.4 with $\tau_0 = 0$.

Thanks to Proposition 2.4.4 with $\tau_0 = \tau/4$ we have that

$$\forall t \leqslant 0,\ \forall x \in \bar{\Omega},\ v\in\mathbb{R}^d, \quad F(t,x,v) = f(t+\tau_0,x,v)$$

is a mild solution of the Boltzmann equation satisfying exactly the same bounds as f in Theorem 2.2.3 and such that F has the property of Proposition 2.4.4 at 0 (note that all the constants depend on τ_0).

Hence, we can apply the first step for t' in $[\tau/4, 3\tau/4]$ and $F(t', x, v)$. This gives us the expected result for $f(t, x, v)$ for $t = t' + \tau_0$ in $[\tau/2, \tau]$.

■

2.7.1 Proof of Theorem 2.2.3

As was mentioned in Section 2.1.2, the main difficulty in the proof is to create a lower bound depending only on the norm of the velocity. This has been achieved thanks to Proposition 2.2.4. If we consider this proposition as the start of an induction then it leads to exactly the same process developed by Mouhot in [78], Section 3. Therefore we will just explain how to go from Proposition 2.2.4 to Theorem 2.2.3, without writing too many details.

First of all, by using the spreading property of the Q^+ operator once again we can grow the lower bound derived in Proposition 2.2.4.

Proposition 2.7.2 *Let f be the mild solution of the Boltzmann equation described in Theorem 2.2.3.*
For all τ in $(0, T)$, there exists $R_0 > 0$ such that

$$\forall n \in \mathbb{N}, \forall t \in \left[\tau - \frac{\tau}{2^{n+1}}, \tau\right], \forall (x, v) \in \bar{\Omega} \times \mathbb{R}^d, f(t, x, v) \geqslant a_n(\tau) \mathbf{1}_{B(0, r_n)}(v),$$

with the induction formulae

$$a_{n+1}(\tau) = \mathrm{cst}\, C_e \frac{a_n^2(\tau) r_n^{d+\gamma} \xi_n^{d/2+1}}{2^{n+1}} \quad \textit{and} \quad r_{n+1} = \sqrt{2}(1 - \xi_n) r_n,$$

where $(\xi_n)_{n\in\mathbb{N}}$ is any sequence in $(0,1)$ and $r_0 = r_V$, $a_0(\tau)$ and C_e only depend on τ, E_f (and $L_f^{p_\gamma}$ if Φ satisfies (2.1.4) with $\gamma < 0$).

Indeed, we take the result in Proposition 2.2.4 to be the first step of our induction and then, for n in $\mathbb{N}$ and $0 < \tau < T$, the Duhamel form of f gives

$$f(t, x, v) \geqslant \int_{\tau - \frac{\tau}{2^{n+1}}}^{\tau - \frac{\tau}{2^{n+2}}} e^{-C_L (t-s)\langle v \rangle^{\gamma^+}} Q^+\left(f(s, X_{s,t}(x, v), \cdot), f(s, X_{s,t}(x, v), \cdot)\right)(V_{s,t}(x, v)) ds,$$

for t in $[\tau - \tau/2^{n+2}, \tau]$.

Using the induction hypothesis together with the spreading property of Q^+ (Lemma 2.4.2) leads us, as in the proofs of Propositions 2.5.1 and 2.6.4, to a bigger ball in velocity, centred at 0. The only issue is to avoid the v-dependence in $\exp\left[-C_L(t-s)\langle v\rangle^{\gamma^+}\right]$ which can easily be achieved as shown at the end of the proof of Proposition 3.2 in [78]. This is exactly the same result as Proposition 3.2 in [78], but with the added uniformity in x.

As in Lemma 3.3 in [78], we can take an appropriate sequence $(\xi_n)_{n\in\mathbb{N}}$ and look at the asymptotic behaviour of $(a_n(\tau))_{n\in\mathbb{N}}$. We obtain the following

$$\forall \tau > 0,\ \exists \rho_\tau, \theta_\tau > 0,\ \forall (x,v) \in \bar{\Omega} \times \mathbb{R}^d, \quad f(t,x,v) \geqslant \frac{\rho_\tau}{(2\pi\theta_\tau)^{d/2}} e^{-\frac{|v|^2}{2\theta}}.$$

Notice that, again, the result is uniform in space, since the previous one was, and that the constants ρ_τ and θ_τ only depend on τ and the physical quantities associated to f.

To conclude, it remains to make the result uniform in time. As noticed in [78], Lemma 3.5, the results we obtained so far do not depend on an explicit form of f_0 but just on uniform bounds and continuity that are satisfied at all times, positions and velocities. Therefore, we can do the same arguments starting at any time and not $t = 0$. So if we take $\tau > 0$ and consider $\tau \leqslant t < T$ we just have to make the proof start at $t - \tau$ to obtain Theorem 2.2.3.

2.8 Exponential lower bound in the non cutoff case: proof of Theorem 2.2.6

In this section we prove the immediate appearance of an exponential lower bound for solutions to the Boltzmann equation (2.1.1) in the case of a collision kernel satisfying the non cutoff property.

The definition of being a mild solution in the case of a non cutoff collision kernel, Definition 2.2.5 and equation (2.2.12), shows that we are in fact dealing with an almost cutoff kernel to which we add a non locally integrable remainder. The strategy will mainly follow what we did in the case of a cutoff collision kernel with the addition of controlling the loss due to the added term.

As in the last section, we shall first prove that solutions to the Boltzmann equation can be uniformly bounded from below by a lower bound depending only on the norm of the velocity and then use the proof given for the non cutoff case in [78]. We will do that by proving the immediate appearance of localised “upheaval points” and spreading them up to the point where we reach a uniform lower bound that includes a ball in velocity centred at the origin. The spreading effect will be done both in the case where the trajectories reach a point far from the boundary and in the case of grazing trajectories. At this point we will spread this lower bound on the norm of the velocity up to the exponential lower bound we expect.

We gather here two lemmas, proved in [78], which we shall use in this section. They control the L^∞-norm of the linear operator S_ε and of the bilinear operator Q^1_ε. We first give a property satisfied by the linear operator S, (2.2.12), which is Corollary 2.2 in [78], where we define

$$m_b = \int_{\mathbb{S}^{d-1}} b\,(\cos\theta)\,(1-\cos\theta)d\sigma = \left|\mathbb{S}^{d-2}\right| \int_0^\pi b\,(\cos\theta)\,(1-\cos\theta)\sin^{d-2}\theta\;d\theta. \tag{2.8.1}$$

Lemma 2.8.1 *Let g be a measurable function on $\mathbb{R}^d$. Then*

$$\forall v\in\mathbb{R}^d,\quad |S[g](v)| \leqslant C^S_g \langle v\rangle^{\gamma^+},$$

where C^S_g is defined by:

1. *If Φ satisfies (2.1.4) with $\gamma \geqslant 0$ or if Φ satisfies (2.1.5), then*

$$C^S_g = \mathrm{cst}\; m_b C_\Phi e_g.$$

2. *If Φ satisfies (2.1.4) with $\gamma\in(-d,0)$, then*

$$C^S_g = \mathrm{cst}\; m_b C_\Phi \left[e_g + l^p_g\right],\quad p > d/(d+\gamma).$$

We will compare the lower bound created by the cutoff part of our kernel to the remaining part Q^1_ε. To do so we need to control its L^∞-norm. This is achieved thanks to Lemma 2.5 in [78], which we recall here.

Lemma 2.8.2 *Let $B=\Phi b$ be a collision kernel satisfying (2.1.3), with Φ satisfying (2.1.4) or (2.1.5) and b satisfying (2.1.6) with $\nu\in[0,2)$. Let f,g be measurable functions on $\mathbb{R}^d$. Then*

1. *If Φ satisfies (2.1.4) with $2+\gamma\geqslant 0$ or if Φ satisfies (2.1.5), then*

$$\forall v\in\mathbb{R}^d,\quad \left|Q^1_b(g,f)(v)\right| \leqslant \mathrm{cst}\; m_b C_\Phi \|g\|_{L^1_{\tilde\gamma}} \|f\|_{W^{2,\infty}} \langle v\rangle^{\tilde\gamma}.$$

2. *If Φ satisfies (2.1.4) with $2+\gamma<0$, then*

$$\forall v\in\mathbb{R}^d,\quad \left|Q^1_b(g,f)(v)\right| \leqslant \mathrm{cst}\; m_b C_\Phi \left[\|g\|_{L^1_{\tilde\gamma}} + \|g\|_{L^p}\right] \|f\|_{W^{2,\infty}} \langle v\rangle^{\tilde\gamma}$$

with $p > d/(d+\gamma+2)$.

2.8.1 A lower bound only depending on the norm of the velocity

In this section we prove the following proposition, which is exactly Proposition 2.2.4 in the non-cutoff framework.

Proposition 2.8.3 *Let f be the mild solution of the Boltzmann equation described in Theorem 2.2.6.*
For all $0 < \tau < T$ there exists $a_0(\tau) > 0$ such that

$$\forall t \in [\tau/2, \tau],\ \forall (x,v) \in \bar{\Omega} \times \mathbb{R}^d, \quad f(t,x,v) \geqslant a_0(\tau) \mathbf{1}_{B(0,r_V)}(v),$$

r_V and $a_0(\tau)$ only depending on E_f, E'_f, W_f (and $L_f^{p_\gamma}$ if Φ satisfies (2.1.4) with $\gamma < 0$).

Proof of Proposition 2.8.3 As before, we would like to create localised "upheaval points" (as the ones created in Proposition 2.4.4) and then extend them. Both steps are done, as in the cutoff case, by induction along the characteristics.

We have the following inequality

$$Q_\varepsilon^+(f,f) + Q_\varepsilon^1(f,f) \geqslant Q_\varepsilon^+(f,f) - \left|Q_\varepsilon^1(f,f)\right|. \tag{2.8.2}$$

From the definition of being a mild solution in the non-cutoff case (Definition 2.2.5), for any $0 < \varepsilon < \varepsilon_0$,

$$\begin{aligned} f(t, X_t(x,v), V_t(x,v)) = {} & f_0(x,v) \exp\left[-\int_0^t \left(L_\varepsilon + S_\varepsilon\right)\left[f(s, X_s(x,v), \cdot)\right](V_s(x,v))\, ds\right] \\ & + \int_0^t \exp\left(-\int_s^t \left(L_\varepsilon + S_\varepsilon\right)\left[f(s', X_{s'}(x,v), \cdot)\right](V_{s'}(x,v))\, ds'\right) \\ & \quad \left(Q_\varepsilon^+ + Q_\varepsilon^1\right)\left[f(s, X_s(x,v), \cdot), f(s, X_s(x,v), \cdot)\right](V_s(x,v))\, ds. \end{aligned} \tag{2.8.3}$$

Due to Lemmas 2.4.1, 2.8.1 and 2.8.2 we find that

$$L_\varepsilon[f] \leqslant C_f n_{b_\varepsilon^{CO}} \langle v \rangle^{\gamma^+}, \quad S_\varepsilon[f] \leqslant C_f m_{b_\varepsilon^{NCO}} \langle v \rangle^{\gamma^+} \tag{2.8.4}$$

and

$$\left|Q_\varepsilon^1(f,f)\right| \leqslant C_f m_{b_\varepsilon^{NCO}} \langle v \rangle^{(2+\gamma)^+} \tag{2.8.5}$$

where $C_f > 0$ is a constant depending on E_f, E'_f, W_f (and $L^{p_\gamma}_f$ if Φ satisfies (2.1.4) with $\gamma < 0$).

The proof of Proposition 2.8.3 is divided into three different inductions that are dealt with in the same way as in the proof of Proposition 2.2.4. Each induction represents a step in the proof: one to create localised initial lower bounds (Lemma 2.4.3), another one to deal with non-grazing trajectories (Proposition 2.5.1) and the final one for grazing trajectories (Proposition 2.6.4). Therefore, we will just point out below the only changes we need to make those inductions work in the non-cutoff case.

In all the inductions in the cutoff case, the key point of the induction was to control at each step quantities of the form

$$\begin{aligned} f(t,x,v) \quad \geqslant \quad & \int_{t_n^{(1)}}^{t_n^{(2)}} \exp\left(-\int_s^t (L_\varepsilon + S_\varepsilon)\left[f(s', X_{s'}(x,v),\cdot)\right](V_{s'}(x,v))\, ds'\right) \\ & \left(Q^+_\varepsilon + Q^1_\varepsilon\right)\left[f(s, X_s(x,v),\cdot), f(s, X_s(x,v),\cdot)\right](V_s(x,v))\, ds, \end{aligned}$$

where $(t_n^{(1)})_{n\in\mathbb{N}}$, $(t_n^{(2)})_{n\in\mathbb{N}}$ are defined differently for grazing and non-grazing trajectories (see proofs of Propositions 2.5.1 and 2.6.4).

Much like those previous induction, and using (2.8.2), (2.8.3) and (2.8.4) – (2.8.5), if $f(t,x,v) \geqslant a_n \mathbf{1}_{B(\bar v, r_n)}$ then

$$f(t,x,v) \geqslant \int_{t_n^{(1)}}^{t_n^{(2)}} e^{-C^\varepsilon_f(R)} \left(a_n^2 Q^+_\varepsilon[\mathbf{1}_{B(\bar v, r_n)}, \mathbf{1}_{B(\bar v, r_n)}] - C_f m_{b_\varepsilon^{NCO}} \langle R\rangle^{(2+\gamma)^+}\right)(V_s(x,v))\, ds,$$

which leads to

$$\begin{aligned} f(t,x,v) \geqslant & \int_{t_n^{(1)}}^{t_n^{(2)}} e^{-C^\varepsilon_f(R)} \\ & \left(a_n^2 \mathrm{cst}\, l_{b_\varepsilon^{CO}} c_\Phi r_n^{d+\gamma} \xi_n^{\frac{d}{2}-1} \mathbf{1}_{B\left(\bar v, r_n\sqrt{2}(1-\xi_n)\right)} - C_f m_{b_\varepsilon^{NCO}} \langle R\rangle^{(2+\gamma)^+}\right)(V_s(x,v))\, ds, \end{aligned} \tag{2.8.6}$$

due to the spreading property of Q^+_ε (see Lemma 2.4.2) and using the shorthand notation $C^\varepsilon_f(R) = C_f(n_{b_\varepsilon^{CO}} + m_{b_\varepsilon^{NCO}})\langle R\rangle^{\gamma^+}$.

To conclude we notice that, thanks to the definitions (2.4.1), (2.2.9) and (2.8.1),

$$l_{b_\varepsilon^{CO}} \geqslant l_b$$

and

$$n_{b_\varepsilon^{CO}} \underset{\varepsilon\to 0}{\sim} \frac{b_0}{\nu}\varepsilon^{-\nu}, \quad m_{b_\varepsilon^{NCO}} \underset{\varepsilon\to 0}{\sim} \frac{b_0}{2-\nu}\varepsilon^{2-\nu} \tag{2.8.7}$$

if ν belongs to $(0,2)$ and

$$n_{b_\varepsilon^{CO}} \underset{\varepsilon\to 0}{\sim} b_0 \left|\log\varepsilon\right|, \quad m_{b_\varepsilon^{NCO}} \underset{\varepsilon\to 0}{\sim} \frac{b_0}{2-\nu}\varepsilon^{2} \tag{2.8.8}$$

for $\nu = 0$.

Thus, at each step of the inductions we just have to redo the proofs done in the cutoff case and choose $\varepsilon = \varepsilon_n$ small enough such that

$$C_f m_{b_{\varepsilon_n}^{NCO}} \langle R\rangle^{(2+\gamma)^+} \leqslant \frac{1}{2} a_n^2 \mathrm{cst}\, l_b c_\Phi r_n^{d+\gamma} \xi_n^{\frac{d}{2}-1}. \tag{2.8.9}$$

Proposition 2.8.3 follows directly from these choices plugged into the study of the cutoff case. ■

2.8.2 Proof of Theorem 2.2.6

Now that we proved the immediate appearance of a lower bound depending only on the norm of the velocity we can spread it up to an exponential lower bound. As in Section 2.7.1, we thoroughly follow the proof of Theorem 2.1 of [78]. The proof in our case is exactly the same induction, starting from Proposition 2.8.3. Therefore we only briefly describe how to construct the expected exponential lower bound. For more details we refer the reader to [78], Section 4.

We start by spreading the initial lower bound (Proposition 2.8.3) by induction where, at each step, we use the spreading property of the $Q^+_{\varepsilon_n}$ operator and fix ε_n small enough to obtain a strictly positive lower bound (see (2.8.9)).

There is, however, a subtlety in the non-cutoff case that we have to deal with. Indeed, at each step of the induction we choose an ε_n of decreasing magnitude, but at the same time in each step the action of the operator $-(Q^-_\varepsilon + Q^2_\varepsilon)$ behaves like (see (2.8.6))

$$\exp\left[-C_f\left(m_{b_{\varepsilon_n}^{NCO}} + n_{b_{\varepsilon_n}^{CO}}\right)(t_n^{(1)} - t_n^{(2)})\langle v\rangle^{\gamma^+}\right].$$

By (2.8.7) – (2.8.8), as ε_n tends to 0 we have that $n_{b_{\varepsilon_n}^{CO}}$ goes to $+\infty$ and so the action of $-(Q^-_\varepsilon + Q^2_\varepsilon)$ seems to decrease the lower bound to 0 exponentially fast. The idea to overcome this difficulty is to find a time interval $t_n^{(1)} - t_n^{(2)} = \Delta_n$ at each step to be sufficiently small to counterbalance the effect of $n_{b_{\varepsilon_n}^{CO}}$.

More precisely, by starting from Proposition 2.8.3 as the first step of our induction, taking

$$t_n^{(1)} = \left(\sum_{k=0}^{n+1} \Delta_k\right)\tau, \quad t_n^{(2)} = \left(\sum_{k=0}^{n} \Delta_k\right)\tau$$

in (2.8.6) and fixing ε_n by (2.8.9) we can prove the following induction property

Proposition 2.8.4 *Let f be the mild solution of the Boltzmann equation described in Theorem 2.2.6.*
For all τ in $(0,T)$ and any sequence $(\Delta_n)_{n\in\mathbb{N}}$ such that $\sum_{n\geqslant 0}\Delta_n = 1$,

$$\forall n \in \mathbb{N}, \forall t \in \left[\left(\sum_{k=0}^{n}\Delta_k\right)\tau, \tau\right], \forall (x,v)\in\bar{\Omega}\times\mathbb{R}^d, f(t,x,v) \geqslant a_n(\tau)\mathbf{1}_{B(0,r_n)}(v),$$

with the induction formulae

$$a_{n+1} = \mathrm{cst}\ \Delta_{n+1}\mathrm{exp}\left[-[\tilde{C}_f a_n^2 r_n^{d+\gamma-\tilde{\gamma}}\xi_n^{d/2-1}]^{-\frac{\nu}{2-\nu}}\left(\sum_{k\geqslant n+1}\Delta_k\right) r_n^{\gamma^+}\right] a_n^2 r_n^{\gamma+d}\xi_n^{d/2+1}$$

if ν is in $(0,2)$,

$$a_{n+1} = \mathrm{cst}\ \Delta_{n+1}\mathrm{exp}\left[-\mathrm{cst}\log[\tilde{C}_f a_n^2 r_n^{d+\gamma-\tilde{\gamma}}\xi_n^{d/2-1}]\left(\sum_{k\geqslant n+1}\Delta_k\right) r_n^{\gamma^+}\right] a_n^2 r_n^{\gamma+d}\xi_n^{d/2+1}$$

if $\nu = 0$ and

$$r_{n+1} = \sqrt{2}r_n(1-\xi_n),$$

where $(\xi_n)_{n\in\mathbb{N}}$ is any sequence in $(0,1)$ and $r_0 = r_V$, $a_0(\tau)$ and $\tilde{C}_f$ depend only on τ, E_f, E'_f, W_f (and $L_f^{p_\gamma}$ if Φ satisfies (2.1.4) with $\gamma<0$).

We emphasize here that the induction formulae are obtained thanks to the use of equivalences (2.8.7) and (2.8.8) inside the exponential term

$$e^{-C_f\left(m_{b^{NCO}_{\varepsilon_n}}+n_{b^{CO}_{\varepsilon_n}}\right)(t_n^{(1)}-t_n^{(2)})\langle R\rangle^{\gamma^+}} \geqslant e^{-C_f\left(m_{b^{NCO}_{\varepsilon_n}}+n_{b^{CO}_{\varepsilon_n}}\right)\left(\sum_{k\geqslant n+1}\Delta_k\right)\langle R\rangle^{\gamma^+}}$$

(see step 2 of proof of Proposition 4.2, Section 4 in [78]).

As we obtain exactly the same induction formulae as in [78], the asymptotic behaviour of the coefficients a_n is the same. Thus, by choosing an appropriate sequence $(\Delta_n)_{n\in\mathbb{N}}$, as done in [78], we can construct the expected exponential lower bound independently of time.

Part II

THE HYDRODYNAMICAL LIMITS OF THE BOLTZMANN EQUATION

Chapter 3

From many-body problems to physics of continua

Scientists work on establishing mathematical descriptions of physical phenomena in order to understand and foresee Nature. Different points of view, different scales, can be considered to translate physical dynamics into equations. However, even though the resulting theories look different in terms of equations and behaviour, they should model the same phenomena but at different scales. In this chapter we explore some physical and mathematical links between particles motion, gas dynamics and fluid mechanics in order to prove the mathematical coherence of the various physical modellings of Nature.

This chapter is far from being an exhaustive overview since it restrains itself to the framework of point particles in the framework of classical mechanics. It however motivates and introduces the important concept of hydrodynamical limits of the Boltzmann equation.

Contents

3.1 Particles, gases and fluids in physics

The study of the motion of large amounts of particles is of great importance in physics and several laws and models had been derived. Whether one wants to describe the movement of nanoscopic particles under the influence of an electromagnetic field or to foresee the evolution of clouds of galaxies, it all comes down to finding a physical theory that agrees with observation.

In this section we briefly present the historical evolutions of the description of natural phenomena in order to understand their distinctive features as well as the links that can be made in between them. The framework presented here is the whole space but bounded domains can be considered with appropiate boundary conditions to be added to the equations.

3.1.1 Newton, Hamiltonian systems and the motion of particles

In 1687, Isaac Newton published his *Principia Mathematica* [87] where he wrote the basis of what we nowadays call the classical mechanics. He described his celebrated second law of motion stating that the net force applied to a body produces an acceleration that is proportional to its mass.

With this tool we are able to describe the motion of N particles of mass m and radius r, evolving in $\mathbb{R}^d$ and subject to an external potential V and a two-body interaction potential Φ. Each of the particle is represented by its position and velocity (x_i, v_i) and the latter couple satisfies the following system of ordinary differential equations, where we put m in the definition of V and Φ.

$$\begin{aligned} \frac{dx_i}{dt} &= v_i, \\ \frac{dv_i}{dt} &= -\nabla_x V(x_i) - \sum_{\substack{j=1 \\ j\neq i}}^{N} \nabla\Phi(x_i - x_j). \end{aligned}$$

This system of nonlinear equations is known as the N body problem, which is very complicated as soon as N is greater than or equal to 3. This difficulty is due to the lack of a sufficient number of conserved quantities along the motion (which is called integrability of the system and imposes constraints on the motion itself) and was already noticed by Poincaré in [92]. However, in this specific case the force is the gradient of a potential and this dynamical system is thus Hamiltonian because with Hamiltonian operator

$$H = \frac{1}{2}\sum_{i=1}^{N} |v_i|^2 + \sum_{i=1}^{N} V(x_i) + \sum_{\substack{j=1 \\ j\neq i}}^{N} \Phi(x_i - x_j).$$

If N is very large then one is only interested in an average behaviour of the group of particles. We therefore turn to the N-particle distribution function F_N in the phase space

$$\mathcal{D}_N = \left\{ (t, x_1, v_1, \ldots, x_N, v_N) \in \mathbb{R}^+ \times \mathbb{R}^{2dN}, \ \forall i \neq j, \quad |x_i - x_j| > r \right\}.$$

In the case of Hamiltonian dynamical systems, the distribution F_N had been shown to satisfy the Liouville equation,

$$\partial_t F_N + \mathrm{Liou}\,(F_N) = 0, \quad \text{in } \mathcal{D}_N, \tag{3.1.1}$$

where the Liouville operator reads

$$\mathrm{Liou} = \sum_{i=1}^{N} \left(\frac{\partial H}{\partial v_i} . \frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i} . \frac{\partial}{\partial v_i} \right).$$

In 1884, Josiah Gibbs [45] emphasized the importance of the Liouville equation as the fundamental equation of statistical mechanics.

3.1.2 Euler, continuous medium and fluid mechanics

The study of the motion of fluids awaited a bit longer and a precise mathematical models appeared only with Leonhard Euler in his *Principes généraux du mouvement des fluids*, [43] published in 1755. Considering a fluid like a continuous medium rather than a group of individual molecules, Euler derived the first equations of fluid dynamics which bares his name and concerns the mass, the momentum and the energy of the fluid. Euler equations link the evolution of the density ρ, the mean velocity u, the inner pression p and the energy E of the fluid. They read

$$\begin{aligned} &\partial_t \rho + \nabla_x \cdot (\rho u) = 0, \\ &\rho \partial_t u + \rho u \cdot \nabla_x u + \nabla_x p = 0, \\ &\partial_t E + \nabla_x \cdot (u\,(E + p)) = 0. \end{aligned} \tag{3.1.2}$$

The first remark we can make is that those equations imply the conservation of total mass, momentum and energy of the fluid which is the minimum that classical mechanics ask for. However, Euler modelled a fluid without friction and thus cannot explain the viscosity phenomena we can observe.

The first step towards modelling of viscid fluids was made by Claude-Louis Navier in 1822 in his *Mémoire sur les lois du mouvement des fluides* [85], where he introduced a shear stress tensor to describe the inner force created by the motion of the fluid. Unfortunately, the way he tackled shear stress did not match real observations and the ultimate improvements to the theory of viscid fluids were achieved by Adhémar Barré de Saint-Venant ([11], written in 1834 and published in 1843) and George Stokes ([103], 1845) by considering a shear stress tensor that is proportional to the gradient of the velocity. This lead to the so-called Navier-Stokes equations for newtonian fluids, where we denote the mean temperature of the fluid by θ,

$$\begin{aligned}&\partial_t\rho + \nabla_x\cdot(\rho u) = 0,\\ &\rho\partial_t u + \rho u\cdot\nabla_x u + \nabla_x p = \nu\Delta_x u,\\ &\partial_t E + \nabla_x\cdot(u\,(E+p)) = \kappa\Delta_x\theta,\end{aligned}\tag{3.1.3}$$

where ν and κ are respectively the dynamic viscosity and the thermal conductivity of the fluid. The interested reader can find a derivation of those equations from the laws of physics as well as other types of fluids in [12].

Remark 3.1.1 *We can think of the Euler equations as the limit of the Navier-Stokes equations when the viscosity of the fluids goes to* 0*.*

Finally, when the shear stress is very important, namely when ν tends to infinity, one obtains the Stokes equations for viscid fluids.

$$\begin{aligned}&\partial_t\rho + \nabla_x\cdot(\rho u) = 0,\\ &\rho\partial_t u + \nabla_x p = \nu\Delta_x u,\\ &\partial_t E = \kappa\Delta_x\theta,\end{aligned}\tag{3.1.4}$$

3.1.3 Maxwell, kinetic theory and gas dynamics

As previously emphasized, the many particles problem is really intricate and its complexity makes it almost impossible to use when one realises that a mole of gas contains more than $6,02\times 10^{23}$ particles or that the Milky Way is constituted of approximately 10^{11} stars.

To overcome this issue, James Clerk Maxwell in 1867 [75] (in a weak formulation based on the physical observables of a system) and Ludwig Boltzmann in 1872 [18][17] developed the founding principles of kinetic theory. This theory proposes to take a statistical approach to model the dynamics of particles when they are so numerous that the individual behaviours are of little interest. Basically, one should try to understand the evolution of the distribution function $f(t,x,v)$ of particles in the phase space as N tends to infinity; the quantity $f(t,x,v)dxdv$ stands for the probability of finding a particles in $[x,x+dx]$ with velocity in $[v,v+dv]$ at time t.

These thoughts lead to several models in statistical physics, depending on which interactions, which systems, are taken into account. In this part we will focus on the Boltzmann equation, already described in Chapter 1, which reads

$$\partial_t f + v \cdot \nabla_x f = Q(f, f). \tag{3.1.5}$$

3.2 Hilbert's sixth problem: the mathematical coherence of models

The different theories briefly described above have been established rather independently from each other and real observations validate them as being the relevant mathematical description of physics. However, they model, at different scales, the same underlying phenomenon that is the interaction between particles.

If we study a system of N particles then looking at their average dynamics when N is large should bring us to the kinetic theory framework for rarefied gases. Moreover, a strongly compressed gas becomes a fluid at very high pressure and therefore there should exist a link between kinetic equations and fluid equations.

At the International Congress of Mathematicians held in Paris in 1900, Hilbert emphasized the importance of mathematically deriving the coherence of all those physical models. More precisely, Hilbert's sixth problem aims at building up a unified description of mechanics, from microscopic atoms to macroscopic continuum. One would like to understand mathematically how macroscopic properties of fluids and gases, such as viscosity or irreversibility, evolving at an observation timescale T_{obs}, can arise from reversible microscopic dynamics, where the mean time between two consecutive collisions is of microscopic timescale T_{coll} (see Figure 3.1).

In this section, we formally study the possible convergences between the different physical models and give some of the existing results in the field that proved these convergences rigorously. We will restrict ourselves to the case when particles moves in a boundary free domain such as $\mathbb{R}^d$ or the torus $\mathbb{T}^d$.

We shall give a more thorough study of the hydrodynamical limits as these will be the purpose of the next two chapters.

3.2.1 From micro to macro models: the law of large numbers

In this section we briefly present some strategies to go directly from Hamiltonian systems to macroscopic dynamics. It deals with a problem that is completely transversal to our work on Boltzmann equation and its hydrodynamical limits and we will thus not go into details but we give a short description for the sake of completeness of our manuscript.

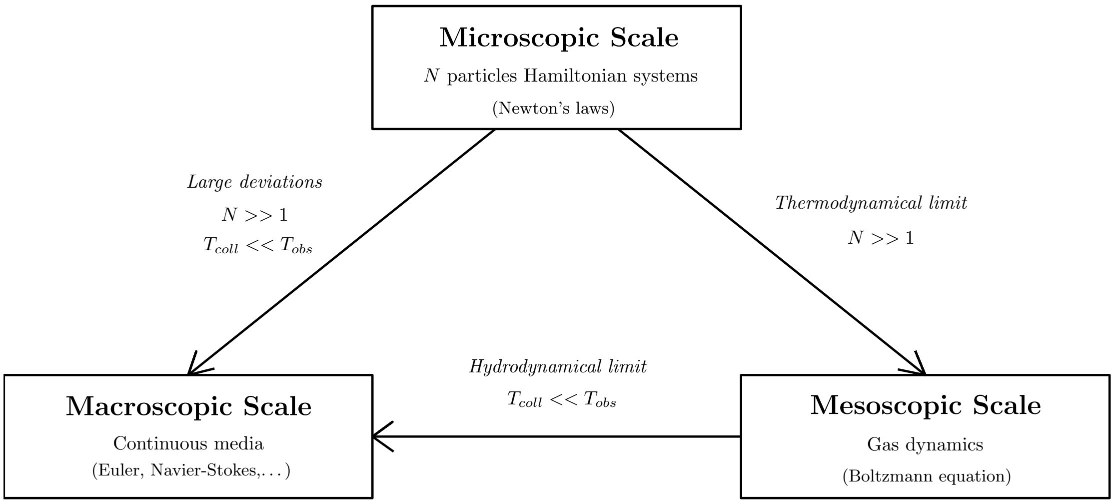

Figure 3.1: Transitions between the different level of description (from [98])

For rigorous proofs and a deeper insight of the theory, we refer the reader to [91], in the case of Euler limit, and to [42][97], for the Navier-Stokes limit. What follows can be found in [98] Chapter 1.

We would like to understand how microscopic properties generate macroscopic dynamics. We start with N particles with positions and velocities (x_i, v_i) which satisfy the Liouville equation (3.1.1). The physical observables of the system of N particles are the mass and momentum densities

$$\begin{aligned} M_N(t,x) &= \frac{1}{N}\sum_{i=1}^{N}\delta(x-x_i(t)), \\ P_N(t,x) &= \frac{1}{N}\sum_{i=1}^{N}v_i(t)\delta(x-x_i(t)), \end{aligned}$$

where δ is the Dirac measure at the origin.

The microscopic dynamics happen much faster and more localized than the fluid mechanics and one thus has to work at different time and space scales. We denote by (x,t) the microscopic variables and by $(\tilde{x},\tilde{t})$ the macroscopic ones. We have a ratio ε between space scale that we will make go to zero:

$$\tilde{x} = \varepsilon x.$$

The ratio between time scales will define the fluid dynamics towards which our Liouville equation converges. Indeed, if we define the typical density $\rho = N/L^3$, with L being a typical macroscopic lenght, we have the three possible outcomes

1. $\rho \sim \varepsilon$ and $\tilde{t} = \varepsilon t$: the number of collision per particle is then finite and this is the Grad limit,

2. $\rho \sim 1$ and $\tilde{t} = \varepsilon t$: the number of collision per particle is of order ε^{-1} and this is the Euler limit,

3. $\rho \sim 1$ and $\tilde{t} = \varepsilon^2 t$: the number of collision per particle is of order ε^{-2} and this is the Diffusive limit.

In all the cases, the goal is to compute the equations satisfied, in a weak sense since we are dealing with probability measures, by $M_N(\tilde{t}/\varepsilon, \tilde{x}/\varepsilon)$ and $P_N(\tilde{t}/\varepsilon, \tilde{x}/\varepsilon)$ and to compute the limiting equations as ε goes to zero. We are looking at probability measures so the convergence has to be understood as a convergence in the sense of the law of large numbers as N tends to infinity, with respect to the density function F_N.

Some results have been rigorously proven in some special settings where we have ergodicity of the system. We refer to the references given at the begin of this section for more details.

3.2.2 The thermodynamical limit: the chaos assumption

As the number N of particles in a system becomes very large the N-body problem is too intricate to offer an interesting description of how the system behaves. Moreover, one is more interested by the global evolution of the system than the actual motion of one particular particle in the case where they are indistinguishable. A statistical approach is preferable and the Liouville equation is easier to handle, see Section 3.1.1.

The derivation from microscopic dynamics to mesoscopic scales is rather hard even for short range interaction potentials. Indeed, it depends exactly on the positions and velocities of particles when we would like to only care about the probability distribution of the latters. In other words, there is no global description of the interacting forces inside the system. Therefore, some assumptions have to be made in order to ensure the statistical stability of the mesoscopic dynamics in the limit $N \to \infty$.

We give here a brief and formal derivation of the Boltzmann equation from the laws on Newton. Most of this section follows closely [30] and [44].

The Liouville equation (3.1.1) in the case of a sole two-body interaction potential Φ_N reads, when the diameter of each particle is denoted by r,

$$\partial_t F_N + \sum_{i=1}^{N} v_i \cdot \nabla_{x_i} F_N - \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \nabla_x \Phi_N(x_i - x_j) \cdot \nabla_{v_i} F_N = 0,$$

in the phase space

$$\mathcal{D}_N = \left\{ (t, x_1, v_1, \dots, x_N, v_N) \in \mathbb{R}^+ \times \mathbb{R}^{2dN},\ \forall i \neq j, \quad |x_i - x_j| > r \right\}.$$

We add on F_N the assumption that particles are indistinguishable, which translates into

$$\forall \sigma \in \mathfrak{S}_N, \quad f(t, x_1, v_1, \dots, x_N, v_N) = f(t, x_{\sigma(1)}, v_{\sigma(1)}, \dots, x_{\sigma(N)}, v_{\sigma(N)}),$$

where $\mathfrak{S}_N$ dentotes the group of permutations of the set $\{1, \dots, N\}$.

We want to extract the average behaviour of a particle, that is to say the first marginal associated to F_N, which we denote by $f_N^{(1)}(t, x_1, v_1)$.

The thermodynamical limit is the resulting equation satisfied by $f_N^{(1)}$ when we let N go to infinity. Looking at the Liouville equation, the main difficulty will be to understand the term $\nabla_x \Phi_N$ in the limit of infinitely many particles. However, as N goes to infinity, the energy of the system has to remain bounded and we thus have to assume that the energy of each interaction *via* Φ_N is small. We therefore need to rescale the potential Φ_N and we present the two ways of doing it.

The mean-field limit:
In that case we consider that the range of the interaction stays macroscopic but that its amplitude decreases like $1/N$. This way we have that $\Phi_N = \Phi/N$ for a given macroscopic potential Φ.

In that case, integrating the Liouville equation against $(x_2, v_2, \dots, x_N, v_N)$ to obtain the equation of the first marginal and taking the limit as N goes to infinity yields

$$\partial_t f + v \cdot \nabla_x f + F \cdot \nabla_v f = 0,$$

where $f = \lim\limits_{N \to \infty} f_N^{(1)}$ and

$$F(t, x) = -\nabla_x \left(\Phi * \int_{\mathbb{R}^d} f(t, x, v) dv \right).$$

This strategy of the mean-field limit generates a lot of interestic mathematical studies. As an example we mention the Vlasov-Poisson equation used to describe plasmas with Coulomb interaction potential (even if the rigorous derivation remains an open problem)

$$\Phi(x) = \frac{q^2}{4\pi\varepsilon_0 |x|},$$

where q is the electric charge of a particle and ε_0 is the vacuum permittivity.

However, we will not deal with this type of rescaling for Φ_N and we refer the interested reader to a review by Golse [47].

Collisional dynamics:
The rescaling of Φ_N we are interested in is the one where interactions become localized in the space variable, acting like collisions between particles. Basically, we suppose that the strenght of the interaction Φ_N stays $O(1)$ but its range is very small. The Liouville equation associated to this problem reads

$$\partial_t F_N + \sum_{i=1}^{N} v_i \cdot \nabla_{x_i} F_N - \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{1}{l} \nabla_x \Phi\left(\frac{x_i - x_j}{l}\right) \cdot \nabla_{v_i} F_N = 0, \tag{3.2.1}$$

where l is the range of interaction of Φ_N and is microscopic. Moreover, this equation has to be satisfied in the following domain

$$\mathcal{D}_N = \Big\{(t, x_1, v_1, \ldots, x_N, v_N) \in \mathbb{R}^+ \times \mathbb{R}^{2dN},\ \forall i \neq j, \quad |x_i - x_j| > l\Big\}.$$

If we integrate the Liouville equation against $(x_2, v_2, \ldots, x_N, v_N)$ we clearly see that, compared to the mean-field limit case where Φ_N implies that each particle feels the average force generated by all the other particles, $f_N^{(1)}$ will depend on $f_N^{(2)}$ *via* the term $l^{-1}\nabla_x\Phi(l^{-1}(x_1 - x_2))$. We thus need to compute the equation satisfied by the second marginal which depends, by the same considerations, of $f_N^{(3)}$. By induction we construct a hierarchy of N equations from the first marginal to the N^{th} one (F_N itself). This system of equations is called the BBGKY hierarchy, from Bogoliubov [16], Born and Green [19], Kirkwood [62][63] and Yvon [115] (see also [30][44][96]).

A requirement to derive the BBGKY hierarchy is to define boundary conditions on $\partial\mathcal{D}_N$, which is the set where at least two particles are in contact, in order to integrate by parts in the integrated Liouville equation. We suppose that the collision between particles (x_i, v_i) and (x_j, v_j) are elastic collisions, that is

1. they are localized in time and space so the positions of the particles remain unchanged and the particles collide at a given time,

2. they are perfectly elastic which means that the momentum and the energy are preserved: if we denote v_i' and v_j' the outcoming velocities we have

$$\begin{cases} v_i' + v_j' = v_i + v_j \\ |v_i'|^2 + |v_j'|^2 = |v_i|^2 + |v_j|^2. \end{cases}$$

Of course, for this elastic collisions to define boundary conditions and physical dynamics, we have to put aside the problematic case when three or more particles collide at the same time or when infinitely many collisions happen in a finite time. Fortunately, the set of initial data leading to such outcome is of Lebesgue measure zero (see Proposition 2.1.1 in [44] for the hard sphere case).

We integrate the Liouville equation (3.2.1) against the last $N-s$ coordinates and take into account the boundary conditions. The complete computations can be found in [44] Chapter 4 thanks to truncated marginals. To simplify here we just assume that it holds true for marginals. The BBGKY hierarchy reads, at least in a weak sense,

$$\partial_t f_N^{(s)} + \sum_{i=1}^{N} v_i \cdot \nabla_{x_i} f_N^{(s)} - \frac{1}{l} \sum_{\substack{i,j=1 \\ j \neq i}}^{N} \nabla_x \Phi \left(\frac{x_i - x_j}{l} \right) \cdot \nabla_{v_i} f_N^{(s)} = \sum_{m=1}^{N-s} m \binom{N-s}{m} Q_l(f_N^{(s+m)}),$$

for all $s \in \{1 \dots N\}$. Here Q_l is an operator encoding the boundary conditions. Namely, it involves the integral over the $N-s$ spherical particles potentially colliding with the free s particles. Therefore, the particles having a diameter l, one can expects a uniform convergence as N tends to infinity if our gas satisfies the Boltzmann-Grad scaling:

$$\lim_{N \to +\infty} N l^{d-1} = O(1).$$

The thermodynamical limit consists in understanding the BBGKY hierarchy when N goes to infinity in the Boltzmann-Grad scaling setting.

As mentionned before, the marginals can only be understood thanks to higher order marginals. To obtain the Boltzmann equation, one has to prove that the limit F_N is of a tensor product form

$$\lim_{N \to +\infty} F_N = \bigotimes_{n \in \mathbb{N}} f,$$

where f is a density function that satisfies the Boltzmann equation. This tensor product has to be understood in the sense of marginal

$$\forall n \in \mathbb{N}, \quad \lim_{N \to +\infty} f_N^{(s)} = \underbrace{f \otimes \cdots \otimes f}_{s \text{ times}}.$$

Such a property is called the chaos assumption and it means that particles asymptotically behave independently of each other, in a weak sense.

An important feature that does not happen in the mean-field case is the fact that the collisional dynamics framework defines a past and a future for the system. Indeed, when two particles bounce against each other, they are no longer independent of each other since the laws of elasticity define their velocities after the collision. The chaos assumption thus implies a choice for the arrow of time.

Deriving Boltzmann equation rigorously from the Liouville equation, and the BBGKY hierarchy, still requires a lot of studies. The first result was due to Lanford [65] where he gave a proof about existence of solution to the BBGKY hierarchy, their convergence and the propagation of the chaos assumption in time. Recently, [44] and [96] filled in the missing details in Lanford's proof in the case of hard spheres (billiard balls with $\Phi = 0$) and short range potentials (Φ with compact support and unbounded near the origin).

Unfortunately, up to now, the proofs hold for very short time, smaller than the mean free time between two consecutive collisions.

Let us briefly mention that another approach has been proposed by Kac [60] to derive the spatially homogeneous Boltzmann equation from a stochastic process underlying the dynamics of particles instead of using Newton's law. In that case, dynamics of velocities are viewed as stochastic processes with jumps standing for collisions. This strategy has been useful to obtain results about the Boltzmann equation, such as insights of Cercignani's conjecture on the entropy decay for entropy-entropy production methods for instance. A recent state of the art about the subject can be found in [76].

3.2.3 The hydrodynamical limits: the Knudsen number

Fluids dynamics are determined by some properties of fluids such as their compressibility and their viscosity. These parameters are expressed in terms of dimensionless coefficients that encode the physical properties of the fluid. For instance, the Mach number $\mathrm{Ma} = u_0/c^*$, where u_0 is the bulk velocity of the flow and c_* is the speed of sound in the medium, determines the compressibility of the fluids and the Reynolds number Re informs about the viscosity of the fluids. The smaller these numbers are, the less compressible or viscous the fluid is.

At the mesoscopic scale of the Boltzmann equation, only microscopic features are governing the dynamics but one expects that the macroscopic properties arise from such dynamics but at different time and space scales. What follows is a gathering of results, thoughts and suggestions made essentially in [46][111][98].

We will not discuss the case of bounded domain as this is more intricate and notably fewer results has been proven. We refer the interested reader to Section 4.4 of [98] or [9] for the particular case of incompressible Euler limit. Note that some results presented here cannot hold true with boundaries because of the existence of a boundary layer phenomenon, as noticed in [73] for compressible Euler limit for which the Prandtl layer occurs.

3.2.3.1 A dimensionless reformulation of the Boltzmann equation

The macroscopic dynamics are visible at a much bigger scale than the microscopic interactions between particles. We therefore consider a macroscopic length scale l_0, which can be the size of the domain where the flow evolves, and an observation time scale t_0, which can be seen as T_{obs} in Figure 3.1.

By choosing a reference temperature θ_0 we define a thermal speed c^* associated to this temperature (see Section 2.2 of [98]). c_* would be the speed of sound in the case of a monoatomic gas. We set new nondimensional variables

$$t' = \frac{t}{t_0}, \quad x' = \frac{x}{l_0}, \quad v' = \frac{v}{c^*},$$

as well as a nondimensional distribution function

$$f'(t', x', v') = \frac{l_0^3 c_0^3}{N_0} f(t, x, v),$$

where N_0 is the average number of particles in a volume l_0^3; $\rho_0 = N_0/l_0^3$ is therefore the mean macroscopic density of the gas.

As noticed in [46] or [98], the Boltzmann operatore Q is expressed in density per unit of time and therefore defines a new microscopic time scale τ_0 in the following sense

$$\int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}} B(v, v_*, \sigma) M_{(\rho_0, 0, \theta_0)}(v) M_{(\rho_0, 0, \theta_0)}(v_*) \, d\sigma dv_* dv = \frac{\rho_0}{\tau_0}.$$

τ_0 is the mean free time between two consecutive collisions of a particle at equilibrium $M_{(\rho_0,0,\theta_0)}$ and thus define the mean free path

$$\lambda_0 = c_* \tau_0.$$

Finally we take a rescaled collisional kernel $B'(v', v'_*, \sigma) = \rho_0 \tau_0 B(v, v_*, \sigma)$ and we obtain a nondimensional form of the Boltzmann equation, where we dropped the prime notations:

$$\mathrm{Ma} \partial_t f + v \cdot \nabla_x f = \frac{1}{\mathrm{Kn}} Q(f, f), \tag{3.2.2}$$

where Ma is the Mach number of the flow and Kn is the Knudsen number, which is the inverse of the average number of collisions for each particle per unit of time. One can choose different length and time scales to study fluctuations around a reference flow instead. In that case one obtains the Strouhal number instead of the Mach number and this choice leads to different hydrodynamical models (see [98]).

Considering fluids as gases where particles are in contact suggests that the Knudsen number governs the convergence from Boltzmann equation to fluid equations. The hydrodynamical limits study the evolution of solutions to the rescaled Boltzmann equation (3.2.2) as Kn goes to zero. Moreover, fluid dynamics essentially amount to the laws of conservation of mass, momentum and energy. We thus expect to derive them only using the conservation laws and the entropy decrease fulfilled by the Boltzmann equation.

3.2.3.2 The compressible Euler limit

For now on we denote the Knudsen number by ε. We look at the family of distributions $(f_\varepsilon)_{\varepsilon>0}$ that satisfy (3.2.2) for all ε in $(0,1]$, that is

$$\varepsilon \mathrm{Ma}\partial_t f_\varepsilon + \varepsilon v \cdot \nabla_x f_\varepsilon = Q(f_\varepsilon, f_\varepsilon).$$

Therefore, at least formally, if f_ε tends to f as ε goes to zero we have that $Q(f,f)=0$ which leads to (see Section 1.1.2.4)

$$f(t,x,v) = M_{\rho,u,\theta}(t,x,v) = \frac{\rho(t,x)}{(2\pi\theta(t,x))^{d/2}} \exp\left(-\frac{|v-u(t,x)|^2}{2\theta(t,x)}\right).$$

Thanks to Section 1.1.2.4 we have the conservation of mass, momemtum and energy for all the f_ε and so this gives in the limit

$$\begin{aligned}
&\mathrm{Ma}\int_{\mathbb{R}^d} M_{\rho,u,\theta}\,dv + \nabla_x\cdot\int_{\mathbb{R}^d} vM_{\rho,u,\theta}\,dv = 0\\
&\mathrm{Ma}\int_{\mathbb{R}^d} vM_{\rho,u,\theta}\,dv + \nabla_x\cdot\int_{\mathbb{R}^d} v\otimes vM_{\rho,u,\theta}\,dv = 0\\
&\mathrm{Ma}\int_{\mathbb{R}^d} \frac{1}{2}|v|^2 M_{\rho,u,\theta}\,dv + \nabla_x\cdot\int_{\mathbb{R}^d} \frac{1}{2}|v|^2 vM_{\rho,u,\theta}\,dv = 0,
\end{aligned}$$

which is easily computed into

$$\begin{aligned}
&\mathrm{Ma}\partial_t\rho + \nabla_x\cdot(\rho u) = 0,\\
&\mathrm{Ma}\partial_t(\rho u) + \nabla_x\cdot(\rho u\otimes u) + \nabla_x(\rho\theta) = 0,\\
&\mathrm{Ma}\partial_t\left(\rho\left(\frac{1}{2}|u|^2 + \frac{d}{2}\theta\right)\right) + \nabla_x\cdot\left(u\rho\left(\frac{1}{2}|u|^2 + \frac{d+2}{2}\theta\right)\right) = 0.
\end{aligned}$$

These equations are the compressible Euler equations for a perfect monoatomic gas where the pressure is $p=\rho\theta$, the thermal energy is $\theta/2$ per degree of freedom and the internal energy is thus $d\theta/2$. We remark here that the H-theorem (see Section 1.1.2.4) leads to

$$\mathrm{Ma}\partial_t\left(\rho\log\frac{\rho}{\theta^{d/2}}\right) + \nabla_x\cdot\left(\rho u\log\frac{\rho}{\theta^{d/2}}\right) \leqslant 0,$$

which is the characterization of physically relevant solutions to the Euler system, known as the Lax admissibility condition.

There are few rigorous results about the derivations above and we refer the reader to Section 6.2 of [98] for a bibliography on the subject. Most of the existing results are valid as long as the compressible Euler theory gives smooth solutions, in other words the proofs fail as soon as a singularity appears in Euler equations, which is a common property of the latter equations (see [101]).

Let us mention here the article by Caflisch [25], extended by Lachowicz [64] to more general initial data by dealing with the problem of initial layer. These articles construct solutions to the Boltzmann equation close to a local Maxwellian $M_{(\rho,u,\theta)}$ the parameters of which satisfy the Euler equations. Their general strategy was to look for solutions to the Boltzmann equation of the form $f_\varepsilon = M_{(\rho,u,\theta)} + \varepsilon g_\varepsilon$ which proved itself to be a powerful method that we shall discuss deeper later.

3.2.3.3 The asymptotic compressible Navier-Stokes limit

Hydrodynamical limits study the asymptotic of the observable quantities of solutions f_ε to (3.2.2) as ε goes to zero. As seen in the section above, if we expand f_ε in terms of ε, the zeroth order term has to be an Eulerian maxwellian $M_{(\rho,u,\theta)}$. Moreover, Remark 3.1.1 seems to consider Navier-Stokes system as a fluctuation around a global equilibrium of Euler system so the natural question is wether an expansion of f_ε gives us the compressible Navier-Stokes equation.

The Hilbert's expansion - or its modified version the Chapman-Enskog's expansion where the variables of g are observables of f_ε and their derivatives - is a formal expansion of f_ε around the Knudsen number

$$f_\varepsilon(t,x,v) = \sum_{i=0}^{+\infty} \varepsilon^i g_i(t,x,v).$$

The goal is now to plug this expression inside the nondimensional form of the Boltzmann equation, to obtain a hierarchy of partial differential equations and to solve them to obtain solutions f_ε of that specific form. The Hilbert's expansion is formal but one can look at a finite expansion with a remainder term that is hoped to be small enough. Mathematical properties of these kind of expansions are detailed in the works of Grad [48][49].

We saw before that $g_0 = M_{(\rho,u,\theta)}$ with (ρ, u, θ) satisfying the Euler equations. The equation one gets at zeroth order is

$$\mathrm{Ma}\partial_t g_0 + v \cdot \nabla_x g_0 = L_{g_0}(g_1),$$

where L_{g_0} is the linearization of the Boltzmann operator around the local Maxwellian g_0. The properties of the linear operator are therefore required but we will not go into them, we refer the reader to [8] for the ones needed in this derivation and to Chapter 4 for a general study. The important point is that we get a form for g_1 which is

$$g_1(t,x,v) = L_{g_0}^{-1}\left(\mathrm{Ma}\partial_t g_0 + v\cdot\nabla_x g_0\right) + \varphi_1(t,x,v), \tag{3.2.3}$$

where ρ_1, u_1 and θ_1 are functions of t and x and φ_1 is in $\mathrm{Ker}\left(L_{g_0}\right)$.

If we define

$$A(v) = v\otimes v - \frac{1}{d}\left|v\right|^2 I_d \quad \text{and} \quad B(v) = \frac{1}{2}v\left[\left|v\right|^2 - (d+2)\right], \tag{3.2.4}$$

then direct computations with $g_0 = M_{(\rho,u,\theta)}$ and the Euler equations give us

$$\mathrm{Ma}\partial_t g_0 + v\cdot\nabla_x g_0 = -\left(\frac{1}{2}A(V) : D(u) + B(V)\cdot\frac{\nabla_x\theta}{\sqrt{\theta}}\right)g_0, \tag{3.2.5}$$

where

$$V = \frac{v-u}{\sqrt{\theta}} \quad \text{and} \quad D(u) = \frac{1}{2}\left(\nabla_x u + (\nabla_x u)^T\right) - \frac{1}{d}(\nabla_x\cdot u)I_d.$$

We notice that $A(V)g_0$ and $B(V)g_0$ are in $\mathrm{Im}\left(L_{g_0}\right)$. Moreover, see [8][46], there exist functions $\alpha,\beta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that

$$L_{g_0}^{-1}\left(A(V)g_0\right) = \rho^{-1}\theta^{\gamma/2}\alpha\left(|V|\right)A(V)g_0 \quad \text{and} \quad L_{g_0}^{-1}\left(Bg_0\right) = \rho^{-1}\theta^{\gamma/2}\beta\left(|V|\right)B(V)g_0. \tag{3.2.6}$$

Combining the latter equalities with (3.2.3) and (3.2.5) yields a precise form for g_1,

$$g_1 = -\rho^{-1}\theta^{\gamma/2}\left(\frac{1}{2}\alpha\left(|V|\right)A(V) : D(u) + \beta\left(,|V|\right)B(V)\cdot\frac{\nabla_x\theta}{\sqrt{\theta}}\right)g_0 + \varphi_1.$$

It remains to compute φ_1 - i.e. its observable quantities (ρ_1, u_1, θ_1) - so we look at the expansion of the Boltzmann equation at first order in ε:

$$\mathrm{Ma}\partial_t g_1 + v\cdot\nabla_x g_1 = L_{g_0}(g_2) + Q(g_1,g_1),$$

for which we still have conservation of mass momentum and energy

$$\int_{\mathbb{R}^d}\psi(v)\left[\mathrm{Ma}\partial_t g_1 + v\cdot\nabla_x g_1\right]\,dv = 0,$$

with $\psi(v) = 1, v, |v|^2$.

This system of conservation laws together with the equations satisfied by (ρ,u,θ) gives that

$$\rho_\varepsilon = \rho + \varepsilon\rho_1, \quad u_\varepsilon = u + \varepsilon u_1, \quad \theta_\varepsilon = \theta + \varepsilon\theta_1$$

satisfy

$$\begin{aligned}
&\mathrm{Ma}\partial_t \rho_\varepsilon + \nabla_x \cdot (\rho_\varepsilon u_\varepsilon) = 0, \\
&\mathrm{Ma}\partial_t (\rho_\varepsilon u_\varepsilon) + \nabla_x \cdot (\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon) + \nabla_x (\rho_\varepsilon \theta_\varepsilon) = \varepsilon \nabla_x \left[\nu(\rho_\varepsilon, \theta_\varepsilon) D(u_\varepsilon)\right], \\
&\mathrm{Ma}\partial_t E_\varepsilon + \nabla_x \cdot (u_\varepsilon (E_\varepsilon + \rho_\varepsilon \theta_\varepsilon)) = \frac{\varepsilon}{2} \nu(\rho_\varepsilon, \theta_\varepsilon) D(u_\varepsilon) : D(u_\varepsilon) + \varepsilon \nabla_x \cdot \left[\kappa(\rho_\varepsilon, \theta_\varepsilon) \nabla_x \theta_\varepsilon\right],
\end{aligned}$$

where the internal energy is the one of a monoatomic gas

$$E_\varepsilon = \rho_\varepsilon \left(\frac{1}{2} |u_\varepsilon|^2 + \frac{d}{2} \theta_\varepsilon\right),$$

and the dynamic viscosity and the thermal conductivity are given by

$$\nu(\rho_\varepsilon, \theta_\varepsilon) = \rho_\varepsilon^{-1} \theta_\varepsilon^{\gamma/2} \left\langle \alpha\left(|V|\right) A(V), A(V) \right\rangle_{L^2_v(g_0)}, \tag{3.2.7}$$

$$\kappa(\rho_\varepsilon, \theta_\varepsilon) = \rho_\varepsilon^{-1} \theta_\varepsilon^{\gamma/2} \left\langle \alpha\left(|V|\right) A(V), A(V) \right\rangle_{L^2_v(g_0)}. \tag{3.2.8}$$

The observables of the system $(\rho_\varepsilon, u_\varepsilon, \theta_\varepsilon)$ satisfy the Navier-Stokes equation with dissipation terms of order ε. Of course, in order to make this rigorous, one has to close the whole system with g_2 and prove that this remainder term is of order ε^2. This can be found formally in [8] and a similar result that the one derived by Caflisch in the case of Euler equation has been proven by De Masi, Esposito and Lebowitz [34], for a Navier-Stokes maxwellian with constant mass and temperature. The interesting point is that the macroscopic viscosity and conductivity arose from microscopic phenomenon described by the linear part of the Boltzmann equation.

To conclude with the Hilbert expansion, one can obtain next orders asymptotic hydrodynamical limits. However, the second order yields the Burnett equations which turned out to be irrelevant physically. For further discussions on these schemes see [48] and [111].

3.2.3.4 The incompressible hydrodynamical limits

The compressible Navier-Stokes equation has been recovered from Boltzmann equation only in an asymptotic regime where the dissipativity tends to 0. One can wonder if we can actually obtain the compressible Navier-Stokes equations with dissipative term of order 1.

Unfortunately, this is impossible. The viscosity of a fluid is measured thanks to its Reynolds number Re and one has the von Karman relation

$$\mathrm{Re} = \frac{\mathrm{Ma}}{\mathrm{Kn}}.$$

Considering the Knudsen number to go to zero therefore implies that either the Reynolds number explodes or that the Mach number goes to zero as well. Therefore, hydrodynamical limits with finite viscosity must be incompressible, since the Mach number measures compressibility.

Fluids equations has been derived from the physics of continua, considering that in arbitrarily small regions of space the fluid is at equilibrium. Euler or Navier-Stokes equations should thus arise from gas dynamics as a perturbation around a global equilibrium, say $M_0 = M_{(1,0,1)}$ without loss of generality. We therefore look at solutions to the Boltzmann equation (3.2.2) of the form

$$f_\varepsilon = M_0 + \delta_\varepsilon h_\varepsilon,$$

where ε still stands for the Knudsen number. We expect that different orders of perturbations δ_ε lead to different hydrodynamical regimes. Indeed, properties of the Boltzmann operator, see Chapter 1, yield

$$\mathrm{Ma}\partial_t h_\varepsilon + v \cdot \nabla_x h_\varepsilon = \frac{1}{\varepsilon} L_{M_0}(h_\varepsilon) + \frac{\delta_\varepsilon}{\varepsilon} Q(h_\varepsilon, h_\varepsilon), \tag{3.2.9}$$

and the term δ_ε emphasized the role the linearity of the equation or, on the contrary, the bilinear part depending on the order of magnitude of δ_ε compared to ε.

In any case, since δ_ε tends to zero as ε go to zero we have that if $h_\varepsilon \to h$ then formally taking the limit as $\varepsilon \to 0$ in (3.2.9) gives

$$L_{M_0}(h) = 0$$

which means that h is a fluctuation of a maxwellian, i.e. of the following form:

$$h(t,x,v) = \left[\rho(t,x) + u(t,x) \cdot v + \left(\frac{|v|^2 - d}{2}\right)\theta(t,x)\right] M_0. \tag{3.2.10}$$

The acoustic limit.
The Mach number has to be related to the Knudsen number if one hopes to obtain a viscous hydrodynamical limit. We briefly mention that in the case where $\mathrm{Ma} = 1$ the Reynolds number tends to infinity whereas the fluid stays compressible. Therefore, one wants to recover the acoustic equations (propagation of waves in the medium) in the limit ε to zero. This is indeed the case as long as $\delta_\varepsilon = \varepsilon^\beta$ with $\beta > 1/2$, see [6][7].

The fluid limits.
In the case Ma tends to zero as ε goes to zero, taking the limit in the local conservation of mass and momemtum associated to (3.2.9) gives

$$\nabla_x \cdot \langle v, h\rangle_{L^2_v} = 0 \quad \text{and} \quad \nabla_x \cdot \langle v \otimes v, h\rangle_{L^2_v} = 0,$$

which stands for the incompressibility criteria and the Boussinesq approximation

$$\nabla_x \cdot u = 0 \quad \text{and} \quad \nabla_x (\rho + \theta) = 0.$$

The Boussinesq relation is an approximation for fluids where density evolutions can be neglected. As a result, in such fluids we can neglect the effects of inner inertia which makes sound waves impossible to develop, which is in adequation with our previous paragraph.

In order to derive the hydrodynamical limit we come back to the conservation laws satisfied by (3.2.9). As an example we consider the conservation of momentum and energy (the conservation of mass being straightforward at least formally) which reads

$$\partial_t \langle v, h_\varepsilon \rangle_{L^2_v} + \frac{1}{\mathrm{Ma}} \nabla_x \cdot \langle v \otimes v, h_\varepsilon \rangle_{L^2_v} = 0,$$

and can be written in terms of A (see (3.2.4)),

$$\partial_t \langle v, h_\varepsilon \rangle_{L^2_v} + \frac{1}{\mathrm{Ma}} \nabla_x \cdot \langle A(v), h_\varepsilon \rangle_{L^2_v} + \nabla_x \left(\frac{1}{\mathrm{Ma}} \langle \frac{1}{d} |v|^2, h_\varepsilon \rangle_{L^2_v} \right) = 0.$$

The last term on the left-hand side seems to explode as ε and Ma tend to 0. However, it is a gradient so if we integrate the equality against a divergence free test function this term disappears. For now on, we consider the computations in a weak sense, integrated against divergence free functions. This way, we can only recover solutions to incompressibles fluid equations in the Leray weak sense [66].

The idea is to use the self-adjointness property of L_{M_0} and express $L_{M_0}(h_\varepsilon)$ thanks to (3.2.9). This gives

$$\lim_{\varepsilon \to 0} \frac{1}{\mathrm{Ma}} \langle A(v), h_\varepsilon \rangle = \lim_{\varepsilon \to 0} \left[\frac{\varepsilon}{\mathrm{Ma}} \langle v \otimes L^{-1}_{M_0}(A(v)), h_\varepsilon \rangle - \frac{\delta_\varepsilon}{\mathrm{Ma}} \langle L^{-1}_{M_0}(A(v)), Q(h_\varepsilon, h_\varepsilon) \rangle \right].$$

The temperature equation is handled the same with the operator B instead of A and to conclude we have to use their properties (3.2.6).

Here we see the importance of the order of magnitude of Ma and δ_ε compared to the Knudsen number ε. The results are the following:

- $\mathrm{Ma} = \delta_\varepsilon = \varepsilon^q$ with $0 < q < 1$ leads to incompressible Euler equations since collisions are faster than the dissipation,
- $\mathrm{Ma} = \varepsilon^q$ with $0 < q < 1$ and $\delta_\varepsilon = \varepsilon^p$ with $p > 1$ leads to incompressible Stokes equation since this time the non-linearity vanishes in the limit,
- $\mathrm{Ma} = \delta_\varepsilon = \varepsilon$ leads to incompressible Navier-Stokes equations because dissipation and collisions occur at the same scale.

A formal proof of these statements is given in [8][46] and a survey of the existing rigorous results can be found in [111]. As noticed before, the physical quantities associated to h_ε converges to weak solutions to the fluid models (Leray sense for Navier-Stokes and dissipative sense for Euler). Rigorous proofs of those derivations require compactness arguments. A strategy that proved itself to be useful is to use the framework of renormalized solutions derived by DiPerna and Lions [38], like in [5].

3.3 A brief description of the following chapters

The problem of hydrodynamical limit is also closely related to the Cauchy problem associated to the linearized Boltzmann equation (3.2.9). Indeed, one would like to have a local or global existence in a perturbative setting with some kind of uniformity in the Knudsen number. Such results exist and we refer to Chapter 4 for a state of the art of the matter.

Furthermore, the methods described above do not give explicit rate of convergence or a constructive way of deriving the limit. Such concerns can be dealt with thanks to the study of the explicit form of the linear semigroup in the Fourier space. This point of view was initiated by Ellis and Pinsky [39] and further developed by Bardos and Ukai [10] to obtain uniform convergence in time. The issue of the initial layer at time $t = 0$ arises from these studies and we refer to Chapter 4 for a deeper understanding of the phenomenon.

The next two chapters deal with the incompressible Navier-Stokes hydrodynamical limit on the torus.

In Chapter 4 we build a Cauchy theory in Sobolev spaces with a maxwellian weight as well as exponential convergence to equilibrium. Then, extending the strategy of [10] to the torus, we prove strong convergence of solutions to the Boltzmann equation towards incompressible Navier-Stokes Leray solutions. Moreover, we give a constructive proof with explicit rates of convergence.

Chapter 5 aims at getting rid of the maxwellian weight as well as derivatives to build a Cauchy theory in Sobolev with polynomial weight. This uses a recent extension theorem for semigroups result by Gualdani, Mischler and Mouhot [51]. Such uniform results should allow to derive solutions to Navier-Stokes equations in large spaces, in particular L^∞_x.

Chapter 4

From Boltzmann to incompressible Navier-Stokes on the torus in Sobolev spaces

We investigate the Boltzmann Equation, depending on the Knudsen number, in the Navier-Stokes perturbative setting on the torus. Using hypocoercivity, we derive a new proof of existence and exponential decay for the solutions of this perturbed equation, with explicit regularity bounds and rates of convergence. These results are uniform in the Knudsen number and thus allow us to obtain a strong derivation of the incompressible Navier-Stokes equations as the Knudsen number tends to 0*. Moreover, our method shows that the smaller the Knudsen number, the less control on the* v*-derivatives of the initial perturbation is needed to have existence and decay. It is also used to deal with other kinetic models. Finally, we show that the study of this hydrodynamical limit is rather different on the torus than the already proven convergences in the whole space as it requires averaging in time, unless the initial layer conditions are satisfied.*

Contents

4.1 Introduction

This chapter deals with the Boltzmann equation in a perturbative setting as the Knudsen number tends to 0. We consider the latter equation to describe the behaviour of rarefied gas particles moving on $\mathbb{T}^d$ (flat torus of dimension $d \geqslant 2$) with velocities in $\mathbb{R}^d$ when the only interactions taken into account are binary collisions. More precisely, the Boltzmann equation describes the time evolution of the distribution of particles in position and velocity. A formal derivation of the Boltzmann equation from Newton's laws under the rarefied gas assumption can be found in [28], while [30] present Lanford's Theorem (see [65] and [44] for detailed proofs) which rigorously proves the derivation in short times.

Consider the following more general form of it, where we denote the Knudsen number by ε .

$$\begin{aligned} \partial_t f + v\cdot\nabla_x f &= \frac{1}{\varepsilon}Q(f,f)\ ,\ \text{on } \mathbb{T}^d\times\mathbb{R}^d \\ &= \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \Phi\left(|v-v_*|\right) b\left(\cos\theta\right)\left[f'f'_*-ff_*\right] dv_* d\sigma, \end{aligned} \tag{4.1.1}$$

where f', f_*, f'_* and f are the values taken by f at v', v_*, v'_* and v respectively. Define:

$$\left\{\begin{aligned} v' &= \frac{v+v_*}{2}+\frac{|v-v_*|}{2}\sigma \\ v'_* &= \frac{v+v_*}{2}-\frac{|v-v_*|}{2}\sigma \end{aligned}\right.\ ,\ \text{and}\quad \cos\theta = \left\langle \frac{v-v_*}{|v-v_*|},\sigma\right\rangle.$$

One can find in [28], [30] or [46] that the global equilibria for the Boltzmann equation are the *Maxwellians* $\mu(v)$, which are gaussian density functions. Without loss of generality we consider only the case of normalized Maxwellians:

$$\mu(v) = \frac{1}{(2\pi)^{\frac{d}{2}}}e^{-\frac{|v|^2}{2}}.$$

The bilinear operator $Q(g,h)$ is given by

$$Q(g,h) = \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \Phi\left(|v-v_*|\right) b\left(\cos\theta\right)\left[h'g'_*-hg_*\right] dv_* d\sigma.$$

4.1.1 The problem and its motivations

The Knudsen number is the inverse of the average number of collisions for each particle per unit of time. Therefore, as reviewed in [111], one can expect a convergence from the Boltzmann model towards the acoustics and the fluid dynamics as the Knudsen number tends to 0. This latter convergence will be specified. However, these different models describe physical phenomenon that do not evolve at the same timescale. As suggested

in previous studies (see [46][111][98]) we can rescale our original equation in time by a factor ε, to get rid of these time scale differences. Moreover, they also suggested that a perturbation of order ε around the global equilibrium $\mu(v)$ should approximate, as the Knudsen number tends to 0, the incompressible Navier-Stokes equations.
Therefore we study the following equation

$$\partial_t f_\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x f_\varepsilon = \frac{1}{\varepsilon^2} Q(f_\varepsilon, f_\varepsilon)\ , \text{ on } \mathbb{T}^d \times \mathbb{R}^d, \tag{4.1.2}$$

under the linearization $f_\varepsilon(t,x,v) = \mu(v) + \varepsilon \mu^{1/2}(v) h_\varepsilon(t,x,v)$. This leads us to the linearized Boltzmann equation

$$\partial_t h_\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x h_\varepsilon = \frac{1}{\varepsilon^2} L(h_\varepsilon) + \frac{1}{\varepsilon} \Gamma(h_\varepsilon, h_\varepsilon). \tag{4.1.3}$$

that we will study thoroughly, and where

$$\begin{cases} L(h) & = \left[Q(\mu, \mu^{\frac{1}{2}} h) + Q(\mu^{\frac{1}{2}} h, \mu) \right] \mu^{-\frac{1}{2}} \\ \Gamma(g,h) & = \frac{1}{2} \left[Q(\mu^{\frac{1}{2}} g, \mu^{\frac{1}{2}} h) + Q(\mu^{\frac{1}{2}} h, \mu^{\frac{1}{2}} g) \right] \mu^{-\frac{1}{2}}. \end{cases}$$

All along this chapter we consider the Boltzmann equation with *hard potential* or *Maxwellian potential* ($\gamma = 0$), that is to say there is a constant $C_\Phi > 0$ such that

$$\Phi(z) = C_\Phi z^\gamma\ ,\ \ \gamma \in [0,1],$$

and a strong form of Grad's *angular cutoff* (see [48]), expressed here by the fact that we assume b to be C^1 with the controls from above

$$\forall z \in [-1,1],\ b(z), b(z') \leqslant C_b,$$

b and Φ being defined in equation (4.1.1).

The aim of the present chapter is to use a constructive method to obtain existence and exponential decay for solutions to the linearized Boltzmann equation (4.1.3), uniformly in the Knudsen number.

Such a uniform result is then used to derive explicit rates of convergence for $(h_\varepsilon)_{\varepsilon>0}$ towards its limit as ε tends to 0. This allows us to prove, and quantify, the convergence from Boltzmann equation to the incompressible Navier-Stokes equations (4.1.4).

4.1.2 Notations

Throughout this chapter, we will use the following notations. For two multi-indexes j and l in $\mathbb{N}^d$ we define:

- $\partial_l^j = \partial_{v_j}\partial_{x_l}$,
- for i in $\{1,\dots,d\}$ we denote by $c_i(j)$ the i^{th} coordinate of j,
- the length of j will be written $|j| = \sum_i c_i(j)$,
- the multi-index δ_{i_0} by : $c_i(\delta_{i_0}) = 1$ if $i = i_0$ and 0 elsewhere.

We will write the spaces we are working on $L^p_{x,v} = L^p\left(\mathbb{T}^d \times \mathbb{R}^d\right)$, $L^p_x = L^p\left(\mathbb{T}^d\right)$ and $L^p_v = L^p\left(\mathbb{R}^d\right)$. The Sobolev spaces $H^s_{x,v}$, H^s_x and H^s_v are defined in the same way and we denote the standard Sobolev norms by $\|\cdot\|^2_{H^s_{x,v}} = \sum\limits_{|j|+|l|\leqslant s} \left\|\partial_l^j \cdot\right\|^2_{L^2_{x,v}}$.

4.1.3 Our strategy and results

The first step of this chpater is to investigate the equation (4.1.3) in order to obtain existence and exponential decay of solutions in Sobolev spaces $H^s_{x,v}$, independently of the Knudsen number ε. Moreover, we want all the required smallness assumptions and rates of convergence to be explicit. Such a result has been proved in [56] by studying independently the behaviour of both microscopic and fluid parts of solutions to (4.1.3), we proposed here another method based on hypocoercivity estimates.

Our strategy is to build a norm on Sobolev spaces which is equivalent to the standard norm and which satisfies a Gronwall type inequality.

First, we construct a functional on $H^s_{x,v}$ by considering a linear combination of $\left\|\partial_l^j \cdot\right\|^2_{L^2_{x,v}}$, for all $|j|+|l| \leqslant s$, together with product terms of the form $\langle \partial^{\delta_i}_{l-\delta_i} \cdot, \partial^0_l \cdot\rangle_{L^2_{x,v}}$. The distortion of the standard norm by the addition of mixed terms is necessary to have a relaxation, due to the hypocoercivity property of the linearized Boltzmann equation (4.1.3) (see [82]).

We then study the flow of this functional along time for solutions to the linearized Boltzmann equation (4.1.3). This flow is controlled by energy estimates and, finally, a non-trivial choice of coefficients in the functional yields an equivalence between the functional and the standard Sobolev norm, as well as a Gronwall type inequality, both of them being independent of ε.

We first apply this strategy to the linear case (i.e. without considering the bilinear remainder term) and prove that it generates a strong semigroup with a spectral gap and, therefore, an exponential relaxation (Theorem 4.2.1). We then extend our method to the full nonlinear model and obtain an a priori estimate on our functional (Proposition 4.2.2).

This estimate enables us to prove the existence of solutions to the Cauchy problem and their exponential decay as long as the initial data is small enough with a smallness not depending on ε (Theorem 4.2.3). We emphasize here that, thanks to the functional we used, the smaller ε the less control is needed on the v-derivatives of the initial data.

However, these results seem to tell us that the v-derivatives of solutions to equation (4.1.3) can blow-up as ε tends to 0. Thus, the last step is to create a new functional, based on the microscopic part of solutions, satisfying the same properties but controlling the v-derivatives as well. The fact that we ask for a control on the microscopic part of solutions to equation (4.1.3) is due to the deep structure of the linear operator L. This leads to the expected exponential decay independently of ε even for those v-derivatives (Theorem 4.2.4).

Finally, the chief aim of the present chapter is to derive explicit rates of convergence from solutions to the linearized Boltzmann equation to the incompressible Navier-Stokes equations.

Theorem 4.2.3 tells us that for all ε we can build a solution h_ε to the linearized Boltzmann equation (4.1.3), as long as the initial perturbation is sufficiently small, independently of ε. We can then consider the sequence $(h_\varepsilon)_{0<\varepsilon\leqslant 1}$ and study its limit. It appears that it converges weakly in $L^\infty_t H^s_x L^2_v$, for $s \geqslant s_0 > d/2$, towards a function h. Furthermore, we have the following form for h (see [10])

$$h(t,x,v) = \left[\rho(t,x) + v.u(t,x) + \frac{1}{2}(|v|^2 - N)\theta(t,x)\right]\mu(v)^{1/2},$$

of which physical observables are weak (in the Leray sense [66]) solutions of the linearized incompressible Navier-Stokes equations (p being the pressure function, ν and κ being constants determined by L, see Theorem 5 in [46])

$$\begin{aligned} \partial_t u - \nu\Delta u + u\cdot\nabla u + \nabla p = 0,& \\ \nabla\cdot u = 0,& \\ \partial_t\theta - \kappa\Delta\theta + u\cdot\nabla\theta = 0,& \end{aligned} \tag{4.1.4}$$

together with the Boussineq relation

$$\nabla(\rho+\theta) = 0. \tag{4.1.5}$$

However, in order to know the initial data of these quantities, we study the Fourier transform on the torus of our linear operator and use Duhamel formula. This gives us a strong convergence result on the time average of h_ε with an explicit rate of convergence in finite time. An interpolation between this finite time convergence and the exponential stability of the global equilibria in Boltzmann and Navier-Stokes equations concludes with a strong convergence for all times (Theorem 4.2.5). The way we tackle this convergence allows us to obtain an explicit form for the limit of $(h_\varepsilon)_{\varepsilon>0}$.

4.1.4 Comparison with existing results

For physical purposes, one may assume that $\varepsilon = 1$ which is a mere normalization and that is why many articles about the linearized Boltzmann equation only deal with this case. The associated Cauchy problem has been worked on over the past fifty years, starting with Grad [50], and it has been studied in different spaces, such as weighted $L^2_v(H^l_x)$ spaces [107] or weighted Sobolev spaces [53][55][114]. Other results have also been proved in $\mathbb{R}^d$ instead of the torus, see for instance [88][1][31], but it will not be the purpose of this chapter.

Our chapter explicitly deals with the general case for ε and we prove results that are uniform in ε, allowing us to consider the hydrodynamical limit as the Knudsen number tends to 0. To solve the Cauchy problem we used an iterative scheme, like in the papers mentioned above, but our strategy yields a condition for the existence of solutions in $H^s_{x,v}$ (without any weight) which is uniform in ε (Theorem 4.6.3). In order to obtain such a result, we had to consider more precise estimates on the bilinear operator Γ, depending on the existence of v-derivatives or not. Bardos and Ukai [10] obtained a similar result in $\mathbb{R}^d$ but in weighted Sobolev spaces and did not prove any decay.

The behaviour of such global in time solutions has also been studied. Guo worked in weighted Sobolev spaces and proved the boundedness of solutions to equation (4.1.3) in [55], as well as an exponential decay (uniform in ε) in [56]. The norm involved in [55][56] is quite intricate and requires a lot of technical computations. To avoid specific and technical calculations, the theory of hypocoercivity (see [81]) focuses on the properties of the Boltzmann operator and which are quite similar to hypoellipticity. This theory has been used in [82] to obtain exponential decay in standard Sobolev spaces in the case $\varepsilon = 1$.

We use the idea of Mouhot and Neumann developed in [82] consisting of considering a functional on $H^s_{x,v}$ involving mixed scalar products. In this chapter we thus construct such a quadratic form, but with coefficient depending on ε. Working in the general case for ε yields new calculations and requires the use of certain orthogonal properties of the bilinear operator Γ to overcome these issues. Moreover, we must construct a new norm out of this functional, which controls the v-derivatives by a factor ε.

The fact that the study yields a norm containing some ε factors prevents us from having a uniform exponential decay for the v-derivatives. We use the idea of Guo, in [56], of looking at the microscopic part of the solution h_ε everytime we look at a differentiation in v. This idea catches the interesting structure of L on its orthogonal part. Combining this idea with our previous strategy fills the gap for the v-derivatives.

Finally, our uniform results enable us to derive a weak convergence in $H^s_x L^2_v$ towards solutions to the incompressible Navier-Stokes equations, together with the Boussineq relation. We then find a way to obtain strong convergence using the ideas of the Fourier study of the linear operator $L - v.\nabla_x$, developed in [39] and [10], combined with Duhamel formula. However, the study done in [10] relies strongly on an argument of stationary phase developed in [109] which is no longer applicable in the torus. Indeed, the Fourier space of $\mathbb{R}^d$ is continuous and so integration by parts can be used in that frequency space. This tool is no longer available in the frequency space of the torus which is discrete.

Theorem 4.2.5 shows that the behaviour of the hydrodynamical limit is quite different on the torus, where an averaging in time is necessary for general initial data. However,we obtain the same relation between the limit at $t = 0$ and the initial perturbation h_{in} and also the existence of an initial layer. That is to say that we have a convergence in $L^2_{[0,T]} = L^2([0,T])$ if and only if the initial perturbation satisfies some physical properties, which appear to be the same as in $\mathbb{R}^d$ studied in [10].

This convergence gives a perturbative result for incompressible Navier-Stokes equations in Sobolev spaces around the steady solution. The regularity of the weak solutions we constructed implies that they are in fact strong solutions (see Serrin [99][100] and Lions [67] Section 2.5). Moreover, our uniform exponential decay for solutions to the linearized Boltzmann equation yields an exponential decay for the perturbative solutions of the incompressible Navier-Stokes equations in higher Sobolev spaces. Such an exponential convergence to equilibrium has been derived in H^1_0 for $d = 2$ or $d = 3$ in [106], or can be deduced from Proposition 3.7 in [72] in higher Sobolev spaces for small initial data. The general convergence to equilibrium can be found in [74] (small initial data) and in [90] but they focus on the general compressible case and no rate of decay is built.

Furthermore, results that do not involve hydrodynamical limits (existence and exponential decay results) are applicable to a larger class of operators. In Appendix 4.*A* we prove that those theorems also hold for other kinetic collisional models such as the linear relaxation, the semi-classical relaxation, the linear Fokker-Planck equation and the Landau equation with hard and moderately soft potential.

4.1.5 Organization of the chapter

Section 4.2 is divided in two different subsections.

As mentionned above, we shall use the hypocoercivity of the Boltzmann equation (4.1.1). This hypocoercivity can be described in terms of technical properties on L and Γ and, in order to obtain more general results, we consider them as a basis of our chapter. Thus, subsection 4.2.1 describes them in detail and a proof of the fact that L and Γ indeed satisfy those properties is given in Appendix 4.*A*. Most of them have been proved in [82] but we require more precise ones to deal with the general case.

The second subsection 4.2.2 is dedicated to a mathematical formulation of the results described in subsection 4.1.3.

As said when we described our strategy (subsection 4.1.3), we are going to study the flow of a functional involving $L^2_{x,v}$-norm of x and v derivatives and mixed scalar products. To control this flow in time we compute energy estimates for each of these terms in a toolbox (section 4.3) which will be used and referred to all along the rest of the chapter. Proofs of those energy estimates are given in Appendix $4.B$.

Finally, sections 4.4, 4.5, 4.6, 4.7 and 4.8 are the proofs respectively of Theorem 4.2.1 (about the strong semigroup property of the linear part of equation (4.1.3)), Proposition 4.2.2 (an a priori estimates on the constructed functional for the full model), Theorem 4.2.3 (existence and exponential decay of solutions to equation (4.1.3)), Theorem 4.2.4 (showing the uniform boundedness of the v-derivatives) and of Theorem 4.2.5 (dealing with the hydrodynamical limit).

We notice here that section 4.6 is divided in two subsection. Subsection 4.6.1 deals with the existence of solutions for all $\varepsilon > 0$ and subsection 4.6.2 proved the exponential decay of those solutions.

4.2 Main results

This section is divided in two parts. The first one translate the hypocoercivity aspects of the Boltzmann operator in terms of mathematical properties for L and Γ. Then, the second one states our results in terms of those assumptions.

4.2.1 Hypocoercivity assumptions

This section is dedicated to the framework and assumptions of the hypocoercivity theory. A state of the art of this theory can be found in [81].

4.2.1.1 Assumptions on the linear operator L

Assumptions in $H^1_{x,v}$:

(H1): Coercivity and general controls

$L : L^2_v \longrightarrow L^2_v$ is a closed and self-adjoint operator with $L = K - \Lambda$ such that:

- Λ is coercive:
  - there exists $\|.\|_{\Lambda_v}$ norm on L^2_v such that

$$\forall h \in L^2_v\,,\ \nu_0^\Lambda \|h\|^2_{L^2_v} \leqslant \nu_1^\Lambda \|h\|^2_{\Lambda_v} \leqslant \langle \Lambda(h), h\rangle_{L^2_v} \leqslant \nu_2^\Lambda \|h\|^2_{\Lambda_v}\,,$$

– Λ has a defect of coercivity regarding its v derivatives:

$$\forall h \in H^1_v\,,\ \langle \nabla_v \Lambda(h), \nabla_v h\rangle_{L^2_v} \geqslant \nu_3^\Lambda \left\|\nabla_v h\right\|^2_{\Lambda_v} - \nu_4^\Lambda \left\|h\right\|^2_{\Lambda_v}.$$

- There exists $C^L > 0$ such that

$$\forall h \in L^2_v\,,\ \forall g \in L^2_v\,,\ \langle L(h), g\rangle_{L^2_v} \leqslant C^L \left\|h\right\|_{\Lambda_v} \left\|g\right\|_{\Lambda_v},$$

where $(\nu_s^\Lambda)_{1\leqslant s\leqslant 4}$ are strictly positive constants depending on the operator and the dimension of the velocities space d.
As in [82], we define a new norm on $L^2_{x,v}$:

$$\left\|.\right\|_\Lambda = \left\|\left\|.\right\|_{\Lambda_v}\right\|_{L^2_x}.$$

(H2): Mixing property in velocity

$$\forall \delta > 0\,,\ \exists C(\delta) > 0\,,\ \forall h \in H^1_v\,,\quad \langle \nabla_v K(h), \nabla_v h\rangle_{L^2_v} \leqslant C(\delta)\left\|h\right\|^2_{L^2_v} + \delta\left\|\nabla_v h\right\|^2_{L^2_v}.$$

(H3): Relaxation to equilibrium
We suppose that the kernel of L is generated by N functions which form an orthonormal basis for $\mathrm{Ker}(L)$:

$$\mathrm{Ker}(L) = \mathrm{Span}\{\phi_1(v), \dots, \phi_N(v)\}.$$

Moreover, we assume that the ϕ_i are of the form $P_i(v)e^{-|v|^2/4}$, where P_i is a polynomial.

Furthermore, denoting by π_L the orthogonal projector in L^2_v on $\mathrm{Ker}(L)$ we assume that we have the following local coercivity property:

$$\exists \lambda > 0\,,\ \forall h \in L^2_v\,,\quad \langle L(h), h\rangle_{L^2_v} \leqslant -\lambda \left\|h^\perp\right\|^2_{\Lambda_v},$$

where $h^\perp = h - \pi_L(h)$ denotes the microscopic part of h (the orthogonal to $\mathrm{Ker}(L)$ in L^2_v).

We are using the same hypothesis as in [82], except that we require the ϕ_i to be of a specific form. This additional requirement allows us to derive properties on the v-derivatives of π_L that we will state in the toolbox section 4.3.

Then we have two more properties on L in order to deal with higher order Sobolev spaces.

Assumptions in $H^s_{x,v}$, $s > 1$:

(H1'): Defect of coercivity for higher derivatives
We assume that L satisfies $(H1)$ along with the following property: for all $s \geqslant 1$, for all $|j| + |l| = s$ such that $|j| \geqslant 1$,

$$\forall h \in H^s_{x,v}\,, \quad \langle \partial^j_l \Lambda(h), \partial^j_l h\rangle_{L^2_{x,v}} \geqslant \nu_5^\Lambda \left\|\partial^j_l h\right\|^2_\Lambda - \nu_6^\Lambda \left\|h\right\|_{H^{s-1}_{x,v}}\,,$$

where ν_5^Λ and ν_6^Λ are strictly positive constants depending on L and d.
We also define a new norm on $H^s_{x,v}$:

$$\|\cdot\|_{H^s_\Lambda} = \left(\sum_{|j|+|l|\leqslant s} \left\|\partial^j_l \cdot\right\|^2_\Lambda\right)^{1/2}.$$

(H2'): Mixing properties

As above, Mouhot and Neumann extended the hypothesis $(H2)$ to higher Sobolev's spaces: for all $s \geqslant 1$, for all $|j| + |l| = s$ such that $|j| \geqslant 1$,

$$\forall \delta > 0\,,\ \exists C(\delta) > 0\,,\ \forall h \in H^s_{x,v}\,, \quad \langle \partial^j_l K(h), \partial^j_l h\rangle_{L^2_{x,v}} \leqslant C(\delta)\left\|h\right\|^2_{H^{s-1}_{x,v}} + \delta \left\|\partial^j_l h\right\|^2_{L^2_{x,v}}.$$

4.2.1.2 Assumptions on the second order term Γ

To solve our problem uniformly in ε we had to precise the hypothesis made in [82] in order to have a deeper understanding of the operator Γ. This lead us to two different assumptions.

(H4): Control on the second order operator

$\Gamma : L^2_v \times L^2_v \longrightarrow L^2_v$ is a bilinear symmetric operator such that for all multi-indexes j and l such that $|j| + |l| \leqslant s$, $s \geqslant 0$,

$$\left|\langle \partial^j_l \Gamma(g,h), f\rangle_{L^2_{x,v}}\right| \leqslant \begin{cases} \mathcal{G}^s_{x,v}(g,h)\left\|f\right\|_\Lambda & ,\ \text{if } j \neq 0 \\ \mathcal{G}^s_{x}(g,h)\left\|f\right\|_\Lambda & ,\ \text{if } j = 0 \end{cases},$$

$\mathcal{G}^s_{x,v}$ and $\mathcal{G}^s_x$ being such that $\mathcal{G}^s_{x,v} \leqslant \mathcal{G}^{s+1}_{x,v}$, $\mathcal{G}^s_x \leqslant \mathcal{G}^{s+1}_x$ and satisfying the following property:

$$\exists s_0 \in \mathbb{N}\,,\ \forall s \geqslant s_0\,,\ \exists C_\Gamma > 0\,, \quad \begin{cases} \mathcal{G}^s_{x,v}(g,h) & \leqslant C_\Gamma \left(\left\|g\right\|_{H^s_{x,v}}\left\|h\right\|_{H^s_\Lambda} + \left\|h\right\|_{H^s_{x,v}}\left\|g\right\|_{H^s_\Lambda}\right) \\ \mathcal{G}^s_{x}(g,h) & \leqslant C_\Gamma \left(\left\|h\right\|_{H^s_x L^2_v}\left\|g\right\|_{H^s_\Lambda} + \left\|g\right\|_{H^s_x L^2_v}\left\|h\right\|_{H^s_\Lambda}\right). \end{cases}$$

(H5): Orthogonality to the Kernel of the linear operator

$$\forall h,\ g \in \mathrm{Dom}(\Gamma) \cap L^2_v\,, \quad \Gamma(g,h) \in \mathrm{Ker}(L)^\perp.$$

4.2.2 Statement of the Theorems

4.2.2.1 Uniform result for the linear Boltzmann equation

For s in $\mathbb{N}^*$ and some constants $(b^{(s)}_{j,l})_{j,l}$, $(\alpha^{(s)}_l)_l$ and $(a^{(s)}_{i,l})_{i,l}$ strictly positive and $0 < \varepsilon \leqslant 1$ we define the following functional on $H^s_{x,v}$, where we emphasize that there is a dependance on ε, which is the key point of our study:

$$\|\cdot\|_{\mathcal{H}^s_\varepsilon} = \left[\sum_{\substack{|j|+|l|\leqslant s\\|j|\geqslant 1}} b^{(s)}_{j,l}\varepsilon^2 \left\|\partial^j_l \cdot\right\|^2_{L^2_{x,v}} + \sum_{|l|\leqslant s} \alpha^{(s)}_l \left\|\partial^0_l \cdot\right\|^2_{L^2_{x,v}} + \sum_{\substack{|l|\leqslant s\\ i,c_i(l)>0}} a^{(s)}_{i,l}\varepsilon\langle \partial^{\delta_i}_{l-\delta_i}\cdot, \partial^0_l\cdot\rangle_{L^2_{x,v}}\right]^{\frac{1}{2}}.$$

We first study the linearized equation (4.1.3), without taking into account the bilinear remainder operator. By letting π_w be the projector in $L^2_{x,v}$ onto $\mathrm{Ker}(w)$ we obtained the following semigroup property for L.

Theorem 4.2.1 *If L is a linear operator satisfying the conditions* (H1'), (H2') *and* (H3) *then there exists $0 < \varepsilon_d \leqslant 1$ such that for all s in $\mathbb{N}^*$,*

1. *for all $0 < \varepsilon \leqslant \varepsilon_d$, $G_\varepsilon = \varepsilon^{-2}L - \varepsilon^{-1}v\cdot\nabla_x$ generates a C^0-semigroup on $H^s_{x,v}$.*
2. *there exist $C^{(s)}_G$, $(b^{(s)}_{j,l})$, $(\alpha^{(s)}_l)$, $(a^{(s)}_{i,l}) > 0$ such that for all $0 < \varepsilon \leqslant \varepsilon_d$:*

$$\|\cdot\|^2_{\mathcal{H}^s_\varepsilon} \sim \left(\|\cdot\|^2_{L^2_{x,v}} + \sum_{|l|\leqslant s}\left\|\partial^0_l\cdot\right\|^2_{L^2_{x,v}} + \varepsilon^2\sum_{\substack{|l|+|j|\leqslant s\\|j|\geqslant 1}}\left\|\partial^j_l\cdot\right\|^2_{L^2_{x,v}}\right),$$

and for all h in $H^s_{x,v}$,

$$\langle G_\varepsilon(h), h\rangle_{\mathcal{H}^s_\varepsilon} \leqslant -C^{(s)}_G \left\|h - \pi_{G_\varepsilon}(h)\right\|^2_{H^s_\Lambda}.$$

This theorem gives us an exponential decay for the semigroup generated by G_ε.

4.2.2.2 Uniform perturbative result for the Boltzmann equation

The next result states that if we add the bilinear remainder operator then it is enough, if ε is small enough, to slightly change our new norm to have a control on the solution.

Proposition 4.2.2 *If L is a linear operator satisfying the conditions* (H1'), (H2') *and* (H3) *and Γ a bilinear operator satisfying* (H4) *and* (H5) *then there exists $0 < \varepsilon_d \leqslant 1$ such that for all s in $\mathbb{N}^*$,*

1. *there exist* $K_0^{(s)}$, $K_1^{(s)}$, $K_2^{(s)}$ $(b_{j,l}^{(s)})$, $(\alpha_l^{(s)})$, $(a_{i,l}^{(s)}) > 0$, *independent of* Γ *and* ε, *such that for all* $0 < \varepsilon \leqslant \varepsilon_d$*:*

$$\|\cdot\|^2_{\mathcal{H}^s_\varepsilon} \sim \left(\|\cdot\|^2_{L^2_{x,v}} + \sum_{|l|\leqslant s} \left\|\partial_l^0 \cdot\right\|^2_{L^2_{x,v}} + \varepsilon^2 \sum_{\substack{|l|+|j|\leqslant s \\ |j|\geqslant 1}} \left\|\partial_l^j \cdot\right\|^2_{L^2_{x,v}} \right),$$

2. *and for all* h_{in} *in* $H^s_{x,v} \cap \mathrm{Ker}(G_\varepsilon)^\perp$ *and all* g *in* $\mathrm{Dom}(\Gamma) \cap H^s_{x,v}$, *if we have a solution* h *in* $H^s_{x,v}$ *to the following equation*

$$\partial_t h + \frac{1}{\varepsilon} v \cdot \nabla_x h = \frac{1}{\varepsilon^2} L(h) + \frac{1}{\varepsilon}\Gamma(g,h),$$

then

$$\frac{d}{dt}\|h\|^2_{\mathcal{H}^s_\varepsilon} \leqslant -K_0^{(s)} \|h\|^2_{H^s_\Lambda} + K_1^{(s)} \left(\mathcal{G}^s_x(g,h)\right)^2 + \varepsilon^2 K_2^{(s)} \left(\mathcal{G}^s_{x,v}(g,h)\right)^2.$$

One can remark that the norm constructed above leaves free the x-derivatives while it controls the v ones by a factor ε.

We want to emphasize here that this result shows that the derivative of the norm is control by the x-derivatives of Γ and the Sobolev norm of Γ, but weakened by a factor ε^2. This is important as our norm $\|.\|^2_{\mathcal{H}^s_\varepsilon}$ controls the $L^2_v(H^s_x)$-norm by a factor of order 1 whereas it controls the whole $H^s_{x,v}$-norm by a multiplicative factor of order $1/\varepsilon$.

Theorem 4.2.3 *Let* Q *be a bilinear operator such that:*

- *the equation* (4.1.2) *admits an equilibrium* $0 \leqslant \mu \in L^1(\mathbb{T}^d \times \mathbb{R}^d)$,
- *the linearized operator* $L = L(h)$ *around* μ *with the scaling* $f = \mu + \varepsilon\mu^{1/2}h$ *satisfies* (H1'), (H2') *and* (H3),
- *the bilinear remaining term* $\Gamma = \Gamma(h,h)$ *in the linearization satisfies* (H4) *and* (H5).

Then there exists $0 < \varepsilon_d \leqslant 1$ *such that for any* $s \geqslant s_0$ *(defined in* (H4) *),*

1. *there exist* $(b_{j,l}^{(s)})$, $(\alpha_l^{(s)})$, $(a_{i,l}^{(s)}) > 0$, *independent of* Γ *and* ε, *such that for all* $0 < \varepsilon \leqslant \varepsilon_d$*:*

$$\|\cdot\|^2_{\mathcal{H}^s_\varepsilon} \sim \left(\|\cdot\|^2_{L^2_{x,v}} + \sum_{|l|\leqslant s} \left\|\partial_l^0 \cdot\right\|^2_{L^2_{x,v}} + \varepsilon^2 \sum_{\substack{|l|+|j|\leqslant s \\ |j|\geqslant 1}} \left\|\partial_l^j \cdot\right\|^2_{L^2_{x,v}} \right),$$

2. *there exist* $\delta_s > 0$, $C_s > 0$ *and* $\tau_s > 0$ *such that for all* $0 < \varepsilon \leqslant \varepsilon_d$*:*

For any distribution $0 \leqslant f_{in} \in L^1(\mathbb{T}^d \times \mathbb{R}^d)$ *with* $f_{in} = \mu + \varepsilon\mu^{1/2}h_{in} \geqslant 0$, h_{in} *in* $\mathrm{Ker}(G_\varepsilon)^\perp$ *and*

$$\|h_{in}\|_{\mathcal{H}^s_\varepsilon} \leqslant \delta_s,$$

there exists a unique global smooth (in $H^s_{x,v}$*, continuous in time) solution* $f_\varepsilon = f_\varepsilon(t,x,v)$ *to* (4.1.2) *which, moreover, satisfies* $f_\varepsilon = \mu + \varepsilon\mu^{1/2}h_\varepsilon \geqslant 0$ *with:*

$$\|h_\varepsilon\|_{\mathcal{H}^s_\varepsilon} \leqslant \|h_{in}\|_{\mathcal{H}^s_\varepsilon} e^{-\tau_s t}.$$

The fact that we are asking h_{in} to be in $\mathrm{Ker}(G_\varepsilon)^\perp$ just states that we want f_{in} to have the same physical quantities as the global equilibrium μ. This is a compulsory requirement as one can easily check that the physical quantities

$$\int_{\mathbb{T}^d\times\mathbb{R}^d} f_\varepsilon(x,v)dxdv, \quad \int_{\mathbb{T}^d\times\mathbb{R}^d} vf_\varepsilon(x,v)dxdv, \quad \int_{\mathbb{T}^d\times\mathbb{R}^d} |v|^2 f_\varepsilon(x,v)dxdv$$

are preserved with time (see [30] for instance).

Notice that the $\mathcal{H}^s_\varepsilon$-norm is this theorem is the same than the one we constructed in Proposition 4.2.2.

4.2.2.3 The boundednes of the v-derivatives

As a corollary we have that the $H^s_x(L^2_v)$-norm decays exponentially independently of ε but that the only control we have on the $H^s_{x,v}$ is

$$\|h_\varepsilon\|_{H^s_{x,v}} \leqslant \frac{\delta_s}{\varepsilon}e^{-\tau_s t}.$$

This seems to tell us that the v-derivatives can blow-up at a rate $1/\varepsilon$. However, Guo, in [56], showed that one can prove that there is no explosion if one controls independently the fluid part and the microscopic part of the solution. This idea, combined with our original one, leads to the construction of a new norm which will only control the microscopic part of the solution whenever we face a derivative in the v variable.

We define the following positive quadratic form

$$\|\cdot\|^2_{\mathcal{H}^s_{\varepsilon\perp}} = \sum_{\substack{|j|+|l|\leqslant s\\ |j|\geqslant 1}} b^{(s)}_{j,l}\left\|\partial^j_l(\mathrm{Id}-\pi_L)\right\|^2_{L^2_{x,v}} + \sum_{|l|\leqslant s}\alpha^{(s)}_l\left\|\partial^0_l\cdot\right\|^2_{L^2_{x,v}} + \sum_{\substack{|l|\leqslant s\\ i,c_i(l)>0}} a^{(s)}_{i,l}\,\varepsilon\langle\partial^{\delta_i}_{l-\delta_i}\cdot,\partial^0_l\cdot\rangle_{L^2_{x,v}}.$$

Theorem 4.2.4 *Under the same conditions as in Theorem 4.2.3, for all* $s \geqslant s_0$*, there exist* $(b^{(s)}_{j,l})$, $(\alpha^{(s)}_l)$, $(a^{(s)}_{i,l}) > 0$ *and* $0 < \varepsilon_d \leqslant 1$ *such that for all* $0 < \varepsilon \leqslant \varepsilon_d$*:*

1. $\|\cdot\|_{\mathcal{H}^s_{\varepsilon\perp}} \sim \|\cdot\|_{H^s_{x,v}}$*, independently of* ε*,*

2. if h_ε is a solution of 4.1.3 in $H^s_{x,v}$ with $\|h_{in}\|_{\mathcal{H}^s_{\varepsilon\perp}} \leqslant \delta'_s$ then

$$\|h_\varepsilon\|_{\mathcal{H}^s_{\varepsilon\perp}} \leqslant \delta'_s e^{-\tau'_s t},$$

where δ'_s and τ'_s are strictly positive constants independent of ε.

This theorem builds up a functional that is equivalent to the standard Sobolev norm, independently of ε. Thus, it gives us the exponential decay of the v-derivatives as well as the decay of the x-derivatives. However, the distorted norm used in Theorem 4.2.3 asked less control on the v-derivatives for the initial data, suggesting that, in the limit as ε goes to zero, almost only the x-variable has to be controlled to obtain existence and exponential decay.

4.2.2.4 The hydrodynamical limit on the torus for Maxwellian particles

Our theorem states that one can really expect a convergence of solutions of collisional kinetic models near equilibrium towards a solution of fluid dynamics equations. Indeed, the smallness assumption on the initial perturbation does not depend on the parameter ε as long as ε is small enough.

We then define the following macroscopic quantities

- the particles density $\rho_\varepsilon(t,x) = \langle \mu(v)^{1/2}, h_\varepsilon(t,x,v)\rangle_{L^2_v}$,
- the mean velocity $u_\varepsilon(t,x) = \langle v\mu(v)^{1/2}, h_\varepsilon(t,x,v)\rangle_{L^2_v}$,
- the temperature $\theta_\varepsilon(t,x) = \dfrac{1}{d}\langle (|v|^2 - d)\mu(v)^{1/2}, h_\varepsilon(t,x,v)\rangle_{L^2_v}$.

The theorem 4.2.3 tells us that, for $s \geqslant s_0$, the sequence $(h_\varepsilon)_{\varepsilon>0}$ converges (up to an extraction) weakly-* in $L^\infty_t(H^s_l L^2_v)$ towards a function h. Such a weak convergence enables us to use the theorem 1.1 of [10], which is a slight modification of the result in [8] to get that

1. h is in $\mathrm{Ker}(L)$, so of the form
$$h(t,x,v) = \left[\rho(t,x) + v.u(t,x) + \frac{1}{2}(|v|^2 - d)\theta(t,x)\right]\mu(v)^{1/2},$$
2. $(\rho_\varepsilon, u_\varepsilon, \theta_\varepsilon)$ converges weakly* in $L^\infty_t(H^s_x)$ towards (ρ, u, θ),
3. (ρ, u, θ) satisfies the incompressible Navier-Stokes equations (4.1.4) as well as the Boussineq equation (4.1.5).

If such a result confirms the fact that one can derive the incompressible Navier-Stokes equations from the Boltzmann equation, it does unfortunately neither give us the continuity of h nor the initial condition verified by (ρ, u, θ), depending on $(\rho_{in}, u_{in}, \theta_{in})$, macroscopic quantities associated to h_{in}. Our next, and final step, is therefore to link the last two triplets and so to understand the convergence $h_\varepsilon \to h$ more deeply. This is the purpose of the next, and last, theorem.

Theorem 4.2.5 *Consider $s \geqslant s_0$ and h_{in} in $H^s_{x,v}$ such that $\|h_{in}\|_{\mathcal{H}^s_\varepsilon} \leqslant \delta_s$.*

Then, $(h_\varepsilon)_{\varepsilon>0}$ exists for all $0 < \varepsilon \leqslant \varepsilon_d$ and converges weakly in $L^\infty_t(H^s_x L^2_v)$ towards h such that $h \in Ker(L)$, with $\nabla_x \cdot u = 0$ and $\rho + \theta = 0$.*

Furthermore, $\int_0^T hdt$ belongs to $H^s_x L^2_v$ and there exists $C > 0$ such that,

$$\left\| \int_0^{+\infty} hdt - \int_0^{+\infty} h_\varepsilon dt \right\|_{H^s_x L^2_v} \leqslant C\sqrt{\varepsilon\, |ln(\varepsilon)|}.$$

One can have a strong convergence in $L^2_{[0,T]} H^s_x L^2_v$ only if h_{in} is in $\mathrm{Ker}(L)$ with $\nabla_x \cdot u_{in} = 0$ and $\rho_{in} + \theta_{in} = 0$ (initial layer conditions).

Moreover, in that case we have

$$\|h - h_\varepsilon\|_{L^2_{[0,+\infty)} H^s_x L^2_v} \leqslant C\sqrt{\varepsilon\, |ln(\varepsilon)|},$$

and for all δ in $[0,1]$, if h_{in} belongs to $H^{s+\delta}_x L^2_v$,

$$\sup_{t\in[0,+\infty)} \|h - h_\varepsilon\|_{H^s_x L^2_v}(t) \leqslant C\varepsilon^{\min(\delta,1/2)}.$$

This theorem gives us strong convergences for $(\rho_\varepsilon, u_\varepsilon, \theta_\varepsilon)$ towards (ρ, u, θ) but above all it gives us that (ρ, u, θ) is the solution to the incompressible Navier-Stokes equations together with the Boussineq equation satisfying the initial conditions:

- $u(0,x) = Pu_{in}(x)$, where $Pu_{in}(x)$ is the divergence-free part of $u_{in}(x)$,
- $\rho(0,x) = -\theta(0,x) = \frac{1}{2}(\rho_{in}(x) - \theta_{in}(x))$.

Finally, we emphasize that in the case of initial data satisfying the initial layer conditions, the strong convergence in time requires a little bit more regularity from the initial data. This fact was already noticed in $\mathbb{R}^d$ (see [10] Lemma 6.1) but overcome by considering weighted norms in velocity.

4.3 Toolbox: fluid projection and a priori energy estimates

In this section we are going to give some inequalities we are going to use and to refer to throughout the sequel. First we start with some properties concerning the projection in L^2_v onto $\mathrm{Ker}(L)$: π_L. Then, because we want to estimate all the terms appearing in the $H^s_{x,v}$-norm to estimate the functionals $\mathcal{H}^s_\varepsilon$ and $\mathcal{H}^s_{\varepsilon\perp}$, we will give upper bound on their time derivatives. The proofs are only technical and the interested reader will find them in Appendix 4.B.

We are assuming there that L is having properties (H1'), (H2') and (H3), that Γ satisfies (H4) and (H5) and that $0 < \varepsilon \leqslant 1$.

4.3.1 Properties concerning the fluid projection π_L

We already know that L is acting on L^2_v, with $\mathrm{Ker}(L) = \mathrm{Span}(\phi_1, \dots, \phi_N)$, with $(\phi_i)_{1\leqslant i\leqslant N}$ an orthonormal family, we obtain directly a useful formula for the orthogonal projection on $\mathrm{Ker}(L)$ in L^2_v, π_L:

$$\forall h \in L^2_v, \quad \pi_L(h) = \sum_{i=1}^N \left(\int_{\mathbb{R}^d} h\phi_i dv\right) \phi_i. \tag{4.3.1}$$

Plus, (H3) states that $\phi_i = P_i(v)e^{-|v|^2/4}$, where P_i is a polynomial. Therefore, direct computations and Cauchy-Schwarz inequality give that π_L is continuous on $H^s_{x,v}$ with

$$\forall s \in \mathbb{N}, \exists C_{\pi s} > 0, \forall h \in H^s_{x,v}, \quad \|\pi_L(h)\|^2_{H^s_{x,v}} \leqslant C_{\pi s} \|h\|^2_{H^s_{x,v}}. \tag{4.3.2}$$

More precisely one can find that for all s in $\mathbb{N}$

$$\forall |j| + |l| = s, \forall h \in H^s_{x,v}, \quad \left\|\partial^j_l \pi_L(h)\right\|^2_{L^2_{x,v}} \leqslant C_{\pi s} \left\|\partial^0_l \pi_L(h)\right\|^2_{L^2_{x,v}}. \tag{4.3.3}$$

Finally, building the Λ-norm one can find that in all the collisional kinetic equations concerned here we have that

$$\exists C_\pi > 0, \forall h \in L^2_{x,v}, \ \|\pi_L(h)\|^2_\Lambda \leqslant C_\pi \|h\|^2_{L^2_{x,v}}. \tag{4.3.4}$$

Then we can also use the properties of the torus to obtain Poincare type inequalities. This can be very useful thanks to the next proposition, which is proved in Appendix 4.B.

Proposition 4.3.1 *Let a and b be in $\mathbb{R}^*$ and consider the operator $G = aL - bv.\nabla_x$ acting on $H^1_{x,v}$.*

If L satisfies (H1) *and* (H3) *then*

$$\mathrm{Ker}(G) = \mathrm{Ker}(L).$$

Remark 4.3.2 *In this proposition, Ker(G) has to be understood as linear combinations with constant coefficients of the functions Φ_i. This subtlety has to be emphasized since in $L^2_{x,v}$, Ker(L) includes all linear combinations of Φ_i but with coefficients being functions of x.*

Therefore, if we define, for $0 < \varepsilon \leqslant 1$:

$$G_\varepsilon = \frac{1}{\varepsilon^2} L - \frac{1}{\varepsilon} v.\nabla_x,$$

then we have a nice desciption of π_{G_ε}:

$$\forall h \in L^2_{x,v},\ \pi_{G_\varepsilon}(h) = \sum_{i=1}^{N} \left(\int_{\mathbb{T}^d} \int_{\mathbb{R}^d} h\phi_i \, dxdv \right) \phi_i.$$

That means that $\pi_{G_\varepsilon}(h)$ is, up to a multiplicative constant, the mean of $\pi_L(h)$ over the torus. We deduce that if h belongs to $\mathrm{Ker}(G_\varepsilon)^\perp$, $\pi_L(h)$ has zero mean on the torus and is an operator not depending on the x variable. Thus we can apply Poincaré inequality on the torus:

$$\forall h \in \mathrm{Ker}(G_\varepsilon)^\perp, \quad \|\pi_L(h)\|^2_{L^2_{x,v}} \leqslant C_p \|\nabla_x \pi_L(h)\|^2_{L^2_{x,v}} \leqslant C_p \|\nabla_x h\|^2_{L^2_{x,v}}. \tag{4.3.5}$$

4.3.2 A priori energy estimates

Our work in this chapter is to study the evolution of the norms involved in the definition of the operators $\mathcal{H}^s_\varepsilon$ and $\mathcal{H}^s_{\varepsilon\perp}$ and to combine them to obtain the results stated above. The Appendix 4.*B* contains the proofs, which are technical computations together with some choices of decomposition, of the following a priori estimates. Note that all the constants K_1, K_{dx} and K_{s-1} used in the inequalities below are independent of ε, Γ and g, and only depend constructively on the constants defined in the hypocoercivity assumptions or in the subsection above. The number e can be any positive real number and will be chosen later.

We would like to study both linear and non-linear models but they appeared to be very similar. In order to avoid long and similar inequalities we will write in parenthesis terms we need to add for the full model.

Let g be a function in $H^s_{x,v}$. We now consider a function h in $\mathrm{Ker}(G_\varepsilon)^\perp \cap H^s_{x,v}$, for some s in $\mathbb{N}^*$, which is solution of the linear (linearized) Boltzmann equation:

$$\partial_t h + \frac{1}{\varepsilon} v.\nabla_x h = \frac{1}{\varepsilon^2} L(h) \left(+ \frac{1}{\varepsilon}\Gamma(g,h) \right).$$

We remind the reader that the following notation is used: $h^\perp = h - \pi_L(h)$.

4.3.2.1 Time evolutions for quantities in $H^1_{x,v}$

We write the $L^2_{x,v}$-norm estimate

$$\frac{d}{dt}\|h\|^2_{L^2_{x,v}} \leqslant -\frac{\lambda}{\varepsilon^2}\left\|h^\perp\right\|^2_\Lambda \left(+\frac{1}{\lambda}\left(\mathcal{G}^0_x(g,h)\right)^2\right). \tag{4.3.6}$$

Then the time evolution of the x-derivatives

$$\frac{d}{dt}\|\nabla_x h\|^2_{L^2_{x,v}} \leqslant -\frac{\lambda}{\varepsilon^2}\left\|\nabla_x h^\perp\right\|^2_\Lambda \left(+\frac{1}{\lambda}\left(\mathcal{G}^1_x(g,h)\right)^2\right), \tag{4.3.7}$$

and of the v-derivatives

$$\begin{aligned}\frac{d}{dt}\|\nabla_v h\|^2_{L^2_{x,v}} \quad \leqslant \quad & \frac{K_1}{\varepsilon^2}\left\|h^\perp\right\|^2_\Lambda + \frac{K_{dx}}{\varepsilon^2}\|\nabla_x h\|^2_{L^2_{x,v}} - \frac{\nu_3^\Lambda}{\varepsilon^2}\|\nabla_v h\|^2_\Lambda \\ & \left(+\frac{3}{\nu_3^\Lambda}\left(\mathcal{G}^1_{x,v}(g,h)\right)^2\right).\end{aligned} \tag{4.3.8}$$

Finally, we will need a control on the scalar product as well, as explained in the strategy subsection 4.1.3. Notice that we have some freedom as e can be any positive number.

$$\begin{aligned}\frac{d}{dt}\langle \nabla_x h, \nabla_v h\rangle_{L^2_{x,v}} \quad \leqslant \quad & \frac{C^L e}{\varepsilon^3}\left\|\nabla_x h^\perp\right\|^2_\Lambda - \frac{1}{\varepsilon}\|\nabla_x h\|^2_{L^2_{x,v}} + \frac{2C^L}{e\varepsilon}\|\nabla_v h\|^2_\Lambda \\ & \left(+\frac{e}{C^L\varepsilon}\left(\mathcal{G}^1_x(g,h)\right)^2\right).\end{aligned} \tag{4.3.9}$$

4.3.2.2 Time evolutions for quantities in $H^s_{x,v}$

We consider multi-indexes j and l such that $|j| + |l| = s$.
As in the previous case, we have a control on the time evolution of the pure x-derivatives,

$$\frac{d}{dt}\left\|\partial^0_l h\right\|^2_{L^2_{x,v}} \leqslant -\frac{\lambda}{\varepsilon^2}\left\|\partial^0_l h^\perp\right\|^2_\Lambda \left(+\frac{1}{\lambda}\left(\mathcal{G}^s_x(g,h)\right)^2\right). \tag{4.3.10}$$

In the case where $|j| \geqslant 1$, that is to say when we have at least one derivative in v, we obtained the following upper bound

$$\begin{aligned}\frac{d}{dt}\left\|\partial_l^j h\right\|_{L^2_{x,v}}^2 \quad &\leqslant \quad -\frac{\nu_5^\Lambda}{\varepsilon^2}\left\|\partial_l^j h\right\|_\Lambda^2 + \frac{3(\nu_1^\Lambda)^2 d}{\nu_5^\Lambda(\nu_0^\Lambda)^2}\sum_{i,c_i(j)>0}\left\|\partial_{l+\delta_i}^{j-\delta_i} h\right\|_\Lambda^2 + \frac{K_{s-1}}{\varepsilon^2}\left\|h\right\|_{H^{s-1}_{x,v}}^2 \\ &\left(+\frac{3}{\nu_5^\Lambda}\left(\mathcal{G}^s_{x,v}(g,h)\right)^2\right).\end{aligned} \tag{4.3.11}$$

We may find useful to consider the particular case where $|j|=1$,

$$\begin{aligned}\frac{d}{dt}\left\|\partial_{l-\delta_i}^{\delta_i} h\right\|_{L^2_{x,v}}^2 \quad &\leqslant \quad -\frac{\nu_5^\Lambda}{\varepsilon^2}\left\|\partial_{l-\delta_i}^{\delta_i} h\right\|_\Lambda^2 + \frac{3\nu_1^\Lambda}{\nu_5^\Lambda\nu_0^\Lambda}\left\|\partial_l^0 h\right\|_{L^2_{x,v}}^2 + \frac{K_{s-1}}{\varepsilon^2}\left\|h\right\|_{H^{s-1}_{x,v}}^2 \\ &\left(+\frac{3}{\nu_5^\Lambda}\left(\mathcal{G}^s_{x,v}(g,h)\right)^2\right).\end{aligned} \tag{4.3.12}$$

Finally we will need the time evolution of the following scalar product:

$$\begin{aligned}\frac{d}{dt}\langle\partial_{l-\delta_i}^{\delta_i} h,\partial_l^0 h\rangle_{L^2_{x,v}} \quad &\leqslant \quad \frac{C^L e}{\varepsilon^3}\left\|\partial_l^0 h^\perp\right\|_\Lambda^2 - \frac{1}{\varepsilon}\left\|\partial_l^0 h\right\|_{L^2_{x,v}}^2 + \frac{2C^L}{e\varepsilon}\left\|\partial_{l-\delta_i}^{\delta_i} h\right\|_\Lambda^2 \\ &\left(+\frac{e}{C^L\varepsilon}\left(\mathcal{G}^s_x(g,h)\right)^2\right),\end{aligned} \tag{4.3.13}$$

where we still have some freedom as e is any positive number.

We just emphasize here that one can see that we were careful about which derivatives are involved in the terms that contain Γ. This is because our operator $\|.\|_{\mathcal{H}^s_\varepsilon}$ controls the $H^s_x(L^2_v)$-norm by a mere constant whereas it controls the entire $H^s_{x,v}$-norm by a factor $1/\varepsilon$.

4.3.2.3 Time evolutions for orthogonal quantities in $H^s_{x,v}$

For the theorem 4.2.4 we are going to need four others inequalities which are a little bit more intricate as they need to know the shape of π_L as described in the subsection above. The proofs are written in Appendix 4.*B* and we are just looking at the whole equation in the setting $g=h$.

We want the time evolution of the v-derivatives of the orthogonal (microscopic) part of h, as suggested in [56] this allows us to really take advantage of the structure of the linear operator L on its orthogonal:

$$\begin{aligned}\frac{d}{dt}\left\|\nabla_v h^\perp\right\|_{L^2_{x,v}}^2 \quad &\leqslant \quad \frac{K_1^\perp}{\varepsilon^2}\left\|h^\perp\right\|_\Lambda^2 + K_{dx}^\perp\left\|\nabla_x h\right\|_{L^2_{x,v}}^2 - \frac{\nu_3^\Lambda}{2\varepsilon^2}\left\|\nabla_v h^\perp\right\|_\Lambda^2 \\ &+\frac{3}{\nu_3^\Lambda}\left(\mathcal{G}^1_{x,v}(h,h)\right)^2.\end{aligned} \tag{4.3.14}$$

Then we can have a new bound for the scalar product used before

$$\begin{aligned}\frac{d}{dt}\langle \nabla_x h, \nabla_v h\rangle_{L^2_{x,v}} \quad \leqslant \quad & \frac{K^\perp e}{\varepsilon^3}\left\|\nabla_x h^\perp\right\|^2_\Lambda + \frac{1}{4C_{\pi 1}C_\pi C_p e\varepsilon}\left\|\nabla_v h^\perp\right\|^2_\Lambda \\ & -\frac{1}{2\varepsilon}\left\|\nabla_x h\right\|^2_{L^2_{x,v}} + \frac{4C_\pi}{\varepsilon}\left(\mathcal{G}^1_{x,v}(h,h)\right)^2, \end{aligned} \qquad (4.3.15)$$

where e is any number greater than 1.

As usual, we may need the same kind of bounds in higher degree Sobolev spaces. The reader may notice that the bounds we are about to write are more intricate than the ones in the previous section because they involve more terms with less derivatives. We consider multi-indexes j and l such that $|j|+|l|=s$. This time we really have to divide in two different cases.

Firstly when $|j|\geqslant 2$,

$$\begin{aligned}\frac{d}{dt}\left\|\partial^j_l h^\perp\right\|^2_{L^2_{x,v}} \leqslant & -\frac{\nu_5^\Lambda}{\varepsilon^2}\left\|\partial^j_l h^\perp\right\|^2_\Lambda + \frac{9(\nu_1^\Lambda)^2 d}{2(\nu_0^\Lambda)^2\nu_5^\Lambda}\sum_{i,c_i(j)>0}\left\|\partial^{j-\delta_i}_{l+\delta_i} h^\perp\right\|^2_\Lambda \\ & + K^\perp_{dl}\sum_{|l'|\leqslant s-1}\left\|\partial^0_{l'} h\right\|^2_{L^2_{x,v}} + \frac{K^\perp_{s-1}}{\varepsilon^2}\left\|h^\perp\right\|^2_{H^{s-1}_{x,v}} + \frac{3}{\nu_5^\Lambda}\left(\mathcal{G}^s_{x,v}(h,h)\right)^2.\end{aligned} \qquad (4.3.16)$$

Then the case when $|j|=1$

$$\begin{aligned}\frac{d}{dt}\left\|\partial^{\delta_i}_{l-\delta_i} h^\perp\right\|^2_{L^2_{x,v}} \quad \leqslant \quad & -\frac{\nu_5^\Lambda}{\varepsilon^2}\left\|\partial^{\delta_i}_{l-\delta_i} h^\perp\right\|^2_\Lambda + K^\perp_{dl}\sum_{|l'|=s}\left\|\partial^0_{l'} h\right\|^2_{L^2_{x,v}} + \frac{K^\perp_{s-1}}{\varepsilon^2}\left\|h^\perp\right\|^2_{H^{s-1}_{x,v}} \\ & + \frac{3}{\nu_5^\Lambda}\left(\mathcal{G}^s_{x,v}(h,h)\right)^2. \end{aligned} \qquad (4.3.17)$$

Finally we give a new version of the control over the scalar product in higher Sobolev's spaces.

$$\begin{aligned}\frac{d}{dt}\langle \partial^{\delta_i}_{l-\delta_i} h, \partial^0_l h\rangle_{L^2_{x,v}} \quad \leqslant \quad & \frac{\tilde{K}^\perp}{\varepsilon^3} e\left\|\partial^0_l h^\perp\right\|^2_\Lambda + \frac{1}{4C_{\pi s}C_\pi d e\varepsilon}\left\|\partial^{\delta_i}_{l-\delta_i} h^\perp\right\|^2_\Lambda - \frac{1}{2\varepsilon}\left\|\partial^0_l h\right\|^2_{L^2_{x,v}} \\ & + \frac{1}{4d\varepsilon}\sum_{|l'|\leqslant s-1}\left\|\partial^0_{l'} h\right\|^2_{L^2_{x,v}} + \frac{2C_\pi}{\varepsilon}\left(\mathcal{G}^s_{x,v}(h,h)\right)^2, \end{aligned} \qquad (4.3.18)$$

for any $e\geqslant 1$.

4.4 Linear case: proof of Theorem 4.2.1

In this section we are looking at the linear equation

$$\partial_t h = G_\varepsilon(h), \text{ on } \mathbb{T}^d \times \mathbb{R}^d.$$

Theorem 4.2.1 will be proved by induction on s. We remind here the operator we will work with on $H^s_{x,v}$

- in the case $s = 1$:

$$\|h\|^2_{\mathcal{H}^1_\varepsilon} = A\,\|h\|^2_{L^2_{x,v}} + \alpha\,\|\nabla_x h\|^2_{L^2_{x,v}} + b\varepsilon^2\,\|\nabla_v h\|^2_{L^2_{x,v}} + a\varepsilon\langle \nabla_x h, \nabla_v h\rangle_{L^2_{x,v}},$$

- in the case $s > 1$:

$$\|h\|^2_{\mathcal{H}^s_\varepsilon} = \sum_{\substack{|j|+|l|\leqslant s\\ |j|\geqslant 1}} b^{(s)}_{j,l}\varepsilon^2 \left\|\partial^j_l h\right\|^2_{L^2_{x,v}} + \sum_{|l|\leqslant s} \alpha^{(s)}_l \left\|\partial^0_l h\right\|^2_{L^2_{x,v}} + \sum_{\substack{|l|\leqslant s\\ i,c_i(l)>0}} a^{(s)}_{i,l}\varepsilon\langle \partial^{\delta_i}_{l-\delta_i} h, \partial^0_l h\rangle_{L^2_{x,v}}.$$

The Theorem 4.2.1 only requires us to choose suitable coefficients that gives us the expected inequality and equivalence.

Consider h_{in} in $H^s_{x,v} \cap \mathrm{Dom}(G_\varepsilon)$. Let h be a solution of $\partial_t h = G_\varepsilon(h)$ on $\mathbb{T}^d \times \mathbb{R}^d$ such that $h(0,\cdot,\cdot) = h_{in}(\cdot,\cdot)$.

Notice that if h_{in} is in $H^s_{x,v} \cap \mathrm{Dom}(G_\varepsilon) \cap \mathrm{Ker}(G_\varepsilon)$ then we have that the associated solution remains the same in time: $\partial_t h = 0$. Therefore the fluid part of a solution does not evolve in time and so the semigroup is identity on $\mathrm{Ker}(G_\varepsilon)$. Besides, we can see directly from the definition and the adjointness property of L that $h \in \mathrm{Ker}(G_\varepsilon)^\perp$ for all t if h_{in} belongs in $\mathrm{Ker}(G_\varepsilon)^\perp$.

Therefore, to prove the theorem it is enough to consider h_{in} in $H^s_{x,v} \cap \mathrm{Dom}(G_\varepsilon) \cap \mathrm{Ker}(G_\varepsilon)^\perp$.

4.4.1 The case $s = 1$

For now on we assume that our operator L satisfies the conditions (H1), (H2) and (H3) and that $0 < \varepsilon \leqslant 1$.

If (H3) holds for L then we have that $\varepsilon^{-2}L$ is a non-positive self-adjoint operator on $L^2_{x,v}$. Moreover, $\varepsilon^{-1}v\cdot\nabla_x$ is skew-symmetric on $L^2_{x,v}$. Therefore the $L^2_{x,v}$-norm decreases along the flow and it can be deduced that G_ε yields a C_0-semigroup on $L^2_{x,v}$ for all positive ε (see [61] for general theory and [107] for its use in our case).

Using the toolbox, which is possible since h is in $\mathrm{Ker}(G_\varepsilon)^\perp$ for all t, we just have to consider the linear combination $A(4.3.6) + \alpha(4.3.7) + b\varepsilon^2(4.3.8) + a\varepsilon(4.3.9)$ to obtain

$$\begin{aligned}\frac{d}{dt}\|h\|^2_{\mathcal{H}^1_\varepsilon} \quad &\leqslant \quad \frac{1}{\varepsilon^2}\left[bK_1-\lambda A\right]\left\|h^\perp\right\|^2_\Lambda+\frac{1}{\varepsilon^2}\left[C^L ea-\lambda\alpha\right]\left\|\nabla_x h^\perp\right\|^2_\Lambda \\ &\quad +\left[\frac{2C^L a}{e}-b\nu_3^\Lambda\right]\|\nabla_v h\|^2_\Lambda+\left[bK_{dx}-a\right]\|\nabla_x h\|^2_{L^2_{x,v}}. \end{aligned} \tag{4.4.1}$$

Then we make the following choices:

1. We fix b such that $-\nu_3^\Lambda b<-1$.
2. We fix A big enough such that $[bK_1-\lambda A]\leqslant -1$.
3. We fix a big enough such that $[bK_{dx}-a]\leqslant -1$.
4. We fix e big enough such that $\left[\frac{2C^L a}{e}-b\nu_3^\Lambda\right]\leqslant -1$.
5. We fix α big enough such that $\left[C^L ea-\lambda\alpha\right]\leqslant -1$ and such that $\begin{cases} a^2 & \leqslant \alpha b \\ b & \leqslant \alpha\end{cases}$.

This leads to, because $0<\varepsilon\leqslant 1$:

$$\frac{d}{dt}\|h\|^2_{\mathcal{H}^1_\varepsilon}\leqslant -\left(\left\|h^\perp\right\|^2_\Lambda+\left\|\nabla_x h^\perp\right\|^2_\Lambda+\|\nabla_v h\|^2_\Lambda+\|\nabla_x h\|^2_{L^2_{x,v}}\right).$$

Finally we can apply the Poincaré inequality (4.3.5) together with the equivalence of the $L^2_{x,v}$-norm and the Λ-norm on the fluid part π_L, equation (4.3.4), to get

$$\exists C,C'>0,\quad \begin{cases} \|h\|^2_\Lambda & \leqslant C\left(\|h^\perp\|^2_\Lambda+\frac{1}{2}\|\nabla_x h\|^2_{L^2_{x,v}}\right), \\ \|\nabla_x h\|^2_\Lambda & \leqslant C'\left(\|\nabla_x h^\perp\|^2_\Lambda+\frac{1}{2}\|\nabla_x h\|^2_{L^2_{x,v}}\right).\end{cases}$$

Therefore we proved the following result:

$$\exists K>0, \forall\, 0<\varepsilon\leqslant 1\,,\quad \frac{d}{dt}\|h\|^2_{\mathcal{H}^1_\varepsilon}\leqslant -C_G^{(1)}\left(\|h\|^2_\Lambda+\|\nabla_{x,v}h\|^2_\Lambda\right).$$

With these constants, $\|.\|_{\mathcal{H}^1_\varepsilon}$ is equivalent to

$$\left(\|h\|^2_{L^2_{x,v}}+\|\nabla_x h\|^2_{L^2_{x,v}}+\varepsilon^2\|\nabla_v h\|^2_{L^2_{x,v}}\right)^{1/2}$$

since $a^2\leqslant \alpha b$ and $b\leqslant\alpha$ and hence:

$$A\|h\|^2_{L^2_{x,v}}+\frac{b}{2}\left(\|\nabla_x h\|^2_{L^2_{x,v}}+\varepsilon^2\|\nabla_v h\|^2_{L^2_{x,v}}\right)\leqslant \|h\|^2_{\mathcal{H}^1_\varepsilon}$$

and

$$\|h\|^2_{\mathcal{H}^1_\varepsilon}\leqslant A\|h\|^2_{L^2_{x,v}}+\frac{3\alpha}{2}\left(\|\nabla_x h\|^2_{L^2_{x,v}}+\varepsilon^2\|\nabla_v h\|^2_{L^2_{x,v}}\right).$$

The results above gives us the expected theorem for $s = 1$.

4.4.2 The induction in higher order Sobolev spaces

Then we assume that the theorem is true up to the integer $s-1$, $s > 1$. Then we suppose that L satisfies (H1'), (H2') and (H3) and we consider ε in $(0,1]$.

Let h_{in} be in $H^s_{x,v} \cap \mathrm{Dom}(G_\varepsilon) \cap \mathrm{Ker}(G_\varepsilon)^\perp$ and h be the solution of $\partial_t h = G_\varepsilon(h)$ such that $h(0,\cdot,\cdot) = h_{in}(\cdot,\cdot)$.

As before, h belongs to $\mathrm{Ker}(G_\varepsilon)^\perp$ for all t and thus we can use the results given by the toolbox.

Thanks to the proof in the case $s = 1$ we know that we are able to handle the case where there is only a difference of one derivative between the number of derivatives in x and in v. Therefore, instead of working with the entire norm of $H^s_{x,v}$, we will look at an equivalent of the Sobolev semi-norm. We define:

$$\begin{aligned}
F_s(t) &= \sum_{\substack{|j|+|l|=s \\ |j|\geqslant 2}} \varepsilon^2 B \left\|\partial^j_l h\right\|^2_{L^2_{x,v}} + B' \sum_{\substack{|l|=s \\ i, c_i(l)>0}} Q_{l,i}(t), \\
Q_{l,i}(t) &= \alpha \left\|\partial^0_l h\right\|^2_{L^2_{x,v}} + b\varepsilon^2 \left\|\partial^{\delta_i}_{l-\delta_i} h\right\|^2_{L^2_{x,v}} + a\varepsilon \langle \partial^{\delta_i}_{l-\delta_i} h, \partial^0_l h\rangle_{L^2_{x,v}},
\end{aligned}$$

where the constants, strictly positive, will be chosen later.

Like in the section above, we shall study the time evolution of every term involved in F_s in order to bound above $dF_s/dt(t)$ with negative coefficients.

4.4.2.1 The time evolution of $Q_{l,i}$

We will first study the time evolution of $Q_{l,i}$ for given $|j| + |l| = s$. The toolbox already gave us all the bounds we need and we just have to gather them in the following way: α(4.3.10) $+ b\varepsilon^2$(4.3.12) $+ a\varepsilon$(4.3.13). This leads to, because $0 < \varepsilon \leqslant 1$,

$$\begin{aligned}
\frac{d}{dt} Q_{l,i}(t) \leqslant\; & \frac{1}{\varepsilon^2}\left[C^L e a - \lambda\alpha\right] \left\|\partial^0_l h^\perp\right\|^2_\Lambda + \left[\frac{2C^L a}{e} - \nu_5^\Lambda b\right] \left\|\partial^{\delta_i}_{l-\delta_i} h\right\|^2_\Lambda \\
& + \left[\frac{3\nu_1^\Lambda}{\nu_5^\Lambda \nu_0^\Lambda} b - a\right] \left\|\partial^0_l h\right\|^2_{L^2_{x,v}} + K_{s-1} b \left\|h\right\|_{H^{s-1}_{x,v}}.
\end{aligned}$$

One can notice that, except for the last term, we have exactly the same kind of bound as in (4.4.1), in the proof of the case $s = 1$. Therefore we can choose α, b, a, e, independently of ε such that it exists $s_Q > 0$ and $C_{s-1} > 0$ such that for all $0 < \varepsilon \leqslant 1$:

- $Q_{l,i}(t) \sim \left\|\partial^0_l h\right\|^2_{L^2_{x,v}} + \varepsilon^2 \left\|\partial^{\delta_i}_{l-\delta_i} h\right\|^2_{L^2_{x,v}}$,

- $\frac{d}{dt}Q_{l,i}(t) \leqslant -K_Q \left(\left\|\partial_l^0 h\right\|_\Lambda^2 + \left\|\partial_{l-\delta_i}^{\delta_i} h\right\|_\Lambda^2 \right) + C_{s-1} \left\|h\right\|_{H^{s-1}_{x,v}},$

where we used (4.3.4) (equivalence of norms $L^2_{x,v}$ and Λ on the fluid part) to get

$$\left\|\partial_l^0 h\right\|_\Lambda^2 \leqslant C' \left(\left\|\partial_l^0 h^\perp\right\|_\Lambda^2 + \left\|\partial_l^0 h\right\|_{L^2_{x,v}}^2 \right).$$

4.4.2.2 The time evolution of F_s and conclusion

The last result about $Q_{l,i}$ gives us that

$$F_s(t) \sim \sum_{|l|=s} \left\|\partial_l^0 h\right\|_{L^2_{x,v}}^2 + \varepsilon^2 \sum_{\substack{|l|+|j|=s \\ |j|\geqslant 1}} \left\|\partial_l^j h\right\|_{L^2_{x,v}}^2 .$$

To study the time evolution of F_s we just need to combine the evolution of $Q_{l,i}$ and the one of $\left\|\partial_l^j h\right\|_{L^2_{x,v}}^2$ which is given in the toolbox by (4.3.11).

$$\begin{aligned} \frac{d}{dt}F_s(t) \quad \leqslant \quad & \sum_{\substack{|j|+|l|=s \\ |j|\geqslant 2}} -\nu_5^\Lambda B \left\|\partial_l^j h\right\|_\Lambda^2 + \sum_{\substack{|j|+|l|=s \\ |j|\geqslant 2}} \frac{3(\nu_1^\Lambda)^2 d}{\nu_5^\Lambda (\nu_0^\Lambda)^2} B\varepsilon^2 \sum_{i,c_i(j)>0} \left\|\partial_{l+\delta_i}^{j-\delta_i} h\right\|_\Lambda^2 \\ & -K_Q B' \sum_{\substack{|l|=s \\ i,c_i(l)>0}} \left(\left\|\partial_l^0 h\right\|_\Lambda^2 + \left\|\partial_{l-\delta_i}^{\delta_i} h\right\|_\Lambda^2 \right) \\ & + \left[\sum_{\substack{|j|+|l|=s \\ |j|\geqslant 2}} K_{s-1}B + \sum_{\substack{|l|=s \\ i,c_i(l)>0}} B'C_{s-1} \right] \left\|h\right\|_{H^{s-1}_{x,v}}^2 . \end{aligned} \tag{4.4.2}$$

Then we choose the following coefficients $B = 2/\nu_5^\Lambda$ and we can rearrange the sums to obtain

$$\begin{aligned} \frac{d}{dt}F_s(t) \quad \leqslant \quad & \sum_{\substack{|j|+|l|=s \\ |j|\geqslant 2}} \left(\frac{6d(\nu_1^\Lambda)^2}{(\nu_5^\Lambda \nu_0^\Lambda)^2}\varepsilon^2 - 2 \right) \left\|\partial_l^j h\right\|_\Lambda^2 + \sum_{\substack{|j|+|l|=s \\ |j|=1}} \left(\frac{6d(\nu_1^\Lambda)^2}{(\nu_5^\Lambda \nu_0^\Lambda)^2}\varepsilon^2 - K_Q B' \right) \left\|\partial_l^j h\right\|_\Lambda^2 \\ & + \sum_{\substack{|j|+|l|=s \\ |j|=0}} (-K_Q B') \left\|\partial_l^j h\right\|_\Lambda^2 + C_+^{(s-1)}(B') \left\|h\right\|_{H^{s-1}_{x,v}} . \end{aligned}$$

Therefore we can choose the remaining coefficients:

1. $\varepsilon_d = \min \left\{ 1, \sqrt{\frac{(\nu_5^\Lambda \nu_0^\Lambda)^2}{6d(\nu_1^\Lambda)^2}} \right\},$

2. we fix B' big enough such that $s_Q B' \geqslant 1$ and $\left(\frac{6d(\nu_1^\Lambda)^2}{(\nu_5^\Lambda \nu_0^\Lambda)^2}\varepsilon_d^2 - K_Q B'\right) \leqslant -1$.

Everything is now fixed in $C_+^{(s-1)}(B')$ and therefore it is just a constant $C_+^{(s-1)}$ that does not depend on ε. Therefore we then have the final result.

$$\forall\, 0 < \varepsilon \leqslant \varepsilon_d\,,\ \frac{d}{dt}F_s(t) \leqslant C_+^{(s-1)}\left\|h\right\|^2_{H^{s-1}_{x,v}} - \left(\sum_{|j|+|l|=s}\left\|\partial_l^j h\right\|^2_\Lambda\right).$$

Then, we know that $\|.\|_\Lambda$ controls the L^2-norm. And therefore:

$$\forall\, 0 < \varepsilon \leqslant \varepsilon_d\,,\ \frac{d}{dt}F_s(t) \leqslant C_+^{(s)}\left(\sum_{|j|+|l|\leqslant s-1}\left\|\partial_l^j h\right\|^2_\Lambda\right) - \left(\sum_{|j|+|l|=s}\left\|\partial_l^j h\right\|^2_\Lambda\right).$$

This inequality is true for all s and therefore we can take a linear combination of the F_s to obtain the following, where C_s is a constant that does not depend on ε since $C_+^{(s)}$ does not depend on it.

$$\forall\, 0 < \varepsilon \leqslant \varepsilon_d\,,\ \frac{d}{dt}\left(\sum_{p=1}^{n} C_p F_p(t)\right) \leqslant -C_G^{(s)}\left(\sum_{|j|+|l||\leqslant s}\left\|\partial_l^j h\right\|^2_\Lambda\right).$$

We can use the induction assumption from rank 1 up to rank $s-1$ to find that this linear combination is equivalent to

$$\|.\|^2_{L^2_{x,v}} + \sum_{|l|\leqslant s}\left\|\partial_l^0 .\right\|^2_{L^2_{x,v}} + \varepsilon^2 \sum_{\substack{|l|+|j|\leqslant s\\ |j|\geqslant 1}}\left\|\partial_l^j .\right\|^2_{L^2_{x,v}}$$

and so fits the expected requirements.

4.5 Estimate for the full equation: proof of Proposition 4.2.2

We will prove that proposition by induction on s. For now on we assume that L satisfies hypothesis (H1'), (H2') and (H3), that Γ satisfies properties (H4) and (H5) and we take g in $H^s_{x,v}$.

So we take h_{in} in $H^s_{x,v}\cap \mathrm{Ker}(G_\varepsilon)^\perp$ and we consider the associated solution, denoted by h, of

$$\partial_t h + \frac{1}{\varepsilon}v\cdot\nabla_x h = \frac{1}{\varepsilon^2}L(h) + \frac{1}{\varepsilon}\Gamma(g,h).$$

One can notice that thanks to (H5) and the self-adjointness of L, h remains in $\mathrm{Ker}(G_\varepsilon)^\perp$ for all times.

Besides, while considering the time evolution we find a term due to G_ε and another due to Γ. Therefore, we will use the results found in the toobox but including the terms in parenthesis.

4.5.1 The case $s = 1$

We want to study the following operator on $H^s_{x,v}$

$$\|h\|^2_{\mathcal{H}^1_\varepsilon} = A\,\|h\|^2_{L^2_{x,v}} + \alpha\,\|\nabla_x h\|^2_{L^2_{x,v}} + b\varepsilon^2\,\|\nabla_v h\|^2_{L^2_{x,v}} + a\varepsilon\langle\nabla_x h, \nabla_v h\rangle_{L^2_{x,v}}.$$

Therefore, using the toolbox we just have to consider the linear combination A(4.3.6)+ α(4.3.7) + $b\varepsilon^2$(4.3.8) + $a\varepsilon$(4.3.9) to yield

$$\begin{aligned}
\frac{d}{dt}\|h\|^2_{\mathcal{H}^1_\varepsilon} \;\leqslant\;& \frac{1}{\varepsilon^2}\left[bK_1 - \lambda A\right]\left\|h^\perp\right\|^2_\Lambda + \frac{1}{\varepsilon^2}\left[C^L ea - \lambda\alpha\right]\left\|\nabla_x h^\perp\right\|^2_\Lambda \\
&+\left[\frac{2C^L a}{e} - b\nu_3^\Lambda\right]\|\nabla_v h\|^2_\Lambda + \left[bK_{dx} - a\right]\|\nabla_x h\|^2_{L^2_{x,v}} \\
&+\frac{A\nu_1^\Lambda}{\nu_0^\Lambda\lambda}\left(\mathcal{G}^0_x(g,h)\right)^2 + \left[\frac{\alpha\nu_1^\Lambda}{\nu_0^\Lambda\lambda} + \frac{\nu_1^\Lambda ea}{C^L\nu_0^\Lambda}\right]\left(\mathcal{G}^1_x(g,h)\right)^2 \\
&+\frac{3\nu_1^\Lambda b}{\nu_0^\Lambda\nu_3^\Lambda}\varepsilon^2\left(\mathcal{G}^1_{x,v}(g,h)\right)^2.
\end{aligned} \tag{4.5.1}$$

One can see that we obtained exactly the same upper bound as in the proof of the previous theorem, equation (4.4.1), adding the terms involving Γ (remember that $\mathcal{G}^s_x$ is increasing in s). Therefore we can make the same choices for A, α, b, a and e, independently of Γ and g, to get that

$$\|h\|^2_{\mathcal{H}^1_\varepsilon} \sim \|h\|^2_{L^2_{x,v}} + \|\nabla_x h\|^2_{L^2_{x,v}} + \varepsilon^2\,\|\nabla_v h\|^2_{L^2_{x,v}},$$

and that, once those parameters are fixed, there exist $K_0^{(1)}$, $K_1^{(1)}$, $K_2^{(1)} > 0$ such that for all $0 < \varepsilon \leqslant 1$,

$$\frac{d}{dt}\|h\|^2_{\mathcal{H}^1_\varepsilon} \leqslant -K_0^{(1)}\left(\|h\|^2_\Lambda + \|\nabla_{x,v}h\|^2_\Lambda\right) + K_1^{(1)}\left(\mathcal{G}^1_x(g,h)\right)^2 + \varepsilon^2 K_2^{(1)}\left(\mathcal{G}^1_{x,v}(g,h)\right)^2,$$

which is the expected result in the case $s = 1$.

4.5.2 The induction in higher order Sobolev spaces

Then we assume that the theorem is true up to the integer $s-1$, $s>1$. Then we suppose that L satisfies (H1'), (H2') and (H3) and we consider ε in $(0,1]$.

Since h_{in} is in $\mathrm{Ker}(G_\varepsilon)^\perp$, h belongs to $\mathrm{Ker}(G_\varepsilon)^\perp$ for all t and so we can use the results given in the toolbox.

As in the proof in the linear case we define:

$$\begin{array}{rcl} F_s(t) & = & \displaystyle\sum_{\substack{|j|+|l|=s \\ |j|\geqslant 2}} \varepsilon^2 B \left\|\partial_l^j h\right\|_{L^2_{x,v}}^2 + B' \sum_{\substack{|l|=s \\ i, c_i(l)>0}} Q_{l,i}(t), \\ Q_{l,i}(t) & = & \alpha \left\|\partial_l^0 h\right\|_{L^2_{x,v}}^2 + b\varepsilon^2 \left\|\partial_{l-\delta_i}^{\delta_i} h\right\|_{L^2_{x,v}}^2 + a\varepsilon \langle \partial_{l-\delta_i}^{\delta_i} h, \partial_l^0 h\rangle_{L^2_{x,v}}, \end{array}$$

where the constants, strictly positive, will be chosen later.

Like in the section above, we shall study the time evolution of every term involved in F_s in order to bound above $dF_s/dt(t)$ with expected coefficients.

4.5.2.1 The time evolution of $Q_{l,i}$

We will first study the time evolution of $Q_{l,i}$ for given $|j|+|l|=s$. The toolbox already gave us all the bounds we need and we just have to gather them in the following way: α(4.3.10) $+ b\varepsilon^2$(4.3.12) $+ a\varepsilon$(4.3.13). This leads to, because $0<\varepsilon\leqslant 1$,

$$\begin{array}{rcl} \dfrac{d}{dt} Q_{l,i}(t) & \leqslant & \dfrac{1}{\varepsilon^2}\left[C^L e a - \lambda\alpha\right] \left\|\partial_l^0 h^\perp\right\|_\Lambda^2 + \left[\dfrac{2C^L a}{e} - \nu_5^\Lambda b\right] \left\|\partial_{l-\delta_i}^{\delta_i} h\right\|_\Lambda^2 \\ & & + \left[\dfrac{3\nu_1^\Lambda}{\nu_5^\Lambda \nu_0^\Lambda} b - a\right] \left\|\partial_l^0 h\right\|_{L^2_{x,v}}^2 + K_{s-1} b \left\|h\right\|_{H^{s-1}_{x,v}} \\ & & + \left[\dfrac{\alpha\nu_1^\Lambda}{\nu_0^\Lambda \lambda} + \dfrac{\nu_1^\Lambda e a}{C^L \nu_0^\Lambda}\right] \left(\mathcal{G}_x^s(g,h)\right)^2 + \dfrac{3\nu_1^\Lambda b}{\nu_0^\Lambda \nu_5^\Lambda} \varepsilon^2 \left(\mathcal{G}_{x,v}^s(g,h)\right)^2. \end{array}$$

One can notice that, except for the term in $\|h\|_{H^{s-1}_{x,v}}$, we have exactly the same kind of bound as in the case $s=1$, given by (4.5.1). Therefore we can choose α, b, a, e, independently of ε, Γ and g such that it exists K_Q, $K_{\Gamma 1}$, $K_{\Gamma 2}>0$ and $C_{s-1}>0$ such that for all $0<\varepsilon\leqslant 1$:

- $Q_{l,i}(t) \sim \left\|\partial_l^0 h\right\|_{L^2_{x,v}}^2 + \varepsilon^2 \left\|\partial_{l-\delta_i}^{\delta_i} h\right\|_{L^2_{x,v}}^2,$

•

$$\begin{aligned}\frac{d}{dt}Q_{l,i}(t) &\leqslant -K_Q\left(\left\|\partial^0_l h\right\|^2_\Lambda + \left\|\partial^{\delta_i}_{l-\delta_i} h\right\|^2_\Lambda\right) + K_{\Gamma 1}\left(\mathcal{G}^s_x(g,h)\right)^2 \\ &\quad + \varepsilon^2 K_{\Gamma 2}\left(\mathcal{G}^s_{x,v}(g,h)\right)^2 + C_{s-1}\left\|h\right\|_{H^{s-1}_{x,v}},\end{aligned}$$

where we used (4.3.4) (equivalence of norms $L^2_{x,v}$ and Λ on the fluid part) to get

$$\left\|\partial^0_l h\right\|^2_\Lambda \leqslant C'\left(\left\|\partial^0_l h^\perp\right\|^2_\Lambda + \left\|\partial^0_l h\right\|^2_{L^2_{x,v}}\right).$$

4.5.2.2 The time evolution of F_s and conclusion

The last result about $Q_{l,i}$ gives us that

$$F_s(t) \sim \sum_{|l|=s}\left\|\partial^0_l h\right\|^2_{L^2_{x,v}} + \varepsilon^2 \sum_{\substack{|l|+|j|=s \\ |j|\geqslant 1}}\left\|\partial^j_l h\right\|^2_{L^2_{x,v}},$$

so it remains to show that F_s satisfies the property describe by the theorem for some B and B'.

To study the time evolution of F_s we just need to combine the evolution of $Q_{l,i}$ and the one of $\left\|\partial^j_l h\right\|^2_{L^2_{x,v}}$ which is given in the toolbox by (4.3.11).

$$\begin{aligned}\frac{d}{dt}F_s(t) \leqslant\; & \sum_{\substack{|j|+|l|=s \\ |j|\geqslant 2}} -\nu^\Lambda_5 B\left\|\partial^j_l h\right\|^2_\Lambda + \sum_{\substack{|j|+|l|=s \\ |j|\geqslant 2}}\frac{3(\nu^\Lambda_1)^2 d}{\nu^\Lambda_5(\nu^\Lambda_0)^2}B\varepsilon^2\sum_{i,c_i(j)>0}\left\|\partial^{j-\delta_i}_{l+\delta_i}h\right\|^2_\Lambda \\ & -K_Q B'\sum_{\substack{|l|=s \\ i,c_i(l)>0}}\left(\left\|\partial^0_l h\right\|^2_\Lambda + \left\|\partial^{\delta_i}_{l-\delta_i}h\right\|^2_\Lambda\right) \\ & + \left[\sum_{\substack{|j|+|l|=s \\ |j|\geqslant 2}} K_{s-1}B + \sum_{\substack{|l|=s \\ i,c_i(l)>0}} B' C_{s-1}\right]\left\|h\right\|^2_{H^{s-1}_{x,v}} \\ & + \sum_{\substack{|l|=s \\ i,c_i(l)>0}} B' K_{\Gamma 1}\left(\mathcal{G}^s_x(g,h)\right)^2 \\ & + \varepsilon^2\left[\sum_{\substack{|l|=s \\ i,c_i(l)>0}} B'K_{\Gamma 2} + \sum_{\substack{|j|+|l|=s \\ |j|\geqslant 2}}\frac{3\nu^\Lambda_1}{\nu^\Lambda_0\nu^\Lambda_5}B\right]\left(\mathcal{G}^s_{x,v}(g,h)\right)^2.\end{aligned} \tag{4.5.2}$$

One can easily see that, apart from the terms including Γ, we have exactly the same bound as in the proof in the linear case, equation (4.4.2). Therefore we can choose B, B' and ε_d like we did, thus independent of Γ and g, to have for all $0 < \varepsilon \leqslant \varepsilon_d$

$$\begin{aligned}\frac{d}{dt}F_s(t) \leqslant \quad & C_+^{(s-1)} \|h\|^2_{H^{s-1}_{x,v}} - \left(\sum_{|j|+|l|=s} \left\|\partial_l^j h\right\|^2_\Lambda\right) \\ & + \tilde{K}_{\Gamma 1} \left(\mathcal{G}_x^s(g,h)\right)^2 + \varepsilon^2 \tilde{K}_{\Gamma 2} \left(\mathcal{G}_{x,v}^s(g,h)\right)^2,\end{aligned}$$

with $C_+^{(s-1)}$, $\tilde{K}_{\Gamma 1}$ and $\tilde{K}_{\Gamma 2}$ positive constants independent of ε, Γ and g.

To conclude we just have to, as in the linear case, take a linear combination of the $(F_p)_{p\leqslant s}$ and use the induction hypothesis (remember that both $\mathcal{G}^p_{x,v}$ and $\mathcal{G}^p_x$ are increasing functions of p) to obtain the expected result: $\forall\, 0 < \varepsilon \leqslant \varepsilon_d$,

$$\begin{aligned}\frac{d}{dt}\left(\sum_{p=1}^n C_p F_p(t)\right) \leqslant \quad - \quad & K_0^{(s)} \left(\sum_{|j|+|l|\leqslant s} \left\|\partial_l^j h\right\|^2_\Lambda\right) + K_1^{(s)} \left(\mathcal{G}_x^s(g,h)\right)^2 \\ + \quad & \varepsilon^2 K_1^{(s)} \left(\mathcal{G}_{x,v}^s(g,h)\right)^2,\end{aligned}$$

with this linear combination being equivalent to

$$\|\cdot\|^2_{L^2_{x,v}} + \sum_{|l|\leqslant s} \left\|\partial_l^0 \cdot\right\|^2_{L^2_{x,v}} + \varepsilon^2 \sum_{\substack{|l|+|j|\leqslant s \\ |j|\geqslant 1}} \left\|\partial_l^j \cdot\right\|^2_{L^2_{x,v}}$$

and so fits the expected requirements.

4.6 Existence and exponential decay: proof of Theorem 4.2.3

One can clearly see that solving the kinetic equation (4.1.2) in the setting $f = \mu + \varepsilon\mu^{1/2}h$ is equivalent to solving the linearized kinetic equation (4.1.3) directly. Therefore we are going to focus only on this linearized equation.

The proof relies on the *a priori* estimate derived in the previous section. We shall use this inequality as a bootstrap to obtain first the existence of solutions thanks to an iteration scheme and then the exponential decay of those solutions, as long as the initial data is small enough.

4.6.1 Proof of the existence of global solutions

4.6.1.1 Construction of solutions to a linearized problem

Here we will follow the classical method that is approximating our solution by a sequence of solutions of a linearization of our initial problem. Then we have to construct a functional on Sobolev spaces for which this sequence can be uniformly bounded in order to be able to extract a convergent subsequence.

Starting from h_0 in $H^s_{x,v} \cap \mathrm{Ker}(G_\varepsilon)^\perp$, to be define later, we define the function h_{n+1} in $H^s_{x,v}$ by induction on $n \geqslant 0$:

$$\left\{ \begin{array}{l} \partial_t h_{n+1} + \dfrac{1}{\varepsilon} v.\nabla_x h_{n+1} = \dfrac{1}{\varepsilon^2} L(h_{n+1}) + \dfrac{1}{\varepsilon}\Gamma(h_n, h_{n+1}) \\ h_{n+1}(0,x,v) = h_{in}(x,v), \end{array} \right. \tag{4.6.1}$$

First we need to check that our sequence is well-defined.

Lemma 4.6.1 *Let L be satisfying assumptions* (H1'), (H2') *and* (H3)*, and let Γ be satisfying assumptions* (H4) *and* (H5)*.*

Then, there exists $0 < \varepsilon_d \leqslant 1$ such that for all $s \geqslant s_0$ (defined in (H4)), there exists $\delta_s > 0$ such that for all $0 < \varepsilon \leqslant \varepsilon_d$, if $\|h_{in}\|_{\mathcal{H}^s_\varepsilon} \leqslant \delta_s$ then the sequence $(h_n)_{n\in\mathbb{N}}$ is well-defined, continuous in time, in $H^s_{x,v}$ and belongs to $\mathrm{Ker}(G_\varepsilon)^\perp$*.*

Proof of Lemma 4.6.1 By induction, let us suppose that for a fixed $n \geqslant 0$ we have constructed h_n in $H^s_{x,v}$, which is true for h_{in}.

Using the previous notation one can see that we are in fact trying to solve the linear equation on the torus:

$$\partial_t h_{n+1} = G_\varepsilon(h_{n+1}) + \frac{1}{\varepsilon}\Gamma(h_n, h_{n+1})$$

with h_{in} as an initial data.

The existence of a solution h_{n+1} has already been shown for each equation covered by the hypocoercivity theory in the case $\varepsilon = 1$ (see papers described in the introduction). It was proved by fixed point arguments applied to the Duhamel's formula. In order not to write several times the same estimates one may use our next lemma 4.6.2 together with the Duhamel's formula (instead of considering directly the time derivative of h_{n+1}) to get a fixed point argument as long as h_{in} is small enough, the smallness not depending on ε.

As shown in the study of the linear part of the linearized model, under assumptions (H1'), (H2') and (H3) G_ε generates a C^0-semigroup on $H^s_{x,v}$, for all $0 < \varepsilon \leqslant \varepsilon_d$. Moreover, hypothesis (H4) shows us that $\Gamma(h_n, \cdot)$ is a bounded linear operator from $(H^s_{x,v}, E(\cdot))$ to $(H^s_{x,v}, \|\cdot\|_{H^s_{x,v}})$. Thus h_{n+1} is in $H^s_{x,v}$.

The belonging to $\mathrm{Ker}(G_\varepsilon)^\perp$ is direct since $\Gamma(h_n, \cdot)$ is in $\mathrm{Ker}(G_\varepsilon)^\perp$ (hypothesis (H5)). ■

Then we have to strongly bound the sequence, at least in short time, to have a chance to obtain a convergent subsequence, up to an extraction.

4.6.1.2 Boundedness of the sequence

We are about to prove the global existence in time of solutions in $C(\mathbb{R}^+, \|.\|_{\mathcal{H}^s_\varepsilon})$.That will give us existence of solutions in standard Sobolev's spaces as long as the initial data is small enough in the sense of the $\mathcal{H}^s_\varepsilon$-norm,which is smaller than the standard $H^s_{x,v}$-norm. To achieve that we define a new functional on $H^s_{x,v}$

$$E(h) = \sup_{t\in\mathbb{R}^+}\left(\|h(t)\|^2_{\mathcal{H}^s_\varepsilon} + \int_0^t \|h(s)\|^2_{H^s_\Lambda}\,ds\right). \tag{4.6.2}$$

Lemma 4.6.2 *Let L be satisfying assumptions* (H1'), (H2') *and* (H3)*, and let Γ be satisfying assumptions* (H4) *and* (H5).

Then there exists $0 < \varepsilon_d \leqslant 1$ such that for all $s \geqslant s_0$ (defined in (H4)) there exists $\delta_s > 0$ independent of ε, such that for all $0 < \varepsilon \leqslant \varepsilon_d$, if $\|h_{in}\|_{\mathcal{H}^s_\varepsilon} \leqslant \delta_s$ then

$$(E(h_n) \leqslant \delta_s) \Rightarrow (E(h_{n+1}) \leqslant \delta_s)\,.$$

Proof of Lemma 4.6.2 We let $t > 0$.
We know that h_{in} belongs to $H^s_{x,v} \cap \mathrm{Ker}(G_\varepsilon)^\perp$. Moreover we have, thanks to Lemma 4.6.1, that (h_n) is well-defined, in $\mathrm{Ker}(G_\varepsilon)^\perp$ and in $H^s_{x,v}$, since $s \geqslant s_0$. Moreover, Γ satisfies (H5). Therefore we can use the Proposition 4.2.2 to write, for $\varepsilon \leqslant \varepsilon_d$ (ε_d being the minimum between the one in Lemma 4.6.1 and the one in Proposition 4.2.2),

$$\frac{d}{dt}\|h_{n+1}\|^2_{\mathcal{H}^s_\varepsilon} \;\leqslant\; -K_0^{(s)}\|h_{n+1}\|^2_{H^s_\Lambda} + K_1^{(s)}\left(\mathcal{G}^s_x(h_n,h_{n+1})\right)^2 + \varepsilon^2 K_2^{(s)}\left(\mathcal{G}^s_{x,v}(h_n,h_{n+1})\right)^2.$$

We can use the hypothesis (H4) and the fact that

$$C_m\left(\|.\|^2_{L^2_{x,v}} + \sum_{|l|\leqslant s}\left\|\partial^0_l .\right\|^2_{L^2_{x,v}} + \varepsilon^2\sum_{\substack{|l|+|j|\leqslant s\\|j|\geqslant 1}}\left\|\partial^j_l .\right\|^2_{L^2_{x,v}}\right) \leqslant \|.\|^2_{\mathcal{H}^s_\varepsilon} \leqslant C_M\|.\|_{H^s_{x,v}}, \tag{4.6.3}$$

to get the following upper bounds:

$$
\begin{aligned}
(\mathcal{G}^s_x(h_n,h_{n+1}))^2 &\leqslant \frac{C_\Gamma^2}{C_m}\left(\|h_n\|^2_{\mathcal{H}^s_\varepsilon}\|h_{n+1}\|^2_{H^s_\Lambda} + \|h_{n+1}\|^2_{\mathcal{H}^s_\varepsilon}\|h_n\|^2_{H^s_\Lambda}\right)\\
\left(\mathcal{G}^s_{x,v}(h_n,h_{n+1})\right)^2 &\leqslant \frac{C_\Gamma^2}{C_m\varepsilon^2}\left(\|h_n\|^2_{\mathcal{H}^s_\varepsilon}\|h_{n+1}\|^2_{H^s_\Lambda} + \|h_{n+1}\|^2_{\mathcal{H}^s_\varepsilon}\|h_n\|^2_{H^s_\Lambda}\right).
\end{aligned}
$$

Therefore we have the following upper bound, where K_1 and K_2 are constants independent of ε:

$$
\begin{aligned}
\frac{d}{dt}\|h_{n+1}\|^2_{\mathcal{H}^s_\varepsilon} &\leqslant -K_0^{(s)}\|h_{n+1}\|^2_{H^s_\Lambda} + K_1\|h_n\|^2_{\mathcal{H}^s_\varepsilon}\|h_{n+1}\|^2_{H^s_\Lambda} + K_2\|h_{n+1}\|^2_{\mathcal{H}^s_\varepsilon}\|h_n\|^2_{H^s_\Lambda}\\
&\leqslant \left[K_1E(h_n) - K_0^{(s)}\right]\|h_{n+1}\|^2_{H^s_\Lambda} + K_2E(h_{n+1})\|h_n\|^2_{H^s_\Lambda}.
\end{aligned}
$$

We consider now that $E(h_n) \leqslant K_0^{(s)}/2K_1$.
We can integrate the equation above between 0 and t and one obtains

$$
\|h_{n+1}\|^2_{\mathcal{H}^s_\varepsilon} + \frac{K_0^{(s)}}{2}\int_0^t \|h_{n+1}\|^2_{H^s_\Lambda}\,ds \leqslant \|h_0\|^2_{\mathcal{H}^s_\varepsilon} + KE(h_{n+1})E(h_n).
$$

This is true for all $t>0$, then we define $C = \min\{1, K_0^{(s)}/2\}$, if $E(h_n) \leqslant C/2K$ we have

$$
E(h_{n+1}) \leqslant \frac{2}{C}\|h_0\|^2_{\mathcal{H}^s_\varepsilon}.
$$

Therefore choosing $M^{(s)} = \min\{C/2K, K_0^{(s)}/2K_1\}$ and $\delta_s \leqslant \min\{M^{(s)}C/2, M^{(s)}\}$ gives us the expected result.

■

4.6.1.3 The global existence of solutions

Now we are able to prove the global existence result:

Theorem 4.6.3 *Let L be satisfying assumptions* (H1'), (H2') *and* (H3)*, and let Γ be satisfying assumptions* (H4) *and* (H5)*.*
Then there exists $0 < \varepsilon_d \leqslant 1$ such that for all $s \geqslant s_0$ (defined in (H4)*), there exists $\delta_s > 0$ and for all $0 < \varepsilon \leqslant \varepsilon_d$:*

If $\|h_{in}\|_{\mathcal{H}^s_\varepsilon} \leqslant \delta_s$ then there exist a solution of (4.1.3) *in $C(\mathbb{R}^+, E(\cdot))$ and it satisfies, for some constant $C>0$,*

$$
E(h) \leqslant C\|h_{in}\|^2_{\mathcal{H}^s_\varepsilon}.
$$

Proof of Theorem 4.6.3 Regarding Lemma 4.6.2, by induction we can strongly bound the sequence $(h_n)_{n\in\mathbb{N}}$, as long as $E(h_0) \leqslant \delta_s$, the constant being defined in Lemma 4.6.2 . Therefore, defining h_0 to be h_{in} at $t=0$ and 0 elsewhere gives us $E(h_0) = \|h_{in}\|_{\mathcal{H}^s_\varepsilon} \leqslant \delta_s$. Thus, we have the boundedness of the sequence $(h_n)_{n\in\mathbb{N}}$ in $L^\infty_t H^s_{x,v} \cap L^1_t H^s_\Lambda$. By compact embeddings into smaller Sobolev's spaces (Rellich theorem) we can take the limit in (4.6.1) as n tends to $+\infty$, since G_ε and Γ are continuous. We obtain h a solution, in $C(\mathbb{R}^+, E(\cdot))$, to

$$\left\{\begin{array}{l} \partial_t h + \frac{1}{\varepsilon} v.\nabla_x h = \frac{1}{\varepsilon^2} L(h) + \frac{1}{\varepsilon}\Gamma(h,h) \\ h(0,x,v) = h_{in}(x,v). \end{array}\right.$$

■

4.6.2 Proof of the exponential decay

The function constructed above, h, is in $\mathrm{Ker}(G_\varepsilon)^\perp$ for all $0 < \varepsilon \leqslant 1$. Moreover, this function is clearly a solution of the following equation:

$$\partial_t h = G_\varepsilon(h) + \frac{1}{\varepsilon}\Gamma(h,h),$$

with Γ satisfying (H5). Therefore, we can use the a priori estimate on solutions of the full perturbative model concerning the time evolution of the $\mathcal{H}^s_\varepsilon$-norm (where we will omit to write the dependence on s for clearness purpose), Proposition 4.2.2.

$$\frac{d}{dt}\|h\|^2_{\mathcal{H}^s_\varepsilon} \leqslant -K_0\|h\|^2_{H^s_\Lambda} + K_1\left(\mathcal{G}^s_x(h,h)\right)^2 + \varepsilon^2 K_2\left(\mathcal{G}^s_{x,v}(h,h)\right)^2.$$

Moreover, using (4.6.3) and hypothesis (H4) to find:

$$\begin{aligned} \left(\mathcal{G}^s_x(h,h)\right)^2 &\leqslant \frac{2C_\Gamma^2}{C_m}\|h\|^2_{\mathcal{H}^s_\varepsilon}\|h\|^2_{H^s_\Lambda} \\ \left(\mathcal{G}^s_{x,v}(h,h)\right)^2 &\leqslant \frac{2C_\Gamma^2}{C_m\varepsilon^2}\|h\|^2_{\mathcal{H}^s_\varepsilon}\|h\|^2_{H^s_\Lambda}. \end{aligned}$$

Hence, K being a constant independent of ε:

$$\frac{d}{dt}\|h\|^2_{\mathcal{H}^s_\varepsilon} \leqslant \left(K\|h\|^2_{\mathcal{H}^s_\varepsilon} - K_0\right)\|h\|^2_{H^s_\Lambda}.$$

Therefore, one can notice that if $\|h_{in}\|^2_{\mathcal{H}^s_\varepsilon} \leqslant K_0/2K$ then we have that $\|h\|^2_{\mathcal{H}^s_\varepsilon}$ is decreasing in time. Hence, because the Λ-norm controls the L^2-norm which controls the $\mathcal{H}$-norm:

$$\begin{aligned} \frac{d}{dt}\|h\|^2_{\mathcal{H}^s_\varepsilon} &\leqslant -\frac{K_0}{2}\|h\|^2_{H^s_\Lambda} \\ &\leqslant -\frac{K_0}{2}\frac{\nu_0^\Lambda}{\nu_1^\Lambda C_M}\|h\|^2_{\mathcal{H}^s_\varepsilon}. \end{aligned}$$

Then we have directly, by Gronwall's lemma and setting $\tau_s = K_0\nu_0^\Lambda/4\nu_1^\Lambda C_M$,

$$\|h\|^2_{\mathcal{H}^s_\varepsilon} \leqslant \|h_{in}\|^2_{\mathcal{H}^s_\varepsilon} e^{-2\tau_s t}$$

as long as $\|h_{in}\|^2_{\mathcal{H}^s_\varepsilon} \leqslant K_0/2K$, which is the expected result with $\delta_s \leqslant \sqrt{K_0/2K}$.

4.7 Exponential decay of v-derivatives: proof of Theorem 4.2.4

In order to prove this theorem we are going to state a proposition giving an a priori estimate on a solution to the equation (4.1.3)

$$\partial_t h + \frac{1}{\varepsilon}v.\nabla_x h = \frac{1}{\varepsilon^2}L(h) + \frac{1}{\varepsilon}\Gamma(h,h).$$

We remind the reader that we work in $H^s_{x,v}$ with the following positive functional

$$\|\cdot\|^2_{\mathcal{H}^s_{\varepsilon\perp}} = \sum_{\substack{|j|+|l|\leqslant s \\ |j|\geqslant 1}} b^{(s)}_{j,l}\left\|\partial^j_l(\mathrm{Id}-\pi_L)\cdot\right\|^2_{L^2_{x,v}} + \sum_{|l|\leqslant s}\alpha^{(s)}_l\left\|\partial^0_l\cdot\right\|^2_{L^2_{x,v}} + \sum_{\substack{|l|\leqslant s \\ i,c_i(l)>0}} a^{(s)}_{i,l}\varepsilon\langle\partial^{\delta_i}_{l-\delta_i}\cdot,\partial^0_l\cdot\rangle_{L^2_{x,v}}.$$

One can notice that if we choose coefficients $(b^{(s)}_{j,l})$, $(\alpha^{(s)}_l)$, $(a^{(s)}_{i,l}) > 0$ such that $\|\cdot\|^2_{\mathcal{H}^s_{1\perp}}$ is equivalent to

$$\sum_{\substack{|j|+|l|\leqslant s \\ |j|\geqslant 1}}\left\|\partial^j_l(\mathrm{Id}-\pi_L)\cdot\right\|^2_{L^2_{x,v}} + \sum_{|l|\leqslant s}\left\|\partial^0_l\cdot\right\|^2_{L^2_{x,v}}$$

then for all ε less than some ε_0, $\|\cdot\|^2_{\mathcal{H}^s_{\varepsilon\perp}}$ is also equivalent to the latter norm with equivalence coefficients not depending on ε.

Moreover, using equation (4.3.3), we have that

$$\left\|\partial^j_l h\right\|^2_{L^2_{x,v}} \leqslant C_{\pi s}\left\|\partial^0_l h\right\|^2_{L^2_{x,v}} + \left\|\partial^j_l h^\perp\right\|^2_{L^2_{x,v}} \leqslant 2C_{\pi s}\left\|\partial^0_l h\right\|^2_{L^2_{x,v}} + \left\|\partial^j_l h\right\|^2_{L^2_{x,v}},$$

and therefore

$$\sum_{\substack{|j|+|l|\leqslant s \\ |j|\geqslant 1}}\left\|\partial^j_l(Id-\pi_L)\right\|^2_{L^2_{x,v}} + \sum_{|l|\leqslant s}\left\|\partial^0_l.\right\|^2_{L^2_{x,v}}$$

is equivalent to the standard Sobolev norm. Thus, we will just construct coefficients $(b_{j,l}^{(s)})$, $(\alpha_l^{(s)})$ and $(a_{i,l}^{(s)})$ so that $\|.\|^2_{\mathcal{H}^s_{1\perp}}$ is equivalent to the latter norm and then for ε small enough we will have the equivalence, not depending on ε, between $\|\cdot\|^2_{\mathcal{H}^s_{\varepsilon\perp}}$ and the $H^s_{x,v}$-norm.

4.7.1 An a priori estimate

In this subsection we will prove the following proposition:

Proposition 4.7.1 *If L is a linear operator satisfying the conditions* (H1'), (H2') *and* (H3) *and Γ a bilinear operator satisfying* (H5) *then there exists $0 < \varepsilon_d \leqslant 1$ such that for all s in $\mathbb{N}^*$,*

1. *for h_{in} in Ker$(G_\varepsilon)^\perp$ if we have h an associated solution of*

$$\partial_t h + \frac{1}{\varepsilon} v \cdot \nabla_x h = \frac{1}{\varepsilon^2} L(h) + \frac{1}{\varepsilon}\Gamma(h,h),$$

2. *there exist $K_0^{(s)}$, $K_1^{(s)}$, $(b_{j,l}^{(s)})$, $(\alpha_l^{(s)})$, $(a_{i,l}^{(s)}) > 0$ such that for all $0 < \varepsilon \leqslant \varepsilon_d$:*

- $\|\cdot\|_{\mathcal{H}^s_{\varepsilon\perp}} \sim \|\cdot\|_{H^s_{x,v}}$,
- $\forall h_{in} \in H^s_{x,v} \cap \mathrm{Ker}(G_\varepsilon)^\perp$,

$$\frac{d}{dt}\|h\|^2_{\mathcal{H}^s_{\varepsilon\perp}} \leqslant -K_0^{(s)}\left(\frac{1}{\varepsilon^2}\left\|h^\perp\right\|^2_{H^s_\Lambda} + \sum_{1\leqslant|l|\leqslant s}\|\partial_l^0 h\|^2_{L^2_{x,v}}\right) + K_1^{(s)}\left(\mathcal{G}^s_{x,v}(h,h)\right)^2.$$

Remark 4.7.2 *We notice here that in front of the microscopic part of h is a negative constant order $-1/\varepsilon^2$ which is the same order than the control derived by Guo in [56] for his dissipation rate.*

We will prove that proposition by induction on s.
So we take h_{in} in $H^s_{x,v} \cap \mathrm{Ker}(G_\varepsilon)^\perp$ and we consider the associated solution of (4.1.3), denoted by h. One can notice that thanks to (H5), h remains in $\mathrm{Ker}(G_\varepsilon)^\perp$ for all times and thus we are allowed to use the inequalities given in the toolbox

4.7.1.1 The case $s = 1$

In that case we have

$$\|h\|^2_{\mathcal{H}^1_{\varepsilon\perp}} = A\|h\|^2_{L^2_{x,v}} + \alpha\|\nabla_x h\|^2_{L^2_{x,v}} + b\left\|\nabla_v h^\perp\right\|^2_{L^2_{x,v}} + a\varepsilon\langle\nabla_x h, \nabla_v h\rangle_{L^2_{x,v}},$$

with A, α, b and a strictly positive.

Therefore we can study the time evolution of that operator acting on h by gathering results given in the toolbox. We simply take A(4.3.6) + α(4.3.7) + b(4.3.14) + $a\varepsilon$(4.3.15)

$$\begin{aligned}
\frac{d}{dt}\|h\|^2_{\mathcal{H}^1_{\varepsilon\perp}} \quad \leqslant \quad & \frac{1}{\varepsilon^2}\left[K_1^{\perp}b-\lambda A\right]\left\|h^{\perp}\right\|^2_{\Lambda}+\frac{1}{\varepsilon^2}\left[K^{\perp}ea-\lambda\alpha\right]\left\|\nabla_x h^{\perp}\right\|^2_{\Lambda} \\
& +\frac{1}{\varepsilon^2}\left[\frac{1}{4C_{\pi 1}C_\pi C_p}\frac{a}{e}-b\frac{\nu_3^{\Lambda}}{2}\right]\left\|\nabla_v h^{\perp}\right\|^2_{\Lambda}+\left[K_{dx}^{\perp}b-\frac{a}{2}\right]\|\nabla_x h\|^2_{L^2_{x,v}} \\
& +K(A,\alpha,b,a)\left(\mathcal{G}^1_{x,v}(h,h)\right)^2, \qquad (4.7.1)
\end{aligned}$$

with s a fonction only depending on the coefficients appearing in hypocoercivity hypothesis and independent of ε.

We directly see that we have exactly the same kind of bound as the one we obtain while working on the a priori estimates for the operator $\|h\|_{\mathcal{H}^1_\varepsilon}$, equation (4.5.1). Therefore we can choose of coefficients A, α, b, e and a in the same way (in the right order) and use the same inequalities to finally obtain the expected result: $\exists s_0,\ K_1>0,\ \forall\, 0<\varepsilon\leqslant 1$,

$$\begin{aligned}
\frac{d}{dt}\|h\|^2_{\mathcal{H}^1_{\varepsilon\perp}} \quad \leqslant \quad & -s_0^{(1)}\left(\frac{1}{\varepsilon^2}\left\|h^{\perp}\right\|^2_{\Lambda}+\frac{1}{\varepsilon^2}\left\|\nabla_x h^{\perp}\right\|^2_{\Lambda}+\frac{1}{\varepsilon^2}\left\|\nabla_v h^{\perp}\right\|^2_{\Lambda}+\|\nabla_x h\|^2_{L^2_{x,v}}\right) \\
& +K_1^{(1)}\left(\mathcal{G}^1_{x,v}(h,h)\right)^2,
\end{aligned}$$

with the constants $s_0^{(1)}$ and $K_1^{(1)}$ independent of ε, and $\|h\|^2_{\mathcal{H}^1_{1\perp}}$ equivalent to $\|h\|^2_{L^2_{x,v}}+\|\nabla_x h\|^2_{L^2_{x,v}}+\left\|\nabla_v h^{\perp}\right\|^2_{L^2_{x,v}}$. Therefore, for all ε small enough we have the expected result in the case $s=1$.

4.7.1.2 The induction in higher order Sobolev spaces

Then we assume that the theorem is true up to the integer $s-1$, $s>1$. Then we suppose that L satisfies (H1'), (H2') and (H3) and we consider ε in $(0,1]$.

Since h_{in} is in $\mathrm{Ker}(G_\varepsilon)^{\perp}$, h belongs to $\mathrm{Ker}(G_\varepsilon)^{\perp}$ for all t and so we can use the results given in the toolbox.

As in the proofs of previous sections, we define on $H^s_{x,v}$:

$$\begin{aligned}
F_s(t) &= \sum_{\substack{|j|+|l|=s \\ |j|\geqslant 2}} B\left\|\partial^j_l h^{\perp}\right\|^2_{L^2_{x,v}}+B'\sum_{\substack{|l|=s \\ i,c_i(l)>0}} Q_{l,i}(t), \\
Q_{l,i}(t) &= \alpha\left\|\partial^0_l h\right\|^2_{L^2_{x,v}}+b\left\|\partial^{\delta_i}_{l-\delta_i}h^{\perp}\right\|^2_{L^2_{x,v}}+a\varepsilon\langle\partial^{\delta_i}_{l-\delta_i}h,\partial^0_l h\rangle_{L^2_{x,v}},
\end{aligned}$$

where the constants, strictly positive, will be chosen later.

Like in the section above, we shall study the time evolution of every term involved in F_s in order to bound above $\frac{dF_s}{dt}(t)$ with expected coefficients. However, in this subsection we will need to control all the $Q_{l,i}$'s in the same time rather than treating them separately as we did in the proof of Proposition (4.2.2), because the toolbox tells us that each $Q_{l,i}$ is controlled by quantities appearing in the others.

4.7.1.3 The time evolution of $\sum Q_{l,i}$

Gathering the toolbox inequalities in the following way: $\alpha(4.3.10)+b(4.3.17)+a\varepsilon(4.3.18)$. This yields, because $0<\varepsilon\leqslant 1$ and $\mathrm{Card}\{i, c_i(l)>0\}\leqslant d$,

$$
\begin{aligned}
\frac{d}{dt}\left(\sum_{\substack{|l|=s\\ i,c_i(l)>0}} Q_{l,i}(t)\right) \leqslant\ & \frac{1}{\varepsilon^2}\left[\tilde{K}^{\perp}ea-\lambda\alpha\right]\sum_{|l|=s}\left\|\partial_l^0 h^{\perp}\right\|_{\Lambda}^2 \\
& +\frac{1}{\varepsilon^2}\left[\frac{1}{4C_{\pi s}C_{\pi}d}\frac{a}{e}-\nu_5^{\Lambda}b\right]\sum_{\substack{|l|=s\\ i,c_i(l)>0}}\left\|\partial_{l-\delta_i}^{\delta_i}h^{\perp}\right\|_{\Lambda}^2 \\
& +\left[K_{dl}^{\perp}db-\frac{a}{2}\right]\sum_{|l|=s}\left\|\partial_l^0 h\right\|_{L^2_{x,v}}^2+\frac{a}{4}\sum_{|l|\leqslant s-1}\left\|\partial_l^0 h\right\|_{L^2_{x,v}}^2 \\
& +\frac{bK_{s-1}^{\perp}}{\varepsilon^2}\left(\sum_{\substack{|l|+|j|=s\\ i,c_i(l)>0}}1\right)\left\|h^{\perp}\right\|_{H^{s-1}_{x,v}}^2+K(\alpha,b,a,e)\left(\mathcal{G}^s_{x,v}(h,h)\right)^2,
\end{aligned}
$$

with s a fonction only depending on the coefficients appearing in hypocoercivity hypothesis and independent of ε.
One can notice that except for the terms in $\|h\|_{H^{s-1}_{x,v}}$ and $\sum_{|l|\leqslant s-1}\left\|\partial_l^0 h\right\|_{L^2_{x,v}}^2$, we have exactly the same bound as in the case $s=1$, equation (4.7.1). Therefore we can choose α, b, a, e, independently of ε and Γ such that there exist $K_0'>0$, $K_1'>0$ and $C_0, C_1>0$ such that for all $0<\varepsilon\leqslant 1$:

- $\sum_{\substack{|l|=s\\ i,c_i(l)>0}} Q_{l,i}(t)\sim\sum_{\substack{|l|=s\\ i,c_i(l)>0}}\left(\left\|\partial_l^0 h\right\|_{L^2_{x,v}}^2+\left\|\partial_{l-\delta_i}^{\delta_i}h^{\perp}\right\|_{L^2_{x,v}}^2\right)$,

•

$$\begin{aligned}\frac{d}{dt}\sum_{\substack{|l|=s\\ i,c_i(l)>0}} Q_{l,i}(t) \leqslant & - K_0'\left(\frac{1}{\varepsilon^2}\sum_{|l|=s}\left\|\partial_l^0 h^\perp\right\|_\Lambda^2 + \frac{1}{\varepsilon^2}\sum_{\substack{|l|=s\\ i,c_i(l)>0}}\left\|\partial_{l-\delta_i}^{\delta_i} h^\perp\right\|_\Lambda^2\right.\\ &\left. + \sum_{|l|=s}\left\|\partial_l^0 h\right\|_{L^2_{x,v}}^2\right)\\ &+ \frac{C_0}{\varepsilon^2}\left\|h^\perp\right\|_{H^{s-1}_{x,v}}^2 + C_1\sum_{|l|\leqslant s-1}\left\|\partial_l^0 h\right\|_{L^2_{x,v}}^2 + K_1'\left(\mathcal{G}^s_{x,v}(h,h)\right)^2.\end{aligned}$$

4.7.1.4 The time evolution of F_s and conclusion

We can finally obtain the time evolution of F_s, using $\frac{d}{dt}\left\|\partial_l^j h^\perp\right\|_{L^2_{x,v}}^2$, equation (4.3.16), so that there is no more ε in front of the Γ term:

$$\begin{aligned}\frac{d}{dt}F_s(t) \leqslant & -B\frac{\nu_5^\lambda}{\varepsilon^2}\sum_{\substack{|j|+|l|=s\\ |j|\geqslant 2}}\left\|\partial_l^j h^\perp\right\|_\Lambda^2 + B\frac{9(\nu_1^\Lambda)^2 d}{2(\nu_0^\Lambda)^2\nu_5^\Lambda}\sum_{\substack{|j|+|l|=s\\ |j|\geqslant 2}}\sum_{i,c_i(j)>0}\left\|\partial_{l+\delta_i}^{j-\delta_i} h^\perp\right\|_\Lambda^2\\ & -K_0'B'\left(\frac{1}{\varepsilon^2}\sum_{|l|=s}\left\|\partial_l^0 h^\perp\right\|_\Lambda^2 + \frac{1}{\varepsilon^2}\sum_{\substack{|l|=s\\ i,c_i(l)>0}}\left\|\partial_{l-\delta_i}^{\delta_i} h^\perp\right\|_\Lambda^2 + \sum_{|l|=s}\left\|\partial_l^0 h\right\|_{L^2_{x,v}}^2\right)\\ & + \left(\sum_{\substack{|j|+|l|=s\\ |j|\geqslant 2}} BK_{dl}^\perp + B'C_1\right)\sum_{|l|\leqslant s-1}\left\|\partial_l^0 h\right\|_{L^2_{x,v}}^2\\ & + \frac{1}{\varepsilon^2}\left[\sum_{\substack{|j|+|l|=s\\ |j|\geqslant 2}} BK_{s-1}^\perp + B'C_0\right]\left\|h^\perp\right\|_{H^{s-1}_{x,v}}^2\\ & + \left[\sum_{\substack{|j|+|l|=s\\ |j|\geqslant 2}}\frac{3B\nu_1^\Lambda}{\nu_0^\Lambda\nu_5^\Lambda} + B'K_1'\right]\left(\mathcal{G}^s_{x,v}(h,h)\right)^2,\end{aligned}$$

Therefore we obtain the same bound (except $\sum_{|l|\leqslant s-1}\left\|\partial_l^0 h\right\|_{L^2_{x,v}}^2$) as in the proof of Proposition 4.2.2, equation (4.5.2), and so by choosing coefficients in the same way we have that there exist $C_+^{(s)} > 0$, $0 < \varepsilon_d \leqslant 1$ and $K_1^{(s*)} > 0$, none of them depending on ε, such that for all $0 < \varepsilon \leqslant \varepsilon_d$:

$$\begin{aligned}\frac{d}{dt}F_s(t) \leqslant \quad & C_+^{(s)}\left(\frac{1}{\varepsilon^2}\sum_{|j|+|l|\leqslant s-1}\left\|\partial_l^j h^\perp\right\|_\Lambda^2 + \sum_{|l|\leqslant s-1}\left\|\partial_l^0 h\right\|_{L^2_{x,v}}^2\right) \\ & -\left(\frac{1}{\varepsilon^2}\sum_{|j|+|l|=s}\left\|\partial_l^j h^\perp\right\|_\Lambda^2 + \sum_{|l|=s}\left\|\partial_l^0 h\right\|_{L^2_{x,v}}^2\right) \\ & +K_1^{(s*)}\left(\mathcal{G}^s_{x,v}(h,h)\right)^2.\end{aligned}$$

This inequality is true for all s and therefore we can take a linear combination of the F_s to obtain the required result. Using the induction hypothesis on F_1 up to F_{s-1} we also have the equivalence of norms.

4.7.2 The exponential decay: proof of Theorem 4.2.4

Thanks to Theorem 4.2.3, we know that we have a solution to the equation (4.1.3) for any given h_{in} small enough in the standard Sobolev norm. Call h the associated solution of $h_{in} \in H^s_{x,v}$ to (4.1.3). Since the existence has been proved we can use the a priori estimate above and the Proposition 4.7.1.
Thus we have

$$\frac{d}{dt}\left\|h\right\|^2_{\mathcal{H}^s_{\varepsilon\perp}} \leqslant -K_0^{(s)}\left(\frac{1}{\varepsilon^2}\left\|h^\perp\right\|^2_{H^s_\Lambda} + \sum_{1\leqslant|l|\leqslant s}\left\|\partial_l^0 h\right\|^2_{L^2_{x,v}}\right) + K_1^{(s)}\left(\mathcal{G}^s_{x,v}(h,h)\right)^2.$$

As before we can use (4.3.4) (equivalence of norms $L^2_{x,v}$ and Λ on the fluid part) to get, for $|l| > 1$,

$$\left\|\partial_l^0 h\right\|^2_\Lambda \leqslant C'\left(\left\|\partial_l^0 h^\perp\right\|^2_\Lambda + \left\|\partial_l^0 h\right\|^2_{L^2_{x,v}}\right),$$

and for the case $|l| \leqslant 1$ we can apply the Poincare inequality (4.3.5) together with the equivalence of the $L^2_{x,v}$-norm and the Λ-norm on the fluid part π_L, (4.3.4) to get

$$\exists C, C' > 0, \begin{cases} \left\|h\right\|^2_\Lambda & \leqslant C\left(\left\|h^\perp\right\|^2_\Lambda + \frac{1}{2}\left\|\nabla_x h\right\|^2_{L^2_{x,v}}\right), \\ \left\|\nabla_x h\right\|^2_\Lambda & \leqslant C'\left(\left\|\nabla_x h^\perp\right\|^2_\Lambda + \frac{1}{2}\left\|\nabla_x h\right\|^2_{L^2_{x,v}}\right).\end{cases}$$

Then we get that

$$\begin{aligned}\frac{d}{dt}\|h\|^2_{\mathcal{H}^s_{\varepsilon\perp}} &\leqslant -K_0^{(s)}\left(\sum_{\substack{|j|+|l|\leqslant s\\|j|\geqslant 1}}\left\|\partial_l^j h^\perp\right\|^2_\Lambda + \sum_{|l|\leqslant s}\left\|\partial_l^0 h\right\|^2_\Lambda\right) + K_1^{(s)}\left(\mathcal{G}^s_{x,v}(h,h)\right)^2\\ &\leqslant -K_0^{(s*)}\|h\|^2_{H^s_\Lambda} + K_1^{(s)}\left(\mathcal{G}^s_{x,v}(h,h)\right)^2.\end{aligned}$$

Then for $s \geqslant s_0$, defined in (H4), and because Γ satisfies (H4) we can write

$$\frac{d}{dt}\|h\|^2_{\mathcal{H}^s_{\varepsilon\perp}} \leqslant \left(K_1^{(s)}C_\Gamma^2\|h\|^2_{H^s_{x,v}} - K_0^{(s*)}\right)\|h\|^2_{H^s_\Lambda}.$$

Because $\|h\|_{\mathcal{H}^s_{\varepsilon\perp}}$ and $\|h\|^2_{H^s_{x,v}}$ are equivalent, independently of ε, we finally have

$$\frac{d}{dt}\|h\|^2_{\mathcal{H}^s_{\varepsilon\perp}} \leqslant \left(K_1^{(s)}C_\Gamma^2 C\|h\|^2_{\mathcal{H}^s_{\varepsilon\perp}} - K_0^{(s*)}\right)\|h\|^2_{H^s_\Lambda}.$$

Therefore if

$$\|h_{in}\|^2_{\mathcal{H}^s_{\varepsilon\perp}} \leqslant \frac{K_0^{(s*)}}{2K_1^{(s)}C_\Gamma^2 C}$$

we have that $\|h\|^2_{\mathcal{H}^s_{\varepsilon\perp}}$ is always decreasing on $\mathbb{R}^+$ and so for all $t > 0$

$$\frac{d}{dt}\|h\|^2_{\mathcal{H}^s_{\varepsilon\perp}} \leqslant -\frac{K_0^{(s*)}}{2K_1^{(s)}C_\Gamma^2 C}\|h\|^2_{H^s_\Lambda}.$$

And the H^s_Λ-norm controls the $H^s_{x,v}$-norm which is equivalent to the $\mathcal{H}^s_{\varepsilon\perp}$-norm. Thus applying Gronwall's lemma gives us the expected exponential decay.

4.8 Incompressible Navier-Stokes Limit: proof of Theorem 4.2.5

In this section we consider $s \geqslant s_0$, $0 < \varepsilon \leqslant \varepsilon_d$ and we take h_{in} in $H^s_{x,v}$ such that $\|h_{in}\|_{\mathcal{H}^s_\varepsilon} \leqslant \delta_s$.

Therefore we know, thanks to theorem 4.2.3, that we have a solution h_ε to the linearized Boltzmann equation

$$\partial_t h_\varepsilon + \frac{1}{\varepsilon}v.\nabla_x h_\varepsilon = \frac{1}{\varepsilon^2}L(h_\varepsilon) + \frac{1}{\varepsilon}\Gamma(h_\varepsilon,h_\varepsilon),$$

with $h_\varepsilon(0,x,v) = h_{in}(x,v)$. Moreover, we also know that (h_ε) tends weakly-* to h in $L^\infty_t(H^s_x L^2_v)$.

The first step towards the proof of Theorem 4.2.5 is to derived a convergence rate in finite time. Then, as described in Section 4.1.3, we shall interpolate this result with the exponential decay behaviour of our solutions in order to obtain a global in time convergence.

4.8.1 A convergence in finite time

In Remark 4.8.13, we define $V_T(\varepsilon)$ and prove the following result

$$\forall T>0, \quad V_T(\varepsilon) = \sup_{t\in[0,T]} \|h_\varepsilon - h\|_{L^\infty_x L^2_v} \to 0, \text{ as } \varepsilon \to 0.$$

Thanks to this remark we can give an explicit convergence in finite time.

Theorem 4.8.1 *Consider $s \geqslant s_0$ and h_{in} in $H^s_{x,v}$ such that $\|h_{in}\|_{\mathcal{H}^s_\varepsilon} \leqslant \delta_s$.*

Then, $(h_\varepsilon)_{\varepsilon>0}$ exists for all $0<\varepsilon\leqslant\varepsilon_d$ and converges weakly in $L^\infty_t(H^s_xL^2_v)$ towards h such that $h\in Ker(L)$, with $\nabla_x\cdot u = 0$ and $\rho+\theta=0$.*

Furthermore, $\int_0^T hdt$ belongs to $H^s_xL^2_v$ and it exists $C>0$ such that for all $T>0$,

$$\left\|\int_0^T hdt - \int_0^T h_\varepsilon dt\right\|_{H^s_xL^2_v} \leqslant C\max\{\sqrt{\varepsilon},\sqrt{T\varepsilon}, TV_T(\varepsilon)\}.$$

One can have a strong convergence in $L^2_{[0,T]}H^s_xL^2_v$ only if h_{in} is in $\mathrm{Ker}(L)$ with $\nabla_x\cdot u_{in}=0$ and $\rho_{in}+\theta_{in}=0$ (initial layer conditions). Moreover, in that case we have, for all $T>0$,

$$\|h-h_\varepsilon\|_{L^2_{[0,T]}H^s_xL^2_v} \leqslant C\max\{\sqrt{\varepsilon},\sqrt{T}V_T(\varepsilon)\},$$

and for all δ in $[0,1]$, if h_{in} belongs to $H^{s+\delta}_xL^2_v$,

$$\sup_{t\in[0,T]}\|h-h_\varepsilon\|_{H^s_xL^2_v}(t) \leqslant C\max\{\varepsilon^{\min(\delta,1/2)}, V_T(\varepsilon)\}.$$

Remark 4.8.2 *We mention here that the obligation of an integration in time for non special initial condition is only due to the linear part $\varepsilon^{-2}L-\varepsilon^{-1}v\cdot\nabla_x$, whereas the case $T=+\infty$ is prevented by the second order term Γ.*

We proved in the linear case, theorem 4.2.1, that the linear operator $G_\varepsilon = \varepsilon^{-2}L - \varepsilon^{-1}v\cdot\nabla_x$ generates a semigroup e^{tG_ε} on $H^s_{x,v}$. Therefore we can use Duhamel's principle to rewrite our equation under the following form, defining $u_\varepsilon = \Gamma(h_\varepsilon, h_\varepsilon)$,

$$\begin{aligned} h_\varepsilon &= e^{tG_\varepsilon} h_{in} + \int_0^t \frac{1}{\varepsilon} e^{(t-s)G_\varepsilon} u_\varepsilon(s) ds \\ &:= U^\varepsilon h_{in} + \Psi^\varepsilon(u_\varepsilon). \end{aligned} \tag{4.8.1}$$

The article by Ellis and Pinsky [39] gives us a Fourier theory in x of the semigroup e^{tG_ε} and therefore we are going to use it to study the strong limit of $U^\varepsilon h_{in}$ and $\Psi^\varepsilon(u_\varepsilon)$ as ε tends to 0. We will denote by $\mathcal{F}_x$ the Fourier transform in x on the torus (which is discrete) and n the discrete variable associated in $\mathbb{Z}^d$.
From [39], we are using Theorem 3.1, rewriten thanks with the Proposition 2.6 and the Appendix II with $\delta = \lambda/4$ in Proposition 2.3, to get the following theorem

Theorem 4.8.3 *There exists $n_0 \in \mathbb{R}^{*+}$, there exists functions*

- $\lambda_j : [-n_0, n_0] \longrightarrow \mathbb{C}$, $-1 \leqslant j \leqslant 2$, C^∞
- $\begin{array}{rccc} e_j : & [-n_0,n_0]\times\mathbb{S}^{d-1} & \longrightarrow & L^2_v \\ & (\zeta,\omega) & \longmapsto & e_j(\zeta,\omega) \end{array}$, $-1 \leqslant j \leqslant d$, *$C^\infty$ in ζ and C^0 in ω,*

such that

1. *for all $-1 \leqslant j \leqslant 2$, $\lambda_j(\zeta) = i\alpha_j\zeta - \beta_j\zeta^2 + \gamma_j(\zeta)$, where $\alpha_j \in \mathbb{R}$, with $\alpha_0 = \alpha_2 = 0$, $\beta_j < 0$ and $|\gamma_j(\zeta)| \leqslant C_\gamma |\zeta|^3$ with $|\gamma_j(\zeta)| \leqslant \frac{\beta_j}{2}|\zeta|^2$,*

2. *for all $-1 \leqslant j \leqslant d$*

   - $e_j(\zeta,\omega) = e_{0j}(\omega) + \zeta e_{1j}(\omega) + \zeta^2 e_{2j}(\zeta,\omega)$,
   - $e_{0-1}(\omega)(v) = e_{01}(-\omega)(v) = A\left(1 - \omega.v + \frac{|v|^2-d}{2}\right)\mu(v)^{1/2}$,

3. *we have $e^{tG_\varepsilon} = \mathcal{F}_x^{-1}\hat{U}(t/\varepsilon^2, \varepsilon n, v)\mathcal{F}_x$ where*

$$\hat{U}(t,n,v) = \sum_{j=-1}^{2} \hat{U}_j(t,n,v) + \hat{U}_R(t,n,v)$$

   with the following properties

   - *for $-1 \leqslant j \leqslant 2$, $\hat{U}_j(t,n,v) = \chi_{|n|\leqslant n_0} e^{t\lambda_j(|n|)} P_j\left(|n|, \frac{n}{|n|}\right)(v)$,*
   - *for $-1 \leqslant j \leqslant 1$, $P_j\left(|n|, \frac{n}{|n|}\right) = e_j\left(|n|, \frac{n}{|n|}\right) \otimes e_j\left(|n|, \frac{-n}{|n|}\right)$,*
   - $P_2\left(|n|, \frac{n}{|n|}\right) = \sum_{j=2}^{d} e_j\left(|n|, \frac{n}{|n|}\right) \otimes e_j\left(|n|, \frac{-n}{|n|}\right)$,
   - *for $-1 \leqslant j \leqslant 2$, $P_j\left(|n|, \frac{n}{|n|}\right) = P_{0j}\left(\frac{n}{|n|}\right) + |n| P_{1j}\left(\frac{n}{|n|}\right) + |n|^2 P_{2j}\left(|n|, \frac{n}{|n|}\right)$,*

- $\sum_{j=-1}^{2} P_{0j} = \pi_L$,
- *it exists* $C_R, \sigma > 0$ *such that for all* $t \in \mathbb{R}^+$ *and all* $n \in \mathbb{Z}^d$,

$$|||\hat{U}_R(t,n,v)|||_{L^2_v} \leqslant C_R e^{-\sigma t}.$$

Remark 4.8.4 *This decomposition of the spectrum of the linear operator is based on a low and high frequencies decomposition. It shows that the spectrum of the whole operator can be viewed as a perturbation of the spectrum of the homogeneous linear operator. It can be divided into large eigenvalues, which are negative and therefore create a strong semigroup property for the remainder term, and small eigenvalues around the origin that are smooth perturbations of the homogeneous ones.*

This theorem gives us all the tools we need to study the convergence as ε tends to 0 since we have an explicit form for the Fourier transform of the semigroup. We also know that this semigroup commutes with the pure x-derivatives. Therefore, studying the convergence in the $L^2_x L^2_v$-norm will be enough to obtain the desired result in the $H^s_x L^2_v$-norm. We are going to prove the following convergences in the different settings stated by Theorem 4.2.5

1. $U^\varepsilon h_{in}$ tends to $V(t,x,v)h_{in}$ with $V(0,x,v)h_{in} = V(0)(h_{in})(x,v)$ where $V(0)$ the projection on the subset of $\mathrm{Ker}(L)$ consisting in functions g such that $\nabla_x \cdot u_g = 0$ and $\rho_g + \theta_g = 0$,
2. $\Psi^\varepsilon(u_\varepsilon)$ converges to $\Psi(h,h)$ with $\Psi(h,h)(t=0) = 0$.

4.8.1.1 Study of the linear part

We remind here that we have

$$U^\varepsilon h_{in} = \mathcal{F}_x^{-1} \hat{U}^\varepsilon(t,n,v)\hat{h}_{in}(n,v)$$

with

$$\begin{aligned}
\hat{U}^\varepsilon(t,n,v) &= \sum_{j=-1}^{2} \hat{U}^\varepsilon_j(t,n,v) + \hat{U}^\varepsilon_R(t,n,v),\\
\hat{U}^\varepsilon_j(t,n,v) &= \chi_{|\varepsilon n| \leqslant n_0} e^{\frac{i\alpha_j t|n|}{\varepsilon} - \beta_j t|n|^2 + \frac{t}{\varepsilon^2}\gamma_j(|\varepsilon n|)} \left[P_{0j}\left(\frac{n}{|n|}\right) + \varepsilon\, |n|\, \tilde{P}_{1j}\left(|\varepsilon n|, \frac{n}{|n|}\right)\right].
\end{aligned}$$

We can decompose $\hat{U}^\varepsilon_j$ into four different terms

$$
\begin{aligned}
\hat{U}_j^\varepsilon(t,n,v) &= e^{\frac{i\alpha_j t|n|}{\varepsilon}-\beta_j t|n|^2} P_{0j}\left(\frac{n}{|n|}\right) \\
&\quad + \chi_{|\varepsilon n|\leqslant n_0} e^{\frac{i\alpha_j t|n|}{\varepsilon}-\beta_j t|n|^2}\left(e^{\frac{t}{\varepsilon^2}\gamma_j(|\varepsilon n|)}-1\right) P_{0j}\left(\frac{n}{|n|}\right) \\
&\quad + \chi_{|\varepsilon n|\leqslant n_0} e^{\frac{i\alpha_j t|n|}{\varepsilon}-\beta_j t|n|^2+\frac{t}{\varepsilon^2}\gamma_j(|\varepsilon n|)} \varepsilon\,|n|\,\tilde{P}_{1j}\left(|\varepsilon n|,\frac{n}{|n|}\right) \\
&\quad + \left(\chi_{|\varepsilon n|\leqslant n_0}-1\right) e^{\frac{i\alpha_j t|n|}{\varepsilon}-\beta_j t|n|^2} P_{0j}\left(\frac{n}{|n|}\right). \\
&= U_{0j}^\varepsilon + U_{1j}^\varepsilon + U_{2j}^\varepsilon + U_{3j}^\varepsilon.
\end{aligned}
\tag{4.8.2}
$$

Remark 4.8.5 *One can notice that U_{00}^ε and U_{02}^ε do not depend on ε, since $\alpha_0 = \alpha_2 = 0$.*

We are going to study each of these four terms in two different lemmas and then add a last lemma to deal with the remainder term $U_R h_{in}$. The lemmas will be proven in Appendix 4.C.

Lemma 4.8.6 *For $\alpha_j \neq 0$ ($j = \pm 1$) we have that it exists $C_0 > 0$ such that for all $T \in [0,+\infty]$*

$$
\left\| \int_0^T U_{0j}^\varepsilon h_{in}\,dt \right\|_{L_x^2 L_v^2}^2 \leqslant C_0 \varepsilon^2 \left\| h_{in} \right\|_{L_x^2 L_v^2}^2 .
$$

Moreover we have a strong convergence in the $L^2_{[0,+\infty)} L_x^2 L_v^2$-norm if and only if h_{in} satisfies $\nabla_x \cdot u_{in} = 0$ and $\rho_{in} + \theta_{in} = 0$. In that case we have $U_{0j}^\varepsilon h_{in} = 0$.

Lemma 4.8.7 *For $-1 \leqslant j \leqslant 2$ and for $1 \leqslant l \leqslant 3$ we have that the three following inequalities hold for U_{lj}^ε*

- $\exists C_l > 0, \forall T > 0, \quad \left\| \int_0^T U_{lj}^\varepsilon h_{in}\,dt \right\|_{L_x^2 L_v^2}^2 \leqslant C_l \varepsilon^2 \left\| h_{in} \right\|_{L_x^2 L_v^2}^2,$
- $\exists C_l' > 0, \quad \left\| U_{lj}^\varepsilon h_{in} \right\|_{L^2_{[0,+\infty)} L_x^2 L_v^2}^2 \leqslant C_l' \varepsilon^2 \left\| h_{in} \right\|_{L_x^2 L_v^2}^2,$
- $\forall \delta \in [0,1], \exists C_\delta^{(l)} > 0, \forall t > 0, \quad \left\| U_{lj}^\varepsilon h_{in}(t) \right\|_{L_x^2 L_v^2}^2 \leqslant C_\delta^{(l)} \varepsilon^{2\delta} \left\| h_{in} \right\|_{H_x^\delta L_v^2}^2 .$

Lemma 4.8.8 *For the remainder term we have the two following inequalities*

- $\exists C_4 > 0, \forall T > 0, \quad \left\| \int_0^T U_R^\varepsilon h_{in} dt \right\|_{L_x^2 L_v^2}^2 \leqslant C_4 T \varepsilon^2 \|h_{in}\|_{L_x^2 L_v^2}^2,$
- $\exists C_4' > 0, \quad \|U_R^\varepsilon h_{in}\|_{L^2_{[0,+\infty)} L_x^2 L_v^2}^2 \leqslant C_4' \varepsilon^2 \|h_{in}\|_{L_x^2 L_v^2}^2,$
- $\forall t_0 > 0, \exists C_r > 0, \forall t > t_0, \quad \|U_R h_{in}(t)\|_{L_x^2 L_v^2}^2 \leqslant \frac{C_r}{\sqrt{t_0}} \varepsilon \|h_{in}\|_{L_x^2 L_v^2}^2.$

Moreover, the strong convergence up to $t_0 = 0$ is possible if and only if h_{in} is in Ker(L). In that case we have

$$\forall \delta \in [0,1], \exists C_\delta^{(R)} > 0, \forall t > 0, \ \|U_R^\varepsilon h_{in}\|_{L_x^2 L_v^2}^2 \leqslant C_\delta^{(R)} \varepsilon^{2\delta} \|h_{in}\|_{H_x^\delta L_v^2}^2 .$$

Therefore, gathering lemmas 4.8.6, 4.8.7 and 4.8.8 and reminding Remark 4.8.5, we proved that, as ε tends to 0, $\left(e^{tG_\varepsilon} h_{in}\right)$ converges to

$$V(t,x,v)h_{in}(x,v) = \mathcal{F}_x^{-1}\left[e^{-\beta_0 t|n|^2} P_{00}\left(\frac{n}{|n|}\right) + e^{-\beta_2 t|n|^2} P_{02}\left(\frac{n}{|n|}\right)\right] \mathcal{F}_x h_{in}. \tag{4.8.3}$$

The convergence is strong when we consider the average in time and is strong in $L_t^2 H_x^s L_v^2$ (and in $C([0,+\infty), H_x^s L_v^2)$ if h_{in} is in $H_x^{s+0} L_v^2$) if an only if both conditions found in Lemma 4.8.6 and Lemma 4.8.8 are satisfied. That is to say h_{in} belongs to Ker(L) with $\nabla_x \cdot u_{in} = 0$ and $\rho_{in} + \theta_{in} = 0$.

Moreover this also allows us to see that $V(0,x,v)h_{in} = V(0)(h_{in})(x,v)$ where $V(0)$ is the projection on the subset of Ker(L) consisting in functions g such that $\nabla_x \cdot u_g = 0$ and $\rho_g + \theta_g = 0$.

4.8.1.2 Study of the bilinear part

We recall here that $u_\varepsilon = \Gamma(h_\varepsilon, h_\varepsilon)$. Therefore, by hypothesis (H5), u_ε belongs to Ker$(L)^\perp$. Then we know that for all $-1 \leqslant j \leqslant 2$, $P_{0j}\left(\frac{n}{|n|}\right)$ is a projection onto a subspace of Ker(L). Therefore we have that, in the Fourier space,

$$P_j\left(|\varepsilon n|, \frac{n}{|n|}\right)\hat{u}_\varepsilon = |\varepsilon n| \, P_{1j}\left(\frac{n}{|n|}\right)\hat{u}_\varepsilon + |\varepsilon n|^2 P_{2j}\left(|\varepsilon n|, \frac{n}{|n|}\right)\hat{u}_\varepsilon.$$

Thus, recalling that

$$\Psi^\varepsilon(u_\varepsilon) = \int_0^t \frac{1}{\varepsilon} e^{(t-s)G_\varepsilon} u_\varepsilon(s) ds,$$

we can decompose it

$$\Psi^\varepsilon(u_\varepsilon) = \sum_{j=-1}^{2} \psi_j^\varepsilon(u_\varepsilon) + \psi_R^\varepsilon(u_\varepsilon),$$

with

$$\begin{aligned}\psi_j^\varepsilon(u_\varepsilon) &= \mathcal{F}_x^{-1}\chi_{|\varepsilon n|\leqslant n_0}\int_0^t e^{\frac{i\alpha_j(t-s)|n|}{\varepsilon}-\beta_j(t-s)|n|^2+\frac{t-s}{\varepsilon^2}\gamma_j(|\varepsilon n|)}\,|n|\left(P_{1j}+\varepsilon\,|n|\,P_{2j}\right)\hat{u}_\varepsilon(s)ds.\\ &:= \psi_{0j}^\varepsilon(u_\varepsilon)+\psi_{1j}^\varepsilon(u_\varepsilon)+\psi_{2j}^\varepsilon(u_\varepsilon)+\psi_{3j}^\varepsilon(u_\varepsilon),\end{aligned}$$

where we have used the same decomposition as in the linear case, equation (4.8.2), substituting t by $t-s$, P_{0j} by $|n|\,P_{1j}$ and $\tilde{P}_{1j}$ by $|n|\,P_{2j}$. And

$$\psi_R^\varepsilon(u_\varepsilon) = \int_0^t \frac{1}{\varepsilon}U_R^\varepsilon(t-s)u_\varepsilon(s)ds.$$

Like the linear case, Remark 4.8.5, ψ_{00}^ε and ψ_{02}^ε do not depend on ε and we are going to prove the convergence towards $\Psi(u) = \mathcal{F}_x^{-1}\left[\psi_{00}^\varepsilon(u)+\psi_{02}^\varepsilon(u)\right]\mathcal{F}_x$, where $u=\Gamma(h,h)$. To establish such a result we are going to study each term in three different lemmas and then a fourth one will deal with the remainder term. The lemmas will be proven in Appendix 4.C.

Lemma 4.8.9 *For $\alpha_j \neq 0$ ($j=\pm 1$) we have the following inequality for ψ_{0j}^ε:*

$$\exists \tilde{C}_0>0, \forall T>0, \qquad \left\|\int_0^T \psi_{0j}^\varepsilon(u_\varepsilon)dt\right\|_{L_x^2L_v^2}^2 \leqslant \tilde{C}_0T^2\varepsilon^2E(h_\varepsilon)^2.$$

Remark 4.8.10 *We know that $(h_\varepsilon)_{\varepsilon>0}$ is bounded in $L_t^\infty H_x^sL_v^2$ (see theorems 4.2.3 and 4.2.4).*

This remark gives us the strong convergence to 0 of the average in time and the strong convergence to 0 without averaging in time as long as h_{in} belongs to $\mathrm{Ker}(L)$ in Lemma 4.8.9.

Lemma 4.8.11 *For $-1\leqslant j\leqslant 2$ and for $1\leqslant l\leqslant 3$ we have that the three following inequalities hold for ψ_{lj}^ε*

- $\exists \tilde{C}_l>0, \forall T>0, \quad \left\|\int_0^T \psi_{lj}^\varepsilon(u_\varepsilon)dt\right\|_{L_x^2L_v^2}^2 \leqslant \tilde{C}_lT\varepsilon^2E(h_\varepsilon)^2,$
- $\exists \tilde{C}_l'>0, \forall T>0, \quad \left\|\psi_{lj}^\varepsilon(u_\varepsilon)\right\|_{L_{[0,T]}^2L_x^2L_v^2}^2 \leqslant \tilde{C}_l'\varepsilon^2E(h_\varepsilon)^2,$
- $\forall\,|\delta|\in[0,1], \exists C_\delta^{(l)}>0, \forall T>0, \quad \left\|\psi_{lj}^\varepsilon(u_\varepsilon)(T)\right\|_{L_x^2L_v^2}^2 \leqslant C_\delta^{(l)}\varepsilon^{2\delta}E(\partial_\delta^0 h_\varepsilon)^2.$

Lemma 4.8.12 *For the remainder term we have the three following inequalities*

- $\exists \tilde{C}_4 > 0, \forall T > 0, \quad \left\| \int_0^T \psi_R^\varepsilon(u_\varepsilon) dt \right\|^2_{L^2_x L^2_v} \leqslant \tilde{C}_4 T \varepsilon E(h_\varepsilon)^2,$
- $\exists \tilde{C}_4' > 0, \forall T > 0, \quad \|\psi_R^\varepsilon(u_\varepsilon)\|^2_{L^2_{[0,T]} L^2_x L^2_v} \leqslant \tilde{C}_4' \varepsilon E(h_\varepsilon)^2,$
- $\exists \tilde{C}_4'' > 0, \forall T > 0, \quad \|\psi_R^\varepsilon(u_\varepsilon)(T)\|^2_{L^2_x L^2_v} \leqslant \tilde{C}_4'' \varepsilon E(h_\varepsilon)^2.$

Gathering all Lemmas 4.8.9, 4.8.11 and 4.8.12 gives us the strong convergence of $\Psi^\varepsilon(u_\varepsilon) - \Psi(u_\varepsilon)$ towards 0, thanks to Remark 4.8.10. It remains to prove that we have indeed the expected convergences of $\Psi(u_\varepsilon)$ towards $\Psi(u)$ as ε tends to 0.

We start this last step by a quick remark relying on Sobolev embeddings and giving us a strong convergence of h_ε towards h in $L^\infty_{[0,T]} L^\infty_x L^2_v$, for $T > 0$.

Remark 4.8.13 *We know that $h_\varepsilon \to h$ weakly-* in $L^\infty_t H^s_x L^2_v$, for $s \geqslant s_0 > d/2$. But we also proved that for all $t > 0$ that $(h_\varepsilon)_\varepsilon$ is bounded in $H^s_x L^2_v$. Therefore the sequence $(\|h_\varepsilon\|_{L^2_v}, \varepsilon > 0)$ is bounded in H^s_x and therefore converges strongly in $H^{s'}_x$ for all $s' < s$. But, by triangular inequality it comes that*

$$\left| \|h_\varepsilon\|_{H^{s'}_x L^2_v} - \|h\|_{H^{s'}_x L^2_v} \right| \leqslant \left\| \|h_\varepsilon\|_{L^2_v} - \|h\|_{L^2_v} \right\|_{H^{s'}_x}.$$

This means that we also have that $\lim_{\varepsilon \to 0} \|h_\varepsilon\|_{H^{s'}_x L^2_v} = \|h\|_{H^{s'}_x L^2_v}$. The space $H^{s'}_x L^2_v$ is a Hilbert space and h_ε tends weakly to h in it, therefore the last result gives us that in fact h_ε tends strongly to h in $H^{s'}_x L^2_v$.

This result is for all $t > 0$ and all $s' \leqslant s$. Furthermore, $s > d/2$ and so we can choose $s' > d/2$. By Sobolev's embedding we obtain that h_ε tends strongly to h in $L^\infty_x L^2_v$, for all $t > 0$. Reminding that $h_\varepsilon \to h$ weakly- in $L^\infty_t H^s_x L^2_v$ and we obtain that we have*

$$\forall T > 0, \quad V_T(\varepsilon) = \sup_{t \in [0,T]} \|h_\varepsilon - h\|_{L^\infty_x L^2_v} \to 0, \text{ as } \varepsilon \to 0.$$

Lemma 4.8.14 *We have the following rate of convergence:*

- $\exists \tilde{C}_5 > 0, \forall T > 0, \left\| \int_0^T \Psi(u_\varepsilon) dt - \int_0^T \Psi(u) dt \right\|^2_{L^2_x L^2_v} \leqslant \tilde{C}_5 T^2 V_T(\varepsilon)^2,$
- $\exists \tilde{C}_5' > 0, \forall T > 0, \|\Psi(u_\varepsilon) - \Psi(u_\varepsilon)\|^2_{L^2_{[0,T]} L^2_x L^2_v} \leqslant \tilde{C}_5' T V_T(\varepsilon)^2,$
- $\exists \tilde{C}_5'' > 0, \forall T > 0, \|\Psi(u_\varepsilon) - \Psi(u_\varepsilon)\|^2_{L^2_x L^2_v}(T) \leqslant \tilde{C}_5'' V_T(\varepsilon)^2.$

Thus, those Lemmas, combined with the study of the linear case (Lemmas 4.8.6, 4.8.7 and 4.8.8) prove the Theorem 4.2.5 with the rate of convergence being the maximum of each rate of convergence. Moreover we have proved

$$h(t,x,v) = V(t,x,v)h_{in}(x,v) + \Psi(t,x,v)(\Gamma(h,h)).$$

4.8.2 Proof of Theorem 4.2.5

Thanks to Theorem 4.8.1 we can control the convergence of h_ε towards h for any finite time T. Then, thanks to the uniqueness property of Theorem 2.1 and the control on the remainder of Theorem 2.3 in [56], in the case of a hard potential collision kernel, one has

$$\forall T > 0, V_T(\varepsilon) \leqslant C_V \varepsilon.$$

Finally, thanks to Theorem 4.2.3, we have the exponential decay for both h_ε and h, leading to

$$\|h_\varepsilon - h\|_{H^s_x L^2_v} \leqslant 2\,\|h_{in}\|_{\mathcal{H}^s_\varepsilon} e^{-\tau_s T}.$$

We define

$$T_M = -\frac{1}{\tau_s}\ln\left(\frac{\varepsilon}{2\,\|h_{in}\|_{\mathcal{H}^s_\varepsilon}}\right)$$

to get that

$$\forall T \geqslant T_M, \quad \|h_\varepsilon - h\|_{H^s_x L^2_v} \leqslant \varepsilon.$$

This conclude the proof Theorem 4.2.5, by applying Theorem 4.8.1 to T_M.

Appendices

4.A Validation of the assumptions

As said in the introduction, all the hypocoercivity theory assumptions hold for several different kinetic models. One can find the proof of the assumptions (H1), (H2), (H3), (H1') and (H2') in [82] directly for the linear relaxation (see also [24]), the semi-classical relaxation (see also [86]), the linear Fokker-Planck equation, the Boltzmann equation with hard potential and angular cutoff and the Landau equation with hard and moderately soft potential (both studied in a constructive way in [4] and [79], for the spectral gaps, see also [53] and [54] for the Cauchy problems):

- The Linear Relaxation

$$\partial_t f + v.\nabla_x f = \frac{1}{\varepsilon}\left[\left(\int_{\mathbb{R}}^{d} f(t,x,v_*)dv_*\right)\mu(v) - f\right],$$

- The Semi-classical Relaxation

$$\partial_t f + v.\nabla_x f = \frac{1}{\varepsilon}\int_{\mathbb{R}^d} \left[\mu(1-\delta f)f_* - \mu_*(1-\delta f_*)f\right] dv_*,$$

- The Linear Fokker-Planck Equation

$$\partial_t f + v.\nabla_x f = \frac{1}{\varepsilon}\nabla_v.\left(\nabla_v f + fv\right),$$

- The Boltzmann Equation with hard potential and angular cutoff

$$\partial_t f + v.\nabla_x f = \frac{1}{\varepsilon}\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} b(cos\theta)|v-v_*|^\gamma \left[f'f'_* - ff_*\right] dv_* d\sigma,$$

- The Landau Equation with hard and moderately soft potential

$$\partial_t f + v.\nabla_x f = \frac{1}{\varepsilon}\nabla_v.\left(\int_{\mathbb{R}^d} \Phi(v-v_*)|v-v_*|^{\gamma+2}\left[f_*(\nabla f) - f(\nabla f)_*\right]\right).$$

Assumption (H4) is clearly satisfied by the first three as in that case we have either $\|.\|_{\Lambda_v} = \|.\|_{L^2_v}$ or $\Gamma = 0$ (see [82]). Moreover, (H5) is obvious in the case of a linear equation. It thus remains to prove properties (H5) for the semi-classical relaxation and (H4) and (H5) for the Boltzmann equation and the Landau equation (since our property (H4) is slightly different from (H4) in [82]).

4.A.1 The semi-classical relaxation

In the case of the semi-classical relaxation, the linearization is slightly different. Indeed, the unique global equilibrium associated to an initial data f_0 is (assuming some initial bounds, see [82])

$$f_\infty = \frac{\kappa_\infty \mu}{1+\delta\kappa_\infty\mu},$$

where κ_∞ depends on f_0.

Thus, we are no longer in the case of a global equilibrium being a Maxwellian. However, a good way of linearizing this equation is (see [82]) considering

$$f = f_\infty + \varepsilon\,\frac{\sqrt{\kappa_\infty\mu}}{1+\delta\kappa_\infty\mu}\,h.$$

Using such a linearization instead of the one used all along this chapter yields the same general equation (4.1.3) with L and Γ satisfying all the requirements (see [82]). Indeed, one may find that $\mathrm{Ker}(L) = \mathrm{Span}\left(f_\infty/\sqrt{\mu}\right)$ and then notice that this is not of the form needed in assumption (H3). However, this is bounded by $e^{-|v|^2/4}$ and therefore we are still able to use the toolbox (section 4.3, thus all the theorems.

Let us look at the bilinear operator to show that it fulfils hypothesis (H5). A straightforward computation gives us the definition of Γ,

$$\Gamma(g,h) = \frac{\delta\sqrt{\kappa_\infty}}{2}\int_{\mathbb{R}^d}\sqrt{\mu_*}\frac{\mu_*-\mu}{1+\varepsilon\kappa_\infty\mu_*}[hg_*+h_*g]dv_*.$$

Then, multiplying by a function f, integrating over $\mathbb{R}^d$ and looking at the change of variable $(v,v_*)\to(v_*,v)$ yields

$$\langle\Gamma(g,h),f\rangle_{L^2_v} = \frac{\delta\sqrt{\kappa_\infty}}{4}\int_{\mathbb{R}^d\times\mathbb{R}^d}(\mu_*-\mu)(gh_*+g_*h)\left[f\frac{\sqrt{\mu_*}}{1+\delta\kappa_\infty\mu_*}-f_*\frac{\sqrt{\mu}}{1+\delta\kappa_\infty\mu}\right]dvdv_*.$$

Therefore, taking f in $\mathrm{Ker}(L)$ gives us the expected property.

4.A.2 Boltzmann operator with angular cutoff and hard potential

Notice that, compared to [82], we defined Γ in a way that it is symmetric which gives us, using the fact that $\mu_*\mu = \mu'_*\mu'$,

$$\Gamma(g,h) = \frac{1}{2}\int_{\mathbb{R}^d\times(S)^{d-1}}B(\mu^{1/2})_*[g'_*h'+g'h'_*-g_*h-gh_*]dv_*d\sigma,$$

4.A.2.1 Orthogonality to Ker(L): (H5)

A well-known property (see [46] for instance) tells us that for all ϕ in L^2_v decreasing fast enough at infinity and for all ψ in L^2_v one has

$$\int_{\mathbb{R}^d}\Gamma(g,h)(v)\psi(v)dv = \frac{1}{8}\int_{(\mathbb{R}^d)^2\times\mathbb{S}^{d-1}}B[g'_*h'+g'h'_*-g_*h-gh_*]$$
$$((\mu^{1/2})_*\psi+(\mu^{1/2})\psi_*-(\mu^{1/2})'_*\psi'-(\mu^{1/2})'\psi'_*)dvdv_*d\sigma.$$

As shown in [30] or [82] we have that $\mathrm{Ker}(L) = \mathrm{Span}(1,v_1,\dots,v_d,|v|^2)\mu^{1/2}$ and therefore taking ψ to be each of these kernel functions gives us (H5).

4.A.2.2 Controlling derivatives: (H4)

To prove (H4) we can define

$$\begin{aligned}\Gamma^+(g,h) &= \int_{\mathbb{R}^d\times(S)^{d-1}} B(\mu^{1/2})_* g'_* h' \, dv_* d\sigma, \\ \Gamma^-(g,h) &= -\int_{\mathbb{R}^d\times(S)^{d-1}} B(\mu^{1/2})_* g_* h \, dv_* d\sigma.\end{aligned}$$

By using the change of variable $u = v - v_*$ we end up with θ being a function of u and σ and $v' = v + f_1(u,\sigma)$ and $v'_* = v + f_2(u,\sigma)$, f_1 and f_2 being functions. Therefore we can make this change of variable, take j and l such that $|j| + |l| \leqslant s$ and differentiate our operator Γ^-.

$$\partial_l^j \Gamma^-(g,h) = -\frac{1}{2} \sum_{\substack{j_0+j_1+j_2=j \\ l_1+l_2=l}} \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} b(cos\theta)|u|^\gamma \partial_0^{j_0}\left(\mu(v-u)^{1/2}\right) \partial_{l_1}^{j_1} g_* \, \partial_{l_2}^{j_2} h \, du d\sigma.$$

Then we can easily compute that, C being a generic constant,

$$\left|\partial_0^{j_0}\left(\mu(v-u)^{1/2}\right)\right| \leqslant C\mu(v-u)^{1/4}.$$

Moreover, we are in the case where $\gamma > 0$ and therefore we have

$$|u|^\gamma \mu(v-u)^{1/4} \leqslant C(1+|v|)^\gamma \mu(v-u)^{1/8}.$$

Combining this and the fact that $|b| \leqslant C_b$ (angular cutoff considered here), multiplying by a function f and integrating over $\mathbb{T}^d \times \mathbb{R}^d$ yields, using Cauchy-Schwarz two times,

$$\begin{aligned}\left|\langle \partial_l^j \Gamma^-(g,h), f\rangle_{L^2_{x,v}}\right| &\leqslant C \sum_{\substack{j_0+j_1+j_2=j \\ l_1+l_2=l}} \int_{\mathbb{T}^d\times\mathbb{R}^d} (1+|v|)^\gamma \left|\partial_{l_2}^{j_2} h\right| |f| \left(\int_{\mathbb{R}^d} \mu_*^{1/8} \left|\partial_{l_1}^{j_1} g_*\right| dv_*\right) dv dx \\ &\leqslant \mathcal{G}^s(g,h) \left\|f\right\|_\Lambda,\end{aligned}$$

with

$$\mathcal{G}^s(g,h) = C \sum_{|j_1|+|l_1|+|j_2|+|l_2|\leqslant s} \left[\int_{\mathbb{T}^d} \left\|\partial_{l_2}^{j_2} h\right\|_{\Lambda_v}^2 \left\|\partial_{l_1}^{j_1} g\right\|_{L^2_v}^2 dx\right]^{1/2}.$$

At that point we can use Sobolev embeddings (see [22], corollary $IX.13$) stating that if $E\left(s_0/2\right) > d/2$ then we have $H_x^{s/2} \hookrightarrow L_x^\infty$.

So, if $|j_1| + |l_1| \leqslant s/2$ we have

$$\begin{aligned}
\left\|\partial^{j_1}_{l_1} g\right\|^2_{L^2_v} &\leqslant \sup_{x\in\mathbb{T}^d} \left\|\partial^{j_1}_{l_1} g\right\|^2_{L^2_v} \leqslant C_s \left\|\left\|\partial^{j_1}_{l_1} g\right\|^2_{L^2_v}\right\|_{H^{s/2}_x} \\
&\leqslant C_s \sum_{|p|\leqslant s/2} \sum_{p_1+p_2=p} \int_{\mathbb{T}^d\times\mathbb{R}^d} \partial^{j_1}_{l_1+p_1} g\, \partial^{j_1}_{l_1+p_2} g\, dvdx \qquad (4.A.1)\\
&\leqslant C_s \left\|g\right\|^2_{H^s_{x,v}},
\end{aligned}$$

by a mere Cauchy-Schwarz inequality.

In the other case, $|j_2| + |l_2| \leqslant s/2$ and by same calculations we show

$$\left\|\partial^{j_2}_{l_2} h\right\|^2_{\Lambda_v} \leqslant C_s \left\|h\right\|^2_{H^s_\Lambda}.$$

Therefore, by just dividing the sum into this two subcases we obtain the result (H4) for Γ^-, noticing that in the case $j = 0$ equation (4.A.1) has no v derivatives and the Cauchy-Schwarz inequality does not create such derivatives so the control is only made by x-derivatives.

The second term Γ^+ is dealt exactly the same way with, at the end (the study of $\mathcal{G}^s$), another change of variable $(v, v_*) \to (v', v'_*)$ which gives the result since $(1 + |v'|)^\gamma \leqslant (1 + |v|)^\gamma + (1 + |v_*|)^\gamma$ if $\gamma > 0$.

4.A.3 Landau operator with hard and moderately soft potential

The Landau operator is used to describe plasmas and for instance in the case of particles interacting via a Coulomb interaction (see [112] for more details). The particular case of Coulomb interaction alone ($\gamma = -3$) will not be studied here as the Landau linear operator has a spectral gap if and only if $\gamma \geqslant -2$ (see [53], for not constructive arguments, [83] for general constructive case and [4] for explicit construction in the case of hard potential $\gamma > 0$) and so only the case $\gamma \geqslant -2$ may be applicable in this study.

We can compute straightforwardly the bilinear symmetric operator associated with the Landau equation:

$$\Gamma(g,h) = \frac{1}{2\sqrt{\mu}} \nabla_v \cdot \int_{\mathbb{R}^d} \sqrt{\mu\mu_*}\Phi(v - v_*) \left[g_*\nabla_v h + h_*\nabla_v g - g(\nabla_v h)_* - h(\nabla_v g)_*\right] dv_*,$$

where $\Phi : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is such that $\Phi(z)$ is the orthogonal projection onto $\mathrm{Span}(z)^\perp$ so

$$\Phi(z)_{ij} = \delta_{ij} - \frac{z_i z_j}{|z|^2},$$

and γ belongs to $[-2, 1]$.

4.A.3.1 Orthogonality to Ker(L): (H5)

Let consider a function ψ in $C^\infty_{x,v}$. A mere integration by part gives us

$$\langle \Gamma(g,h),\psi\rangle_{L^2_v} = -\frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} \nabla_v\left(\frac{\psi}{\sqrt{\mu}}\right)\cdot\left(\sqrt{\mu\mu_*}\Phi(v-v_*)[G]\right)dv_*dv,$$

where

$$G = g_*\nabla_v h + h_*\nabla_v g - g(\nabla_v h)_* - h(\nabla_v g)_*.$$

Then the change of variable $(v,v_*)\to(v_*,v)$ only changes $\nabla_v(\psi/\sqrt{\mu})$ to $\left[\nabla_v(\psi/\sqrt{\mu})\right]_*$ and G becomes $-G$. Therefore we finally obtain

$$\langle \Gamma(g,h),\psi\rangle_{L^2_v} = \frac{1}{4}\int_{\mathbb{R}^d\times\mathbb{R}^d} \sqrt{\mu\mu_*}\Phi(v-v_*)[G]\cdot\left[\left(\nabla_v\left(\frac{\psi}{\sqrt{\mu}}\right)\right)_* - \nabla_v\left(\frac{\psi}{\sqrt{\mu}}\right)\right]dv_*dv.$$

As shown in [30] or [82] we have that $\mathrm{Ker}(L) = \mathrm{Span}(1,v_1,\dots,v_d,|v|^2)\mu^{1/2}$. Computing the term inside brackets for each of these functions gives us 0 or, in the case $|v|^2\sqrt{\mu}$, $2(v_*-v)$.
However, by definition, $\Phi(v-v_*)[G]$ belongs to $\mathrm{Span}(v-v_*)^\perp$ and therefore $\Phi(v-v_*)[G]\cdot(v_*-v)=0$. So Γ indeed satisfies (H5).

4.A.3.2 Controlling derivatives: (H4)

The article [53] gives us directly the expected result in its Theorem 3, equation (35) with $\theta=0$. The case where there are only x-derivatives is also included if one takes $\beta=0$.

4.B Proofs of the results given in the toolbox

We used the estimates given by the toolbox throughout this chapter. This appendix is to prove all of them. It is divided in two parts. The first one is dedicated to the proof of the equality between null spaces whereas the second part deals with the time derivatives inequalities.

4.B.1 Proof of Proposition 4.3.1:

We are about to prove the following proposition.

Proposition 4.B.1 *Let a and b be in $\mathbb{R}^*$ and consider the operator $G = aL - bv\cdot\nabla_x$ acting on $H^1_{x,v}$.*

If L satisfies (H1) *and* (H3) *then*

$$\mathrm{Ker}(G) = \mathrm{Ker}(L).$$

To prove this result we will need a lemma.

Lemma 4.B.2 *Let $f : \mathbb{T}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}$ be continuous on $\mathbb{T}^d \times \mathbb{R}^d$ and differentiable in x. If $v \cdot \nabla_x f(x,v) = 0$ for all (x,v) in $\mathbb{T}^d \times \mathbb{R}^d$ then f does not depend on x.*

Proof of Lemma 4.*B*.2 Fix x in $\mathbb{T}^d$ and v $\mathbb{Q}$-free in $\mathbb{R}^d$.
For y in $\mathbb{R}^d$ we will denote by $\bar{y}$ its equivalent class in $\mathbb{T}^d$.

$$\text{Define} \quad g : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ t & \longmapsto & f(x \bar{+} tv, v) \end{array}.$$

We find easily that g is differentiable on $\mathbb{R}$ and that $g'(t) = v.\nabla_x f(x,v) = 0$ on $\mathbb{R}$. Therefore:

$$\forall t \in \mathbb{R}, f(x \bar{+} tv, v) = f(x,v).$$

However, a well-known property about the torus is that the set $\{x + nv, n \in \mathbb{Z}\}$ is dense in $\mathbb{T}^d$ for all x in $\mathbb{T}^d$ and v $\mathbb{Q}$-free in $\mathbb{R}^d$. This combined with the last result and the continuity of f leads to:

$$\forall y \in \mathbb{T}^d, \quad f(y,v) = f(x,v).$$

To conclude it is enough to see that the set of $\mathbb{Q}$-free vector in $\mathbb{R}^d$ is dense in $\mathbb{R}^d$ and then, by continuity of f in v:

$$\forall y \in \mathbb{T}^d, \forall v \in \mathbb{R}^d, \quad f(y,v) = f(x,v).$$

■

Now we have all the tools to prove the proposition about the kernel of operators.

Proof of Proposition 4.*B*.1 Since L satisfies (H1) we know that L acts on L^2_v and that its Kernel functions ϕ_i only depend on v. Thus, we have directly the first inclusion

$$\mathrm{Ker}(L) \subset \mathrm{Ker}(G).$$

Then, let us consider h in $H^1_{x,v}$ such that $G(h) = 0$.
Because the transport operator $v \cdot \nabla_x$ is skew-symmetric in $L^2_{x,v}$ we have

$$0 = \langle G(h), h \rangle_{L^2_{x,v}} = a \int_{\mathbb{T}^d} \langle L(h), h \rangle_{L^2_v} dx.$$

However, because L satisfies (H3) we obtain:

$$0 \geqslant \lambda \int_{\mathbb{T}^d} \|h(x,.) - \pi_L(h(x,.))\|_{\Lambda_v}^2 \, dx.$$

But λ is strictly positive and thus:

$$\forall x \in \mathbb{T}^d, \quad h(x,\cdot) = \pi_L(h(x,\cdot)) = \sum_{i=1}^d c_i(x)\phi_i.$$

Finally we have, by assumption, $G(h) = 0$ and because $h(x,\cdot)$ belongs to $\mathrm{Ker}(L)$ for all x in $\mathbb{T}^d$ we end up with

$$\forall (x,v) \in \mathbb{T}^d \times \mathbb{R}^d, \quad v \cdot \nabla_x h(x,v) = 0.$$

By applying the lemma above we then obtain that h does not depend on x. But $(\phi_i)_{1\leqslant i \leqslant d}$ is an orthonormal family, basis of $\mathrm{Ker}(L)$, and therefore we find that for all i, c_i does not depend on x.
So,we have proved that:

$$\forall (x,v) \in \mathbb{T}^d \times \mathbb{R}^d, \ h(x,v) = \sum_{i=1}^d c_i \phi_i(v).$$

Therefore, h belongs to $\mathrm{Ker}(L)$ and only depends on x. ■

4.B.2 A priori energy estimates

In this subsection we derive all the inequalities we used. Therefore, we assume that L satisfies (H1'), (H2') and (H3) while Γ has the properties (H4) and (H5), and we pick g in $H^s_{x,v}$. We consider h in $H^s_{x,v} \cap \mathrm{Ker}(G_\varepsilon)^\perp$ and we assume that h is a solution to (4.1.3):

$$\partial_t h + \frac{1}{\varepsilon} v.\nabla_x h = \frac{1}{\varepsilon^2} L(h) + \frac{1}{\varepsilon}\Gamma(g,h).$$

In the toolbox, we wrote inequalities on function which were solutions of the linear equation. As the reader may notice, we will deal with the second order operator just by applying the first part of (H4) and Young's inequality. Such an inequality only provides two positive terms, and thus by just setting Γ equal to 0 in the next inequalities we get the expected bounds in the linear case (not the sharpest ones though). Therefore we will just describe the more general case and the linear one is included in it.

4.B.2.1 Time evolution of pure x-derivatives

The operators L and Γ only act on the v variable. Thus, for $0 \leqslant |l| \leqslant s$, ∂_l^0 commutes with L and $v \cdot \nabla_x$. Remind that $v \cdot \nabla_x$ is skew-symmetric in $L^2_{x,v}(\mathbb{T}^d \times \mathbb{R}^d)$ and therefore we can compute

$$\frac{d}{dt}\left\|\partial_l^0 h\right\|^2_{L^2_{x,v}} = \frac{2}{\varepsilon^2}\langle L(\partial_l^0 h), \partial_l^0 h\rangle_{L^2_{x,v}} + \frac{2}{\varepsilon}\langle \partial_l^0 \Gamma(g,h), \partial_l^0 h\rangle_{L^2_{x,v}}.$$

We can then use hypothesis (H3) to obtain

$$\frac{2}{\varepsilon^2}\langle L(\partial_l^0 h), \partial_l^0 h\rangle_{L^2_{x,v}} \leqslant -\frac{2\lambda}{\varepsilon^2}\left\|(\partial_l^0 h)^\perp\right\|^2_\Lambda.$$

We also use (H3) to get $(\partial_l^0 h)^\perp = \partial_l^0 h^\perp$.

To deal with the second scalar product, we will use hypothesis (H4) and (H5), which is still valid for $\partial_l^0\Gamma$ since π_L only acts on the v variable, followed by a Young inequality with some $D_1 > 0$. This yields

$$\begin{aligned}
\frac{2}{\varepsilon}\langle \partial_l^0 \Gamma(g,h), \partial_l^0 h\rangle_{L^2_{x,v}} &= \frac{2}{\varepsilon}\langle \partial_l^0 \Gamma(g,h), \partial_l^0 h^\perp\rangle_{L^2_{x,v}} \\
&\leqslant \frac{2}{\varepsilon}\mathcal{G}^s_x(g,h)\left\|\partial_l^0 h^\perp\right\|_\Lambda \\
&\leqslant \frac{D_1}{\varepsilon}\left(\mathcal{G}^s_x(g,h)\right)^2 + \frac{1}{D_1\varepsilon}\left\|\partial_l^0 h^\perp\right\|^2_\Lambda.
\end{aligned}$$

Gathering the last two upper bounds we obtain

$$\frac{d}{dt}\left\|\partial_l^0 h\right\|^2_{L^2_{x,v}} \leqslant \left[\frac{1}{D_1\varepsilon} - \frac{2\lambda}{\varepsilon^2}\right]\left\|\partial_l^0 h^\perp\right\|^2_\Lambda + \frac{D_1}{\varepsilon}\left(\mathcal{G}^s_x(g,h)\right)^2.$$

Finally, taking $D_1 = \varepsilon/\lambda$ gives us inequalities (4.3.6), (4.3.7) and (4.3.10).

4.B.2.2 Time evolution of $\|\nabla_v h\|^2_{L^2_{x,v}}$

For that term we get, by applying the equation satisfied by h, the following:

$$\frac{d}{dt}\left\|\nabla_v h\right\|^2_{L^2_{x,v}} = \frac{2}{\varepsilon^2}\langle \nabla_v L(h), \nabla_v h\rangle_{L^2_{x,v}} - \frac{2}{\varepsilon}\langle \nabla_v(v\cdot\nabla_x h), \nabla_v h\rangle_{L^2_{x,v}} + \frac{2}{\varepsilon}\langle \nabla_v \Gamma(g,h), \nabla_v h\rangle_{L^2_{x,v}}.$$

And by writing the second term on the right-hand side of the equality and integrating by part in x, we have

$$\langle \nabla_v(v\cdot\nabla_x h), \nabla_v h\rangle_{L^2_{x,v}} = \langle \nabla_x h, \nabla_v h\rangle_{L^2_{x,v}}.$$

Therefore the following holds:

$$\frac{d}{dt}\|\nabla_v h\|^2_{L^2_{x,v}} = \frac{2}{\varepsilon^2}\langle\nabla_v L(h),\nabla_v h\rangle_{L^2_{x,v}} - \frac{2}{\varepsilon}\langle\nabla_x h,\nabla_v h\rangle_{L^2_{x,v}} + \frac{2}{\varepsilon}\langle\nabla_v\Gamma(g,h),\nabla_v h\rangle_{L^2_{x,v}}.$$

Then we have by (H1) that $L = K - \Lambda$ and we can estimate each component thanks to (H1) and (H2):

$$\begin{array}{rcl}
-\langle\nabla_v\Lambda(h),\nabla_v h\rangle_{L^2_{x,v}} & \leqslant & \nu_4^\Lambda \|h\|^2_{L^2_{x,v}} - \nu_3^\Lambda\|\nabla_v h\|^2_\Lambda\,, \\
\langle\nabla_v K(h),\nabla_v h\rangle_{L^2_{x,v}} & \leqslant & C(\delta)\|h\|^2_{L^2_{x,v}} + \delta\|\nabla_v h\|^2_{L^2_{x,v}}\,,
\end{array}$$

where δ is a strictly positive real that we will choose later.

Finally, for a $D > 0$ that we will choose later, we have the following upper bound, by Cauchy-Schwarz inequality:

$$-\frac{2}{\varepsilon}\langle\nabla_x h,\nabla_v h\rangle_{L^2_{x,v}} \leqslant \frac{D}{\varepsilon}\|\nabla_x h\|^2_{L^2_{x,v}} + \frac{\nu_1^\Lambda}{D\nu_0^\Lambda\varepsilon}\|\nabla_v h\|^2_\Lambda\,,$$

using the fact that $\|.\|^2_{L^2_{x,v}} \leqslant \frac{\nu_1^\Lambda}{\nu_0^\Lambda}\|.\|^2_\Lambda$. Finally, another Young inequality gives us a control on the last scalar product, for a $D_2 > 0$ to be chosen later

$$\frac{2}{\varepsilon}\langle\nabla_v\Gamma(g,h),\nabla_v h\rangle_{L^2_{x,v}} \leqslant \frac{D_2}{\varepsilon}\left(\mathcal{G}^1_{x,v}(g,h)\right)^2 + \frac{1}{D_2\varepsilon}\|\nabla_v h\|^2_\Lambda\,.$$

We gather here the last three inequalities to obtain our global upper bound:

$$\begin{array}{rcl}
\frac{d}{dt}\|\nabla_v h\|^2_{L^2_{x,v}} & \leqslant & \frac{1}{\varepsilon^2}\left(2\nu_4^\Lambda + 2C(\delta)\right)\|h\|^2_{L^2_{x,v}} + \frac{D}{\varepsilon}\|\nabla_x h\|^2_{L^2_{x,v}} \\
& & + \left(\frac{2\nu_1^\Lambda\delta}{\nu_0^\Lambda\varepsilon^2} - \frac{2\nu_3^\Lambda}{\varepsilon^2} + \frac{\nu_1^\Lambda}{D\varepsilon\nu_0^\Lambda} + \frac{1}{D_2\varepsilon}\right)\|\nabla_v h\|^2_\Lambda + \frac{D_2}{\varepsilon}\left(\mathcal{G}^1_{x,v}(g,h)\right)^2.
\end{array}$$

We can go even further since we have $\|h\|^2_{L^2_{x,v}} = \left\|h^\perp\right\|^2_{L^2_{x,v}} + \|\pi_L(h)\|^2_{L^2_{x,v}}$.
But because h is in $\mathrm{Ker}(G_\varepsilon)^\perp$ we can use the toolbox and the equation (4.3.5) about the Poincaré inequality:

$$\|\pi_L(h)\|^2_{L^2_{x,v}} \leqslant C_p\|\nabla_x h\|^2_{L^2_{x,v}}\,.$$

This last inequality yields:

$$\begin{aligned}
\frac{d}{dt}\left\|\nabla_v h\right\|^2_{L^2_{x,v}} \leqslant\ & \frac{\nu_1^\Lambda}{\nu_0^\Lambda \varepsilon^2}\left(2\nu_4^\Lambda + 2C(\delta)\right)\left\|h^\perp\right\|^2_\Lambda + \left[\frac{C_p}{\varepsilon^2}\left(2\nu_4^\Lambda + 2C(\delta)\right) + \frac{D}{\varepsilon}\right]\left\|\nabla_x h\right\|^2_{L^2_{x,v}} \\
& + \left[\frac{2\nu_1^\Lambda \delta}{\nu_0^\Lambda \varepsilon^2} - \frac{2\nu_3^\Lambda}{\varepsilon^2} + \frac{\nu_1^\Lambda}{D\varepsilon\nu_0^\Lambda} + \frac{1}{D_2\varepsilon}\right]\left\|\nabla_v h\right\|^2_\Lambda + \frac{D_2}{\varepsilon}\left(\mathcal{G}^1_{x,v}(g,h)\right)^2.
\end{aligned}$$

Therefore, we can choose $\delta = \nu_0^\Lambda \nu_3^\Lambda / 6\nu_1^\Lambda$, $D = 3\nu_1^\Lambda \varepsilon / \nu_0^\Lambda \nu_3^\Lambda$ and $D_2 = 3\varepsilon/\nu_3^\Lambda$ to get the equation (4.3.8).

4.B.2.3 Time evolution of $\langle \nabla_x h, \nabla_v h\rangle_{L^2_{x,v}}$

In the same way, and integrating by part in x then in v we obtain the following equality:

$$\begin{aligned}
&\frac{d}{dt}\langle \nabla_x h, \nabla_v h\rangle_{L^2_{x,v}} \\
&\qquad = \frac{2}{\varepsilon^2}\langle L(\nabla_x h), \nabla_v h\rangle_{L^2_{x,v}} - \frac{2}{\varepsilon}\langle \nabla_v(v\cdot\nabla_x h), \nabla_x h\rangle_{L^2_{x,v}} + \frac{2}{\varepsilon}\langle \nabla_x \Gamma(g,h), \nabla_v h\rangle_{L^2_{x,v}}.
\end{aligned}$$

By writing explicitly $\langle \nabla_v(v\cdot\nabla_x h), \nabla_x h\rangle_{L^2_{x,v}}$ and by integrating by part one can show that the following holds:

$$\langle \nabla_v(v.\nabla_x h), \nabla_x h\rangle_{L^2_{x,v}} = \frac{1}{2}\left\|\nabla_x h\right\|^2_{L^2_{x,v}}.$$

Therefore we have an explicit formula for that term and we can find the time derivative of the scalar product being:

$$\frac{d}{dt}\langle \nabla_x h, \nabla_v h\rangle_{L^2_{x,v}} = \frac{2}{\varepsilon^2}\langle L(\nabla_x h), \nabla_v h\rangle_{L^2_{x,v}} - \frac{1}{\varepsilon}\left\|\nabla_x h\right\|^2_{L^2_{x,v}} + \frac{2}{\varepsilon}\langle \nabla_x \Gamma(g,h), \nabla_v h\rangle_{L^2_{x,v}}.$$

We can bound above the first term in the right-hand side of the equality thanks to (H1) and then Cauchy-Schwarz in x, with a constant $\eta > 0$ to be define later.

$$\begin{aligned}
\frac{2}{\varepsilon^2}\langle L(\nabla_x h), \nabla_v h\rangle_{L^2_{x,v}} &= \frac{2}{\varepsilon^2}\langle L(\nabla_x h^\perp), \nabla_v h\rangle_{L^2_{x,v}} \\
&\leqslant \frac{C^L}{\varepsilon^2}\int_{\mathbb{T}^d} 2\left\|\nabla_x h^\perp\right\|_{\Lambda_v}\left\|\nabla_v h\right\|_{\Lambda_v} dx \\
&\leqslant \frac{C^L\eta}{\varepsilon^2}\left\|\nabla_x h^\perp\right\|^2_\Lambda + \frac{C^L}{\eta\varepsilon^2}\left\|\nabla_v h\right\|^2_\Lambda.
\end{aligned}$$

Then applying hypothesis (H4) and Young's inequality one more time with a constant $D_3 > 0$ one may find

$$\frac{2}{\varepsilon}\langle \nabla_x \Gamma(g,h), \nabla_v h\rangle_{L^2_{x,v}} \leqslant \frac{D_3}{\varepsilon}\left(\mathcal{G}^1_x(g,h)\right)^2 + \frac{1}{D_3\varepsilon}\left\|\nabla_v h\right\|^2_\Lambda.$$

Hence we end up with the following inequality:

$$\begin{aligned}\frac{d}{dt}\langle \nabla_x h, \nabla_v h\rangle_{L^2_{x,v}} \leqslant \quad & \frac{C^L\eta}{\varepsilon^2}\left\|\nabla_x h^\perp\right\|^2_\Lambda + \left(\frac{C^L}{\eta\varepsilon^2}+\frac{1}{D_3}\right)\|\nabla_v h\|^2_\Lambda - \frac{1}{\varepsilon}\|\nabla_x h\|^2_{L^2_{x,v}} \\ & + \frac{D_3}{\varepsilon}\left(\mathcal{G}^1_x(g,h)\right)^2.\end{aligned}$$

Now define $\eta = e/\varepsilon$, $e > 0$, and $D_3 = e/C^L$ to obtain equation (4.3.9).

4.B.2.4 Time evolution of $\left\|\partial^j_l h\right\|^2_{L^2_{x,v}}$ for $|j| \geqslant 1$ and $|j| + |l| = s$

This term is the only term far from what we already did since we are mixing more than one derivative in x and one derivative in v in general. By simply differentiating in time and integrating by part we find the following equality.

$$\begin{aligned}\frac{d}{dt}\left\|\partial^j_l h\right\|^2_{L^2_{x,v}} &= \frac{2}{\varepsilon^2}\langle \partial^j_l L(h), \partial^j_l h\rangle_{L^2_{x,v}} - \frac{2}{\varepsilon}\langle \partial^j_l(v.\nabla_x h), \partial^j_l h\rangle_{L^2_{x,v}} \\ &\quad + \frac{2}{\varepsilon}\langle \partial^j_l\Gamma(g,h), \partial^j_l h\rangle_{L^2_{x,v}} \\ &= \frac{2}{\varepsilon^2}\langle \partial^j_l L(h), \partial^j_l h\rangle_{L^2_{x,v}} - \frac{2}{\varepsilon}\sum_{i, c_i(j)>0}\langle \partial^j_l h, \partial^{j-\delta_i}_{l+\delta_i} h\rangle_{L^2_{x,v}} \\ &\quad + \frac{2}{\varepsilon}\langle \partial^j_l\Gamma(g,h), \partial^j_l h\rangle_{L^2_{x,v}}.\end{aligned}$$

We can then apply Cauchy-Schwarz for the terms inside the sum symbol. For each we can use a $D_{i,l,s} > 0$ but because they play an equivalent role we will take the same $D > 0$, that we will choose later:

$$-\frac{2}{\varepsilon}\langle \partial^j_l h, \partial^{j-\delta_i}_{l+\delta_i} h\rangle_{L^2_{x,v}} \leqslant \frac{\nu_1^\Lambda}{D\nu_0^\Lambda\varepsilon}\left\|\partial^j_l h\right\|^2_\Lambda + \frac{D}{\varepsilon}\left\|\partial^{j-\delta_i}_{l+\delta_i} h\right\|^2_{L^2_{x,v}}.$$

Then we can use (H1') and (H2'), with a $\delta > 0$ we will choose later, to obtain

$$\frac{2}{\varepsilon^2}\langle \partial^j_l L(h), \partial^j_l h\rangle_{L^2_{x,v}} \leqslant \frac{2}{\varepsilon^2}(C(\delta)+\nu_6^\Lambda)\|h\|^2_{H^{s-1}_{x,v}} + \frac{2}{\varepsilon^2}\left(\frac{\delta\nu_1^\Lambda}{\nu_0^\Lambda} - \nu_5^\Lambda\right)\left\|\partial^j_l h\right\|^2_\Lambda.$$

Finally, applying (H4) and Young's inequality with a constant $D_2 > 0$ we obtain

$$\frac{2}{\varepsilon}\langle \partial^j_l\Gamma(g,h), \partial^j_l h\rangle_{L^2_{x,v}} \leqslant \frac{D_2}{\varepsilon}\left(\mathcal{G}^s_{x,v}(g,h)\right)^2 + \frac{1}{D_2\varepsilon}\left\|\partial^j_l h\right\|^2_\Lambda.$$

Combining these three inequality we find an upper bound for the time evolution. Here we also use the fact that the number of i such that $c_i(j) > 0$ is less or equal to d.

$$
\begin{aligned}
\frac{d}{dt}\left\|\partial_l^j h\right\|_{L^2_{x,v}}^2 \quad \leqslant \quad & \left[\frac{\nu_1^\Lambda d}{D\varepsilon\nu_0^\Lambda}+\frac{2}{\varepsilon^2}\left(\frac{\delta\nu_1^\Lambda}{\nu_0^\Lambda}-\nu_5^\Lambda\right)+\frac{1}{D_2\varepsilon}\right]\left\|\partial_l^j h\right\|_\Lambda^2 \\
& +\frac{D}{\varepsilon}\sum_{i,c_i(j)>0}\left\|\partial_{l+\delta_i}^{j-\delta_i}h\right\|_{L^2_{x,v}}^2+\frac{2}{\varepsilon^2}(C(\delta)+\nu_6^\Lambda)\left\|h\right\|_{H^{s-1}_{x,v}}^2 \\
& +\frac{D_2}{\varepsilon}\left(\mathcal{G}^s_{x,v}(g,h)\right)^2.
\end{aligned}
$$

Hence, we obtain equations (4.3.11) and (4.3.12) by taking $D = 3\nu_1^\Lambda\varepsilon/\nu_0^\Lambda\nu_5^\Lambda$, $D_2 = 3\varepsilon/\nu_5^\Lambda$ and $\delta = \nu_0^\Lambda\nu_5^\Lambda/6\nu_1^\Lambda$. Also note that in (4.3.11) we used $\left\|\partial_{l+\delta_i}^{j-\delta_i}h\right\|_{L^2_{x,v}}^2 \leqslant \frac{\nu_1^\Lambda}{\nu_0^\Lambda}\left\|\partial_{l+\delta_i}^{j-\delta_i}h\right\|_\Lambda^2$.

4.B.2.5 Time evolution of $\langle\partial_{l-\delta_i}^{\delta_i}h,\partial_l^0 h\rangle_{L^2_{x,v}}$

With no more calculations, we can bound this term in the same way we did for $\langle\nabla_x h,\nabla_v h\rangle$. Here we get

$$
\begin{aligned}
\frac{d}{dt}\langle\partial_{l-\delta_i}^{\delta_i}h,\partial_l^0 h\rangle_{L^2_{x,v}} \quad \leqslant \quad & \frac{C^L\eta}{\varepsilon^2}\left\|\partial_l^0 h^\perp\right\|_\Lambda^2+\left[\frac{C^L}{\eta\varepsilon^2}+\frac{1}{\varepsilon D_3}\right]\left\|\partial_{l-\delta_i}^{\delta_i}h\right\|_\Lambda^2-\frac{1}{\varepsilon}\left\|\partial_l^0 h\right\|_{L^2_{x,v}}^2 \\
& +\frac{D_3}{\varepsilon}\left(\mathcal{G}^s_x(g,h)\right)^2.
\end{aligned}
$$

Now define $\eta = e/\varepsilon$, $e>0$, and $D_3 = e/C^L$ to obtain equation (4.3.13).

In the next paragraphs, we are setting $g = h$.

4.B.2.6 Time evolution of $\left\|\nabla_v h^\perp\right\|_{L^2_{x,v}}^2$

By simply differentiating norm and using (H5) to get $\Gamma(h,h)^\perp = \Gamma(h,h)$, we compute

$$
\frac{d}{dt}\left\|\nabla_v h^\perp\right\|_{L^2_{x,v}}^2 = 2\langle\nabla_v(G_\varepsilon(h))^\perp,\nabla_v h^\perp\rangle_{L^2_{x,v}}+\frac{2}{\varepsilon}\langle\nabla_v\Gamma(h,h),\nabla_v h^\perp\rangle_{L^2_{x,v}}.
$$

By applying (H4) and Young's inequality to the second term on the right-hand side, with a constant $D_2>0$, and controlling the $L^2_{x,v}$-norm by the Λ-norm we obtain:

$$
\frac{2}{\varepsilon}\langle\nabla_v\Gamma(h,h),\nabla_v h^\perp\rangle_{L^2_{x,v}} \leqslant \frac{D_2}{\varepsilon}\left(\mathcal{G}^1_{x,v}(h,h)\right)^2+\frac{1}{\varepsilon D_2}\left\|\nabla_v h^\perp\right\|_\Lambda^2.
$$

Then we have to control the first term. Just by writing it and decomposing terms in projection onto $\mathrm{Ker}(L)$ and onto its orthogonal we yield:

$$\begin{aligned}
2\langle \nabla_v (G_\varepsilon(h))^\perp, \nabla_v h^\perp\rangle_{L^2_{x,v}} &= \frac{2}{\varepsilon^2}\langle \nabla_v L(h), \nabla_v h^\perp\rangle_{L^2_{x,v}} - \frac{2}{\varepsilon}\langle \nabla_v (v\cdot\nabla_x h)^\perp, \nabla_v h^\perp\rangle_{L^2_{x,v}} \\
&= \frac{2}{\varepsilon^2}\langle \nabla_v L(h^\perp), \nabla_v h^\perp\rangle_{L^2_{x,v}} - \frac{2}{\varepsilon}\langle \nabla_x h, \nabla_v h^\perp\rangle_{L^2_{x,v}} \\
&\quad -\frac{2}{\varepsilon}\langle v\cdot\nabla_v\nabla_x \pi_L(h), \nabla_v h^\perp\rangle_{L^2_{x,v}} \\
&\quad +\frac{2}{\varepsilon}\langle \nabla_v \pi_L(v\cdot\nabla_x h), \nabla_v h^\perp\rangle_{L^2_{x,v}}.
\end{aligned}$$

Then we can control the first term on the right-hand side thanks to (H1) and (H2), $\delta > 0$ to be chosen later:

$$\frac{2}{\varepsilon^2}\langle \nabla_v L(h^\perp), \nabla_v h^\perp\rangle_{L^2_{x,v}} \leqslant \frac{2(C(\delta)+\nu_4^\Lambda)\nu_\Lambda^1}{\nu_0^\Lambda\varepsilon^2}\left\|h^\perp\right\|_\Lambda^2 + \frac{2}{\varepsilon^2}\left(\frac{\nu_1^\Lambda\delta}{\nu_0^\Lambda} - \nu_3^\Lambda\right)\left\|\nabla_v h^\perp\right\|_\Lambda^2.$$

We apply Cauchy-Schwarz inequality to the next term, with D to be chosen later:

$$-\frac{2}{\varepsilon}\langle \nabla_x h, \nabla_v h^\perp\rangle_{L^2_{x,v}} \leqslant \frac{D}{\varepsilon}\left\|\nabla_x h\right\|^2_{L^2_{x,v}} + \frac{\nu_1^\Lambda}{\nu_0^\Lambda D\varepsilon}\left\|\nabla_v h^\perp\right\|_\Lambda^2.$$

For the third term we are going to apply Cauchy-Schwarz inequality and then use the property (H3). The latter property tells us that the functions in Ker(L) are of the form a polynomial in v times $e^{-|v|^2/4}$. This fact combined with the shape of π_L, equation (4.3.1), shows us that we can control, by a mere Cauchy-Schwarz inequality, the third term. Then the property (4.3.3) yields the following upper bound:

$$\begin{aligned}
-\frac{2}{\varepsilon}\langle v\cdot\nabla_v\nabla_x\pi_L(h), \nabla_v h^\perp\rangle_{L^2_{x,v}} &\leqslant \frac{\tilde{D}}{\varepsilon}\left\|v\cdot\nabla_v\pi_L(\nabla_x h)\right\|^2_{L^2_{x,v}} + \frac{1}{\tilde{D}\varepsilon}\left\|\nabla_v h^\perp\right\|^2_{L^2_{x,v}} \\
&\leqslant \frac{\tilde{D}C_{\pi 1}}{\varepsilon}\left\|\nabla_x h\right\|^2_{L^2_{x,v}} + \frac{\nu_1^\Lambda}{\nu_0^\Lambda\tilde{D}\varepsilon}\left\|\nabla_v h^\perp\right\|_\Lambda^2.
\end{aligned}$$

Finally, we first use equation (4.3.3) controling the v-derivatives of π_L and then see that the norm of $\pi_L(v.f)$ is easily controled by the norm of f (just use (H3) and the definition of π_L (4.3.1) and apply Cauchy-Schwarz inequality) by a factor $C_{\pi 1}$ (increase this constant if necessary in (4.3.3)):

$$\begin{aligned}\frac{2}{\varepsilon}\langle\nabla_v\pi_L(v.\nabla_x h),\nabla_v h^\perp\rangle_{L^2_{x,v}} &\leqslant \frac{D'}{\varepsilon}\|\nabla_v\pi_L(v.\nabla_x h)\|^2_{L^2_{x,v}}+\frac{1}{\varepsilon D'}\left\|\nabla_v h^\perp\right\|^2_{L^2_{x,v}}\\ &\leqslant \frac{D'C_{\pi 1}}{\varepsilon}\|\pi_L(v.\nabla_x h)\|^2_{L^2_{x,v}}+\frac{1}{\varepsilon D'}\left\|\nabla_v h^\perp\right\|^2_{L^2_{x,v}}\\ &\leqslant \frac{D'C_{\pi 1}^2}{\varepsilon}\|\nabla_x h\|^2_{L^2_{x,v}}+\frac{\nu_1^\Lambda}{\nu_0^\Lambda\varepsilon D'}\left\|\nabla_v h^\perp\right\|^2_{L^2_{x,v}}.\end{aligned}$$

We then gather all those bounds to get the last upper bound for the time derivative of the v-derivative.

$$\begin{aligned}\frac{d}{dt}\left\|\nabla_v h^\perp\right\|^2_{L^2_{x,v}} \leqslant\; & \frac{\nu_1^\Lambda}{\nu_0^\Lambda\varepsilon^2}\left(2\nu_4^\Lambda+2C(\delta)\right)\left\|h^\perp\right\|^2_\Lambda+\left[\frac{D}{\varepsilon}+\frac{D'C_{\pi 1}^2}{\varepsilon}+\frac{\tilde{D}C_{\pi 1}}{\varepsilon}\right]\|\nabla_x h\|^2_{L^2_{x,v}}\\ &+\left[\frac{2\nu_1^\Lambda\delta}{\nu_0^\Lambda\varepsilon^2}-\frac{2\nu_3^\Lambda}{\varepsilon^2}+\frac{\nu_1^\Lambda}{\varepsilon\nu_0^\Lambda}\left(\frac{1}{D}+\frac{1}{D'}+\frac{1}{\tilde{D}}\right)+\frac{1}{\varepsilon D_2}\right]\left\|\nabla_v h^\perp\right\|^2_\Lambda\\ &+\frac{D_2}{\varepsilon}\left(\mathcal{G}^1_{x,v}(h,h)\right)^2.\end{aligned}$$

Therefore we obtain (4.3.14) by taking $D=D'=\tilde{D}=9\nu_1^\Lambda\varepsilon/\nu_0^\Lambda\nu_3^\Lambda$, $\delta=\nu_0^\Lambda\nu_3^\Lambda/6\nu_1^\Lambda$ and $D_2=3\varepsilon/\nu_3^\Lambda$.

4.B.2.7 A new time evolution of $\langle\nabla_x h,\nabla_v h\rangle_{L^2_{x,v}}$

By integrating by part in x then in v we obtain the following equality on the evolution of the scalar product:

$$\frac{d}{dt}\langle\nabla_x h,\nabla_v h\rangle_{L^2_{x,v}}=2\langle\nabla_x G_\varepsilon(h),\nabla_v h\rangle_{L^2_{x,v}}+\frac{2}{\varepsilon}\langle\nabla_v\Gamma(h,h),\nabla_x h\rangle_{L^2_{x,v}}.$$

We will bound above the first term as in the previous case and for the second term involving Γ we use (H4) and Young's inequality with a constant $D_3>0$:

$$2\langle\nabla_v\Gamma(h,h),\nabla_x h\rangle_{L^2_{x,v}}\leqslant D_3\left(\mathcal{G}^1_{x,v}(h,h)\right)^2+\frac{1}{D_3}\|\nabla_x h\|^2_\Lambda.$$

We decompose $\nabla_x h$ thanks to π_L and we use (4.3.4) to control the fluid part of it,

$$2\langle\nabla_v\Gamma(h,h),\nabla_x h\rangle_{L^2_{x,v}}\leqslant D_3\left(\mathcal{G}^1_{x,v}(h,h)\right)^2+\frac{1}{D_3}\left\|\nabla_x h^\perp\right\|^2_\Lambda+\frac{C_\pi}{D_3}\|\nabla_x h\|^2_{L^2_{x,v}}.$$

Finally we obtain an upper bound for the time-derivative:

$$
\begin{aligned}
\frac{d}{dt}\langle \nabla_x h, \nabla_v h\rangle_{L^2_{x,v}} \leqslant & \left[\frac{C^L\eta}{\varepsilon^2} + \frac{1}{\varepsilon D_3}\right] \left\|\nabla_x h^\perp\right\|^2_\Lambda + \frac{C^L}{\eta\varepsilon^2}\left\|\nabla_v h\right\|^2_\Lambda + \left[\frac{C_\pi}{\varepsilon D_3} - \frac{1}{\varepsilon}\right]\left\|\nabla_x h\right\|^2_{L^2_{x,v}} \\
& + \frac{D_3}{\varepsilon}\left(\mathcal{G}^1_{x,v}(h,h)\right)^2.
\end{aligned}
$$

But now, we can use the properties (4.3.3) and (4.3.4) of the projection π_L to go further.

$$
\begin{aligned}
\left\|\nabla_v h\right\|^2_\Lambda & \leqslant 2\left\|\nabla_v h^\perp\right\|^2_\Lambda + 2\left\|\nabla_v \pi_L(h)\right\|^2_\Lambda \\
& \leqslant 2\left\|\nabla_v h^\perp\right\|^2_\Lambda + 2C_{\pi 1}C_\pi\left\|\pi_L(h)\right\|^2_{L^2_{x,v}} \\
& \leqslant 2\left\|\nabla_v h^\perp\right\|^2_\Lambda + 2C_{\pi 1}C_\pi C_p\left\|\nabla_x h\right\|^2_{L^2_{x,v}},
\end{aligned}
$$

where we used Poincare inequality (4.3.5) because h is in $\mathrm{Ker}(G_\varepsilon)^\perp$.

Hence we have a final upper bound for the time derivative:

$$
\begin{aligned}
\frac{d}{dt}\langle \nabla_x h, \nabla_v h\rangle_{L^2_{x,v}} \leqslant & \left[\frac{C^L\eta}{\varepsilon^2} + \frac{1}{\varepsilon D_3}\right]\left\|\nabla_x h^\perp\right\|^2_\Lambda \\
& + \frac{2C^L}{\eta\varepsilon^2}\left\|\nabla_v h^\perp\right\|^2_\Lambda + \left[\frac{2C^L C_{\pi 1}C_\pi C_p}{\varepsilon^2\eta} + \frac{C_\pi}{\varepsilon D_3} - \frac{1}{\varepsilon}\right]\left\|\nabla_x h\right\|^2_{L^2_{x,v}} \\
& + \frac{D_3}{\varepsilon}\left(\mathcal{G}^1_{x,v}(h,h)\right)^2.
\end{aligned}
$$

Thus, setting $\eta = 8eC^L C_{\pi 1}C_\pi C_p/\varepsilon$ with $e \geqslant 1$ and $D_3 = 4C_\pi$ we obtain equation (4.3.15).

4.B.2.8 Time evolution of $\left\|\partial^j_l h^\perp\right\|^2_{L^2_{x,v}}$, $j \geqslant 1$ and $|j| + |l| = s$

We have the following time evolution:

$$
\frac{d}{dt}\left\|\partial^j_l h^\perp\right\|^2_{L^2_{x,v}} = 2\langle \partial^j_l (G_\varepsilon(h))^\perp, \partial^j_l h^\perp\rangle_{L^2_{x,v}} + \frac{2}{\varepsilon}\langle \partial^j_l \Gamma(h,h), \partial^j_l h^\perp\rangle_{L^2_{x,v}}.
$$

As above, we apply (H4) for the last term on the right hand side, with a constant $D_2 > 0$,

$$
2\langle \partial^j_l \Gamma(h,h), \partial^j_l h^\perp\rangle_{L^2_{x,v}} \leqslant D_2\left(\mathcal{G}^s_{x,v}(h,h)\right)^2 + \frac{1}{D_2}\left\|\partial^j_l h^\perp\right\|^2_\Lambda.
$$

Then we evaluate the first term on the right-hand side.

$$
\begin{aligned}
2\langle \partial_l^j (G_\varepsilon(h))^\perp, \partial_l^j h^\perp\rangle_{L^2_{x,v}} &= \frac{2}{\varepsilon^2}\langle \partial_l^j L(h), \partial_l^j h^\perp\rangle_{L^2_{x,v}} - \frac{2}{\varepsilon}\langle \partial_l^j (v.\nabla_x h)^\perp, \partial_l^j h^\perp\rangle_{L^2_{x,v}} \\
&= \frac{2}{\varepsilon^2}\langle \partial_l^j L(h^\perp), \partial_l^j h^\perp\rangle_{L^2_{x,v}} - \frac{2}{\varepsilon}\langle v\cdot \partial_l^j \pi_L(\nabla_x h), \partial_l^j h^\perp\rangle_{L^2_{x,v}} \\
&\quad -\frac{2}{\varepsilon}\sum_{i, c_i(j)>0} \langle \partial_{l+\delta_i}^{j-\delta_i} h, \partial_l^j h^\perp\rangle_{L^2_{x,v}} \\
&\quad +\frac{2}{\varepsilon}\langle \partial_l^j \pi_L(v\cdot\nabla_x h), \partial_l^j h^\perp\rangle_{L^2_{x,v}}.
\end{aligned}
$$

Then we shall bound each of these four terms on the right-hand side.
We can first use the properties (H1') and (H2') of L to get, for some δ to be chosen later,

$$
\frac{2}{\varepsilon^2}\langle \partial_l^j L(h^\perp), \partial_l^j h^\perp\rangle_{L^2_{x,v}} \leqslant \frac{2}{\varepsilon^2}\left(C(\delta)+\nu_6^\Lambda\right)\left\|h^\perp\right\|^2_{H^{s-1}_{x,v}} + \frac{2}{\varepsilon^2}\left(\frac{\nu_1^\Lambda\delta}{\nu_0^\Lambda} - \nu_5^\Lambda\right)\left\|\partial_l^j h^\perp\right\|^2_\Lambda .
$$

For the three remaining terms we will apply Cauchy-Schwarz inequality and use the properties of π_L concerning v-derivatives and multiplications by a polynomial in v.
First

$$
\begin{aligned}
-\frac{2}{\varepsilon}\langle v\cdot \partial_l^j \pi_L(\nabla_x h), \partial_l^j h^\perp\rangle_{L^2_{x,v}} &\leqslant \frac{D}{\varepsilon}\left\|v\cdot\partial_l^j \pi_L(\nabla_x h)\right\|^2_{L^2_{x,v}} + \frac{1}{D\varepsilon}\left\|\partial_l^j h^\perp\right\|^2_{L^2_{x,v}} \\
&\leqslant \frac{DC_{\pi s}}{\varepsilon}\left\|\partial_l^0(\nabla_x h)\right\|^2_{L^2_{x,v}} + \frac{\nu_1^\Lambda}{\nu_0^\Lambda D\varepsilon}\left\|\partial_l^j h^\perp\right\|^2_\Lambda \\
&\leqslant \begin{cases} \dfrac{DC_{\pi s}}{\varepsilon}\displaystyle\sum_{|l'|=s}\left\|\partial_{l'}^0 h\right\|^2_{L^2_{x,v}} + \dfrac{\nu_1^\Lambda}{\nu_0^\Lambda D\varepsilon}\left\|\partial_l^j h^\perp\right\|^2_\Lambda, \ |j|=1 \\ \dfrac{DC_{\pi s}}{\varepsilon}\displaystyle\sum_{|l'|\leqslant s-1}\left\|\partial_{l'}^0 h\right\|^2_{L^2_{x,v}} + \dfrac{\nu_1^\Lambda}{\nu_0^\Lambda D\varepsilon}\left\|\partial_l^j h^\perp\right\|^2_\Lambda, \ |j|>1, \end{cases}
\end{aligned}
$$

where we used that $|l| = |s| - |j|$. Then

$$
-\frac{2}{\varepsilon}\langle \partial_{l+\delta_i}^{j-\delta_i} h, \partial_l^j h^\perp\rangle_{L^2_{x,v}} \leqslant \frac{D'}{\varepsilon}\left\|\partial_{l+\delta_i}^{j-\delta_i} h\right\|^2_{L^2_{x,v}} + \frac{\nu_1^\Lambda}{\nu_0^\Lambda D'\varepsilon}\left\|\partial_l^j h^\perp\right\|^2_\Lambda
$$

In the case where $|j|>1$ we can also use that $\left\|\partial_{l+\delta_i}^{j-\delta_i} h\right\|^2_{L^2_{x,v}}$ can be decomposed thanks to π_L and its orthogonal projector. Then the fluid part is controlled by the x-derivatives only.
And finally

$$
\begin{aligned}
\frac{2}{\varepsilon}\langle \partial_l^j \pi_L(v\cdot\nabla_x h), \partial_l^j h^\perp\rangle_{L^2_{x,v}} &\leqslant \frac{\tilde{D}}{\varepsilon}\left\|\partial_l^j \pi_L(v\cdot\nabla_x h)\right\|^2_{L^2_{x,v}} + \frac{1}{\tilde{D}\varepsilon}\left\|\partial_l^j h^\perp\right\|^2_{L^2_{x,v}} \\
&\leqslant \frac{\tilde{D}C_{\pi s}}{\varepsilon}\left\|\partial_l^0\nabla_x h\right\|^2_{L^2_{x,v}} + \frac{\nu_1^\Lambda}{\tilde{D}\nu_0^\Lambda\varepsilon}\left\|\partial_l^j h^\perp\right\|^2_\Lambda \\
&\leqslant \begin{cases} \dfrac{\tilde{D}C_{\pi s}}{\varepsilon}\displaystyle\sum_{|l'|=s}\left\|\partial_{l'}^0 h\right\|^2_{L^2_{x,v}} + \dfrac{\nu_1^\Lambda}{\nu_0^\Lambda\tilde{D}\varepsilon}\left\|\partial_l^j h^\perp\right\|^2_\Lambda, & \text{if } |j|=1 \\[2ex] \dfrac{\tilde{D}C_{\pi s}}{\varepsilon}\displaystyle\sum_{|l'|\leqslant s-1}\left\|\partial_{l'}^0 h\right\|^2_{L^2_{x,v}} + \dfrac{\nu_1^\Lambda}{\nu_0^\Lambda\tilde{D}\varepsilon}\left\|\partial_l^j h^\perp\right\|^2_\Lambda, & \text{if } |j|>1, \end{cases}
\end{aligned}
$$

We are now able to combine all those estimates to get an upper bound of the time-derivative we are looking at. We can also give to different bounds, depending on the size $|j|$. We also used that the number of i such that $c_i(j)>0$ is less than d.

In the case $|j|>1$,

$$
\begin{aligned}
\frac{d}{dt}\left\|\partial_l^j h^\perp\right\|^2_{L^2_{x,v}} \leqslant\ & \left[\frac{2}{\varepsilon^2}\left(\frac{\nu_1^\Lambda\delta}{\nu_0^\Lambda} - \nu_5^\Lambda\right) + \frac{\nu_1^\Lambda}{\nu_0^\Lambda\varepsilon}\left(\frac{1}{D} + \frac{d}{D'} + \frac{1}{\tilde{D}}\right) + \frac{1}{D_2}\right]\left\|\partial_l^j h^\perp\right\|^2_\Lambda \\
& + \frac{D'\nu_1^\Lambda}{2\nu_0^\Lambda\varepsilon}\sum_{i,c_i(j)>0}\left\|\partial_{l+\delta_i}^{j-\delta_i} h^\perp\right\|^2_\Lambda \\
& + \left[\frac{DC_{\pi s}}{2\varepsilon} + \frac{D'C_{\pi s}}{\varepsilon} + \frac{\tilde{D}C_{\pi s}}{\varepsilon}\right]\sum_{|l'|\leqslant s-1}\left\|\partial_{l'}^0 h\right\|^2_{L^2_{x,v}} \\
& + \frac{2(C(\delta)+\nu_6^\Lambda)}{\varepsilon^2}\left\|h^\perp\right\|^2_{H^{s-1}_{x,v}} \\
& + \frac{D_2}{\varepsilon}\left(\mathcal{G}^s_{x,v}(h,h)\right)^2.
\end{aligned}
$$

And in the case $|j|=1$,

$$
\begin{aligned}
\frac{d}{dt}\left\|\partial_{l-\delta_i}^{\delta_i} h^\perp\right\|^2_{L^2_{x,v}} \leqslant\ & \left[\frac{2}{\varepsilon^2}\left(\frac{\nu_1^\Lambda\delta}{\nu_0^\Lambda} - \nu_5^\Lambda\right) + \frac{\nu_1^\Lambda}{\nu_0^\Lambda\varepsilon}\left(\frac{1}{D} + \frac{1}{D'} + \frac{1}{\tilde{D}}\right) + \frac{1}{D_2}\right]\left\|\partial_{l-\delta_i}^{\delta_i} h^\perp\right\|^2_\Lambda \\
& + \left[\frac{DC_{\pi s}}{\varepsilon} + \frac{D'}{\varepsilon} + \frac{\tilde{D}C_{\pi s}}{\varepsilon}\right]\sum_{|l'|=s}\left\|\partial_{l'}^0 h\right\|^2_{L^2_{x,v}} \\
& + \frac{2(C(\delta)+\nu_6^\Lambda)}{\varepsilon^2}\left\|h^\perp\right\|^2_{H^{s-1}_{x,v}} \\
& + \frac{D_2}{\varepsilon}\left(\mathcal{G}^s_{x,v}(h,h)\right)^2.
\end{aligned}
$$

By taking $D=\tilde{D}=9\nu_1^\Lambda\varepsilon/\nu_0^\Lambda\nu_5^\Lambda$, $D_2=3\varepsilon/\nu_5^\Lambda$, $\delta=\nu_0^\Lambda\nu_5^\Lambda/6\nu_1^\Lambda$ and $D'=9\nu_1^\Lambda\varepsilon/\nu_0^\Lambda\nu_5^\Lambda$, if

$|j| = 1$, or $D' = 9\nu_1^\Lambda d\varepsilon/\nu_0^\Lambda \nu_5^\Lambda$, if $|j| > 1$, we obtain (4.3.16) and (4.3.17).

4.B.2.9 A new time evolution of $\langle \partial^{\delta_i}_{l-\delta_i} h, \partial^0_l h \rangle_{L^2_{x,v}}$

By integrating by part in x then in v we obtain the following equality on the evolution of the scalar product.

$$\frac{d}{dt}\langle \partial^{\delta_i}_{l-\delta_i} h, \partial^0_l h \rangle_{L^2_{x,v}} = 2\langle \partial^{\delta_i}_{l-\delta_i} G_\varepsilon(h), \partial^0_l h \rangle_{L^2_{x,v}} + \frac{2}{\varepsilon}\langle \partial^{\delta_i}_{l-\delta_i} \Gamma(h,h), \partial^0_l h \rangle_{L^2_{x,v}}.$$

We will bound above the first term as in the previous case and for the second term involving Γ we use (H4) and Young's inequality with a constant $D_3 > 0$. Moreover, we decompose $\partial^0_l h$ into its fluid part and its microscopic part and we apply (4.3.4) on the fluid part. This yields

$$2\langle \partial^{\delta_i}_{l-\delta_i} \Gamma(h,h), \partial^0_l h \rangle_{L^2_{x,v}} \leqslant D_3 \left(\mathcal{G}^s_{x,v}(h,h)\right)^2 + \frac{1}{D_3}\left\|\partial^0_l h^\perp\right\|^2_\Lambda + \frac{C_\pi}{D_3}\left\|\partial^0_l h\right\|^2_{L^2_{x,v}}.$$

Finally we obtain an upper bound for the time-derivative:

$$\begin{aligned}
\frac{d}{dt}\langle \partial^{\delta_i}_{l-\delta_i} h, \partial^0_l h \rangle_{L^2_{x,v}} \leqslant\ & \left[\frac{C^L\eta}{\varepsilon^2} + \frac{1}{D_3}\right]\left\|\partial^0_l h^\perp\right\|^2_\Lambda + \frac{C^L}{\eta\varepsilon^2}\left\|\partial^{\delta_i}_{l-\delta_i} h\right\|^2_\Lambda + \left(\frac{C_\pi}{\varepsilon D_3} - \frac{1}{\varepsilon}\right)\left\|\partial^0_l h\right\|^2_{L^2_{x,v}} \\
& + \frac{D_3}{\varepsilon}\left(\mathcal{G}^s_{x,v}(h,h)\right)^2.
\end{aligned}$$

Now we can use the properties of π_L concerning the v-derivatives, equation (4.3.3), the equivalence of norm under the projection π_L, equation (4.3.4), and Poincare inequality get the following upper bound:

$$\begin{aligned}
\left\|\partial^{\delta_i}_{l-\delta_i} h\right\|^2_\Lambda &\leqslant 2\left\|\partial^{\delta_i}_{l-\delta_i} h^\perp\right\|^2_\Lambda + 2\left\|\partial^{\delta_i}_{l-\delta_i} \pi_L(h)\right\|^2_\Lambda \\
&\leqslant 2\left\|\partial^{\delta_i}_{l-\delta_i} h^\perp\right\|^2_\Lambda + 2C_{\pi s}C_\pi\left\|\partial^0_{l-\delta_i}(h)\right\|^2_{L^2_{x,v}} \\
&\leqslant 2\left\|\partial^{\delta_i}_{l-\delta_i} h^\perp\right\|^2_\Lambda + 2C_{\pi s}C_\pi \sum_{|l'|\leqslant s-1}\left\|\partial^0_{l'} h\right\|^2_{L^2_{x,v}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{d}{dt}\langle\partial^{\delta_i}_{l-\delta_i}h,\partial^0_l h\rangle_{L^2_{x,v}} \quad\leqslant\quad & \left[\frac{C^L\eta}{\varepsilon^2}+\frac{1}{D_3}\right]\left\|\partial^0_l h^\perp\right\|^2_\Lambda+\frac{2C^L}{\eta\varepsilon^2}\left\|\partial^{\delta_i}_{l-\delta_i}h^\perp\right\|^2_\Lambda \\ & +\left(\frac{C_\pi}{\varepsilon D_3}-\frac{1}{\varepsilon}\right)\left\|\partial^0_l h\right\|^2_{L^2_{x,v}}+\frac{2C^LC_{\pi s}C_\pi}{\eta\varepsilon^2}\sum_{|l'|\leqslant s-1}\left\|\partial^0_{l'}h\right\|^2_{L^2_{x,v}} \\ & +\frac{D_3}{\varepsilon}\left(\mathcal{G}^s_{x,v}(h,h)\right)^2.\end{aligned}$$

We finally define $\eta = 8eC^LC_{\pi s}C_\pi d/\varepsilon$, with $e>1$, and $D_3 = 2C_\pi$ to yield equation (4.3.18).

4.C Proof of the hydrodynamical limit lemmas

In this section we are going to prove all the different lemmas used in section 9.

All along the demonstration we will use this inequality:

$$\forall t>0,\ k\in\mathbb{N}^*,\ q\geqslant 0,\ p>0,\ \ t^qk^{2p}e^{-atk^2}\leqslant C_p(a)t^{q-p}. \qquad (4.C.1)$$

4.C.1 Study of the linear part

4.C.1.1 Proof of Lemma 4.8.6

Fix T in $[0,+\infty]$. By integrating we compute

$$\begin{aligned}\int_0^T U^\varepsilon_{0j}h_{in}dt &= \sum_{n\in\mathbb{Z}^d-\{0\}}e^{in.x}\left[\int_0^T e^{\frac{i\alpha_j t|n|}{\varepsilon}-\beta_j t|n|^2}dt\right]P_{0j}\left(\frac{n}{|n|}\right)\hat{h}_{in}(n,v) \\ &= \sum_{n\in\mathbb{Z}^d-\{0\}}e^{in.x}\frac{\varepsilon}{i\alpha_j|n|-\varepsilon\beta_j|n|^2}\left[e^{\frac{i\alpha_j T|n|}{\varepsilon}-\beta_j T|n|^2}-1\right]P_{0j}\hat{h}_{in}(n,v).\end{aligned}$$

The Fourier transform is an isometry in L^2_x and therefore

$$\left\|\int_0^T U^\varepsilon_{0j}h_{in}dt\right\|^2_{L^2_xL^2_v}\leqslant\varepsilon^2\sum_{n\in\mathbb{Z}^d-\{0\}}\frac{2}{\alpha_j^2|n|^2+\varepsilon^2\beta_j^2|n|^4}\left\|P_{0j}\left(\frac{n}{|n|}\right)\hat{h}_{in}(n,\cdot)\right\|^2_{L^2_v}.$$

Finally, we know that, like e_{0j}, P_{0j} is continuous on the compact $\mathbb{S}^{d-1}$ and so is bounded. But the latter is a linear operator acting on L^2_v and therefore it is bounded by

M_{0j} in the operator norm on L^2_v. Thus

$$\begin{aligned}\left\|\int_0^T U^\varepsilon_{0j} h_{in} dt\right\|^2_{L^2_x L^2_v} &\leqslant \varepsilon^2 \frac{M^2_{0j}}{\alpha_j^2} \sum_{n\in\mathbb{Z}^d-\{0\}} \left\|\hat{h}_{in}(n,\cdot)\right\|^2_{L^2_v} \\ &\leqslant \varepsilon^2 \frac{M^2_{0j}}{\alpha_j^2} \left\|h_{in}(\cdot,\cdot)\right\|^2_{L^2_x L^2_v},\end{aligned}$$

which is the expected result.

Now, let us look at the L^2_x-norm of this operator, to see how the torus case is different from the case $\mathbb{R}^d$ studied in [39] and [10].
Consider a direction n_1 in the Fourier transform space of the torus and define $\phi_{n_1} = \mathcal{F}_x^{-1}\left(e^{in_1}\right)$. We have the following equality

$$\langle U^\varepsilon_{0j} h_{in}, \phi_{n_1}\rangle_{L^2_x} = \langle \hat{U}^\varepsilon_{0j} \hat{h}_{in}, \hat{\phi}_{n_1}\rangle_{L^2_n} = e^{\frac{i\alpha_j t|n_1|}{\varepsilon} - \beta_j t|n_1|^2} P_{0j}\left(\frac{n_1}{|n_1|}\right)\hat{h}_{in}(n_1,v).$$

If we do not integrate in time, one can easily see that this expression cannot have a limit as ε tends to 0 if $P_{0j}\left(\frac{n_1}{|n_1|}\right)\hat{h}_{in}(n_1,v) \neq 0$, and so we cannot even have a weak convergence. The difference with the whole space case is this possibility to single out one mode in the frequency space in the case of the torus. This leads to the possible existence of periodic function at a given frequency, the norm of which will never decrease. This is impossible in the case of a continuous Fourier space, as in $\mathbb{R}^d$, and well described by the Riemann-Lebesgue lemma.

Therefore we have a convergence without averaging in time if and only if

$$P_{0j}\left(\frac{n_1}{|n_1|}\right)\hat{h}_{in}(n_1,v) = 0,$$

for all $j = \pm 1$ and all direction n_1. This means that for all $j = \pm 1$ and all n_1, $\langle e_{0j}\left(\frac{n_1}{|n_1|}\right), \hat{h}_{in}\rangle_{L^2_v} = 0$. By the expression known (see theorem 4.8.3) of $e_{0\pm 1}$, this is true if and only if $\nabla_x \cdot u_{in} = 0$ and $\rho_{in} + \theta_{in} = 0$.

4.C.1.2 Proof of Lemma 4.8.7

This lemma deals with three different terms and we study them one by one because they behaviour are quite different.

The term U^ε_{1j}: We remind that we have

$$\hat{U}^\varepsilon_{1j}\hat{h}_{in} = \chi_{|\varepsilon n|\leqslant n_0} e^{\frac{i\alpha_j t|n|}{\varepsilon} - \beta_j t|n|^2}\left(e^{\frac{t}{\varepsilon^2}\gamma_j(|\varepsilon n|)} - 1\right) P_{0j}\left(\frac{n}{|n|}\right)\hat{h}_{in}(n,v).$$

If we take $T > 0$, by Parseval identity we get

$$\left\| \int_0^T U^\varepsilon_{1j} h_{in} dt \right\|^2_{L^2_x L^2_v} = \sum_{n \in \mathbb{Z}^d - \{0\}} \chi_{|\varepsilon n \leqslant n_0|} \left| \int_0^T e^{\frac{i\alpha_j t|n|}{\varepsilon} - \beta_j t |n|^2} \left(e^{\frac{t}{\varepsilon^2}\gamma_j(|\varepsilon n|)} - 1 \right) dt \right|^2 \left\| P_{0j} \hat{h}_{in} \right\|^2_{L^2_v}.$$

But then we can use the fact that $|e^a - 1| \leqslant |a| e^{|a|}$, the inequalites satisfied by γ_j and the computational inequality (4.C.1) to obtain

$$\begin{aligned} \left| \int_0^T e^{\frac{i\alpha_j t|n|}{\varepsilon} - \beta_j t |n|^2} \left(e^{\frac{t}{\varepsilon^2}\gamma_j(|\varepsilon n|)} - 1 \right) dt \right| &\leqslant C_\gamma \varepsilon \int_0^T t\,|n|^3 \, e^{-\frac{t\beta_j}{2}|n|^2} dt \\ &\leqslant C_\gamma \varepsilon C_{3/2}\left(\frac{\beta_j}{4}\right) \int_0^T \frac{1}{\sqrt{t}} e^{-\frac{t\beta_j}{4}|n|^2} dt \\ &\leqslant C_\gamma \varepsilon C_{3/2}\left(\frac{\beta_j}{4}\right) \int_0^{+\infty} \frac{1}{\sqrt{t}} e^{-\frac{t\beta_j}{4}} dt, \end{aligned}$$

which is independent of n and is written $I\varepsilon$. Therefore we have the expected inequality, by using the continuity of P_{0j},

$$\left\| \int_0^T U^\varepsilon_{1j} h_{in} dt \right\|^2_{L^2_x L^2_v} \leqslant \varepsilon^2 I^2 M_{0j}^2 \left\| h_{in} \right\|^2_{L^2_x L^2_v}.$$

The last two inequalities we want to show comes from Parseval's identity, the properties of γ_j and the computational inequality (4.C.1):

$$\begin{aligned} \left\| U^\varepsilon_{1j} h_{in} \right\|^2_{L^2_x L^2_v} &= \sum_{n \in \mathbb{Z}^d - \{0\}} \chi_{|\varepsilon n| \leqslant n_0} e^{-2\beta_j t |n|^2} \left| e^{\frac{t}{\varepsilon^2}\gamma_j |\varepsilon n|} - 1 \right|^2 \left\| P_{0j}\left(\frac{n}{|n|}\right) \hat{h}_{in} \right\|^2_{L^2_v} \\ &\leqslant M_{0j}^2 C_\gamma^2 \varepsilon^2 \sum_{n \in \mathbb{Z}^d - \{0\}} \chi_{|\varepsilon n| \leqslant n_0} t^2 \, |n|^6 \, e^{-\beta_j t |n|^2} \left\| \hat{h}_{in} \right\|^2_{L^2_v} \\ &\leqslant M_{0j}^2 C_\gamma^2 \varepsilon^2 C_2 \left(\frac{\beta_j}{2}\right) \sum_{n \in \mathbb{Z}^d - \{0\}} \chi_{|\varepsilon n| \leqslant n_0} |n|^2 \, e^{-\frac{\beta_j t}{2}|n|^2} \left\| \hat{h}_{in} \right\|^2_{L^2_v}. \end{aligned} \tag{4.C.2}$$

Finally, if we integrate in t between 0 and $+\infty$ we obtain the expected second inequality of the lemma. If we merely bound $e^{-\frac{\beta_j t}{2}|n|^2}$ by one and use the fact that $\chi_{|\varepsilon n| \leqslant n_0} \leqslant 1$ and $\chi_{|\varepsilon n| \leqslant n_0} \varepsilon^2 |n|^2 \leqslant n_0^2$ we obtain the third inequality of the lemma for $\delta = 1$ and $\delta = 0$. Then by interpolation we obtain the general case for $0 \leqslant \delta \leqslant 1$.

The term U^ε_{2j}: Fix $T > 0$. By Parseval's identity we have

$$\begin{aligned}\left\|\int_0^T U_{2j}^\varepsilon h_{in} dt\right\|_{L^2_x L^2_v}^2 &= \sum_{n\in\mathbb{Z}^d} \chi_{|\varepsilon n|\leqslant n_0} \left|\int_0^T e^{\frac{i\alpha_j t|n|}{\varepsilon}-\beta_j t|n|^2+\frac{t}{\varepsilon^2}\gamma_j(|\varepsilon n|)} dt\right|^2 |\varepsilon n|^2 \left\|\tilde{P}_{1j}\hat{h}_{in}\right\|_{L^2_v}^2 \\ &\leqslant \sum_{n\in\mathbb{Z}^d-\{0\}} \frac{4}{\beta_j^2 |n|^4} |\varepsilon n|^2 \left\|\tilde{P}_{1j}\left(|\varepsilon n|, \frac{n}{|n|}\right)\hat{h}_{in}\right\|_{L^2_v}^2,\end{aligned}$$

where we used the inequalities satisfied by γ and integration in time.
Then, $\tilde{P}_{1j}$ is continuous on the compact $[-n_0, n_0]\times\mathbb{S}^{d-1}$ and so is bounded, as an operator acting on L^2_v, by $M_{1j} > 0$. Hence, Parseval's identity offers us the first inequality of the lemma.

The last two inequalities are just using Parseval's identity and the continuity of $\tilde{P}_{1j}$. Indeed,

$$\begin{aligned}\left\|U_{2j}^\varepsilon h_{in}\right\|_{L^2_x L^2_v}^2 &= \sum_{n\in\mathbb{Z}^d-\{0\}} \chi_{|\varepsilon n|\leqslant n_0} \left| e^{\frac{i\alpha_j t|n|}{\varepsilon}-\beta_j t|n|^2+\frac{t}{\varepsilon^2}\gamma_j(|\varepsilon n|)}\right|^2 |\varepsilon n|^2 \left\|\tilde{P}_{1j}(n)\hat{h}_{in}\right\|_{L^2_v}^2 \\ &\leqslant M_{1j}^2\varepsilon^2 \sum_{n\in\mathbb{Z}^d-\{0\}} \chi_{|\varepsilon n|\leqslant n_0} |n|^2 e^{-t\beta_j|n|^2} \left\|\hat{h}_{in}\right\|_{L^2_v}^2.\end{aligned}$$

We recognize here the same form of inequality (4.C.2). Thus, we obtain the last two inequalities of the statement in the same way.

The term U_{3j}^ε: We remind the reader that

$$\hat{U}_{3j}^\varepsilon = \left(\chi_{|\varepsilon n|\leqslant n_0} - 1\right) e^{\frac{i\alpha_j t|n|}{\varepsilon}-\beta_j t|n|^2} P_{0j}\left(\frac{n}{|n|}\right).$$

We have the following inequality

$$\left|\chi_{|\varepsilon n|\leqslant n_0} - 1\right| \leqslant \frac{\varepsilon n}{n_0}.$$

Therefore, replacing $\tilde{P}_{1j}$ by $\frac{1}{n_0}P_{0j}$ and β_j by $2\beta_j$ (since $\frac{t}{\varepsilon^2}\gamma_j(|\varepsilon n|) \leqslant \frac{t\beta_j}{2}|n|^2$) in the proof made for U_{2j}^ε we obtain the expected three inequalities for $U_{3j}^\varepsilon h_{in}$, the last one only with $\delta = 1$.
To have the last inequality in δ, it is enough to bound $\left|\chi_{|\varepsilon n|\leqslant n_0} - 1\right|$ by 1 and then using the continuity of P_{0j} to have the result for $\delta = 0$. Finally, we interpolate to get the general result for all $0 \leqslant \delta \leqslant 1$.

4.C.1.3 Proof of Lemma 4.8.8

Thanks to Theorem 4.8.3 we have that

$$\|U_R^\varepsilon h_{in}\|^2_{L^2_xL^2_v} = \left\|\hat{U}_R(t/\varepsilon^2, \varepsilon n, v)\hat{h}_{in}\right\|^2_{L^2_nL^2_v} \leqslant C_R^2 e^{-2\frac{\sigma t}{\varepsilon^2}} \|h_{in}\|^2_{L^2_xL^2_v}.$$

But then we have, thanks to the technical lemma 4.C.1, that $e^{-2\frac{\sigma t}{\varepsilon^2}} \leqslant C_{1/2}(2\sigma)\frac{\varepsilon}{\sqrt{t}}$, which gives us the last two inequalities we wanted. For the first inequality, a mere Cauchy-Schwartz inequality yields

$$\left\|\int_0^T U_R^\varepsilon h_{in} dt\right\|^2_{L^2_xL^2_v} \leqslant T\int_0^T \|U_R^\varepsilon h_{in}\|^2_{L^2_xL^2_v} dt,$$

which gives us the first inequality by integrating in t.

Now, let us suppose that we have the strong convergence down to $t=0$. At $t=0$ we can write that $e^{tG_\varepsilon} = \mathrm{Id}$ and therefore that:

$$\mathrm{Id} = \chi_{|\varepsilon n|\leqslant n_0}\sum_{j=-1}^{2} P_j\left(|\varepsilon n|, \frac{n}{|n|}\right) + \hat{U}_R(0,\varepsilon n, v).$$

We have the strong convergence down to 0 as ε tends to 0. Therefore, taking the latter equality at $\varepsilon = 0$ we have, because $\sum_{j=-1}^{2} P_{0j} = \pi_L$,

$$\hat{U}_R(0,0,v) = \mathrm{Id} - \pi_L.$$

Then $\hat{U}_R\hat{h}_{in}$ tends to 0 as ε tends to 0 in $C([0,+\infty), L^2_xL^2_v)$ if and only if h_{in} belongs to $\mathrm{Ker}(L)$.

In that case, we can use the proof of Lemma 6.2 of [10] in which they noticed that

$$U_R^\varepsilon(t,x,v) = e^{tG_\varepsilon}U_R^\varepsilon(0,x,v) = e^{tG_\varepsilon}\left[\mathcal{F}_x^{-1}\left(Id - \chi_{|\varepsilon n|\leqslant n_0}\sum_{j=-1}^{2} P_j(\varepsilon n)\right)\mathcal{F}_x\right].$$

Thanks to that new form we have that, if $h_{in} = \pi_L(h_{in})$,

$$U_R^\varepsilon(t,x,v)h_{in} = e^{tG_\varepsilon}\left[\mathcal{F}_x^{-1}\left((1-\chi_{|\varepsilon n|\leqslant n_0}) - |\varepsilon n|\,\chi_{|\varepsilon n|\leqslant n_0}\sum_{j=-1}^{2}\tilde{P}_{1j}(\varepsilon n)\right)\hat{h}_{in}\right],$$

because $\pi_L = \sum_{j=-1}^{2} P_{0j}$.

Therefore we can redo the same estimates we worked out in the previous lemmas and use the same interpolation method to get the result stated in Lemma 4.8.8.

4.C.2 Study of the bilinear part

4.C.2.1 A simplification without loss of generality

All the terms we are about to study, apart from the remainder term, are of the following form

$$\psi_{ij}^{\varepsilon}(u_\varepsilon) = \int_0^t \sum_{n\in\mathbb{Z}^d-\{0\}} g(t,s,k,x)P(n)\hat{u}_\varepsilon(s,k,v)ds,$$

with $P(n)$ being a projector in L^2_v, bounded uniformely in n.

Looking at the dual definition of the norm of a function in $L^2_{x,v}$, we can consider f in $C_c^\infty\left(\mathbb{T}^d\times\mathbb{R}^d\right)$ such that $\|f\|_{L^2_{x,v}} = 1$ and take the scalar product with $\psi_{ij}^{\varepsilon}(u_\varepsilon)$. This yields, since P is a projector and thus symmetric,

$$\begin{aligned}\langle\psi_{ij}^{\varepsilon}(u_\varepsilon), f\rangle_{L^2_{x,v}} &= \int_{\mathbb{T}^d}\int_0^t \sum_{n\in\mathbb{Z}^d-\{0\}} g(t,s,k,x)\langle P(n)\hat{u}_\varepsilon, f\rangle_{L^2_v} ds \\ &= \int_{\mathbb{T}^d}\int_0^t \sum_{n\in\mathbb{Z}^d-\{0\}} g(t,s,k,x)\langle \hat{u}_\varepsilon, P(n)f\rangle_{L^2_v} ds. \end{aligned} \tag{4.C.3}$$

We are working in $L^2_xL^2_v$ in order to simplify computations as they are exactly the same in higher Sobolev spaces. Therefore, we can assume that hypothesis $(H4)$ is still valid in L^2_v without loss of generality. This means

$$\langle \hat{u}_\varepsilon, P(n)f\rangle_{L^2_v} \leqslant \|h\|_{L^2_xL^2_v}\|h\|_{\Lambda_v}\|P(n)f\|_{\Lambda_v}. \tag{4.C.4}$$

Finally, in terms of Fourier coefficients in x, $P(n)$ is a projector in L^2_v and uniformely bounded in n as an operator in L^2_v.
Thus, combining (4.C.4) and the definition of the functional E, (4.6.2), we see that

$$\int_0^T \left\|\hat{f}_\varepsilon\right\|^2_{L^2_xL^2_v} dt$$

is a continuous operator from $C(\mathbb{R}^+, L^2_xL^2_v, E(\cdot))$ to $C(\mathbb{R}^+, L^2_xL^2_v, \|\cdot\|_{L^2_xL^2_v})$. Looking at (4.C.3), we can consider without loss of generality that the following holds (even for the remainder term) for all $T>0$:

$$\psi_{ij}^{\varepsilon}(u_\varepsilon) = \int_0^t \sum_{n\in\mathbb{Z}^d-\{0\}} g(t,s,k,x)\hat{f}_\varepsilon(s,k,v)ds,$$

with
$$\int_0^T \left\| \hat{f}_\varepsilon \right\|^2_{L^2_x L^2_v} dt \leqslant M_{ij} E(h_\varepsilon)^2.$$

4.C.2.2 Proof of Lemma 4.8.9

For the first inequality, fix $T > 0$ and integrate by part in t to obtain

$$\begin{aligned}
\int_0^T \psi^\varepsilon_{0j}(u_\varepsilon) dt &= \sum_{n\in\mathbb{Z}^d - \{0\}} e^{in.x} \int_0^T \left(\int_0^t e^{i\frac{\alpha_j(t-s)}{\varepsilon}|n| - (t-s)\beta_j |n|^2} |n| \, \hat{f}_\varepsilon(s) ds \right) dt \\
&= \sum_{n\in\mathbb{Z}^d - \{0\}} e^{in.x} \frac{\varepsilon}{i\alpha_j |n| - \varepsilon\beta_j |n|} \left[\int_0^T \left(e^{i\frac{\alpha_j(T-s)}{\varepsilon}|n| - (T-s)\beta_j |n|^2} - 1 \right) \hat{f}_\varepsilon(s) ds \right].
\end{aligned}$$

Finally we can use Parseval's identity

$$\begin{aligned}
\left\| \int_0^T \psi^\varepsilon_{0j}(u_\varepsilon) dt \right\|^2_{L^2_x L^2_v} &\leqslant \sum_{n\in\mathbb{Z}^d - \{0\}} \frac{\varepsilon^2}{\varepsilon^2 \beta_j^2 |n|^2 + \alpha_j^2} T \int_0^T 2 \left\| \hat{f}_\varepsilon(s,n,v) \right\|^2_{L^2_v} ds \\
&\leqslant \frac{2M^2_{1j}}{\alpha_j^2} T \varepsilon^2 E(h_\varepsilon)^2,
\end{aligned}$$

where we used the subsection above and Parseval's identity again. This is exactly the expected result.

4.C.2.3 Proof of Lemma 4.8.11

We divide this proof in three paragraphes, each of them studying a different term.

The term ψ^ε_{1j}: We will just prove the last two inequalities and then merely applying Cauchy-Schwarz inequality will lead to the first one.
Fix $t > 0$. By a change of variable we can write

$$\psi^\varepsilon_{1j}(u_\varepsilon) = \sum_{n\in\mathbb{Z}^d - \{0\}} e^{ik.x} \chi_{|\varepsilon n| \leqslant n_0} \int_0^t e^{\frac{i\alpha_j s}{\varepsilon}|n| - \beta_j s |n|^2} \left(e^{\frac{s}{\varepsilon^2}\gamma_j(|\varepsilon n|)} - 1 \right) |n| \, \hat{f}_\varepsilon(t-s) ds.$$

By the study made in the proof of Lemma 4.8.7 we have that

$$\begin{aligned}
&\left| \int_0^t e^{\frac{i\alpha_j s}{\varepsilon}|n| - \beta_j s |n|^2} \left(e^{\frac{s}{\varepsilon^2}\gamma_j(|\varepsilon n|)} - 1 \right) |n| \, \hat{f}_\varepsilon(t-s) ds \right| \\
&\qquad \leqslant C_\gamma |n|^4 \varepsilon \int_0^t s e^{-\frac{\beta_j s}{2}|n|^2} \left| \hat{f}_\varepsilon(t-s) \right| ds.
\end{aligned}$$

Then we use the computational inequality (4.C.1) and a Cauchy-Schwarz to obtain

$$\begin{aligned}\left|\int_0^t e^{\frac{i\alpha_j s}{\varepsilon}|n|-\beta_j s|n|^2}\left(e^{\frac{s}{\varepsilon^2}\gamma_j(|\varepsilon n|)}-1\right)|n|\,\hat{u}_\varepsilon ds\right| \\ &\leqslant \varepsilon C_\gamma C_1\left(\frac{\beta_j}{4}\right)|n|^2\int_0^t e^{-\frac{\beta_j s}{4}|n|^2}\left|\hat{f}_\varepsilon\right| ds \\ &\leqslant \varepsilon C_\gamma C_1\left(\frac{\beta_j}{4}\right)|n|^2\sqrt{\frac{4}{\beta_j|n|^2}}\left[\int_0^t e^{-\frac{\beta_j(t-s)}{4}|n|^2}\left|\hat{f}_\varepsilon\right|^2 ds\right]^{1/2}. \quad (4.C.5)\end{aligned}$$

We can obtain the result by using Parseval's identity, denoting C a constant independent of ε and T, the continuity of P_{1j} and the computational inequality (4.C.1).

$$\left\|\psi_{1j}^\varepsilon(u_\varepsilon)(t)\right\|_{L^2_xL^2_v}^2 \leqslant C\sum_{n\in\mathbb{Z}^d-\{0\}}\chi_{|\varepsilon n|\leqslant n_0}\varepsilon^2|n|^2\int_0^t e^{-\frac{\beta_j(t-s)s}{4}|n|^2}\left\|\hat{f}_\varepsilon(s)\right\|_{L^2_v}^2 ds.$$

If we merely bound $e^{-\frac{\beta_j(t-s)}{2}|n|^2}$ by one and use the fact that $\chi_{|\varepsilon n|\leqslant n_0}\leqslant 1$ and $\chi_{|\varepsilon n|\leqslant n_0}\varepsilon^2|n|^2\leqslant n_0^2$ we obtain the third inequality of the lemma for $\delta=1$ and $\delta=0$. Then by interpolation we obtain the general case for $0\leqslant\delta\leqslant 1$.

If we integrate in t between 0 and a fixed $T>0$, a mere integration by part yields the expected control on the $L^2_{t,x,v}$-norm. Finally, from the latter control and a Cauchy-Schwarz inequality we deduce the first inequality.

The term ψ_{2j}^ε: As in the case ψ_{1j}^ε, we are going to prove the third inequality only. Fix $T>0$, a change of variable gives us

$$\psi_{2j}^\varepsilon(u_\varepsilon)=\sum_{n\in\mathbb{Z}^d-\{0\}}e^{ik.x}\chi_{|\varepsilon n|\leqslant n_0}\int_0^T e^{\frac{i\alpha_j s}{\varepsilon}|n|-\beta_j s|n|^2+\frac{s}{\varepsilon^2}\gamma_j(|\varepsilon n|)}\varepsilon|n|^2\hat{f}_\varepsilon(T-s)ds.$$

We can see that

$$\left|\int_0^T e^{\frac{i\alpha_j s}{\varepsilon}|n|-\beta_j s|n|^2+\frac{s}{\varepsilon^2}\gamma_j(|\varepsilon n|)}\varepsilon|n|^2\hat{f}_\varepsilon(T-s)ds\right|\leqslant\varepsilon|n|^2\int_0^T e^{-\frac{\beta_j s}{2}|n|^2}\left|\hat{f}_\varepsilon(T-s)\right|ds.$$

This bound is of the same form as equation (4.C.5). Therefore we have the same result.

The term ψ^{ε}_{3j}: As above, we will show the third inequality only.
Fix $T > 0$, we can write

$$\psi^{\varepsilon}_{3j}(u_{\varepsilon}) = \sum_{n\in\mathbb{Z}^d-\{0\}} e^{ik.x}(\chi_{|\varepsilon n|\leqslant n_0} - 1)\int_0^T e^{\frac{i\alpha_j s}{\varepsilon}|n| - \beta_j s|n|^2}\,|n|\,\hat{f}_{\varepsilon}(T-s,n,v)ds.$$

Looking at the fact that $\left|\chi_{|\varepsilon n|\leqslant n_0} - 1\right| \leqslant \frac{\varepsilon|n|}{n_0}$, we find the same kind of inequality as equation (4.C.5). Thus, we reach the same result.

4.C.2.4 Proof of Lemma 4.8.12

We remind the reader that

$$\Psi^{\varepsilon}_R(u_{\varepsilon}) = \int_0^t \frac{1}{\varepsilon}U^{\varepsilon}_R(t-s)f_{\varepsilon}(s)ds,$$

and that, by Theorem 4.8.3,

$$\|U^{\varepsilon}_R f_{\varepsilon}\|^2_{L^2_xL^2_v} \leqslant C_R^2 e^{-2\frac{\sigma t}{\varepsilon^2}}\,\|f_{\varepsilon}\|^2_{L^2_xL^2_v}.$$

Hence, a Cauchy-Schwarz inequality gives us the third inequality for $\|\psi^{\varepsilon}_R(u_{\varepsilon})(T)\|^2_{L^2_xL^2_v}$, and then the two others inequality stated above.

4.C.2.5 Proof of Lemma 4.8.14

We remind the reader that

$$\Psi(u) = \mathcal{F}_x^{-1}\left[\psi^{\varepsilon}_{00}(u) + \psi^{\varepsilon}_{02}(u)\right]\mathcal{F}_x.$$

As above, and because in that case $\alpha_j = 0$, we can write $\psi^{\varepsilon}_{0j}(u_{\varepsilon} - u)(T)$, for some $T > 0$, and apply a Cauchy-Schwarz inequality:

$$\begin{aligned}
\left\|\psi^{\varepsilon}_{0j}(u_{\varepsilon} - u)\right\|^2_{L^2_xL^2_v}(T) &= \sum_{n\in\mathbb{Z}^d-\{0\}} |n|^2\int_{\mathbb{R}^d}\left|\int_0^T e^{-s\beta_j|n|^2}P_{1j}\hat{\Gamma}(h_{\varepsilon} - h, h_{\varepsilon} + h)ds\right|^2 dv \\
&\leqslant \frac{M_{1j}^2}{\beta_j^2}\operatorname*{Sup}_{t\in[0,T]}\|\Gamma(h_{\varepsilon} - h, h_{\varepsilon} + h)\|^2_{L^2_xL^2_v}.
\end{aligned}$$

But because $\mathbb{T}^d$ is bounded in $\mathbb{R}^d$ and thanks to (H4) and the boundedness of $(h_{\varepsilon})_{\varepsilon}$ and h (both bounded by M) in $H^s_xL^2_v$ (Theorem 4.2.3), we can have the following control:

$$\|\Gamma(h_{\varepsilon} - h, h_{\varepsilon} + h)\|^2_{L^2_xL^2_v} \leqslant 4M^2C_{\Gamma}^2\mathrm{Volume}(\mathbb{T}^d)\,\|h_{\varepsilon} - h\|_{L^{\infty}_xL^2_v}.$$

Therefore we obtain the last inequality and the first two just come from Cauchy-Schwarz inequality.

Chapter 5

The Incompressible Navier-Stokes limit in polynomial weighted spaces

We study the Boltzmann equation on the d-dimensional torus in a perturbative setting around a global equilibrium under the Navier-Stokes linearisation. We use a recent functional analysis breakthrough to prove that the linear part of the equation generates a C^0*-semigroup with exponential decay in Sobolev spaces with polynomial weight, independently on the Knudsen number. Finally we show a Cauchy theory and an exponential decay for the perturbed Boltzmann equation, uniformly in the Knudsen number, in Sobolev spaces with polynomial weight. The polynomial weight is almost optimal and furthermore, this result only requires derivatives in the space variable and allows to connect to solutions to the incompressible Navier-Stokes equations in these spaces.*

This is a joint work with Sara Merino-Aceituno and Clément Mouhot, both from the University of Cambridge.

Contents

5.1 Introduction

This chapter deals with the Boltzmann equation in a perturbative setting as the Knudsen number tends to zero. This equation rules the dynamics of rarefied gas particles moving on the flat torus in dimension d, $\mathbb{T}^d$, when the only interactions taken into account are binary collisions. More precisely, the Boltzmann equation describes the time evolution of the distribution $f = f(t,x,v)$ of particles in position x and velocity v. A formal derivation of the Boltzmann equation from Newton's laws under the rarefied gas assumption can be found in [28], while [30] presents Lanford's Theorem (see [65] and [44] for detailed proofs) which rigorously proves the derivation in short times.

We denote the Knudsen number by ε and the Boltzmann equation reads

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} Q(f,f) \ , \text{ on } \mathbb{T}^d \times \mathbb{R}^d,$$

where Q is the Boltzmann collision operator given by

$$Q(f,f) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} B\left(|v - v_*|, \cos\theta\right) \left[f' f'_* - f f_*\right] dv_* d\sigma.$$

The Boltzmann kernel operator B encodes the physics of the collision process and f', f_*, f'_* and f are the values taken by f at v', v_*, v'_* and v respectively, where

$$\begin{cases} v' = \dfrac{v+v_*}{2} + \dfrac{|v-v_*|}{2}\sigma \\ v'_* = \dfrac{v+v_*}{2} - \dfrac{|v-v_*|}{2}\sigma \end{cases}, \text{ and } \quad \cos\theta = \left\langle \frac{v-v_*}{|v-v_*|}, \sigma \right\rangle.$$

.

The Boltzmann collision operator comes from a symmetric bilinear operator $Q(g,h)$ defined by

$$Q(g,h) = \frac{1}{2}\int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} B\left(|v - v_*|, \cos\theta\right) \left[h' g'_* + h'_* g' - h g_* - h_* g\right] dv_* d\sigma.$$

It is well-known (see [28], [30] or [46] for example) that the global equilibria for the Boltzmann equation are the *Maxwellians*, which are gaussian density functions depending only on the v variable. Without loss of generality we consider only the case of normalized Maxwellians:

$$\mu(v) = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{-\frac{|v|^2}{2}}.$$

In this chapter we will assume that the Boltzmann collision kernel is of the following form

$$B\left(|v-v_*|,\cos\theta\right)=\Phi\left(|v-v_*|\right)b\left(\cos\theta\right), \tag{5.1.1}$$

with Φ and b positive functions. This hypothesis is satisfied for all physical model and is more convenient to work with but do not impede the generality of our results.

We also restrict ourselves to the case of *hard potential* or *Maxwellian potential* ($\gamma=0$), that is to say there is a constant $C_\Phi>0$ such that

$$\Phi(z)=C_\Phi z^\gamma,\quad \gamma\in[0,1], \tag{5.1.2}$$

with a strong form of Grad's *angular cutoff* (see [48]), expressed here by the fact that we assume b to be C^1 with the controls from above

$$\forall z\in[-1,1],\quad b(z),\ b(z')\leqslant C_b. \tag{5.1.3}$$

5.1.1 The problem and its motivations

The Knudsen number is the inverse of the average number of collisions for each particle per unit of time. Therefore, as reviewed in [111], one can expect a convergence, in some sense, from the Boltzmann model towards the acoustics and the fluids dynamics as the Knudsen number tends to 0. However, these different models describe physical phenomena that do not evolve at the same timescale and the right rescaling to approximate the incompressible Navier-Stokes equation (see [8][46][111][98]) is the following equation

$$\partial_t f_\varepsilon+\frac{1}{\varepsilon}v\cdot\nabla_x f_\varepsilon=\frac{1}{\varepsilon^2}Q(f_\varepsilon,f_\varepsilon)\ ,\ \text{on } \mathbb{T}^d\times\mathbb{R}^d, \tag{5.1.4}$$

under the linearization $f_\varepsilon(t,x,v)=\mu(v)+\varepsilon h_\varepsilon(t,x,v)$. This leads to the perturbed Boltzmann equation

$$\partial_t h_\varepsilon+\frac{1}{\varepsilon}v\cdot\nabla_x h_\varepsilon=\frac{1}{\varepsilon^2}\mathcal{L}(h_\varepsilon)+\frac{1}{\varepsilon}Q(h_\varepsilon,h_\varepsilon), \tag{5.1.5}$$

where we defined

$$\mathcal{L}(h)=2Q(\mu,h).$$

The hydrodynamical limit of the perturbed equation is the system of equations satisfied by the limit, as ε tends to 0, of the hydrodynamical fluctuations that are the following physical observables of h_ε:

$$\begin{aligned}
\rho_\varepsilon(t,x) &= \int_{\mathbb{R}^d} h_\varepsilon(t,x,v)\,dv,\\
u_\varepsilon(t,x) &= \int_{\mathbb{R}^d} v h_\varepsilon(t,x,v)\,dv,\\
\theta_\varepsilon(t,x) &= \frac{1}{d}\int_{\mathbb{R}^d}(|v|^2-d)h_\varepsilon(t,x,v)\,dv.
\end{aligned}$$

Note that $(\rho_\varepsilon, u_\varepsilon, \theta_\varepsilon)$ are the linearised fluctuations of the mass, momemtum and the thermal energy around the global equilibrium μ.

In our perturbative framework, previous studies [8][10][23] (and also Chapter 4) show that the hydrodynamical limits ρ, u and θ are the weak (in the Leray sense [66]) solutions of the linearized incompressible Navier-Stokes equations:

$$\begin{aligned} \partial_t u - \nu \Delta u + u \cdot \nabla u + \nabla p &= 0, \\ \nabla \cdot u &= 0, \\ \partial_t \theta - \kappa \Delta \theta + u \cdot \nabla \theta &= 0, \end{aligned} \tag{5.1.6}$$

where p is the pressure function and ν and κ are constants determined by L (see [8] or [46] Theorem 5). They also satisfy the Boussineq relation

$$\nabla(\rho + \theta) = 0. \tag{5.1.7}$$

The aim of the present chapter is to use a constructive method to obtain existence and exponential decay for solutions to the perturbed Boltzmann equation (5.1.4), uniformly in the Knudsen number. One will thus be allowed to extract a converging (at least weakly) subsequence of h_ε converging to the incompressible Navier-Stokes equations [10][8][23] (see also Chapter 4). Such uniform results have been obtained on the torus in Sobolev spaces with exponential weight $H^s_{x,v}\left(\mu^{-1/2}\right)$ in [56][23] and the present work improves this strong weight to a polynomial weight without the need of derivatives in the velocity variable (see also Chapter 4).

5.1.2 Existing results

The first part of our work is to prove that the linear part of the Boltzmann equation

$$\mathcal{G}_\varepsilon = \frac{1}{\varepsilon^2}\mathcal{L} - \frac{1}{\varepsilon} v \cdot \nabla_x$$

generates a strongly continuous semigroup with an exponential decay in Lebesgue and Sobolev spaces with polynomial weight, namely $1 + |v|^k$ for some k large enough.

It has been known for long that the linear Boltzmann operator $\mathcal{L}$ is a self-adjoint non positive linear operator in the space $L^2_v\left(\mu^{-1/2}\right)$. Moreover it has a spectral gap λ_0. This has been proved in [27][48][49] with non constructive methods for hard potential with cutoff and in [14][15] in the Maxwellian case. These results were made constructive in [4][79] for more general collision operators. One can easily extend this spectral gap to Sobolev spaces $H^s_v\left(\mu^{-1/2}\right)$ (see for instance [51] Section 4.1).

The next step is to see if the latter properties about $\mathcal{L}$ in the velocity space can be transposed when one adds the skew-symmetric transport operator $-v\cdot\nabla_x$. The first results were obtained in [107] where $\mathcal{G}_1$ was proven to generate a strong continuous semigroup in $L^2_v H^s_x\left(\mu^{-1/2}\right)$ and in $L^\infty_v H^s_x\left(\mu^{-1/2}(1+|v|)^k\right)$, for s and k large enough. Then [82] obtained constructively this result in $H^s_{x,v}\left(\mu^{-1/2}\right)$ using hypocoercivity properties of the Boltzmann linear operator. Finally, a recent breakthrough proving abstract extension of semigroups [51] showed that $\mathcal{G}_1$ generates a C^0-semigroup in all the Sobolev spaces of the form $W^{\alpha,q}_v W^{\beta,p}_x(m)$, for m being an exponential weight (including maxwellian density if $q = p = 2$) or a polynomial weight $(1+|v|)^k$, as long as $\alpha \leqslant \beta$ and k is large enough depending on q (with $k > 2$ in the case $q = 1$).

The full Boltzmann equation perturbed around a global equilibrium $\mu(v)$ (5.1.5) has also been studied in the case $\varepsilon = 1$. The associated Cauchy problem has been worked on over the past fifty years, starting with Grad [50], and it has been studied in different spaces, such as $L^2_v H^s_x\left(\mu^{-1/2}\right)$ spaces [107] or $H^s_{x,v}\left(\mu^{-1/2}(1+|v|)^k\right)$ [55][114]. The Cauchy theory was then extended to $H^s_{x,v}\left(\mu^{-1/2}\right)$ where an exponential trend to equilibrium has also been obtained. This was obtained using hypocoercivity properties of the linear operator [82] or nonlinear estimates on fluid and microscopic parts of the equation [56]. Recently, [51] proved existence and uniqueness for (5.1.5) in more the general spaces $\left(W^{\alpha,1}_v \cap W^{\alpha,q}_v\right) W^{\beta,p}_x\left(1+|v|\right)^k\big)$ for $\alpha \leqslant \beta$ and β and k large enough with explicit thresholds. This result therefore gets rid of the exponential weight needed in the previous studies.

All the results presented above hold in the case of the torus. We refer the reader interested in the Cauchy problem, both for the torus and the whole space, to the review [110].

For physical purposes, these studies for $\varepsilon = 1$ are relevant since mere rescalings or changes of physical units changes (5.1.4) to the case where the Knudsen number equals 1. However, if one wants to study the hydrodynamical limits of the Boltzmann equation, one needs to obtain explicit dependencies on the Knudsen number. Using hypocoercivity methods [23] (see also Chapter 4) gave a constructive uniform approach on the semigroup generated by $\mathcal{G}_\varepsilon$ in $H^s_{x,v}\left(\mu^{-1/2}\right)$ and its exponential decay. The study of the full perturbed Boltzmann equation (5.1.5) taking into account the dependencies on the Knudsen number has been obtained [56][23] in the same spaces $H^s_{x,v}\left(\mu^{-1/2}\right)$, for s large enough (see also Chapter 4). More precisely, for initial data sufficiently close to μ there exists a unique non-negative solution to (5.1.4) and it decays exponentially fast towards its equilibrium. The smallness assumption was proven to be independent of the Knudsen number as well as the rate of decay and the methods used in [23] (see also Chapter 4) are constructive.

5.1.3 Our contributions and strategy

The present work brings two major improvements.

In the spirit of [51], we first prove that $\mathcal{G}_\varepsilon$ generates a strong continuous semigroup in Sobolev spaces $W^{\alpha,1}_v W^{\beta,p}_x \left(1+|v|)^k\right)$ for $\alpha \leqslant \beta$ and β and k large enough with explicit thresholds. It is done by starting from existing results in $H^s_{x,v}\left(\mu^{-1/2}\right)$ and then decomposing the linear operator $\mathcal{G}_\varepsilon$ into a dissipative part and a regularising part that is then treated in more and more regular spaces up to the space where the semigroup properties have been derived in previous articles. We thus improve the existing result [23], also given in Chapter 4. Our main contribution is an adapted version of the abstract extension theorem developed in [51] that takes into account the dependencies on the Knudsen number as well as a careful study of the dissipative and the regularising parts of the operator $\mathcal{G}_\varepsilon$.

The second contribution of this chapter is the solution to the Cauchy problem with exponential trend to equilibrium, independently on ε, in spaces

$$W^{\alpha,1}_v W^{\beta,1}_x \left(1+|v|^{2+0}\right) \quad \text{and} \quad W^{\alpha,1}_v H^{\beta}_x \left(1+|v|^{2+0}\right),$$

for β large enough and all $\alpha \leqslant \beta$. First, this result makes the recent study [51] uniform in the Knudsen number. Second, it improves the Cauchy theory developed uniformly in ε in [56][23] by dropping the exponential weight and the v-derivatives. Moreover, one can notice that the polynomial weight is almost the optimal one for the Boltzmann equation (conservation of mass and energy).

The main issue to obtain uniform results is that the bilinear operator $\varepsilon^{-1}Q$ cannot be treated as a mere perturbation that evolves under the flow of $S_{\mathcal{G}_\varepsilon}$, the semigroup generated by $\mathcal{G}_\varepsilon$, since the latter has an exponential decay of order $O(1)$ that is negligeable compared to $O(\varepsilon^{-1})$ as ε tends to zero. We develop an analytic point of view about the extension theorem in [51] and include the bilinear term. More precisely, we decompose the perturbed equation (5.1.5) into a hierarchy of equations taking place in spaces that have more and more regularity up to $H^s_{x,v}\left(\mu^{-1/2}\right)$ where estimates had been derived in [23] (see also Chapter 4). At each step we use the dissipative part of the linear operator to control the remainder term $\varepsilon^{-1}Q$ whereas the regularising part is controlled in spaces with higher regularity.

5.1.4 Organization of the chapter

Section 5.2 first introduces the different notations and definitions we will use throughout the chapter and then states the precise theorems we prove in this work. Section 5.2.2 deals with the semigroup generated by the full linear operator $\varepsilon^{-2}\mathcal{L} - \varepsilon^{-1} v \cdot \nabla_x$ whereas Section 5.2.3 is dedicated to the full Boltzmann equation.

The full linear part of the Boltzmann operator is proven to generate a strongly continuous semigroup in Lebesgue and Sobolev spaces with polynomial weight in Section 5.3.

We start with Section 5.3.1, a thorough description of our strategy and a version of the extension theorem of [51] that takes into account the dependencies in ε.

We show in this section that $\varepsilon^{-2}\mathcal{L} - \varepsilon^{-1}v \cdot \nabla_x$ can be decompose into a regularising operator in the velocity variable (Section 5.3.2) and a dissipative one (Section 5.3.3).

We then combine the last two properties to gain regularity both in space and velocity (Section 5.3.4) to finally prove the existence and exponential decay of the associated semigroup (Section 5.3.5).

The last section, Section 5.4, proves existence, uniqueness and exponential decay of solutions to the perturbed Boltzmann equation (5.1.5).

Section 5.4.1 gives a new point of view on the extension we used to generate the semigroup associated to $\varepsilon^{-2}\mathcal{L} - \varepsilon^{-1}v \cdot \nabla_x$ and how it can be used with the bilinear operator. This strategy is developed through Sections 5.4.2 and 5.4.3 and it leads to the proof of the exponential decay towards equilibrium in Section 5.4.4.

5.2 Main results

5.2.1 Notations

We gather here the notations we will use throughout this chapter.

Function spaces. We first define the following shorthand notation,

$$\langle \cdot \rangle = \sqrt{1 + |\cdot|^2}.$$

The convention we choose is to index the space by the name of the concerned variable so we have, for p in $[1, +\infty]$,

$$L^p_{[0,T]} = L^p\left([0,T]\right), \quad L^p_x = L^p\left(\mathbb{T}^d\right), \quad L^p_v = L^p\left(\mathbb{R}^d\right).$$

Let p and q be in $[1, +\infty)$, α and β in $\mathbb{N}$ and $m : \mathbb{R}^d \longrightarrow \mathbb{R}^+$ a strictly positive measurable function. For any multi-indexes $j = (j_1, \dots, j_d)$ and $l = (l_1, \dots, l_d)$ in $\mathbb{N}^d$ we denote the $(j,l)^{th}$ partial derivative by

$$\partial^j_l = \partial^l_x \partial^j_v.$$

We define the space $W_v^{\alpha,q}W_x^{\beta,p}(m)$ by the norm

$$\|f\|_{W_v^{\alpha,q}W_x^{\beta,p}(m)} = \sum_{\substack{|j|\leqslant\alpha,|l|\leqslant\beta \\ |l|+|j|\leqslant\max(\alpha,\beta)}} \left\| \left(\partial_l^j f\right) m \right\|_{L_v^q L_x^p},$$

where we used the Lebesgue norm

$$\|g\|_{L_v^q L_x^p} = \left[\int_{\mathbb{R}^d} \left(\int_{\mathbb{T}^d} |f(x,v)|^p \, dx\right)^{q/p} dv\right]^{1/q}.$$

Linear Boltzmann operator. First we use a writing convention. This chapter aims at extending results known in a small space E, namely $H^s_{x,v}\left(\mu^{-1/2}\right)$ with s sufficiently large, into a larger space $\mathcal{E}$, namely Lebesgue and Sobolev spaces with polynomial weight. We will use curly letters for operators in $\mathcal{E}$ and their non-curly equivalent to denote their restriction to E. For instance, we will denote

$$\mathcal{L}|_E = L.$$

The linear Boltzmann operator L has several properties we will use throughout this chapter (see [28][30][112][51] for instance).

L is a closed self-adjoint operator in $L^2_v\left(\mu^{-1/2}\right)$ with kernel

$$\mathrm{Ker}\,(L) = \mathrm{Span}\,\{\phi_0(v),\dots,\phi_{d+1}(v)\}\,\mu,$$

where $\phi_0(v) = 1$, for $i = 1,\dots,d$ we defined $\phi_i(v) = v_i$ and $\phi_{d+1} = \left(|v|^2 - d\right)/\sqrt{2d}$. The family $(\phi_i)_{0\leqslant i\leqslant d+1}$ is an orthonormal basis of $\mathrm{Ker}\,(L)$ in $L^2_v\left(\mu^{-1/2}\right)$ and we denote π_L the orthogonal projection onto $\mathrm{Ker}\,(L)$ in $L^2_v\left(\mu^{-1/2}\right)$:

$$\pi_L(h) = \sum_{i=0}^{d+1} \left(\int_{\mathbb{R}^d} h(u)\phi_i(u)\,du\right)\phi_i(v)\mu(v), \tag{5.2.1}$$

and we define $\pi_L^\perp = \mathrm{Id} - \pi_L$. We will also denote the full linear Boltzmann operator by

$$\mathcal{G}_\varepsilon = \frac{1}{\varepsilon^2}L - \frac{1}{\varepsilon}v\cdot\nabla_x.$$

For s in $\mathbb{N}$ we will use the convention

$$(\mathcal{G}_\varepsilon)|_{H^s_{x,v}\left(\mu^{-1/2}\right)} = G_\varepsilon.$$

It has been proven ([23] Proposition 3.1 or see Proposition 4.3.1 in Chapter 4) that the kernel of G_ε does not depend on ε and that its generators in $L^2_{x,v}\left(\mu^{-1/2}\right)$ are the same than the ones of $\mathrm{Ker}\,(L)$. We therefore have that the orthogonal projection onto $\mathrm{Ker}\,(G_\varepsilon)$ in $L^2_{x,v}\left(\mu^{-1/2}\right)$ is given by

$$\Pi_G(h) = \Pi_{G_\varepsilon}(h) = \sum_{i=0}^{d+1}\left(\int_{\mathbb{T}^d\times\mathbb{R}^d} h(x,u)\phi_i(u)\,dxdu\right)\phi_i(v)\mu(v), \tag{5.2.2}$$

and we define $\Pi_G^\perp = \mathrm{Id} - \Pi_G$.

Note that for a function h in $L^2_{x,v}\left(\mu^{-1/2}\right)$ we have that

$$\forall (x,v)\in\mathbb{T}^d\times\mathbb{R}^d, \quad \Pi_G(h)(x,v) = \int_{\mathbb{T}^d}\pi_L(h(x_*,\cdot))(v)\,dx_*.$$

5.2.2 Results about the full linear part

We first deal with $\mathcal{G}_\varepsilon$, the linear part of the perturbed Boltzmann operator. We prove that it generates a strongly continuous semigroup with an exponential decay in Lebesgue and Sobolev spaces with a weight $\langle v\rangle^k$ as long as k is large enough. The precise statement is the following.

Theorem 5.2.1 *Let B be a Boltzmann collision kernel satisfying* (5.1.1)-(5.1.2)-(5.1.3)*. There exists $0<\varepsilon_d\leqslant 1$ such that for all p, q in $[1,+\infty]$, all α, β in $\mathbb{N}$ with $\alpha\leqslant\beta$ and all $k>k_q^*$, where*

$$k_q^* = \frac{3+\sqrt{49-48/q}}{2} + \gamma\left(1-\frac{1}{q}\right), \tag{5.2.3}$$

with γ defined in (5.1.2)*,*

1. *for all $0<\varepsilon\leqslant\varepsilon_d$, $\mathcal{G}_\varepsilon = \varepsilon^{-2}\mathcal{L} - \varepsilon^{-1}v\cdot\nabla_x$ generates a C^0-semigroup, $S_{\mathcal{G}_\varepsilon}(t)$, on $W^{\alpha,q}_v W^{\beta,p}_x\left(\langle v\rangle^k\right)$,*

2. *for all $\tau>0$, there exist $C_{\mathcal{G}}(\tau),\lambda_0>0$, such that for all $0<\varepsilon\leqslant\varepsilon_d$ and for all h_{in} in $W^{\alpha,q}_v W^{\beta,p}_x\left(\langle v\rangle^k\right)$, for all $t\geqslant\tau$*

$$\left\|S_{\mathcal{G}_\varepsilon}(t)(h_{in}) - \Pi_{\mathcal{G}}(h_{in})\right\|_{W^{\alpha,q}_v W^{\beta,p}_x\left(\langle v\rangle^k\right)} \leqslant C_{\mathcal{G}}(\tau)e^{-\lambda_0 t}\left\|h_{in} - \Pi_{\mathcal{G}}(h_{in})\right\|_{W^{\alpha,q}_v W^{\beta,p}_x\left(\langle v\rangle^k\right)},$$

where $\Pi_{\mathcal{G}}$ is the spectral projector onto $\mathrm{Ker}\,(\mathcal{G}_\varepsilon)$ which is given, for all ε, by

$$\Pi_{\mathcal{G}}(g) = \sum_{i=0}^{d+1}\left(\int_{\mathbb{T}^d\times\mathbb{R}^d} g\phi_i\,dxdv\right)\phi_i\mu. \tag{5.2.4}$$

The constants ε_d, $C_{\mathcal{G}}(\tau)$ and λ_0 are constructive and only depends on d, p, q, k, α, β and the kernel of the Boltzmann operator.

We refer to [61] and [51] Section 2 for definitions and properties of spectral projectors.

Remark 5.2.2 *We can make a couple of remarks about this theorem.*

1. *It has been proven in [23] Section* 3 *(or Chapter 4 Section 4.3), that in $H^1_{x,v}(\mu^{-1/2})$,* $\mathrm{Ker}\,(G_\varepsilon)$ *does not depend on ε if ε is positive and we therefore can define $\Pi_G = \Pi_{G_\varepsilon}$. During the proof of Theorem 5.2.1 we will show that $(\Pi_{\mathcal{G}_\varepsilon})|_{H^s_{x,v}(\mu^{-1/2})} = \Pi_{G_\varepsilon}$ and thus $\Pi_{\mathcal{G}}$ is well-defined and is independent of ε and given by (5.2.2).*

2. *As noticed in [51], the rate of decay λ_0 can be taken equal to the spectral gap of $\mathcal{L}|_{H^s_{x,v}(\mu^{-1/2})}$ (see [23] or Chapter 4), for s as large as wanted, when k is big enough (and we obtained a constructive threshold).*

3. *Finally, we emphasize that in the case $q = 1$, the result holds for all $k > 2$. This is almost the minimal regularity $L^2_v\left(1+|v|^2\right)$ for the Boltzmann equation, that is solutions with bounded mass and energy.*

5.2.3 Existence, uniqueness and trend to equilibrium

A physically relevant requirement for solutions to the Boltzmann equation are that their mass, momentum and energy are preserved with time. This is also an *a priori* property of the Boltzmann equation on the torus (see [112] Chapter 1 Section 2 for instance) which reads

$$\forall t \geqslant 0, \quad \int_{\mathbb{T}^d\times\mathbb{R}^d} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} f_\varepsilon(t,x,v)\,dxdv = \int_{\mathbb{T}^d\times\mathbb{R}^d} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} f_\varepsilon(0,x,v)\,dxdv.$$

If one expects trend to the equilibrium $\mu(v)$ for the solutions $f_\varepsilon = \mu + \varepsilon h_\varepsilon$ of the Boltzmann equation (5.1.4) then it must be that

$$\forall t \geqslant 0, \quad \int_{\mathbb{T}^d\times\mathbb{R}^d} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} h_\varepsilon(t,x,v)\,dxdv = 0,$$

that is $\Pi_{\mathcal{G}_\varepsilon}(h_\varepsilon(t,\cdot,\cdot)) = 0$ for all t, which is a property that is indeed preserved along time for solution to the perturbed Boltzmann equation (5.1.5), see [23] or Chapter 4 for instance.

We hence state the following theorem answering the Cauchy problem and the exponential convergence towards the equilibrium μ.

Theorem 5.2.3 *Let B be a Boltzmann collision kernel satisfying* (5.1.1)-(5.1.2)-(5.1.3) *and let $p = 1$ or $p = 2$.*
There exists $0 < \varepsilon_d \leqslant 1$ and β_0 in $\mathbb{N}$ such that

- *for all α, β in $\mathbb{N}$ such that $\beta \geqslant \beta_0$ and $\alpha \leqslant \beta$ and for all $k > 2$ define*

$$\mathcal{E}^p = W_v^{\alpha,1} W_x^{\beta,p} \left(\langle v \rangle^k\right),$$

- *for any λ_0' in $(0, \lambda_0)$ (λ_0 defined in Theorem 5.2.1) there exist $C_{\alpha,\beta}$, $\eta_{\alpha,\beta} > 0$ such that for any $0 < \varepsilon \leqslant \varepsilon_d$, for any distribution $0 \leqslant f_{in} = \mu + \varepsilon h_{in}$:*

If

(i) *h_{in} is in $\mathrm{Ker}(\mathcal{G}_\varepsilon)^\perp$ in $\mathcal{E}^p$,*

(ii) $\|h_{in}\|_{\mathcal{E}^p} \leqslant \eta_{\alpha,\beta}$,

then there exists a unique global solution $f_\varepsilon = f_\varepsilon(t,x,v)$ to (5.1.4) *in $\mathcal{E}^p$ which, moreover, satisfies $f_\varepsilon = M + \varepsilon h_\varepsilon \geqslant 0$ with:*

- *h_ε belongs to $\mathrm{Ker}(\mathcal{G}_\varepsilon)^\perp$ for all times,*
-
$$\|h_\varepsilon\|_{\mathcal{E}^p} \leqslant C_{\alpha,\beta} \|h_{in}\|_{\mathcal{E}^p} e^{-\lambda_0' t}.$$

The constants $C_{\alpha,\beta}$ and $\eta_{\alpha,\beta}$ are constructive and depends only on α, β, k, d, λ_0' and the kernel of the Boltzmann operator.

5.3 The linear part: a C^0-semigroup in spaces with polynomial weight, proof of Theorem 5.2.1

In this section we focus on the linear part of the perturbed Boltzmann equation in $W_v^{\alpha,q} W_x^{\beta,p}\left(\langle v \rangle^k\right)$. We thus consider the following equation:

$$\partial_t h = \mathcal{G}_\varepsilon(h). \tag{5.3.1}$$

5.3.1 Strategy of the proof

If we denote $\mathcal{E} = W_v^{\alpha,q} W_x^{\beta,p}\left(\langle v\rangle^k\right)$ and $E = H^s_{x,v}\left(\mu^{-1/2}\right)$ we have that $E \subset \mathcal{E}$, dense with continuous embedding for s large enough. [23] Theorem 2.1 (with the norm of Theorem 2.4) states that $G_\varepsilon = (\mathcal{G}_\varepsilon)|_E$ generates a strongly continuous semigroup in E with exponential decay (these results are given in Chapter 4 Section 4.2). Theorem 5.2.1 can therefore be understood as the possibility to extend properties of G_ε in a small space E to $\mathcal{G}_\varepsilon$ in a larger space $\mathcal{E}$.

This issue of extending spectral gap properties as well as semigroup properties has been first tackled by Mouhot to obtain constructive rates of convergence to equilibrium for the homogeneous Boltzmann equation [80]. Recently, Gualdani, Mischler and Mouhot [51] proposed a more abstract approach that allows to deal with the full linear operator. In their work, they proved that if some conditions on $\mathcal{G}_\varepsilon$ and G_ε are satisfied then we can pass on some semigroup properties from E to $\mathcal{E}$. The main argument of the proof of Theorem 5.2.1 is to show that we can use their result in our setting, independently of ε.

To be more precise, we give below a modified version of their main functional analysis theorem which is combination of Theorem 2.13 and Lemma 2.17 where we added dependencies on ε.

We refer to [51] Section 2 for the definition of hypodissipativity (roughly speaking it is a dissipative property in a different norm on a Banach space) and the definition of the convolution of two semigroups of operators (denoted by the symbol $(*)$). In the sequel we will use $\mathscr{C}(E)$ for the set of closed operators on E and $\mathscr{B}(E)$ for the set of bounded operators on E. For any operator G in $\mathscr{C}(E)$ we denote $\mathrm{R}(G)$ its range and $\Sigma(G)$ its spectrum.

Theorem 5.3.1 (Modified extension theorem from [51]) *Let ε be a parameter such that $0 < \varepsilon \leqslant 1$.*
Let $E, \mathcal{E}$ be two Banach spaces with $E \subset \mathcal{E}$ dense with continuous embedding, and consider G_ε in $\mathscr{C}(E)$, $\mathcal{G}_\varepsilon$ in $\mathscr{C}(\mathcal{E})$ with $(\mathcal{G}_\varepsilon)|_E = G_\varepsilon$ and $a > 0$.

We assume the following

(A1) *G_ε generates a semigroup S_{G_ε} on E, $G_\varepsilon + a$ is hypodissipative on $R\left(\mathrm{Id} - \Pi_{G_\varepsilon,a}\right)$ and*

$$\Sigma\left(G_\varepsilon\right) \cap \{z \in \mathbb{C},\ \mathrm{Re}(z) > -a\} = \{0\} \quad \textit{is a discrete eigenvalue.}$$

(A2) *There exists $\mathcal{A}_\varepsilon, \mathcal{B}_\varepsilon$ in $\mathscr{C}(\mathcal{E})$ such that $\mathcal{G}_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon$ (with corresponding restrictions $A_\varepsilon, B_\varepsilon$ on E) and there exist some "intermediate spaces" (not necessarily ordered)*

$$E = \mathcal{E}_J,\ \mathcal{E}_{J-1}, \ldots,\ \mathcal{E}_2,\ \mathcal{E}_1 = \mathcal{E}$$

such that, still denoting $\mathcal{B}_\varepsilon := (\mathcal{B}_\varepsilon)|_{\mathcal{E}_j}$ and $\mathcal{A}_\varepsilon := (\mathcal{A}_\varepsilon)|_{\mathcal{E}_j}$

(i) $\left(\mathcal{B}_\varepsilon + a/\varepsilon^2\right)$ *is hypodissipative on* $\mathcal{E}_j$*;*

(ii) $\mathcal{A}_\varepsilon \in \mathscr{B}\left(\mathcal{E}_j\right)$ *with* $\|\mathcal{A}_\varepsilon\|_{\mathscr{B}(\mathcal{E}_j)} \leqslant C_A/\varepsilon^2$*;*

(iii) there are some constants $l_0, l_1 \in \mathbb{N}^*$, $C \geqslant 1$, $K \in \mathbb{R}$, $\alpha \in [0,1)$ *such that*

$$\forall t \geqslant 0, \quad \|T_{l_0}(t)\|_{\mathscr{B}(\mathcal{E}_j,\mathcal{E}_{j+1})} \leqslant C\frac{e^{Kt/\varepsilon^2}}{\varepsilon^{l_1}t^\alpha},$$

for $1 \leqslant j \leqslant J-1$, *with the notation* $T_l := \left(\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}\right)^{(*l)}$.

Then $\mathcal{G}_\varepsilon$ *is hypodissipative in* $\mathcal{E}$ *and for all* $a' < a$ *there exists* $n = n(a') \geqslant 1$ *and some positive constants* $C_{a'}$ *and* $C'_{a'}$ *such that*

$$\|T_n(t)\|_{\mathscr{B}(\mathcal{E})} \leqslant \frac{C_{a'}}{\varepsilon^{nl_1/l_0}}e^{-a't/\varepsilon^2}; \tag{5.3.2}$$

$$S_{\mathcal{G}_\varepsilon}(t) = S_{G_\varepsilon}(t)\Pi_{\mathcal{G}} + \sum_{l=0}^{n-1}(-1)^l\left(\mathrm{Id}-\Pi_{\mathcal{G}}\right)S_{\mathcal{B}_\varepsilon} * T_l(t) + (-1)^n\left[\left(\mathrm{Id}-\Pi_{\mathcal{G}}\right)S_{G_\varepsilon}\right] * T_n(t); \tag{5.3.3}$$

$$\left|\left|S_{\mathcal{G}_\varepsilon}(t) - S_{G_\varepsilon}(t)\Pi_{\mathcal{G}} - (-1)^n\left[\left(\mathrm{Id}-\Pi_{\mathcal{G}}\right)S_{G_\varepsilon}\right] * T_n(t)\right|\right|_{\mathscr{B}(\mathcal{E})} \leqslant C'_{a'}\frac{t^n}{\varepsilon^{n(2+l_1/l_0)}}e^{-a't/\varepsilon^2}, \tag{5.3.4}$$

where $\Pi_{\mathcal{G}}$ *has been defined in* (5.2.4).

We will use Theorem 5.3.1 to directly prove Theorem 5.2.1. Indeed, [23] Theorem 2.1 states that G_ε generates a strongly continuous semigroup with exponential decay in $E = H^s_{x,v}\left(\mu^{-1/2}\right)$, which is the required assumption $(A1)$ (properties about the spectral gap of the spectrum can be found in [4]). Therefore if $\mathcal{G}_\varepsilon$ fulfils hypothesis $(A2)$ then it generates a strongly continuous semigroup, with an exponential decay of order a' for all $a' < a$, since for all α, β, $\eta > 0$, all $t > t_0 > 0$ and all $0 < 2\eta' < \eta$,

$$\frac{t^\alpha}{\varepsilon^{2\beta}}e^{-\eta\frac{t}{\varepsilon^2}} \leqslant C_{\beta,\eta'}t^{\alpha-\beta}e^{-(\eta-\eta')\frac{t}{2\varepsilon^2}} \leqslant C_{\beta,\eta'}t^{\alpha-\beta}e^{-(\eta-\eta')t} \leqslant C_{t_0,\alpha,\beta,\eta'}e^{-(\eta-2\eta')t}, \tag{5.3.5}$$

for $0 < \varepsilon \leqslant 1$.

5.3.2 Decomposition of the operator and assumption $(A2)(ii)$

In this section we find a decomposition $\mathcal{G}_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon$ that will fit the requirements $(A1) - (A2)$ of Theorem 5.3.1. This decomposition has been found in [51] in the case $\varepsilon = 1$. We will use exactly the same operators but including the dependencies in ε. All the results presented in the rest of this section are true for $\varepsilon = 1$ (see [51] Section 4) so we will try to relate as much as possible our computations with the ones for $\varepsilon = 1$.

For δ in $(0,1)$, to be chosen later, we consider $\Theta_\delta = \Theta_\delta(v, v_*, \sigma)$ in C^∞ that is bounded by one on the set

$$\left\{|v| \leqslant \delta^{-1} \quad \text{and} \quad 2\delta \leqslant |v-v_*| \leqslant \delta^{-1} \quad \text{and} \quad |\cos\theta| \leqslant 1-2\delta\right\}$$

and whose support is included in

$$\left\{|v| \leqslant 2\delta^{-1} \quad \text{and} \quad \delta \leqslant |v-v_*| \leqslant 2\delta^{-1} \quad \text{and} \quad |\cos\theta| \leqslant 1-\delta\right\}.$$

We define the splitting

$$\mathcal{G}_\varepsilon = \mathcal{A}_\varepsilon^{(\delta)} + \mathcal{B}_\varepsilon^{(\delta)},$$

with

$$\mathcal{A}_\varepsilon^{(\delta)} h(v) = \frac{1}{\varepsilon^2}\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \Theta_\delta \left[\mu'_* h' + \mu' h'_* - \mu h_*\right] b\left(\cos\theta\right)|v-v_*|^\gamma \, d\sigma dv_*$$

and

$$\mathcal{B}_\varepsilon^{(\delta)} h(v) = \mathcal{B}_{2,\varepsilon}^{(\delta)} h(v) - \frac{1}{\varepsilon^2}\nu(v)h(v) - \frac{1}{\varepsilon} v\cdot\nabla_x h(v),$$

where

$$\mathcal{B}_{2,\varepsilon}^{(\delta)} h(v) = \frac{1}{\varepsilon^2}\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} (1-\Theta_\delta) \left[\mu'_* h' + \mu' h'_* - \mu h_*\right] b\left(\cos\theta\right)|v-v_*|^\gamma \, d\sigma dv_*$$

and $\nu(v)$ is the standard collision frequency

$$\nu(v) = \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} b\left(\cos\theta\right)|v-v_*|^\gamma \mu_* \, d\sigma dv_*.$$

Note that there exists ν_0, $\nu_1 > 0$ such that

$$\forall v\in\mathbb{R}^d, \quad \nu_0(1+|v|^\gamma) \leqslant \nu(v) \leqslant \nu_1(1+|v|^\gamma). \tag{5.3.6}$$

We have that

$$\mathcal{A}_\varepsilon^{(\delta)} = \frac{1}{\varepsilon^2}\mathcal{A}_1^{(\delta)} \quad \text{and} \quad \mathcal{B}_{2,\varepsilon}^{(\delta)} = \frac{1}{\varepsilon^2}\mathcal{B}_{2,1}^{(\delta)}.$$

We therefore obtain the following controls on $\mathcal{A}_\varepsilon^{(\delta)}$.

Proposition 5.3.2 *For all $0<\varepsilon<\varepsilon^d$, for any q in $[1,+\infty]$ and $\alpha \geqslant 0$, the operator $\mathcal{A}_\varepsilon^{(\delta)}$ maps L^q_v into $W^{\alpha,q}_v$ with compact support.*
There exists $C_{\delta,\alpha,q}, R_\delta > 0$ independent of ε such that

$$\forall h\in L^q_v,\ \mathrm{supp}\left(\mathcal{A}_\varepsilon^{(\delta)} h\right) \subset B(0,R_\delta), \quad \left\|\mathcal{A}_\varepsilon^{(\delta)} h\right\|_{W^{\alpha,q}_v} \leqslant \frac{C_{\delta,\alpha,q}}{\varepsilon^2}\left\|h\right\|_{L^q_v}.$$

Moreover, for any p *in* $[1, +\infty]$ *and for all* h *in* $L^q_v L^p_x$,

$$\left\| \mathcal{A}^{(\delta)}_\varepsilon h \right\|_{L^q_v L^p_x} \leqslant \left\| \mathcal{A}^{(\delta)}_\varepsilon \left(\|h\|_{L^p_x} \right) \right\|_{L^q_v}$$

Remark 5.3.3 *We notice here that this Proposition gives the point* $(A2)(ii)$ *of Theorem* 5.3.1 *if the* $\mathcal{E}_j$ *are Sobolev spaces.*

Proof of Proposition 5.3.2 The kernel of the operator $\mathcal{A}^{(\delta)}_\varepsilon$ is of compact support so its Carleman representation (see [27]) gives the existence of $k^{(\delta)}$ in $C^\infty_c\left(\mathbb{R}^d \times \mathbb{R}^d\right)$ such that

$$\mathcal{A}^{(\delta)}_\varepsilon h(v) = \frac{1}{\varepsilon^2} \int_{\mathbb{R}^d} k^{(\delta)}(v, v_*) h(v_*)\, dv_*, \tag{5.3.7}$$

and therefore the control on $\left\| \mathcal{A}^{(\delta)}_\varepsilon h \right\|_{W^{\alpha,q}_v}$ is straightforward.

The control of $\left\| \mathcal{A}^{(\delta)}_\varepsilon h \right\|_{L^q_v L^p_x}$ comes directly from Minkowski's integral inequality which states

$$\left[\int_{\mathbb{T}^d} \left(\int_{\mathbb{R}^d} k^{(\delta)}(v, v_*) h(x, v_*) dv_* \right)^p dx \right]^{1/p} \leqslant \int_{\mathbb{R}^d} \left(\int_{\mathbb{T}^d} k^{(\delta)}(v, v_*)^p h(x, v_*)^p dx \right)^{1/p} dv_*.$$

■

5.3.3 Dissipativity estimates for $\mathcal{B}^{(\delta)}_\varepsilon$, assumption $(A2)(i)$

One can find in [51] proof of Lemma 4.14 case $(W2)$ and $(W3)$ the following estimate on the operator $\mathcal{B}^{(\delta)}_\varepsilon$ in the case $\varepsilon = 1$.

Lemma 5.3.4 *For all* p, q *in* $[1, +\infty]$, *for all* $k > 2$ *and for any* δ *in* $(0,1)$ *and all* h *in* $L^q_v L^p_x\left(\langle v \rangle^k\right)$,

$$\int_{\mathbb{R}^d} \langle v \rangle^{kq} \|h\|^{q-p}_{L^p_x} \left(\int_{\mathbb{T}^d} sgn(h)\, |h|^{p-1}\, \mathcal{B}^{(\delta)}_1 h\, dx \right) dv \leqslant \left[\Lambda_{k-\gamma/q',q}(\delta) - 1 \right] \|h\|^q_{L^q_v L^p_x\left(\langle v \rangle^k \nu^{1/q}\right)},$$

where q' *is the conjugate exponent of* $1/q$ *and* $\Lambda_{k,q}(\delta)$ *is a constructive constant such that*

$$\lim_{\delta \to 0} \Lambda_{k,q}(\delta) = \phi_q(k) = \left(\frac{4}{k+2} \right)^{1/q} \left(\frac{4}{k-1} \right)^{1-1/q}.$$

Remark 5.3.5 *As noticed in [51] Remark* 4.3*, the quantity $\phi_q(k)$ is strictly less than one for k bigger than a constant k_q^{**}. The constant k_q^* we are considering is not optimal and is such that $\phi_q(k-\gamma/q') < 1$, where q' is the conjugate exponent of q. This appearance of $k-\gamma/q'$ is due to a loss of weight of order $\nu^{-1/q'}$ in the estimate of the spectral gap, see proof of Proposition* 5.3.6.

In the case of the Boltzmann operator with hard potential and angular cutoff, point $(A2)(i)$ is fulfilled by $\mathcal{B}_\varepsilon^{(\delta)}$ for δ small enough. This is the purpose of the following lemma. We recall here that $\nu_0 = \inf_{v\in\mathbb{R}^d}(\nu(v)) > 0$ and that we define

$$\|\cdot\|_{W_v^{\alpha,q}W_x^{\beta,p}\left(\langle v\rangle^k\right)} = \sum_{\substack{|l|+|j|\leqslant \max(\alpha,\beta)\\ |j|\leqslant\alpha,|l|\leqslant\beta}} \left\|\partial_l^j\cdot\right\|_{L_v^qL_x^p\left(\langle v\rangle^k\right)}.$$

Proposition 5.3.6 *Consider p,q in $[1,+\infty]$, $k>k_q^*$, defined by* (5.2.3)*, and α, β in $\mathbb{N}$ such that $\alpha\leqslant\beta$.*
Then there exists $\delta_{k,q}$ in $(0,1)$ such that for all $0<\delta\leqslant\delta_{k,q}$ there exists $\lambda_0=\lambda_0(k,q,\delta)$ in $(0,\nu_0)$ such that for all $0<\varepsilon\leqslant 1$,

- *$\lambda_0(k,q,\delta)$ tends to $\lambda_0^*(k,q)$ as δ goes to 0,*
- *$\lambda_0^*(k,q)$ tends to ν_0 when k goes to $+\infty$,*
- *$\left(\mathcal{B}_\varepsilon^{(\delta)}+\lambda_0/\varepsilon^2\right)$ is dissipative in $W_v^{\alpha,q}W_x^{\beta,p}\left(\langle v\rangle^k\right)$.*

Proof of Proposition 5.3.6 Let h_0 be in $W_v^{\alpha,q}W_x^{\beta,p}\left(\langle v\rangle^k\right)$ and considert h to be a solution to the linear equation

$$\partial_t h = \mathcal{B}_\varepsilon^{(\delta)}h = \mathcal{B}_{2,\varepsilon}^{(\delta)}h - \frac{1}{\varepsilon^2}\nu h - \frac{1}{\varepsilon}v\cdot\nabla_x h, \tag{5.3.8}$$

with initial value h_0.

Since the x-derivative commutes with the equation we can consider only the case when $\beta=\alpha$. The proof is split into two parts. First we prove Proposition 5.3.6 in the case $\alpha=0$ and then we study the case with v-derivatives.

Step 1: the case $\alpha=0$. Take p,q in $[1,+\infty)$.
We recall that

$$\|h\|_{L_v^qL_x^p\left(\langle v\rangle^k\right)} = \left[\int_{\mathbb{R}^d}\left(1+|v|^k\right)^q\left(\int_{\mathbb{T}^d}|h|^p\,dx\right)^{q/p}dv\right]^{1/q}.$$

Therefore we can compute

$$\begin{aligned}\frac{d}{dt}\|h\|_{L^q_v L^p_x(\langle v\rangle^k)} = \|h\|^{1-q}_{L^q_v L^p_x(\langle v\rangle^k)} \\ \times \int_{\mathbb{R}^d}\left(1+|v|^k\right)^q \|h\|^{q-p}_{L^p_x}\left(\int_{\mathbb{T}^d}\operatorname{sgn}(h)\,|h|^{p-1}\,\mathcal{B}^{(\delta)}_\varepsilon h\,dx\right)dv.\end{aligned} \tag{5.3.9}$$

Observing that

$$\int_{\mathbb{T}^d}\operatorname{sgn}(h)\,|h|^{p-1}\,v\cdot\nabla_x h\,dx = \frac{1}{p}v\cdot\int_{\mathbb{T}^d}\nabla_x\left(|h|^p\right)\,dx = 0,$$

we deduce

$$\begin{aligned}\frac{d}{dt}\|h\|_{L^q_v L^p_x(\langle v\rangle^k)} = \|h\|^{1-q}_{L^q_v L^p_x(\langle v\rangle^k)} \\ \times\frac{1}{\varepsilon^2}\int_{\mathbb{R}^d}\left(1+|v|^k\right)^q \|h\|^{q-p}_{L^p_x}\left(\int_{\mathbb{T}^d}\operatorname{sgn}(h)\,|h|^{p-1}\,\mathcal{B}^{(\delta)}_1 h\,dx\right)dv.\end{aligned}$$

We can therefore use Lemma 5.3.4 which leads to

$$\frac{d}{dt}\|h\|_{L^q_v L^p_x(\langle v\rangle^k)} \leqslant -\frac{1}{\varepsilon^2}\left[1-\Lambda_{k-\gamma/q',q}(\delta)\right]\|h\|^q_{L^q_v L^p_x\left(\langle v\rangle^k\nu^{1/q}\right)}\|h\|^{1-q}_{L^q_v L^p_x(\langle v\rangle^k)}, \tag{5.3.10}$$

We already noticed that $\Lambda_{k-1/q',q}(\delta)$ is strictly less than 1 for δ smaller than some $\delta_{k,q}$ (see Remark 5.3.5). Therefore, because $\nu(v)\geqslant\nu_0$ for all v we have that for all δ smaller than $\delta_{k,q}$ the following holds,

$$\frac{d}{dt}\|h\|_{L^q_v L^p_x(\langle v\rangle^k)} \leqslant -\frac{\nu_0}{\varepsilon^2}\left[1-\Lambda_{k-\gamma/q',q}(\delta)\right]\|h\|_{L^q_v L^p_x(\langle v\rangle^k)},$$

This concludes the proof of Proposition 5.3.6 for $\alpha=0$ and $1\leqslant p,q<+\infty$. The cases $p=\infty$ and $q=\infty$ are respectively dealt with by taking the limit $p\to\infty$ and $q\to\infty$ which is possible since $\delta_{k,q}$ is independent of p and can be chosen to converge to a strictly positive constant when q goes to ∞, thanks to the definition of $\Lambda_{k,q}(\delta)$.

Step 2: the case with v-derivatives. Take p,q in $[1,+\infty]$ and $\alpha=\beta=1$.

Since the x-derivative commutes with (5.3.8) the equation satisfied by h, we have that (5.3.10) holds for x-derivatives. Notice that $1-q\leqslant 0$ gives

$$\begin{aligned}\frac{d}{dt}\left(\|h\|_{L^q_v L^p_x(\langle v\rangle^k)}+\|\nabla_x h\|_{L^q_v L^p_x(\langle v\rangle^k)}\right) \\ \leqslant -\frac{\nu_0^{1-1/q}}{\varepsilon^2}\left[1-\Lambda_{k-\gamma/q',q}(\delta)\right]\left(\|h\|_{L^q_v L^p_x\left(\langle v\rangle^k\nu^{1/q}\right)}+\|\nabla_x h\|_{L^q_v L^p_x\left(\langle v\rangle^k\nu^{1/q}\right)}\right).\end{aligned} \tag{5.3.11}$$

Applying a v-derivatives to (5.3.8) yields

$$\begin{aligned}\partial_t \nabla_v h &= \mathcal{B}_\varepsilon^{(\delta)}(\nabla_v h) + \left(\nabla_v B_\varepsilon^{(\delta)}\right)(h) \\ &= \mathcal{B}_\varepsilon^{(\delta)}(\nabla_v h) - \frac{1}{\varepsilon}\nabla_x h + \mathcal{R}_\varepsilon^{(\delta)}(h),\end{aligned}$$

where $\mathcal{R}_\varepsilon^{(\delta)}(h) = \left(\nabla_v \mathcal{B}_{2,\varepsilon}^{(\delta)}\right)(h) - \frac{1}{\varepsilon^2}\nabla_v(\nu)h = \frac{1}{\varepsilon^2}\mathcal{R}_1^{(\delta)}(h)$.

From (5.3.11), our computations in Step 1 with $\delta \leqslant \delta_{k,q}$ and the following norm

$$\|h\|_{W_v^{1,q}W_x^{1,p}(\langle v\rangle^k)_\eta} = \|h\|_{L_v^qL_x^p(\langle v\rangle^k)} + \|\nabla_x h\|_{L_v^qL_x^p(\langle v\rangle^k)} + \eta\,\|\nabla_v h\|_{L_v^qL_x^p(\langle v\rangle^k)},$$

with $\eta > 0$ to be fixed later, we obtain

$$\begin{aligned}&\frac{d}{dt}\|h\|_{W_v^{1,q}W_x^{1,p}(\langle v\rangle^k)_\eta} \\ &\leqslant -\frac{\nu_0^{1-1/q}}{\varepsilon^2}\left[1-\Lambda_{k-\gamma/q',q}(\delta)\right]\left(\|h\|_{L_v^qL_x^p(\langle v\rangle^k\nu^{1/q})} + \|\nabla_x h\|_{L_v^qL_x^p(\langle v\rangle^k\nu^{1/q})}\right) \\ &\quad - \eta\frac{\nu_0^{1-1/q}}{\varepsilon^2}\left[1-\Lambda_{k-\gamma/q',q}(\delta)\right]\|\nabla_v h\|_{L_v^qL_x^p(\langle v\rangle^k\nu^{1/q})} \\ &\quad - \frac{\eta}{\varepsilon}\|\nabla_v h\|^{1-q}_{L_v^qL_x^p(\langle v\rangle^k)}\int_{\mathbb{R}^d}\left(\langle v\rangle^k\right)^q\|\nabla_v h\|_{L_x^p}^{q-p}\left(\int_{\mathbb{T}^d}\mathrm{sgn}(h)\,|\nabla_v h|^{p-1}\,\nabla_x h\,dx\right)dv \\ &\quad + \frac{\eta}{\varepsilon^2}\|\nabla_v h\|^{1-q}_{L_v^qL_x^p(\langle v\rangle^k)}\int_{\mathbb{R}^d}\left(\langle v\rangle^k\right)^q\|\nabla_v h\|_{L_x^p}^{q-p}\left(\int_{\mathbb{T}^d}\mathrm{sgn}(h)\,|\nabla_v h|^{p-1}\,\mathcal{R}_1^{(\delta)}(h)\,dx\right)dv.\end{aligned}$$

We take the absolute value and use Hölder inequality twice on the last two terms which makes the terms in $\nabla_v h$ disappear, and this gives

$$\begin{aligned}&\frac{d}{dt}\|h\|_{W_v^{1,q}W_x^{1,p}(\langle v\rangle^k)_\eta} \\ &\leqslant -\frac{\nu_0^{1-1/q}}{\varepsilon^2}\left[1-\Lambda_{k-\gamma/q',q}(\delta)\right]\left(\|h\|_{L_v^qL_x^p(\langle v\rangle^k\nu^{1/q})} + \eta\,\|\nabla_v h\|_{L_v^qL_x^p(\langle v\rangle^k\nu^{1/q})}\right) \\ &\quad + \frac{1}{\varepsilon^2}\left(\varepsilon\eta\nu_0^{-1/q} - \nu_0^{1-1/q}\left[1-\Lambda_{k-\gamma/q',q}(\delta)\right]\right)\|\nabla_x h\|_{L_v^qL_x^p(\langle v\rangle^k\nu^{1/q})} \\ &\quad + \frac{\eta}{\varepsilon^2}\left\|\mathcal{R}_1^{(\delta)}(h)\right\|_{L_v^qL_x^p(\langle v\rangle^k)}.\end{aligned}$$

One can find in [51] proof of Lemma 4.14 case $(W2)$ and $(W3)$ the following estimate

$$\left\|\mathcal{R}_1^{(\delta)}(h)\right\|_{L_v^qL_x^p(\langle v\rangle^k)} \leqslant C_\delta\,\|h\|_{L_v^qL_x^p(\langle v\rangle^k\nu^{1/q})},$$

where $C_\delta > 0$ is a constant only depending on δ.

Because $\varepsilon \leqslant 1$, this latter estimates yields

$$\begin{aligned}\frac{d}{dt}\|h\|_{W^{1,q}_v W^{1,p}_x(\langle v\rangle^k)_\eta} \leqslant & \frac{1}{\varepsilon^2}\left(C_\delta\eta - \nu_0^{1-1/q}\left[1-\Lambda_{k-\gamma/q',q}(\delta)\right]\right)\|h\|_{L^q_v L^p_x(\langle v\rangle^k\nu^{1/q})}\\ &+\frac{1}{\varepsilon^2}\left(\eta\nu_0^{-1/q} - \nu_0^{1-1/q}\left[1-\Lambda_{k-\gamma/q',q}(\delta)\right]\right)\|\nabla_x h\|_{L^q_v L^p_x(\langle v\rangle^k\nu^{1/q})}\\ &-\eta\frac{\nu_0^{1-1/q}}{\varepsilon^2}\left[1-\Lambda_{k-\gamma/q',q}(\delta)\right]\|\nabla_v h\|_{L^q_v L^p_x(\langle v\rangle^k\nu^{1/q})},\end{aligned} \tag{5.3.12}$$

which concludes the proof if we take η small enough in terms of δ, for $\delta \leqslant \delta_{k,q}$.

The case where $1 < \alpha = \beta$ is dealt with in the same way with the norm

$$\|h\|_{W^{\alpha,q}_v W^{\alpha,p}_x(\langle v\rangle^k)_\eta} = \sum_{0\leqslant|j|+|l|\leqslant\alpha}\eta^{|j|}\left\|\partial^j_l h\right\|_{L^q_v L^p_x(\langle v\rangle^k)},$$

with η small enough in terms of δ. ■

5.3.4 Estimates on the iterated convolution product, assumption $(A2)(iii)$

In order to use Theorem 5.3.1, it remains to show that our equation (5.3.1) satisfies hypothesis $(A2)(iii)$, that is we need to control the iterated quantities $T_l := \left(\mathcal{A}^{(\delta)}_\varepsilon S_{\mathcal{B}^{(\delta)}_\varepsilon}\right)^{(*l)}$ for some l in $\mathbb{N}$. The following proposition describes such controls when $p=1$.

Proposition 5.3.7 *Consider $k > k^*_q$, defined by (5.2.3), and s in $\mathbb{N}$.*
For any δ in $(0,\delta_{k,q}]$ and any λ'_0 in $(0,\lambda_0)$ ($\delta_{k,q}$ and λ_0 defined in Proposition 5.3.6), there exists $C_1 = C_1(\lambda'_0,\delta) > 0$ and $R = R(\delta) > 0$ such that for any $t \geqslant 0$,

$$\forall n \in \mathbb{N}, \quad \mathrm{supp}\, T_n(t)h \subset K := B(0,R)$$

and

$$\forall s \geqslant 1, \quad \|T_1(t)h\|_{W^{s+1,1}_{x,v}(K)} \leqslant C_1\frac{e^{-\frac{\lambda'_0}{\varepsilon^2}t}}{\varepsilon^2 t}\|h\|_{W^{s+1,1}_v W^s_x(\langle v\rangle^k)}, \tag{5.3.13}$$

$$\forall s \geqslant 0, \quad \|T_2(t)h\|_{W^{s+1/2,1}_{x,v}(K)} \leqslant C_1\frac{e^{-\frac{\lambda'_0}{\varepsilon^2}t}}{\varepsilon^4}\|h\|_{W^{s,1}_{x,v}(\langle v\rangle^k)}. \tag{5.3.14}$$

Proof of Proposition 5.3.7

Most of the proof is an adaptation of [51] proof of Lemma 4.19 to keep track of the dependencies on ε. We will refer to it when we are using some of its computations.

Control of $T_1(t)h$: The x-derivatives commutes with $T_1(t)$ and therefore it is enough to consider h in $W^{s,1}_v W^{1,1}_x(\langle v\rangle^k)$, with $s \geqslant 1$, and to control $\|T_1(t)h\|_{W^{s+1,1}_v W^{1,1}_x(K)}$. This gives

$$\|T_1(t)h\|_{W^{s+1,1}_v W^{1,1}_x(K)} \leqslant \|T_1(t)h\|_{W^{s+1,1}_v L^1_x(K)} + \|\nabla_x T_1(t)h\|_{W^{s+1,1}_v L^1_x(K)}. \tag{5.3.15}$$

The first term is easily dealt with thanks to the estimate on $\mathcal{A}^{(\delta)}_\varepsilon$, Proposition 5.3.2, and the dissipativity property of $\mathcal{B}^{(\delta)}_\varepsilon$, Proposition 5.3.6,

$$\|T_1(t)h\|_{W^{s+1,1}_v L^1_x(K)} = \left\|\mathcal{A}^{(\delta)}_\varepsilon S_{\mathcal{B}^{(\delta)}_\varepsilon} h\right\|_{W^{s+1,1}_v L^1_x(K)} \leqslant \frac{C}{\varepsilon^2} e^{-\frac{\lambda_0}{\varepsilon^2}t} \|h\|_{L^1_v L^1_x(\langle v\rangle^k)}. \tag{5.3.16}$$

For the second term, define $f(t) = S_{\mathcal{B}^{(\delta)}_\varepsilon} h$ and

$$D_t = \varepsilon^{-1} t \nabla_x + \nabla_v. \tag{5.3.17}$$

By direct computations we have that

$$\varepsilon^{-1} t \nabla_x T_1(t)h = \mathcal{A}^{(\delta)}_\varepsilon (D_t f) - \left(\nabla_v \mathcal{A}^{(\delta)}_\varepsilon\right) f,$$

which leads to, by Proposition 5.3.2,

$$\varepsilon^{-1} t \|\nabla_x T_1(t)h\|_{W^{s+1,1}_v L^1_x(K)} \leqslant \frac{C}{\varepsilon^2}\left[\|D_t f\|_{L^1_{x,v}(\langle v\rangle^k)} + \|f\|_{L^1_{x,v}(\langle v\rangle^k)}\right]. \tag{5.3.18}$$

The dissipativity property of $\mathcal{B}^{(\delta)}_\varepsilon$, in particular (5.3.10) with $q=1$, yields

$$\frac{d}{dt}\|f\|_{L^1_{x,v}(\langle v\rangle^k)} \leqslant -\frac{\lambda_0}{\varepsilon^2}\|f\|_{L^1_{x,v}(\langle v\rangle^k \nu)}. \tag{5.3.19}$$

Direct computations yields

$$\partial_t (D_t f) = \mathcal{B}^{(\delta)}_\varepsilon (D_t f) + \frac{1}{\varepsilon^2}\mathcal{J}^{(\delta)} f,$$

where

$$\mathcal{J}^{(\delta)} = \nabla_v\left(\mathcal{B}^{(\delta)}_1(\cdot)\right) - \mathcal{B}^{(\delta)}_1(\nabla_v(\cdot)) \tag{5.3.20}$$

is independent of ε and satisfies (see [51] proof of Lemma 4.19) for all g in $L^1_v\left(\langle v\rangle^k \nu\right)$

$$\left\|\mathcal{J}^{(\delta)} g\right\|_{L^1_v(\langle v\rangle^k)} \leqslant C_\delta \|g\|_{L^1_v(\langle v\rangle^k \nu)}.$$

In the same way as proof of Proposition 5.3.6 we obtain

$$\frac{d}{dt}\left\|D_t f\right\|_{L^1_{x,v}\left(\langle v\rangle^k\right)} \leqslant -\frac{\lambda_0}{\varepsilon^2}\left\|D_t f\right\|_{L^1_{x,v}\left(\langle v\rangle^k \nu\right)} + \frac{C_\delta}{\varepsilon^2}\left\|f\right\|_{L^1_v\left(\langle v\rangle^k \nu\right)}. \tag{5.3.21}$$

We then consider λ_0' in $(0, \lambda_0)$ and define $\eta = (\lambda_0 - \lambda_0')/C_\delta$. We compute, with (5.3.19),

$$\frac{d}{dt}\left[e^{\frac{\lambda_0'}{\varepsilon^2}t}\left(\eta\left\|D_t f\right\|_{L^1_{x,v}\left(\langle v\rangle^k\right)} + \left\|f\right\|_{L^1_{x,v}\left(\langle v\rangle^k\right)}\right)\right] \leqslant 0,$$

and thus

$$\left\|D_t f\right\|_{L^1_{x,v}\left(\langle v\rangle^k\right)} + \left\|f\right\|_{L^1_{x,v}\left(\langle v\rangle^k\right)} \leqslant \eta^{-1} e^{-\frac{\lambda_0'}{\varepsilon^2}t}\left\|h\right\|_{W^{1,1}_v L^1_x\left(\langle v\rangle^k\right)}. \tag{5.3.22}$$

To conclude we plug (5.3.22) into (5.3.18) and we combine it with (5.3.16) into (5.3.15). This yields, because $s \geqslant 1$,

$$\left\|T_1(t)h\right\|_{W^{s+1,1}_v W^{1,1}_x(K)} \leqslant C\frac{e^{-\frac{\lambda_0'}{\varepsilon^2}t}}{\varepsilon^2 t}\left\|h\right\|_{W^{s,1}_v L^1_x\left(\langle v\rangle^k\right)},$$

which implies the expected result (5.3.13) because $T_1(t)$ commutes with x-derivatives.

Control of $T_2(t)h$: For $s \geqslant 0$ we can interpolate (for interpolation theory in Sobolev spaces see [13] Chapters 6) between (5.3.16) and (5.3.13) to get

$$\left\|T_1(t)h\right\|_{W^{s+1/2,1}_{x,v}(K)} \leqslant C\frac{e^{-\frac{\lambda_0'}{\varepsilon^2}t}}{\varepsilon^2\sqrt{t}}\left\|h\right\|_{W^{s,1}_v L^1_x\left(\langle v\rangle^k\right)}.$$

Then, we firstly use the inequality above and secondly (5.3.16) to obtain

$$\begin{aligned}
\left\|T_2(t)h\right\|_{W^{s+1/2,1}_{x,v}(K)} &\leqslant \int_0^t \left\|T_1(t-s)T_1(s)h\right\|_{W^{s+1/2,1}_{x,v}(K)}\,ds \\
&\leqslant \frac{C}{\varepsilon^4}e^{-\frac{\lambda_0' t}{\varepsilon^2}}\left(\int_0^t \frac{e^{-\frac{\lambda_0-\lambda_0'}{\varepsilon^2}s}}{\sqrt{t-s}}\,ds\right)\left\|h\right\|_{W^{s,1}_v L^1_x\left(\langle v\rangle^k\right)},
\end{aligned}$$

which is the expected result (5.3.14).

■

The aim is to link our space $L^q_v L^p_x\left(\langle v\rangle^k\right)$ to the space $H^s_{x,v}\left(\mu^{-1/2}\right)$. We thus state the following control on the iterated convolution in the case $p = 2$.

Proposition 5.3.8 *Consider $k > k_q^*$, defined by (5.2.3), and s in $\mathbb{N}$.*
For any δ in $(0, \delta_{k,q}]$ there exists $C_2 = C_2(\delta) > 0$ and $R = R(\delta) > 0$ such that for any $t \geqslant 0$,

$$\forall n \in \mathbb{N}, \quad \operatorname{supp} T_n(t)h \subset K := B(0, R)$$

and

$$\forall s \geqslant 0, \quad \|T_2(t)h\|_{H^{s+1/2}_{x,v}(K)} \leqslant \frac{C_T}{\varepsilon^{5/2}} \|h\|_{H^s_{x,v}(\langle v\rangle^k)}. \tag{5.3.23}$$

Proof of Proposition 5.3.8 Consider h in $W^{s,2}_{x,v}(\langle v\rangle^k)$, s in $\mathbb{N}$.

This Proposition is easier than when $p = 1$ because there exists velocity averaging lemmas in this framework, as discussed in [51] Remark 4.21. The x-derivative commutes with T_1 and therefore we suppose there is no derivative in space.

Define $f(t) = S_{\mathcal{B}^{(\delta)}_\varepsilon}(t)(h)$ so that f is solution to the kinetic equation

$$\partial_t f + \frac{1}{\varepsilon} v \cdot \nabla_x f = s_\varepsilon(t,x,v),$$

with $s_\varepsilon(t,x,v) = -\varepsilon^{-2}\nu f + \varepsilon^{-2}\mathcal{B}^{(\delta)}_{2,\varepsilon} f$.

Let j be a multi-index such that $|j| \leqslant s$. We apply ∂^j_0 to the latter equation, which gives

$$\partial_t\left(\partial^j_0 f\right) + \frac{1}{\varepsilon} v \cdot \nabla_x \left(\partial^j_0 f\right) = \partial^j_0 s_\varepsilon(t,x,v) + \frac{1}{\varepsilon}\sum_{|i|+|l|=|j|} a_{i,l}\partial^i_l f, \tag{5.3.24}$$

where $a_{i,j}$ are non-negative numbers.

A classical averaging lemma (see [20] Lemma 1 and [21] in which we emphasize the dependencies in ε) reads, for (5.3.24) with $\partial^j_0 f(0,x,v) = \partial^j_0 h(x,v)$, for all ψ in $\mathcal{D}\left(\mathbb{R}^d\right)$

$$\begin{aligned} &\left\| \int_{\mathbb{R}^d} \partial^j_0 f(t,x,v)\psi(v)\,dv \right\|_{L^2_t\left(H^{1/2}_x\right)} \\ &\quad \leqslant \frac{C}{\sqrt{\varepsilon}}\left(\left\|\partial^j_0 h(x,v)\right\|_{L^2_{x,v}} + \left\|\partial^j_0 s_\varepsilon\right\|_{L^2_{t,x,v}} + \frac{1}{\varepsilon}\left\| \sum_{|i|+|l|=|j|} a_{i,l}\partial^i_l f\right\|_{L^2_{t,x,v}}\right). \end{aligned} \tag{5.3.25}$$

We use [51], Lemmas 4.4 and 4.7, in order to bound the terms involving $\mathcal{B}^{(\delta)}_{2,\varepsilon} = \varepsilon^{-2}\mathcal{B}^{(\delta)}_{2,1}$ we have that

$$\|s_\varepsilon\|_{H^s_{x,v}(\langle v\rangle^k)} \leqslant \frac{1}{\varepsilon^2}\|s_1\|_{H^s_{x,v}(\langle v\rangle^k)} \leqslant \frac{C}{\varepsilon^2}\|f\|_{H^s_{x,v}(\langle v\rangle^k\nu)} \leqslant \frac{C}{\varepsilon^2}e^{-\frac{\lambda_0}{\varepsilon^2}t}\|h\|_{H^s_{x,v}(\langle v\rangle^k\nu)},$$

where the last inequality comes from the hypodissipativity properties of $S_{\mathcal{B}_\varepsilon}(t)$, see Proposition 5.3.6.

Using the dissipativity properties of $S_{\mathcal{B}_\varepsilon}(t)$ one more time we deduce that

$$\|T_1(t)h\|_{L^2_t\left(H^{s+1/2}_{x,v}(\langle v\rangle^k)\right)} \leqslant \frac{C}{\varepsilon^{5/2}}\|h\|_{H^s_{x,v}(\langle v\rangle^k\nu)}. \tag{5.3.26}$$

To conclude we notice that $\int_0^t T_1(t-s)T_1(s)\,ds$ is a continuous linear operator on the Hilbert space $H^{s+1/2}_{x,v}(K)$ and thus we can see it as an element of $H^{s+1/2}_{x,v}(K)$ by Riesz's representation theorem. Hence, thanks to Cauchy-Schwartz,

$$\begin{aligned}
\|T_2(t)h\|_{H^{s+1/2}_{x,v}(K)} &= \left\| \left(\int_0^t T_1(t-s)T_1(s)\,ds \right)(h) \right\|_{H^{s+1/2}_{x,v}(K)} \\
&\leqslant \|h\|_{H^s_{x,v}(\langle v\rangle^k \nu)} \int_0^t \|T_1(t-s)T_1(s)\|_{\mathscr{B}\left(H^s_{x,v}(\langle v\rangle^k\nu), H^{s+1/2}_{x,v}(K)\right)}\,ds \\
&\leqslant \|h\|_{H^s_{x,v}(\langle v\rangle^k \nu)} \left(\int_0^t \|T_1(t-s)\|^2_{\mathscr{B}\left(H^s_{x,v}(K), H^{s+1/2}_{x,v}(K)\right)}\,ds \right)^{1/2} \\
&\qquad \times \left(\int_0^t \|T_1(s)\|^2_{\mathscr{B}\left(H^s_{x,v}(\langle v\rangle^k\nu), H^s_{x,v}(K)\right)}\,ds \right)^{1/2} \\
&\leqslant \|h\|_{H^s_{x,v}(\langle v\rangle^k \nu)} \frac{C}{\varepsilon^{5/2}} \left(\int_0^t \frac{C_A}{\varepsilon^2} e^{-\frac{\lambda_0'}{\varepsilon^2}s}\,ds \right)^{1/2} \\
&\leqslant \frac{C}{\varepsilon^{5/2}} \|h\|_{H^s_{x,v}(\langle v\rangle^k \nu)},
\end{aligned}$$

where we used Proposition 5.3.2 and the fact that $S_{\mathcal{B}^{(\delta)}_\varepsilon}$ is a contraction semigroup on $H^s_{x,v}$ with spectral gap λ_0'/ε^2.

■

5.3.5 Proof of Theorem 5.2.1

As we explained it in Section 5.3.1, the proof of Theorem 5.2.1 is direct from the application of Theorem 5.3.1. This theorem is clearly applicable in our case and we emphasize it through the extreme case of no derivative in space or velocity variables.

Indeed, we consider s in $\mathbb{N}$ to be chosen big enough later. We define $\mathcal{E} = L^q_v L^p_x\left(\langle v\rangle^k\right)$ and $E = H^s_{x,v}\left(\mu^{-1/2}\right)$ and we have $E \subset \mathcal{E}$ for s big enough (dense with continuous embedding). Indeed, in the case $q \geqslant 2$ and $p \geqslant 2$, standard Sobolev embeddings (see [22] Section $IX.3.$) imply $E \subset L^q_v L^p_x\left(\mu^{-1/2}\right)$. In the case $p < 2$ we have, on the torus, $L^2_x \subset L^p_x$ and $H^s_x \subset L^2_x$ by the same Sobolev embeddings. Finally, in the case $q < 2$ we have that $L^2_v\left(\mu^{-1/2}\right) \subset L^q_v\left(\langle v\rangle^k\right)$ (it can be done by a mere Cauchy-Schwarz inequality) and the same Sobolev embeddings give $H^s_v\left(\mu^{-1/2}\right) \subset L^2_v\left(\mu^{-1/2}\right)$.

On the torus we have the following embedding: $L^p_x \subset L^1_x$. Thanks to Proposition 5.3.2 and Proposition 5.3.6 we obtain (same arguments as (5.3.16))

$$\|T_1(t)h\|_{\mathcal{E}} \leqslant C \left\| A^{(\delta)}_\varepsilon S_{\mathcal{B}^{(\delta)}_\varepsilon} h \right\|_{L^q_v L^1_x(K)} \leqslant \frac{C}{\varepsilon^2} e^{-\frac{\lambda_0}{\varepsilon^2}t} \|h\|_{L^1_v L^1_x(\langle v\rangle^k)}. \tag{5.3.27}$$

We therefore define $\mathcal{E}_2 = L^1_v L^1_x(\langle v\rangle^k)$.

Then we define by $\mathcal{E}_j = W^{(j-2)/2,1}_{x,v}(\langle v\rangle^k)$ for j from 2 to m with m big enough such that $W^{(m-1)/2,1}_{x,v}(\langle v\rangle^k) \subset L^2_{x,v}(\langle v\rangle^k)$. Then we denote $\mathcal{E}_j = H^{(j-m-1)/2}_{x,v}(\langle v\rangle^k)$ up $J-1$ where $H^{(J-m-2)/2}_{x,v}(\langle v\rangle^k) \subset E$.

Point $(A1)$ of Theorem 5.3.1 is satisfied thanks to [4] and [23] Theorem 2.1 (with the norm of Theorem 2.4), point $(A2)(i)$ by Proposition 5.3.6 and point $(A2)(ii)$ by Proposition 5.3.2. Finally, point $(A2)(iii)$ is given by (5.3.27) for $\mathcal{E}$ and $\mathcal{E}_1$, then by Proposition 5.3.7 (5.3.14) up to $\mathcal{E}_m$ and by Proposition 5.3.8 from $\mathcal{E}_m$ to $\mathcal{E}_J$ and E.

5.4 An *a priori* estimate for the full perturbed equation: proof of Theorem 5.2.3

In this section we work in $W^{\alpha,1}_v H^\beta_x\left(\langle v\rangle^k\right)$ or in $W^{\alpha,1}_v W^{\beta,1}_x\left(\langle v\rangle^k\right)$, with $\alpha \leqslant \beta$ on the full perturbed Boltzmann equation

$$\partial_t h = \mathcal{G}_\varepsilon(h) + \frac{1}{\varepsilon} Q(h,h).$$

5.4.1 Description of the problem and notations

When $\varepsilon = 1$, the linear part $\mathcal{G}_\varepsilon$ has the same order of magnitude than the bilinear term Q in the linearized Boltzmann equation (5.1.5). In this case, Theorem 5.2.1 suffices to obtain existence and exponential decay since the contraction property of the semigroup $S_{\mathcal{G}_1}$ controls the bilinear part for small initial data (see [51]).

In the general case, $S_{\mathcal{G}_\varepsilon}$ only generates a semigroup with a spectral gap of order 1, insufficient to control $\varepsilon^{-1}Q$. However, [56][23] (and Chapter 4) show that a careful study of $\varepsilon^{-1}Q$ compared to G_ε yields existence and exponential decay of solutions to (5.1.5) in $H^s_{x,v}\left(\mu^{-1/2}\right)$ for s large enough (see Theorem 5.4.7 for an adapted version of this result). Our strategy is to use the same kind of ideas as when we extended the semigroup properties from $H^\beta_{x,v}\left(\mu^{-1/2}\right)$ to $W^{\alpha,1}_v H^\beta_x\left(\langle v\rangle^k\right)$ and $W^{\alpha,1}_v W^{\beta,1}_x\left(\langle v\rangle^k\right)$ but including the bilinear term. Namely, we shall decompose the partial differential equation (5.1.5) into a system of partial differential equations from $W^{\alpha,1}_v H^\beta_x\left(\langle v\rangle^k\right)$ or $W^{\alpha,1}_v W^{\beta,1}_x\left(\langle v\rangle^k\right)$ to $H^\beta\left(\mu^{-1/2}\right)$ and use the perturbative estimates of [23] (that are given in Chapter 4).

As noticed in Remark 2.16 of [51], Theorem 5.3.1 extending the semigroup generated by $\mathcal{G}_\varepsilon$ in $H^s\left(\mu^{-1/2}\right)$ to $L^1_v L^\infty_x\left(\langle v\rangle^k\right)$ can be interpreted as a decomposition of

$$\partial_t f = \mathcal{G}_\varepsilon f,$$

into a system of partial differential equations, involving operators $\mathcal{G}_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon$ (defined in Section 5.3.2), with $f = f^1 + \cdots + f^J$ satisfying

- f^1 is in $L^1_v L^\infty_x \left(\langle v\rangle^k\right)$ and $f^1_{in} = f_{in}$ in $Ker(\mathcal{G}_\varepsilon)^\perp$,
- for all $2 \leqslant j \leqslant J-1$, f^j is in $\mathcal{E}_j$ and $f^j_{in} = 0$,
- f^J is in $H^s\left(\mu^{-1/2}\right)$, $f^J_{in} = 0$ and in that space we can use the contraction property of $S_{\mathcal{G}_\varepsilon}$.

We will decompose the linearized Boltzmann equation in a similar way than the one explained above. We shall define a sequence of spaces $(\mathcal{E}_j)_{1\leqslant j\leqslant J}$. In each space $\mathcal{E}_j$, $1 \leqslant j \leqslant J-1$, a piece of the bilinear term, of order ε^{-1}, will be added and controlled by the dissipativity property of $\mathcal{B}^{(\delta)}_\varepsilon$, of order ε^{-2}. Contrary to the study in the linear case, the bilinear operator generates terms involving functions in all the spaces $\mathcal{E}_j$ which have to be compared and controlled. This imposes to construct $(\mathcal{E}_j)_{1\leqslant j\leqslant J}$ as a nested sequence.

The difficult part of the linear operator, namely $\mathcal{A}^{(\delta)}_\varepsilon$, enjoys a regularising effect and could therefore be treated in more regular spaces. Of course, our decomposition will be much easier since we solely want to go from an exponential weight into a polynomial weight Sobolev spaces, without losing any derivatives in x or v.

In order to shorten notations we define, for $p = 1, 2$ and k to be defined later,

$$\mathcal{E}^p = W^{\alpha,1}_v W^{\beta,p}_x \left(\langle v\rangle^k\right) \quad \text{and} \quad E = H^\beta_{x,v}\left(\mu^{-1/2}\right). \tag{5.4.1}$$

We take h_{in} in $\mathcal{E}^p$ and we decompose the partial differential equation,

$$\partial_t h = \mathcal{G}_\varepsilon(h) + \frac{1}{\varepsilon}Q(h,h) = \mathcal{A}^{(\delta)}_\varepsilon(h) + \mathcal{B}^{(\delta)}_\varepsilon(h) + \frac{1}{\varepsilon}Q(h,h)$$

into an equivalent system of partial differential equations for the following decomposition

$$h(t,x,v) = h^0(t,x,v) + h^1(t,x,v), \tag{5.4.2}$$

with

1. In $\mathcal{E}^p$, $h^0_{t=0} = h_{in}$ and

$$\partial_t h^0 = \mathcal{B}^{(\delta)}_\varepsilon(h^0) + \frac{1}{\varepsilon}Q(h^0,h^0) + \frac{2}{\varepsilon}Q\left(h^0,h^1\right), \tag{5.4.3}$$

2. In E, $h^1_{t=0} = 0$ and

$$\partial_t h^1 = \mathcal{G}_\varepsilon(h^1) + \frac{1}{\varepsilon}Q(h^1,h^1) + \mathcal{A}^{(\delta)}_\varepsilon(h^0). \tag{5.4.4}$$

The aim of this Section is to establish the following estimate of solutions to the system (5.4.3) − (5.4.4).

Theorem 5.4.1 *Let $p = 1$ or $p = 2$. There exist β_0 in $\mathbb{N}$ and ε_d in $(0, 1]$ depending on d and the kernel of the Boltzmann operator such that:*
For all $\beta \geqslant \beta_0$, for any δ in $(0, \delta_{k,1}]$ and any λ_0' in $(0, \lambda_0)$ ($\delta_{k,1}$ and λ_0 defined in Proposition 5.3.6) there exist C_β, $\eta_\beta > 0$ such that for any $0 < \varepsilon \leqslant \varepsilon_d$ and h_{in} in $\mathcal{E}^p$,
if

(i) $\|h_{in}\|_{\mathcal{E}^p} \leqslant \eta_\beta$,

(ii) (h^0, h^1) *is solution to the system* $(5.4.3) - (5.4.4)$,

then

$$\left\|h^0 + h^1\right\|_{\mathcal{E}^p} \leqslant C_\beta \left\|h_{in}\right\|_{\mathcal{E}^p} e^{-\lambda_0' t}.$$

The constants C_β and η_β are constructive and depends only on β, d, δ, λ_0' and the kernel of the Boltzmann operator.

Remark 5.4.2 (Link with Theorem 5.2.3) *The existence and uniqueness for the perturbed Boltzmann equation (5.1.5) in $\mathcal{E}^p$ has been proved for $\varepsilon = 1$, that is equivalent of ε fixed with constant depending on it, in [51] Theorems* 5.3 *and* 5.5 *respectively. The constants, as well as the smallness assumption on the initial data, in the theorem above are independent of ε and therefore this a priori result combined with existence and uniqueness developed in [51] and in [23] (for existence and uniqueness in E, see statements in Chapter 4) implies the existence and uniqueness independently of ε which is Theorem 5.2.3.*

The next subsections deal with the estimates one can get for solutions to the system $(5.4.3) - (5.4.4)$. We study each of them independently and the *a priori* exponential decay will be a straightforward application of these results together with a maximum principle argument.

Section 5.4.2 focuses on the *a priori* study of the equation in $\mathcal{E}^p$. Section 5.4.3 deals with (5.4.4) in E. Finally, Section 5.4.4 gathers the previous results to prove Theorem 5.4.1.

5.4.2 Study of equation (5.4.3) in $\mathcal{E}$

In this section we prove the following general proposition about the equation taking place in $\mathcal{E}^p = W_v^{\alpha,1} W_x^{\beta,p}\left(\langle v \rangle^k\right)$, for $p = 1$ or $p = 2$. We define the shorthand notation

$$\mathcal{E}^p_\nu = W_v^{\alpha,1} W_x^{\beta,p}\left(\langle v \rangle^k \nu\right).$$

Proposition 5.4.3 *Let $p = 1$ or $p = 2$ and $0 < \varepsilon \leqslant 1$. Let $k > k_1^* = 2$, $\beta > 2d/p$. Let h_{in} be in $\mathcal{E}^p$ and h^1 in $\mathcal{E}^p_\nu$.*
For any δ in $(0, \delta_{k,1}]$ and any λ_0' in $(0, \lambda_0)$ ($\delta_{k,1}$ and λ_0 defined in Proposition 5.3.6) there exist $\eta_0 > 0$ such that
if

(i) $\|h_{in}\|_{\mathcal{E}^p} \leqslant \eta_0\,, \quad \|h^1\|_{\mathcal{E}^p_\nu} \leqslant \eta_0,$

(ii) h^0 *satisfies* $h^0_{t=0} = h_{in}$ *and is solution to*

$$\partial_t h^0 = \mathcal{B}_\varepsilon^{(\delta)}(h^0) + \frac{1}{\varepsilon}Q(h^0, h^0) + \frac{2}{\varepsilon}Q\left(h^0, h^1\right),$$

then

$$\|h^0\|_{\mathcal{E}^p} \leqslant e^{-\frac{\lambda_0'}{\varepsilon^2}t}\,\|h_{in}\|_{\mathcal{E}^p}\,.$$

The constant η_0 is constructive and depends only on δ, λ_0' and the kernel of the Boltzmann operator.

We need to control the bilinear term Q, which is given by the following lemma.

Lemma 5.4.4 *For all $p = 1, 2$ and α, β in $\mathbb{N}$ such that $\beta > 2d/p$, there exists $C_{\beta,p} > 0$ such that all f and g*

$$\|Q(f, g)\|_{\mathcal{E}^p} \leqslant C_{\beta,p}\left(\|g\|_{\mathcal{E}^p_\nu}\|f\|_{\mathcal{E}^p} + \|g\|_{\mathcal{E}^p}\|f\|_{\mathcal{E}^p_\nu}\right).$$

This lemma has been proved in Lemma 5.16 in [51], which is adapted from interpolation results in [3] or duality arguments as in [84] Theorem 2.1.

Proof of Proposition 5.4.3

Consider δ in $(0, \delta_{k,1}]$ and λ_0' in $(0, \lambda_0)$. Take $p = 1$ or $p = 2$ and $\beta > 2d/p$.

We have that

$$\partial_t h^0 = \mathcal{B}_\varepsilon^{(\delta)}(h^0) + \frac{1}{\varepsilon}Q(h^0, h^0) + \frac{2}{\varepsilon}Q\left(h^0, h^1\right).$$

Thanks to the dissipativity of property of $\mathcal{B}_\varepsilon^{(\delta)}$, more precisely the proof of Lemma 5.3.6, we have

$$\begin{aligned}\frac{d}{dt}\|h^0\|_{\mathcal{E}^p} &\leqslant -\frac{\lambda_0}{\varepsilon^2\nu_0}\|h^0\|_{\mathcal{E}^p_\nu} + \frac{1}{\varepsilon}\left|\langle Q(h^0, h^0) + 2Q(h^0, h^1), h^0\rangle_{\mathcal{E}^p}\right| \\ &\leqslant -\frac{\lambda_0}{\varepsilon^2\nu_0}\|h^0\|_{\mathcal{E}^p_\nu} + \frac{1}{\varepsilon}\left\|Q(h^0, h^0) + 2Q(h^0, h^1)\right\|_{\mathcal{E}^p},\end{aligned}$$

where we used the scalar product notation to refer to the product operator appearing in $W_v^{\alpha,1}W_x^{\beta,p}$ when one differentiates $\|h\|_{W_v^{\alpha,1}W_x^{\beta,p}(\langle v\rangle^k)}$ (of the same form as (5.3.9)). For the second inequality we used Hölder inequality between L_x^p and $L_x^{p/(p-1)}$ inside the product operator:

$$\int_{\mathbb{T}^d} \operatorname{sgn}(h^0)\,|h^0|^{p-1}\,F(h^0)\,dx \leqslant \|h^0\|_{L_x^p}^{p-1}\,\|F(h^0)\|_{L_x^p}\,.$$

Then estimating Q using Lemma 5.4.4 yields

$$\frac{d}{dt}\,\|h^0\|_{\mathcal{E}^p} \leqslant -\frac{1}{\varepsilon^2}\left[\frac{\lambda_0}{\nu_0} - 2\varepsilon C_{\beta,p}\left(\|h^0\|_{\mathcal{E}^p} + \frac{2}{\nu_0}\,\|h^1\|_{\mathcal{E}_\nu^p}\right)\right]\|h^0\|_{\mathcal{E}_\nu^p}\,, \tag{5.4.5}$$

we recall $\nu_0 = \inf\limits_{v\in\mathbb{R}^d}(\nu(v)) > 0$.

Therefore, if

$$\|h^1\|_{\mathcal{E}_\nu^p} \leqslant \varepsilon^{-1}\frac{(\lambda_0-\lambda_0')}{8C_{\beta,p}} \quad \text{and} \quad \|h_{t=0}\|_{\mathcal{E}^p} \leqslant \varepsilon^{-1}\frac{(\lambda_0-\lambda_0')}{4\nu_0 C_{\beta,p}},$$

then $\|h^0\|_{\mathcal{E}^p}$ is always decreasing in time with

$$\frac{d}{dt}\,\|h^0\|_{\mathcal{E}^p} \leqslant -\frac{\lambda_0'}{\varepsilon^2\nu_0}\,\|h^0\|_{\mathcal{E}_\nu^p}\,,$$

which hence yields the expected exponential decay by Grönwall Lemma.

■

5.4.3 Study of equations (5.4.4) in E

In the space $E = H_{x,v}^\beta\left(\mu^{-1/2}\right)$, solutions to the perturbed Boltzmann equation enjoy an exponential decay. More precisely, [23] derived a precise Grönwall that we will now use to obtain estimates on the solution h^1. We will use the following shorthand notation

$$E_\nu = H_{x,v}^\beta\left(\mu^{-1/2}\nu^{1/2}\right)$$

In this section we use the previous theorem to obtain exponential decay of h^1 in E. This result is stated in the following proposition, where C_t^0 denotes the space of time-continuous functions.

Proposition 5.4.5 *Let $p=1$ or $p=2$, $0<\varepsilon\leqslant\varepsilon^d\leqslant 1$, $\beta\geqslant s_0$ and $\alpha\leqslant\beta$ (ε_d and s_0 being constructive constants that will be defined in Theorem 5.4.7).*
Let h_{in} be in $\mathcal{E}^p$ and h^0 in $C_t^0\mathcal{E}^p$.
For any δ in $(0,\delta_{k,1}]$ and any λ_0' in $(0,\lambda_0)$ ($\delta_{k,1}$ and λ_0 defined in Proposition 5.3.6) there

exist $\eta_1, C_1 > 0$ *such that*
if

(i) $\|h_{in}\|_{\mathcal{E}^p} \leqslant \eta_1$,

(ii) there exists $C_0 > 0$ *such that* $\|h^0\|_{\mathcal{E}^p} \leqslant C_0 e^{-\frac{\lambda_0+\lambda_0'}{2\varepsilon^2}t}\|h_{in}\|_{\mathcal{E}^p}$,

(iii) h^1 *satisfies* $h^1_{t=0} = 0$ *and is solution to*

$$\partial_t h^1 = \mathcal{G}_\varepsilon(h^1) + \frac{1}{\varepsilon}Q(h^1, h^1) + \mathcal{A}_\varepsilon^{(\delta)}(h^0)$$

then

$$\|h^1\|_E \leqslant C_1 e^{-\lambda_0' t}\|h_{in}\|_{\mathcal{E}^p}$$

The constants C_1 *and* η_1 *are constructive and depends only on* δ, λ_0' *and the kernel of the Boltzmann operator.*

In order to prove Proposition 5.4.5 we need a new control on the bilinear term.

For any operator $F : E \times E \longrightarrow E$, we will say that F satisfies the property (H) if the following holds.

Property (H)**:**

1. for all g^1, g^2 in E we have $\pi_L\left(F(g^1, g^2)\right) = 0$, where π_L is the orthogonal projection on $\mathrm{Ker}\,(L)$ in $L^2_v\left(\mu^{-1/2}\right)$ (see (5.2.1)),
2. for all $s' > 0$ there exists $\mathcal{F}_F^{s'} : E \times E \longrightarrow \mathbb{R}^+$ such that for all multi-indexes j and l such that $|j| + |l| \leqslant s'$,
$$\left|\langle \partial_l^j F(g^1, g^2), g^3\rangle_{L^2_{x,v}\left(\mu^{-1/2}\right)}\right| \leqslant \mathcal{F}_F^{s'}(g^1, g^2)\,\|g^3\|_{L^2_{x,v}\left(\mu^{-1/2}\nu^{1/2}\right)},$$
with $\mathcal{F}_F^{s'} \leqslant \mathcal{F}_F^{s'+1}$.

Lemma 5.4.6 *The Boltzmann linear operator* Q *satisfies the property* (H) *with*

$$\forall s > d,\ \exists C_s > 0, \quad \mathcal{F}_Q^s(g, h) \leqslant C_s\left[\|f\|_E\|g\|_{E_\nu} + \|f\|_{E_\nu}\|g\|_E\right].$$

The latter control on the bilinear part is from [23] Appendix $A.2$ (see Chapter 4 Appendix $4.A$).

Proof of Proposition 5.4.5 We state below the estimate derived in [23] (note that this is a version of [23] Theorem 2.4 extended by estimates proved in [23] Propositions 2.2 and 7.1).

Theorem 5.4.7 *There exist $0 < \varepsilon_d \leqslant 1$ and s_0 in $\mathbb{N}$ such that for any $s \geqslant s_0$ and any λ_0'' in $(0, \lambda_0)$ there exists δ_s, $C_s > 0$ such that,*

- *for any h_{in} in $H^s_{x,v}\left(\mu^{-1/2}\right)$ with*
$$\|h_{in}\|_{H^s_{x,v}\left(\mu^{-1/2}\right)} \leqslant \delta_s,$$
- *for any operator $F : H^s_{x,v}\left(\mu^{-1/2}\right) \times H^s_{x,v}\left(\mu^{-1/2}\right) \longrightarrow H^s_{x,v}\left(\mu^{-1/2}\right)$ satisfying the property* (H)*;*

Then for all $0 < \varepsilon \leqslant \varepsilon_d$ and for all g^1, g^2 in $H^s_{x,v}\left(\mu^{-1/2}\right)$, if h is a solution to

$$\begin{cases} \partial_t h = G_\varepsilon(h) + \dfrac{1}{\varepsilon} F(g^1, g^2) \\ h_{t=0} = h_{in}, \end{cases}$$

and h is in $\operatorname{Ker}(G_\varepsilon)$ for all time, then

$$\forall t \in \mathbb{R}^+, \quad \frac{d}{dt}\|h\|^2_{H^s_{x,v}\left(\mu^{-1/2}\right)} \leqslant -\frac{2\lambda_0''}{\nu_0^2}\|h\|^2_{H^s_{x,v}\left(\mu^{-1/2}\nu\right)} + C_s\left(\mathcal{F}^s_F(g^1, g^2)\right)^2.$$

Now, let λ'' be in $(0, \lambda_0)$, $s \geqslant s_0$ and $0 < \varepsilon \leqslant \varepsilon_d$.
The proof of Proposition 5.4.5 will be done in two steps. First we study the projection of h^1 onto $\operatorname{Ker}(G_\varepsilon)$ and then its orthogonal part.

Estimate on the projection part. We have that, see the decomposition (5.4.2), that $h^1 = h - h^0$ with h solution to the perturbed Boltzmann equation and thus satisfying $\Pi_{\mathcal{G}}(h) = 0$. We therefore have that

$$\Pi_{\mathcal{G}}(h^1) = -\Pi_{\mathcal{G}}(h^0).$$

Moreover, Theorem 5.2.1 tells us that $\Pi_{\mathcal{G}}$ and Π_G coincide on E and thus

$$\Pi_G(h^1) = -\Pi_{\mathcal{G}}(h^0),$$

and assumption (ii) together with the shape of $\Pi_{\mathcal{G}}$ (see (5.2.4)), there exists a constant $C_\Pi > 0$, depending only on the dimension d and s and the constant C_0, such that

$$\left\|\Pi_G(h^1)\right\|_{E_\nu} \leqslant C_\Pi e^{-\frac{\lambda_0+\lambda_0'}{2\varepsilon^2}t}\|h_{in}\|_{\mathcal{E}^p}. \tag{5.4.6}$$

Estimate on the orthogonal part. Applying $\Pi_G^\perp = \mathrm{Id} - \Pi_G$, the orthogonal projection onto $(\mathrm{Ker}\,(G_\varepsilon))^\perp$ in $L^2_{x,v}\left(\mu^{-1/2}\right)$, to the differential equation satisfied by h^1 yields

$$\begin{aligned}\partial_t\left(\Pi_G^\perp(h^1)\right) &= G_\varepsilon(h^1) + \Pi_G^\perp\left(\frac{1}{\varepsilon}Q(h^1,h^1) + \mathcal{A}_\varepsilon^{(\delta)}(h^0)\right)\\ &= G_\varepsilon\left(\Pi_G^\perp(h^1)\right) + \Pi_G^\perp\left(\frac{1}{\varepsilon}Q(h^1,h^1) + \mathcal{A}_\varepsilon^{(\delta)}(h^0)\right). \end{aligned} \tag{5.4.7}$$

Moreover, we have by definition of Π_G and π_L (see (5.2.4) and (5.2.1)) that

$$(\pi_L(h) = 0) \implies (\Pi_G(h) = 0)$$

and therefore

$$\Pi_G^\perp\left(Q(h^1,h^1)\right) = Q(h^1,h^1),$$

since Q satisfies property (H).1. by Lemma 5.4.6. Plugging the latter equality into (5.4.7) gives

$$\partial_t\left(\Pi_G^\perp(h^1)\right) = G_\varepsilon\left(\Pi_G^\perp(h^1)\right) + \frac{1}{\varepsilon}Q(h^1,h^1) + \Pi_G^\perp\left(\mathcal{A}_\varepsilon^{(\delta)}(h^0)\right).$$

By definition, $\Pi_G^\perp(h^1)$ is in $(\mathrm{Ker}\,(G_\varepsilon))^\perp$ for all time and thanks to the control on the Boltzmann operator Q in E (Lemma 5.4.6), we are able to use Theorem 5.4.7 with $\lambda_0 > \lambda_0'$ to which we have to add the source term $\Pi_G^\perp\left(\mathcal{A}_\varepsilon^{(\delta)}(h^0)\right)$. This yields the following differential inequality, where we denote by C any positive constant independent of ε,

$$\begin{aligned}&\frac{d}{dt}\left\|\Pi_G^\perp(h^1)\right\|_E^2 \\ &\quad\leqslant -\frac{2\lambda_0''}{\nu_0^2}\left\|\Pi_G^\perp(h^1)\right\|_{E_\nu}^2 + C\left(\mathcal{F}_Q^s(h^1,h^1)\right)^2 + \left|\langle \Pi_G^\perp\left(\mathcal{A}_\varepsilon^{(\delta)}(h^0)\right), \Pi_G^\perp(h^1)\rangle_E\right| \\ &\quad\leqslant -\frac{2\lambda_0''}{\nu_0^2}\left\|\Pi_G^\perp(h^1)\right\|_{E_\nu}^2 + C\left\|h^1\right\|_E^2\left\|h^1\right\|_{E_\nu}^2 + \left\|\Pi_G^\perp\left(\mathcal{A}_\varepsilon^{(\delta)}(h^0)\right)\right\|_E\left\|\Pi_G^\perp(h^1)\right\|_E,\end{aligned} \tag{5.4.8}$$

where we used a Cauchy-Schwarz inequality on the last term on the right-hand side.

Then we can decompose $h^1 = \Pi_G(h^1) + \Pi_G^\perp(h^1)$ to get first

$$\begin{aligned}\left\|h^1\right\|_E^2\left\|h^1\right\|_{E_\nu}^2 \leqslant& 4\left\|\Pi_G^\perp(h^1)\right\|_E^2\left\|\Pi_G^\perp(h^1)\right\|_{E_\nu}^2 + \frac{8}{\nu_0^2}\left\|\Pi_G(h^1)\right\|_{E_\nu}^2\left\|\Pi_G^\perp(h^1)\right\|_{E_\nu}^2\\ &+\frac{4}{\nu_0^2}\left\|\Pi_G(h^1)\right\|_{E_\nu}^4,\end{aligned}$$

into which we can plug the control on $\left\|\Pi_G(h^1)\right\|_{E_\nu}^2$ we derived in (5.4.6) to obtain, with

$\|h_{in}\| \leqslant \eta_1$,

$$\left\|h^1\right\|_E^2 \left\|h^1\right\|_{E_\nu}^2 \leqslant 4 \left\|\Pi_G^\perp(h^1)\right\|_E^2 \left\|\Pi_G^\perp(h^1)\right\|_{E_\nu}^2 + C\eta_1^2 \left\|\Pi_G^\perp(h^1)\right\|_{E_\nu}^2 + Ce^{-\frac{2(\lambda_0+\lambda_0')}{\varepsilon^2}t} \left\|h_{in}\right\|_{\mathcal{E}^p}^4. \tag{5.4.9}$$

And finally, this inequality together with assumption (ii) gives the existence of a constant $C_A > 0$ such that

$$\left\|\Pi_G^\perp\left(\mathcal{A}_\varepsilon^{(\delta)}(h^0)\right)\right\|_E \left\|\Pi_G^\perp(h^1)\right\|_E \leqslant \frac{C_A}{\varepsilon^2} \left\|h_{in}\right\|_{\mathcal{E}^p} e^{-\frac{\lambda_0+\lambda_0'}{2\varepsilon^2}t} \left\|\Pi_G^\perp(h^1)\right\|_E. \tag{5.4.10}$$

We plug (5.4.9) and (5.4.10) into (5.4.8) and obtain, with C and C' being positive constants independent of ε,

$$\begin{aligned}\frac{d}{dt}\left\|\Pi_G^\perp(h^1)\right\|_E^2 \leqslant &- \left[\frac{2\lambda_0''}{\nu_0^2} - \left(4\left\|\Pi_G^\perp(h^1)\right\|_E^2 + C\eta_1^2\right)\right]\left\|\Pi_G^\perp(h^1)\right\|_{E_\nu}^2 \\ &+ C'\left(\left\|h_{in}\right\|_{\mathcal{E}^p}^4 + \frac{1}{\varepsilon^2}\left\|h_{in}\right\|_{\mathcal{E}^p}\left\|\Pi_G^\perp(h^1)\right\|_E\right) e^{-\frac{\lambda_0+\lambda_0'}{2\varepsilon^2}t}.\end{aligned}$$

We now choose η_1 sufficiently small so that

$$C\eta_1^2 \leqslant \frac{\lambda_0'' - \lambda_0'}{\nu_0^2},$$

which in turns implies

$$\begin{aligned}\frac{d}{dt}\left\|\Pi_G^\perp(h^1)\right\|_E^2 \leqslant &- \left[\frac{\lambda_0''+\lambda_0'}{\nu_0^2} - 4\left\|\Pi_G^\perp(h^1)\right\|_E^2\right]\left\|\Pi_G^\perp(h^1)\right\|_{E_\nu}^2 \\ &+ C'\left(\left\|h_{in}\right\|_{\mathcal{E}^p}^4 + \frac{1}{\varepsilon^2}\left\|h_{in}\right\|_{\mathcal{E}^p}\left\|\Pi_G^\perp(h^1)\right\|_E\right) e^{-\frac{\lambda_0+\lambda_0'}{2\varepsilon^2}t}.\end{aligned} \tag{5.4.11}$$

We define

$$\eta_* = \frac{\lambda_0'' - \lambda_0'}{4\nu_0^2}.$$

We have that $h^1_{t=0} = 0$ so we can define

$$t_0 = \sup\{t > 0, \quad \left\|\Pi_G^\perp(h^1)\right\|_E^2 < \eta_*\}.$$

Suppose that $t_0 < +\infty$, we therefore have for all t in $[0, t_0]$

$$\frac{d}{dt}\left\|\Pi_G^\perp(h^1)\right\|_E^2 \leqslant -\frac{2\lambda_0'}{\nu_0^2}\left\|\Pi_G^\perp(h^1)\right\|_{E_\nu}^2 + C'\left(\left\|h_{in}\right\|_{\mathcal{E}^p}^4 + \frac{\sqrt{\eta_*}}{\varepsilon^2}\left\|h_{in}\right\|_{\mathcal{E}^p}\right) e^{-\frac{\lambda_0+\lambda_0'}{2\varepsilon^2}t},$$

which gives

$$\forall t \in [0,t_0], \quad \frac{d}{dt}\left\|\Pi_G^\perp(h^1)\right\|_E^2 \leqslant -2\lambda_0' \left\|\Pi_G^\perp(h^1)\right\|_E^2 + C'\left(\|h_{in}\|_{\mathcal{E}^p}^4 + \frac{\sqrt{\eta_*}}{\varepsilon^2}\|h_{in}\|_{\mathcal{E}^p}\right)e^{-\frac{\lambda_0+\lambda_0'}{2\varepsilon^2}t},$$

and by Gronwall lemma with $\Pi_G^\perp(h^1)_{(t=0)} = 0$,

$$\begin{aligned}\forall t \in [0,t_0], \quad \left\|\Pi_G^\perp(h^1)\right\|_E^2 &\leqslant C'\left(\|h_{in}\|_{\mathcal{E}^p}^4 + \frac{\sqrt{\eta_*}}{\varepsilon^2}\|h_{in}\|_{\mathcal{E}^p}\right)\left(\int_0^t e^{-\frac{\lambda_0+\lambda_0'}{2\varepsilon^2}s}\, e^{2\lambda_0's}\,ds\right)e^{-2\lambda_0't} \\ &\leqslant C'\left(\varepsilon^2\|h_{in}\|_{\mathcal{E}^p}^4 + \sqrt{\eta_*}\,\|h_{in}\|_{\mathcal{E}^p}\right)\left(\int_0^{+\infty} e^{-\frac{\lambda_0-\lambda_0'}{2}u}\,du\right)e^{-2\lambda_0't},\end{aligned}$$

where we used the change of variable $u = \varepsilon^{-2}s$ and we considered $\varepsilon \leqslant 1/4$ (which only amounts to decreasing ε_d).

Hence, there exists $K > 0$ independent of ε such that

$$\forall t \in [0,t_0], \quad \left\|\Pi_G^\perp(h^1)\right\|_E^2 \leqslant K(\eta_1^4 + \eta_1\sqrt{\eta_*}).$$

If we thus chose η_1 sufficiently small such that $(\eta_1^4 + \eta_1\sqrt{\eta_*})K < \eta_*/2$ we reach a contradiction when t goes to t_0 since $\left\|\Pi_G^\perp(h^1)\right\|_E^2(t_0) \geqslant \eta_*$. Therefore, choosing η_1 small enough independently on ε implies first that $t_0 = +\infty$ and second that

$$\forall t \in [0,+\infty), \quad \left\|\Pi_G^\perp(h^1)\right\|_E^2 \leqslant C\,\|h_{in}\|_{\mathcal{E}^p}^2\, e^{-2\lambda_0't}. \tag{5.4.12}$$

End of the proof. By just decomposing h^1 into its projection and orthogonal part and using the estimates (5.4.6) and (5.4.12) gives the expected exponential decay for h^1 in E.
∎

5.4.4 Proof of Theorem 5.4.1

Let $p = 1$ or $p = 2$, λ'' be in $(0,\lambda_0)$, $\beta \geqslant \beta_0 = s_0$ and $0 < \varepsilon \leqslant \varepsilon_d$. All the constants used in this section are the ones constructed in Proposition 5.4.3 with $(\lambda_0+\lambda_0')/2$ and Proposition 5.4.5 with λ_0'.

E is continuously embedded in $\mathcal{E}_\nu^p$ because $L_v^2\left(\mu^{-1/2}\right) \subset L_v^2\left(\langle v\rangle^k\right)$ (mere Cauchy-Schwarz inequality) and $L_x^2 \subset L_x^1$ because $\mathbb{T}^d$ is bounded. Hence, there exists $C_{E,\mathcal{E}} > 0$ such that

$$\frac{1}{\nu_0}\|\cdot\|_{\mathcal{E}^p} \leqslant \|\cdot\|_{\mathcal{E}_\nu^p} \leqslant C_{E,\mathcal{E}}\|\cdot\|_E\,. \tag{5.4.13}$$

We define

$$\eta = \min\left(\eta_0, \eta_1, \frac{\eta_0}{2C_{E,\mathcal{E}}C_1}\right),$$

and we assume $\|h_{in}\|_{\mathcal{E}^p} \leqslant \eta$. Since $h^1_{t=0} = 0$ we also define

$$t_0 = \sup\{t > 0, \quad \left\|h^1\right\|_{\mathcal{E}^p_\nu} < \eta_0\}.$$

Suppose that $t_0 < +\infty$. Then, thanks to Proposition 5.4.3 we have that

$$\forall t \in [0, t_0], \quad \left\|h^0\right\|_{\mathcal{E}^p} \leqslant \|h_{in}\|_{\mathcal{E}^p} e^{-\frac{\lambda_0+\lambda_0'}{2\varepsilon^2}t}.$$

We can thus apply Proposition 5.4.5 and get

$$\forall t \in [0, t_0], \quad \left\|h^1\right\|_E \leqslant C_1 \|h_{in}\|_{\mathcal{E}^p} e^{-\lambda_0' t} \leqslant C_1 \eta \leqslant \frac{\eta_0}{2C_{E,\mathcal{E}}},$$

which is in contradiction with the definition of t_0 thanks to (5.4.13). Therefore $t_0 = +\infty$ and we have the expected exponential decay stated in Theorem 5.4.1 for all time.

Part III

A QUANTIC VERSION OF BOLTZMANN EQUATION FOR GASES OF BOSONS AND FERMIONS

Chapter 6

The homogeneous Boltzmann-Nordheim equation for bosons: local existence and uniqueness

The Boltzmann-Nordheim equation is a modification, based on physical considerations, of the Boltzmann equation that describes the dynamics of the distribution of particles in a quantum gas composed by bosons or fermions. We investigate the homogeneous Boltzmann-Nordheim equation for the particular case of bosons. We solve existence and uniqueness locally in time for any initial data that are bounded and with finite mass and energy, without any assumption of isotropy. We also show that moments of all order appear immediately for such solutions. Finally, we discuss the phenomenon of Bose-Einstein condensate in a gas of bosons at low temperature and the recent results associated to it.

Contents

6.1 Introduction

This chapter deals with the dynamics of the distribution of particles in time and velocity, $f(t,v) \geqslant 0$ in $\mathbb{R}^+ \times \mathbb{R}^d$ ($d \geqslant 2$), for a dilute homogeneous quantum gas of bosons. In greater generality, the dynamics of particles undergoing binary collisions in quantum statistics is given by the Boltzmann-Nordheim equation

$$\begin{aligned}
\partial_t f &= Q(f)\,, \text{ on } \mathbb{R}^+ \times \mathbb{R}^d \\
&= \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} B\,(v, v_*, \theta)\left[f'(1+\alpha f)f'_*(1+\alpha f_*) - f(1+\alpha f')f_*(1+\alpha f'_*)\right] dv_* d\sigma,
\end{aligned}$$

where f', f_*, f'_* and f are the values taken by f at v', v_*, v'_* and v respectively and B is the collision kernel which encodes the physical properties of the collision process. Define:

$$\left\{\begin{aligned}
v' &= \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma \\
v'_* &= \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma
\end{aligned}\right., \text{ and } \quad \cos\theta = \left\langle \frac{v-v_*}{|v-v_*|}, \sigma \right\rangle.$$

This equation has been derived by Nordheim (see [89]) using quantum statistics considerations. Basically, when $\alpha = 0$ one recovers exactly the Boltzmann equation which rules the dynamics of particles in a dilute gas when only elastic binary collisions are taken into account. The main difference with the Boltzmann-Nordheim equation is that in quantum statistics the probability of two particles colliding not only depends on the number of particles undergoing the collision but also the number of particles already in the final state the latter collision yields. In the case of fermions ($\alpha = -1$), this probability decreases and in the case of bosons it increases ($\alpha = 1$).

The collision kernel $B \geqslant 0$ contains all the information about the interaction between two particles with velocities v and v_*, and is determined by physics. We can mention here that one can derive this type of equations from Newton mechanics (coupled with quantum effects in the case of the Boltzmann-Nordheim equation) at least formally, see [28] or [30] for the classical mechanics case and [89] or [32] in the quantum case. However, if mathematically rigorous derivations are known for small times for the classical Boltzmann equation (Landford's theorem, see [65] or more recently [44][96]), we do not have, at the moment, the same kind of proof for the Boltzmann-Nordheim equation.

6.1.1 The problem and its motivations

All along this chapter we will assume that the collision kernel B can be decomposed as

$$B(v, v_*, \theta) = \Phi\left(|v - v_*|\right) b\left(\cos\theta\right),$$

which is a common assumption as it is more convenient and also covers a wide range of physical applications.

Moreover, we will consider only kernels with hard potentials or Maxwellian potentials ($\gamma = 0$ hereinbelow), that is to say there is a constant $C_\Phi > 0$ such that

$$\Phi(z) = C_\Phi z^\gamma \,, \ \ \gamma \in [0,1], \tag{6.1.1}$$

and satisfying Grad's *angular cutoff* (see [48]), expressed here by the fact that we assume $b \circ \cos$ to be continuous on $(0,\pi)$ and to be integrable on the sphere:

$$l_b = \int_{\mathbb{S}^{d-1}} b\left(\cos\theta\right) d\sigma = \left|\mathbb{S}^{d-2}\right| \int_0^\pi b\left(\cos\theta\right)\sin^{d-2}\theta \, d\theta < \infty. \tag{6.1.2}$$

All those assumptions allow us to rewrite the Boltzmann-Nordheim equation with $\alpha = 1$, into the equation we are going to study

$$\partial_t f = C_\Phi \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} |v-v_*|^\gamma b\left(\cos\theta\right)\left[f'f'_*(1+f+f_*) - ff_*(1+f'+f'_*)\right] dv_* d\sigma. \tag{6.1.3}$$

with the following decomposition

$$\partial_t f = Q^+(f) - fQ^-(f)$$

where we defined

$$\begin{aligned} Q^+(f) &= C_\Phi \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} |v-v_*|^\gamma b\left(\cos\theta\right) f'f'_*(1+f+f_*) dv_* d\sigma, & (6.1.4)\\ Q^-(f) &= C_\Phi \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} |v-v_*|^\gamma b\left(\cos\theta\right) f_*(1+f'+f'_*) dv_* d\sigma. & (6.1.5)\end{aligned}$$

In this chapter, we are first interested in the existence and uniqueness properties of (6.1.3). Then, we shall understand and quantify the possible appearance, in finite time, of a Bose-Einstein condensate in a gas of bosons. This condensate is a concentration of mass in velocity at the mean velocity. In mathematical terms, this can be seen as the appearance of a dirac function in the solution of the equation (6.1.3), noticeable by a blow-up in finite time.

Such a concentration is physically expected, based on various experiments and numerical simulations (see [40] for an overview of these results), as long as the temperature T of the gas is below a critical temperature $T_c(M_0)$ which depends on the mass M_0 of the bosonic gas.

6.1.2 *A priori* expectations for the creation of a Bose-Einstein condensate

In this section, we use some properties of the Boltzmann-Nordheim equation for bosons to understand why a concentration phenomenon is expected. We emphasize here that everything is done *a priori* and should not be considered as a rigorous proof.

The first thing to notice is the symmetry property of the Boltzmann-Nordheim operator.

Lemma 6.1.1 *Let f be such that $Q(f)$ is well-defined. Then for all $\Psi(v)$ we have*

$$\int_{\mathbb{R}^d} Q(f)\Psi \, dv = \frac{C_\Phi}{2} \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} q(f)(v,v_*) \left[\Psi'_* + \Psi' - \Psi_* - \Psi\right] \, d\sigma dv dv_*,$$

with

$$q(f)(v,v_*) = |v-v_*|^\gamma b\,(cos\,\theta)\, f f_* \left(1 + f' + f'_*\right).$$

This result is well-known for the Boltzmann equation and is simply a play with the changes of variables $(v,v_*) \to (v_*,v)$ and $(v,v_*) \to (v',v'_*)$ and the symmetries of the operator $q(f)$. A straightforward consequence is the *a priori* conservation of mass, momentum and energy for a solution of (6.1.3), f, associated to an initial data f_0, that is

$$\int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} f(v)\,dv = \int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} f_0(v)\,dv \tag{6.1.6}$$

The entropy associated to (6.1.3) is the following operator

$$S(f) = \int_{\mathbb{R}^d} \left[(1+f)\mathrm{log}(1+f) - f\mathrm{log}(f)\right]\,dv$$

which is, *a priori*, always increasing in time. It has been proven (see [58]) that for given mass M_0, momentum v_0 and energy E_0, there exists a unique maximizer of S of mass M_0, momentum v_0 and energy E_0 and this maximizer is of the form

$$F_{BE}(v) = m_0\delta(v-v_0) + \frac{1}{e^{\frac{\beta}{2}\left(|v-v_0|^2-\mu\right)} - 1}, \tag{6.1.7}$$

with $m_0 \geqslant 0$, β in $(0,+\infty]$ is the inverse of the equilibrium temperature and $-\infty < \mu \leqslant 0$ is the chemical potential. Moreover, the following equality is satisfied: $\mu.m_0 = 0$.
Besides, functions of the form F_{BE} fulfilling the same constraints are the only maximizers of

the entropy. Therefore, for a given initial data f_0, the associated solution of the Boltzmann-Nordheim equation (6.1.3) should converge, in some sense, to the function F_{BE} associated to the physical quantities of f_0. Hence the appearance of a dirac function at v_0 if $m_0 \neq 0$.

One can find in [69] or [41] that for a given (M_0, v_0, E_0) we have that $m_0 = 0$ if and only if

$$M_0 \leqslant \frac{\zeta(3/2)}{(\zeta(5/2))^{3/5}} \left(\frac{4\pi}{3}\right)^{3/5} E_0^{3/5}. \tag{6.1.8}$$

According to [32], Chap. 2, the kinetic temperature of a bosonic gas is given by

$$T = \frac{m}{3k_B}\frac{E_0}{M_0}$$

which implies that, by plugging it into (6.1.8), $m_0 = 0$ if and only if $T \geqslant T_c(M_0)$ where we compute

$$T_c(M_0) = \frac{m\zeta(5/2)}{2\pi k_B \zeta(3/2)} \left(\frac{M_0}{\zeta(3/2)}\right)^{2/3}.$$

In the equations above, k_B is the constant of Boltzmann.

Initial data satisfying (6.1.8) are called subcritical (or critical in case of equality).

Therefore, for low temperature $T < T_c(M_0)$ we expect our solution to split into a regular part and a dirac mass at v_0 as it converges towards its equilibrium F_{BE} with $m_0 \neq 0$. Spohn, in [102], used this idea of a splitting into a regular and a singular part to derive a physical quantitative study of the Bose-Einstein condensate and its interactions with the normal fluid, in the case of radially symmetric (isotropic) solutions.

6.1.3 Comparison with previous results

The first theorem of the present chapter deals with local-in-time existence and uniqueness of solutions to the bosonic Boltzmann-Nordheim equation for bounded initial datum f_0 with bounded mass and energy (second moment).

The issue of existence and uniqueness for the homogeneous bosonic Boltzmann-Nordheim equation has been studied recently, especially by X. Lu [69][70][71] and M. Escobedo and J. J. L. Velázquez [40][41]. However, all those studies focused on the case of radially symmetric solutions $f(t,v) = f(t,|v|^2)$ and in the case of hard potential with angular cut-off.

In his papers [69] and [70], X. Lu developped a global-in-time Cauchy theory for isotropic initial data with bounded mass and energy and extended the concept of solutions for isotropic distributions. In these cases he proved existence and uniqueness of radially symmetric solutions that preserve mass and energy. Moreover, he showed the boundedness of moments of order $s > 2$ as long as the initial data has a moment of order

s.

Very recently, M. Escobedo and J. J. L. Velázquez in [40] used an idea developped by Carleman for the Boltzmann equation ([26]) in order to obtain a result of uniqueness and existence locally in time for radially symmetric solutions in the spaces $L^\infty(1+|v|^{6+0})$. We discussed above that the creation of the Bose-Einstein condensate leads to a blow-up in finite time. Therefore one cannot expect more than local-in-time results in L^∞-spaces.

The *a priori* conservation of mass, momentum and energy seems to imply that the most natural space to tackle the Cauchy problem is L^1_2, the space of positive functions with bounded mass and energy. This was indeed the case for the homogeneous Boltzmann equation (see [68] and [77]). However, our quick look at the Bose-Einstein condensate told us that one may physically expect that a solution to (6.1.3) is bounded up to the appearance of a blow-up. Moreover, the L^∞-norm is of great importance in the study of the Boltzmann-Nordheim operator in order to be able to deal with the trilinear part of the operator Q. Therefore it seems that the natural framework of the homogeneous Boltzmann-Nordheim equation for bosons is $L^1_2 \cap L^\infty$.

The present work shows a local-in-time existence and uniqueness result for initial data in $L^1_2 \cap L^\infty$ without any isotropic requirement. Along the way, it also proves the immediate appearance of moments of all orders for these solutions.

The issue of the creation of a condensate of Bose-Einstein has been extensively studied experimentally and numerically in physics (see [40] for references on these results). Mathematically, a formal derivation of some properties of this condensate as well as its interactions with the regular part of the bosonic part has been studied in [102] in an isotropic framework.

In the series of papers [69][70][71], X. Lu proved, with not entirely constructive methods, a condensate phenomenon in the limit t goes to infinity. Indeed, he proved that the isotropic solutions he constructed tend to the regular part of their associated equilibrium F_{BE} (see (6.1.7)). But for low temperatures, the regular part of F_{BE} does not have the same mass than the initial solutions. This loss of mass proves the creation of a singular part in the limit. As mentionned in [71], this argument does not require the solution to be isotropic and the condensate Lu catches is to be understood as a concentration phenomenon in the limit t goes to infinity. This limiting behaviour neither prove nor prevent the creation of a Bose-Einstein condensate in finite time.

The appearance of Bose-Einstein in finite time has been mathematically shown in a recent breakthrough [40][41]. In the article [40] the authors showed that if the initial data is isotropic in $L^\infty(1+|v|^{6+0})$ and satisfies some properties about its distribution of mass near $|v|^2 = 0$ then the associated isotropic solution is only define in finite time and its L^∞-norm blows up. They achieve this work thanks to a thorough study of the concentration phenomenon occuring in a bosonic gas. The article [41] proves that supercritical initial data indeed satisfy the blow-up assumptions in the case of radially symmetric solutions.

6.1.4 Our strategy

We tackle the issue of the existence of solutions with an approximative scheme (see Section 6.7). More precisely, we truncate the Boltzmann-Nordheim operator Q and solve the associated differential equation using a Euler scheme. The sequence of functions we obtain is then proved to be weakly compact and goes to a solution of (6.1.3). The key ingredients are a new control on the operator Q^+ for high and small relative velocities $v - v_*$ as well as an extended version of Povzner inequality (see Section 6.3).

The proof of the uniqueness follows very closely the proof of uniqueness developped by S. Mischler and B. Wennberg in [77] for the homogeneous Boltzmann equation. Our extended version of Povzner inequality matches the main features of their proof. The main issue is the control of terms of the form $|v - v_*|^{2+\gamma}$ that appear when one studies the evolution of the energy of solutions. This is achieve by the fact that bounded solutions of (6.1.3) happen to have more regularity (see Proposition 6.4.1) and thanks to an explicit control on the explosion at $t = 0$ of the moment of order $2 + \gamma$ of solutions to (6.1.3) (see Proposition 6.5.5). The speed of the blow-up is exactly the one required to use a Nagumo's type uniqueness criterion in small times. The uniqueness for later time uses a Gronwall-type lemma which is available thanks to the boundedness of the moment of order $2 + \gamma$ whenever $t > 0$ (see Section 6.5).

6.1.5 Organisation of the chapter

Section 6.2 is dedicated to the statement and the description of the main results proved in this chapter.

The first problem we shall deal with is the uniqueness result. As said when we described our strategy (Section 6.1.4), this part requires the control of a little bit more than the L^∞-norm as well as the control of moments of order greater than 2.

A very important tool is an extended version of the Povzner inequality (first derived in [94]) and we shall use it throughout this chapter. The statement of this lemma and its proof are given in Section 6.3.

Section 6.4 focuses on an *a priori* boundedness property of solutions to the bosonic Boltzmann-Nordheim equation. Proposition 6.4.1 will allow us to control terms of the form $|v|^\gamma f(t,v)$ in L^∞.

The next section, Section 6.5, deals with the moments of solutions to (6.1.3). It is divided in two subsections. The first one is dedicated to the immediate appearance of bounded moments of all order, see Proposition 6.5.1. Then, Section 6.5.2 quantifies the explosion near $t = 0$ of the moment of order $2 + \gamma$.

Finally, Section 6.6 proves the uniqueness of bounded solutions preserving mass and energy.

Then we turn to the proof of existence of such bounded, mass and energy preserving solutions in Section 6.7. We construct our sequence of approximations in Section 6.7.2 and derive some of their properties. Section 6.7.3 shows that this sequence converges toward a mass-preserving solution of (6.1.3) and finally Section 6.7.4 proves that this limit is also energy-preserving.

6.2 Main results

We begin with the notations we shall use all along the chapter.

We are going to use spaces in the v and the t variables. Therefore, to shorten notations, we will index by v or t the spaces we are working on. The subscript v will always refer as $\mathbb{R}^d$, for instance $L^1_v = L^1(\mathbb{R}^d)$, $L^\infty_{[0,T],v} = L^\infty([0,T]\times\mathbb{R}^d)$. Moreover, we define

$$L^1_{2,v} = \left\{ f \in L^1_v, \quad \left\| (1+|v|^2) f \right\|_{L^1_v} < +\infty \right\}.$$

Finally, we denote, for all s and t in $\mathbb{R}^+$,

$$M_s(t) = \int_{\mathbb{R}^d} |v|^s \, f(t,v)\, dv. \tag{6.2.1}$$

The first main theorem is the Cauchy problem for the Boltzmann-Nordheim equation for bosons.

Theorem 6.2.1 *Let $f_0(v)$ be in $L^1_{2,v} \cap L^\infty_v$.*
Then there exists $T_0 > 0$, depending only on C_Φ, l_b, γ, $\|f_0\|_{L^1_{2,v}}$ and $\|f_0\|_{L^\infty_v}$, such that there exists a unique f in $L^\infty_{loc}\left([0,T_0), L^1_{2,v}\cap L^\infty_v\right)$ solution on (6.1.3) on $[0,T_0)\times\mathbb{R}^d$ that preserves mass and energy.

Moreover, this solution satisfies

- $T_0 = +\infty$ *or* $\displaystyle\lim_{T\to T_0^-} \|f\|_{L^\infty_{[0,T]\times\mathbb{R}^d}} = +\infty$,
- *f preserves the momentum of f_0,*
- *for all $s > 0$ and for all $0 < T < T_0$,*

$$M_s(t) \in L^\infty_{loc}\left([T, T_0)\right).$$

- *for all $T < T_0$,*

$$\sup_{[0,T]\times\mathbb{R}^d}\left(f(t,v)+\int_0^t(1+|v|^\gamma)\,f(s,v)\,ds\right)<\infty.$$

Remark 6.2.2 *We empasize here that moments appear as soon as t is strictly positive. However, we only get that $M_s(t)$ is in $L^\infty_{loc}([T,T_0))$. This is slightly weaker than the result derived in [77] for the Boltzmann equation but it is explained by the fact that at T_0 we can obtain a blow-up of the L^∞-norm. The latter norm is not required for the control of the bilinear Boltzmann operator but is of great importance for the trilinear part of the Boltzmann-Nordheim operator.*

Let us mention here that Theorem 6.2.1 implies a Bose-Einstein concentration phenomenon as time goes to infinity for subcritical initial data if they are globally defined. Indeed, Lu ([71] Theorem 2) proved in the case $T < T_c(M_0)$ that distributional solutions (not necessarily isotropic) with finite mass and energy present a concentration phenomenon in the limit t goes to infinity.

The latter argument is however non explicit and does not prove any blow-up in finite time whereas [40] gives the appearance of a Bose-Einstein condensate in finite time in the isotropic setting. A work in progress is the proof of the creation of a condensate in finite time in our more general framework.

6.3 An extended version of a Povzner-type inequality

This section is dedicated to proving a refinement of a result in [77], which extends a Povzner-type inequality (see [94]) which captures the geometry of the collisions inside the Boltzmann kernel. The statement of the lemma is very close to Lemma 2.2 in [77].

Lemma 6.3.1 *Assume that $b(\theta)$ is a locally bounded function and consider $F \geqslant 1$ a function in $L^\infty(\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1})$.*
For a given function ψ let

$$K_\psi(v,v_*)=\int_{\mathbb{S}^{d-1}}F(v,v_*,\sigma)b(\theta)\left(\psi(\left|v'_*\right|^2)+\psi(\left|v'\right|^2)-\psi(|v_*|^2)-\psi(|v|^2)\right)\,d\sigma.$$

Then one can write $K_\psi(v,v_*)=G_\psi(v,v_*)-H_\psi(v,v_*)$, where G_ψ and H_ψ satisfies the following inequalities (where we omit the subscript ψ).

Let $\chi(v,v_*)=1-\mathbf{1}_{\{|v|<|v_*|<2|v|\}}$ then *i) If $\psi(x) = x^{1+\alpha}$ with $\alpha > 0$ then*

$$|G(v, v_*)| \leqslant C_G \alpha \left(|v|\,|v_*|\right)^{1+\alpha}$$

and

$$H(v, v_*) \geqslant C_H \alpha \left(|v|^{2+2\alpha} + |v_*|^{2+2\alpha}\right) \chi(v, v_*).$$

ii) If $\psi(x) = x^{1+\alpha}$ with $-1 < \alpha < 0$ then

$$|G(v, v_*)| \leqslant C_G\,|\alpha|\left(|v|\,|v_*|\right)^{1+\alpha}$$

and

$$-H(v, v_*) \geqslant C_H\,|\alpha| \left(|v|^{2+2\alpha} + |v_*|^{2+2\alpha}\right) \chi(v, v_*).$$

iii) If ψ is a positive convex function that can be written $\psi(x) = x\phi(x)$, where ϕ is concave, increasing to infinity, and such that for any $\varepsilon > 0$ and any α in $(0, 1)$, it satisfies $(\phi(x) - \phi(\alpha x))\,x^{\varepsilon} \to \infty$ as $x \to \infty$. Then, for all $\varepsilon > 0$,

$$|G(v, v_*)| \leqslant C_G\,|v|\,\phi\left(|v|^2\right)|v_*|\,\phi\left(|v_*|^2\right)$$

and

$$H(v, v_*) \geqslant C_H \left(|v|^{2-\varepsilon} + |v_*|^{2-\varepsilon}\right) \chi(v, v_*).$$

In addition, there is a constant $C > 0$ such that $\phi'(x) \leqslant C/(1+x)$ implies $G(v, v_*) \leqslant C_G\,|v|\,|v_*|$.

The constants in the Lemma depends on α, ψ, ε, b and $\|F\|_{L^{\infty}_{v,v_*,\sigma}}$.

Remark 6.3.2 *As noticed in [77], the operator H_ψ can be taken monotonous in ψ in the following sense. If $\psi_1 - \psi_2 \geqslant 0$ is convex then $H_{\psi_1} - H_{\psi_2} \geqslant 0$. This property will prove itself really useful to apply Lemma 6.3.1 to truncated sequences converging to convex functions.*

Proof of Lemma 6.3.1 The proof of this result has been done in Lemma 2.2 of [77] in the case where $F = 1$. Therefore, our goal will be to compare our new operators H and G with H_1 and G_1 (obtained when $F = 1$) in each of the three cases.

We start with the term H which is quite straightforward. We define H to be the following operator, which coincides with H_1 in [77] when $F=1$,

$$H(v,v_*) = 4\int_0^{\pi/2} \left[\bar{b}(\theta)\bar{F}(v,v_*,\theta) + \bar{b}(\pi/2-\theta)\bar{F}(v,v_*,\pi/2-\theta)\right] \\ \times \left[\cos^2\!\theta\, \psi\left(|v|^2\right) + \sin^2\!\theta\, \psi\left(|v_*|^2\right) - \psi\left(|v|^2\cos^2\!\theta + |v_*|^2\sin^2\!\theta\right)\right]\, d\theta, \tag{6.3.1}$$

where $\bar{b}(\theta) = b(\theta)\,|d\sigma|\,(\theta)$ and $\bar{F}(v,v_*,\theta) = \int_{-\pi}^{\pi} F(v,v_*,\sigma)\,d\omega$ where (θ,ω) are spherical coordinates parametrising $\mathbb{S}^{d-1}$ (ω is then a $(d-2)$-uple of angles).

The core of the proof is the fact that the term

$$\cos^2\!\theta\, \psi\left(|v|^2\right) + \sin^2\!\theta\, \psi\left(|v_*|^2\right) - \psi\left(|v|^2\cos^2\!\theta + |v_*|^2\sin^2\!\theta\right)$$

keeps the same sign if ψ is convex (positive sign) and if ψ is concave (negative sign). We have that

$$1 \leqslant F(v,v_*,\theta) \leqslant \|F\|_{L^\infty_{v,v_*,\sigma}}$$

and therefore when ψ is convex we have

$$H(v,v_*) \geqslant H_1(v,v_*)$$

and if ψ is concave

$$H(v,v_*) \leqslant \|F\|_{L^\infty_{v,v_*,\sigma}} H_1(v,v_*).$$

This yields the expected inequalities $i)$, $ii)$ and $iii)$ for the operator H since they hold true for H_1.

The proof for the operator G is more intricate and we shall write it in dimension $d=3$ for sake of simplicity.
We follow the proof in [77] and we parametrise the sphere $\mathbb{S}^2$ by

$$\mathbb{S}^2 = \{(\theta,\omega), \quad -\pi \leqslant \omega \leqslant \pi,\ 0 \leqslant \theta \leqslant \pi/2\}$$

with the measure

$$d\sigma = 4\sin\theta\cos\theta\; d\theta d\omega.$$

To shorten notation we define, for a given v and a given v_*

$$Y(\theta) = |v|^2\cos^2\theta + |v_*|^2\sin^2\theta,$$

$$Z(\theta) = 2\,|v|\,|v_*|\sin\theta\cos\theta.$$

In these coordinates, with the notations above, we have geometrically that

$$\begin{cases} \left|v'\right|^2 = Y(\theta) + \tau Z(\theta)\cos\omega \\ \left|v'_*\right|^2 = Y(\pi/2-\theta) - \tau Z(\theta)\cos\omega, \end{cases}$$

where τ denotes the sine of the angle between the vector v and $v*$.

With these notations we obtain

$$K(v,v_*) = G(v,v_*) - H(v,v_*)$$

where H is given by (6.3.1) (after the change of variable $\theta \to \pi/2-\theta$) and

$$G(v,v_*) = 4\int_0^{\pi/2} \bar{b}(\theta)\int_{-\pi}^{\pi} F(v,v_*,\sigma)\left[\psi\left(Y(\theta)+\tau Z(\theta)\cos\omega\right) - \psi\left(Y(\theta)\right)\right]\, d\omega d\theta$$
$$+\,4\int_0^{\pi/2} \bar{b}(\theta)\int_{-\pi}^{\pi} F(v,v_*,\sigma)\left[\psi\left(Y(\pi/2-\theta)-\tau Z(\theta)\cos\omega\right) - \psi\left(Y(\pi/2-\theta)\right)\right]\, d\omega d\theta.$$

The two terms on the right-hand side will be treated the same way and therefore we focus only on

$$I = 4\int_0^{\pi/2} \bar{b}(\theta)\int_{-\pi}^{\pi} F(v,v_*,\sigma)\left[\psi\left(Y(\theta)+\tau Z(\theta)\cos\omega\right) - \psi\left(Y(\theta)\right)\right]\, d\omega d\theta.$$

Since ψ is increasing in all the cases we have that $\psi\left(Y(\theta)+\tau Z(\theta)\cos\omega\right) - \psi\left(Y(\theta)\right)$ is positive when $-\pi/2 \leqslant \omega \leqslant \pi/2$ and negative elsewhere on $[-\pi,\pi]$. Thus,

$$\begin{aligned} |I| &\leqslant 4\,\|F\|_{L^\infty_{v,v_*,\sigma}} \int_0^{\pi/2} \bar{b}(\theta)\int_{-\pi}^{\pi} \left|\psi\left(Y(\theta)+\tau Z(\theta)\cos\omega\right) - \psi\left(Y(\theta)\right)\right| d\omega d\theta \\ &= 8\,\|F\|_{L^\infty_{v,v_*,\sigma}} \int_0^{\pi/2} \bar{b}(\theta)\int_{0}^{\pi} \left|\psi\left(Y(\theta)+\tau Z(\theta)\cos\omega\right) - \psi\left(Y(\theta)\right)\right| d\omega d\theta \\ &= 8\,\|F\|_{L^\infty_{v,v_*,\sigma}} \int_0^{\pi/2} \bar{b}(\theta) \\ &\qquad \times \int_0^{\pi/2} \left[\psi\left(Y(\theta)+\tau Z(\theta)\cos\omega\right) - \psi\left(Y(\theta)-\tau Z(\theta)\cos\omega\right)\right]\, d\omega d\theta, \end{aligned} \tag{6.3.2}$$

where we just made the change of variable $\omega \to \pi-\omega$ on $[\pi/2,\pi]$.

Upper bound in cases $i)$ **and** $ii)$**.** In these cases, we have that ψ is twice differentiable and therefore we can integrate by part twice in the integral with respect to ω. The first time we consider 1 to be the derivative of ω to get

$$|I| \leqslant 8 \|F\|_{L^\infty_{v,v_*,\sigma}} \int_0^{\pi/2} \bar{b}(\theta)\tau Z \times \int_0^{\pi/2} \left(\omega \sin\omega \left[\psi'(Y+\tau Z\cos\omega) + \psi'(Y-\tau Z\cos\omega)\right]\right) \, d\omega d\theta,$$

and in the second integration by part considers $\omega\sin\omega$ as a derivative to get

$$\begin{aligned}|I| \leqslant & 8 \|F\|_{L^\infty_{v,v_*,\sigma}} \int_0^{\pi/2} \bar{b}(\theta)\tau^2 Z^2 \int_0^{\pi/2} (\sin\omega - \omega\cos\omega) \\ & \qquad \times \left[\psi''(Y+\tau Z\cos\omega) - \psi''(Y-\tau Z\cos\omega)\right] \, d\omega d\theta \\ & + 16 \|F\|_{L^\infty_{v,v_*,\sigma}} \int_0^{\pi/2} \bar{b}(\theta)\tau Z(\theta)\psi'(Y(\theta)) \, d\theta.\end{aligned} \tag{6.3.3}$$

On $[0,\pi/2]$, $\sin\omega - \omega\cos\omega$ is positive and thus

$$\begin{aligned}&(\sin\omega - \omega\cos\omega)\left[\psi''(Y+\tau Z\cos\omega) - \psi''(Y-\tau Z\cos\omega)\right] \\ &\quad \leqslant (\sin\omega - \omega\cos\omega)\left[\left|\psi''(Y+\tau Z\cos\omega)\right| + \left|\psi''(Y-\tau Z\cos\omega)\right|\right],\end{aligned}$$

which is the integrand dealt with in the case $F=1$ in Lemma 2.2 in [77]. Hence, (6.3.3) becomes

$$|I| \leqslant \|F\|_{L^\infty_{v,v_*,\sigma}} |I_1| + 16(1+\alpha) \|F\|_{L^\infty_{v,v_*,\sigma}} \int_0^{\pi/2} \bar{b}(\theta)\tau Z(\theta) Y(\theta)^\alpha \, d\theta. \tag{6.3.4}$$

It only remains to control the last integral which can be achieve thanks to the fact that for θ in $[0,\pi/2]$,

$$Z(\theta)Y(\theta)^\alpha \leqslant |v|\,|v_*| \left(|v|^2 + |v_*|^2\right)^\alpha. \tag{6.3.5}$$

In the case $-1<\alpha<0$ we have easily that (6.3.5) yields

$$Z(\theta)Y(\theta)^\alpha \leqslant 2^\alpha \left(|v|^2 + |v_*|^2\right)^{1+\alpha},$$

which, combined with (6.3.4) gives us the expected inequality in point ii).

In the case $\alpha>0$ we use (6.3.5) in two different ways. First of all we notice the following

$$\forall\, \frac{|v|}{2} \leqslant |v_*| \leqslant 2\,|v|, \quad Z(\theta)Y(\theta)^\alpha \leqslant 2^{2+\alpha}5^\alpha \left(|v|\,|v_*|\right)^{\alpha+1}. \tag{6.3.6}$$

Then basic computations yields

$$\forall \varepsilon>0,\ \forall v, v_*, \quad |v|\,|v_*| \left(|v|^2 + |v_*|^2\right)^\alpha \leqslant \frac{1}{\epsilon^\alpha}\left(|v|\,|v_*|\right)^{1+\alpha} + \varepsilon\left(|v|^2 + |v_*|^2\right)^{1+\alpha}. \tag{6.3.7}$$

To conclude in that case we gather (6.3.6) and (6.3.7) to obtain that

$$\forall \varepsilon > 0, \quad Z(\theta)Y(\theta)^\alpha \leqslant C_\varepsilon \left(|v|\,|v_*|\right)^{1+\alpha} + \varepsilon \left(|v|^2 + |v_*|^2\right)^{1+\alpha} \chi(v, v_*).$$

This last bound combined with (6.3.4) gives the inequality of point i), up to the fact that we choose ε small enough so that the second term in the right-hand side of the inequality above can be included in the inequality satisfied by H, which only leads to a slight change of definition for H in that case.

Upper bound in cases iii)**.** We start from (6.3.2)

$$\begin{aligned}|I| \leqslant & 8 \left\|F\right\|_{L^\infty_{v,v_*,\sigma}} \\ & \times \int_0^{\pi/2} \bar{b}(\theta) \int_0^{\pi/2} \left[\psi\left(Y(\theta) + \tau Z(\theta)\cos\omega\right) - \psi\left(Y(\theta) - \tau Z(\theta)\cos\omega\right)\right] \, d\omega d\theta.\end{aligned}$$

In case iii) we consider $\psi(x) = x\phi(x)$ with ψ being convex and ϕ being concave. Therefore, the latters are almost everywhere differentiable with

$$\begin{aligned}\psi\left(Y - \tau Z\cos\omega\right) &\geqslant \psi\left(Y\right) - \tau Z\cos\omega\psi'(Y),\\ \phi\left(Y + \tau Z\cos\omega\right) &\leqslant \phi\left(Y\right) + \tau Z\cos\omega\phi'(Y).\end{aligned}$$

Hence, developing every term in (6.3.2) yields

$$\begin{aligned}|I| \leqslant & 8 \left\|F\right\|_{L^\infty_{v,v_*,\sigma}} \\ & \times \int_0^{\pi/2} \bar{b}(\theta) \int_0^{\pi/2} \left[2\tau Z\cos\omega\phi(Y) + 2\tau Z\cos\omega Y\phi'(Y) + \tau^2 Z^2 \cos^2\omega\phi'(Y)\right] \, d\omega d\theta.\end{aligned}$$

We recognize here the term I for $F = 1$, see proof of Lemma 2.2 in [77]. Therefore

$$|I| \leqslant 2 \left\|F\right\|_{L^\infty_{v,v_*,\sigma}} |I_1| \,,$$

and hence iii) follows directly from the case where $F = 1$.

This concludes the proof of Lemma 6.3.1. ■

6.4 *A priori* control on the $L^\infty_v\left((1+|v|^\gamma)L^1_t\right)$

This section is dedicated to proving an *a priori* estimate in the L^∞_v space for solutions to (6.1.3), in small times. We cannot expect more than small times as we know from [40] that, even for radially symmetric solutions, there exists solutions with a blow-up in finite time.

We will prove the following result

Proposition 6.4.1 *Let $f_0(v)$ be in $L^1_{2,v} \cap L^\infty_v$.*
Let f be a non-negative solution of (6.1.3) in $L^\infty_{[0,T_0)}\left(L^1_{2,v} \cap L^\infty_v\right)$, with initial value f_0, satisfying the conservation of mass and energy.
Then for all $0 \leqslant T < T_0$ there exists $C_T > 0$ such that following controls holds

$$\sup_{[0,T]\times\mathbb{R}^d} \left(f(t,v) + \int_0^t (1+|v|^\gamma)\, f(s,v)\, ds \right) \leqslant C_T.$$

6.4.1 Some properties of the Boltzmann-Nordheim operator

Here we gather and prove some useful properties about the positive operator Q^- and Q^+. First, we have the following control on the negative part

Lemma 6.4.2 *Let $f \geqslant 0$ be in $L^1_{2,v}$. Then there exists $C_\gamma > 0$ (given by (6.4.2)) such that*

$$\forall v \in \mathbb{R}^d, \quad Q^-(f)(v) \geqslant C_\Phi l_b \left(1+|v|^\gamma\right) \|f\|_{L^1_v} - C_\Phi C_\gamma l_b \|f\|_{L^1_{2,v}}.$$

Proof of Lemma 6.4.2 We have that

$$Q^-(f)(v) = C_\Phi \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} |v-v_*|^\gamma\, b(\cos\theta) f_* \left[1 + f'_* + f'\right]\, dv_* d\sigma.$$

We supposed that f is positive, thus

$$Q^-(f)(v) \geqslant C_\Phi \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} |v-v_*|^\gamma\, b(\cos\theta) f_*\, dv_* d\sigma$$

Since γ is in $[0,1]$, we know that the following triangular inequality holds

$$\left(|v|^\gamma - |v_*|^\gamma\right) \leqslant |v-v_*|^\gamma \leqslant \left(|v|^\gamma + |v_*|^\gamma\right). \tag{6.4.1}$$

This yields

$$\begin{aligned} Q^-(f)(v) &\geqslant C_\Phi l_b \int_{\mathbb{R}^d} \left((1+|v|^\gamma) - (1+|v_*|^\gamma)\right) f_*\, dv_* \\ &\geqslant C_\Phi l_b \left[(1+|v|^\gamma)\, \|f\|_{L^1_v} - C_\gamma \int_{\mathbb{R}^d} (1+|v_*|^2) f_*\, dv_* \right], \end{aligned}$$

because $\gamma \leqslant 1$ and so there exists $C_\gamma > 0$ such that for all $x \geqslant 0$,

$$(1+x^\gamma) \leqslant C_\gamma (1+x^2). \tag{6.4.2}$$

■

Remark 6.4.3 *If $\int_{\mathbb{R}^d} f\ln f\, dv$ was finite then it would be possible to lower bound $Q^-(f)$ by a quantity that is strictly positive. Unfortunately, this quantity decreases in the case of the classical Boltzmann equation whereas the decrease of entropy for Boltzmann-Nordheim is given by the decrease of*

$$\int_{\mathbb{R}^d} \left[(1+f)\ln(1+f) - f\ln f\right]\, dv,$$

which does not bring any knowledge about a non-concentration property for f.

Moreover we also have the following general bound on the positive part

Lemma 6.4.4 *Let f and h be in $L^1_{2,v} \cap L^\infty_v(1+|v|^\gamma)$. Then we have that for all $\lambda > 0$ there exists $C(\lambda) > 0$ such that*

$$\lim_{\lambda \to 0} C(\lambda) = 0$$

and such that

$$\begin{aligned}\Big\| C_\Phi \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} &|v-v_*|^\gamma b\left(\cos\theta\right) f' h'_* dv_* d\sigma \Big\|_{L^\infty_v} \\ &\leqslant C_\gamma C_\Phi l_b \left\|h\right\|_{L^1_{2,v}} \left[C(\lambda)\left\|(1+|v|^\gamma)\, f\right\|_{L^\infty_v} + \frac{2^{d-2}}{\lambda^{d-1}} C_\gamma \left\|f\right\|_{L^1_{2,v}}\right],\end{aligned} \tag{6.4.3}$$

and

$$\begin{aligned}\Big\| C_\Phi \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} &|v-v_*|^\gamma b\left(\cos\theta\right) f' h'_* dv_* d\sigma \Big\|_{L^\infty_v} \\ &\leqslant \left(1+|v|^\gamma\right) C_\gamma C_\Phi l_b \left\|h\right\|_{L^1_{2,v}} \left[C(\lambda)\left\|f\right\|_{L^\infty_v} + \frac{2^{d-2}}{\lambda^{d-1}} \left\|f\right\|_{L^1_v}\right],\end{aligned} \tag{6.4.4}$$

where C_γ has been defined in (6.4.2) and $C(\lambda)$ is given by (6.4.7).

Proof of Lemma 6.4.4 The L^∞_v-norm is intricate and for this purpose we write the operator under another form. We use the Carleman representation of the operator (see [27]), which uses the final velocities after a collision, v' and v'_*, as the parameters we integrate against:

$$\begin{aligned}\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} &|v-v_*|^\gamma b\left(\cos\theta\right) f' h'_* dv_* d\sigma \\ &= C_\Phi \int_{\mathbb{R}^d} dv' \int_{E_{vv'}} dv'_* \, \frac{1}{|v-v'|^{d-1}} \tilde{B}\left(2v - v' - v'_*, \frac{v'-v'_*}{|v'-v'_*|}\right) f' h'_*.\end{aligned}$$

In this form we have that $E_{vv'}$ is the hyperplane orthogonal to $v - v'$ going throught v and the new operator $\tilde{B}$ is such that

$$\tilde{B}(z,\omega) = 2^{d-2}\left\langle \frac{z}{|z|},\omega\right\rangle^{d-2} B(z,\sigma).$$

With this new representation we have that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} |v-v_*|^\gamma b\left(\cos\theta\right) f' h'_* dv_* d\sigma\right| \\
&\qquad\qquad \leqslant 2^{d-2} C_\Phi l_b \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{\left(1+|v'|^\gamma\right)\left(1+|v'_*|^\gamma\right)}{|v-v'|^{d-1}} f' h'_*\, dv' dv'_* \\
&\qquad\qquad \leqslant 2^{d-2} C_\Phi l_b C_\gamma \left\|h\right\|_{L^1_{2,v}} \int_{\mathbb{R}^d} \frac{\left(1+|v'|^\gamma\right) f'}{|v-v'|^{d-1}}\, dv'. \qquad (6.4.5)
\end{aligned}$$

Now we are going to split our integral into velocities far from v and velocities close to v. Let us consider $\lambda > 0$,

$$\int_{\mathbb{R}^d} \frac{\left(1+|v'|^\gamma\right) f'}{|v-v'|^{d-1}}\, dv' \leqslant \|(1+|v|^\gamma) f\|_{L^\infty_v} \int_{|v-v'|\leqslant\lambda} \frac{dv'}{|v-v'|^{d-1}} + \frac{C_\gamma}{\lambda^{d-1}} \|f\|_{L^1_{2,v}}. \qquad (6.4.6)$$

The function $1/\left|x\right|^{d-1}$ is integrable near 0 and therefore we can define

$$C(\lambda) = 2^{d-2}(1+\lambda)\int_{|x|\leqslant\lambda} \frac{dv'}{|x|^{d-1}}, \qquad (6.4.7)$$

which fulfils the requirements of (6.4.3) in Lemma 6.4.4.

In (6.4.6), instead of taking $\|(1+|v|^\gamma)f\|_{L^\infty_v}$ we could also use

$$\left(1+\left|v'\right|^\gamma\right) \leqslant \left(1+\left|v-v'\right|^\gamma\right)\left(1+|v|^\gamma\right)$$

and the fact that $0 \leqslant \gamma \leqslant 1$. Then taking the L^∞_v-norm of f leads to the expected (6.4.4) with $C(\lambda)$ described by (6.4.7). ■

6.4.2 A priori estimate: proof of Proposition 6.4.1

Let f be a solution of the Boltzmann-Nordheim equation (6.1.3), satisfying the assumptions of Proposition 6.4.1. This means that

$$\forall (t,v) \in [0,T_0)\times\mathbb{R}^d, \quad f(t,v) = f_0(v) + \int_0^t Q\left(f(s,\cdot)\right)(v)\, ds. \qquad (6.4.8)$$

We consider $0 \leqslant T < T_0$ and we define the following quantities, for $0 \leqslant t < T$ and v in $\mathbb{R}^d$:

$$\begin{aligned} e(f)(t,v) &= \int_0^t (1+|v|^\gamma)\, f(s,v)\, ds, \\ E(f)(t,v) &= f(t,v) + \int_0^t (1+|v|^\gamma)\, f(s,v)\, ds. \end{aligned}$$

In (6.4.8), we apply to $Q(f)$ the Lemmas 6.4.2 and 6.4.4 (with λ_1 and λ_2 to be defined later) together with the conservations laws satisfied by f (6.1.6) to get

$$\begin{aligned} f(t,v) \leqslant \|f_0\|_{L^\infty_v} &- C_\Phi l_b \|f_0\|_{L^1_v} \int_0^t (1+|v|^\gamma)\, f(s,v)\, ds \\ &+ C_0 \int_0^t f(s,v)\, ds \\ &+ C_0 \left[C(\lambda_1) \sup_{|u-v|\leqslant\lambda_1} (e(f)(t,u)) + \frac{2^{d-2}C_\gamma}{\lambda_1^{d-1}} T \|f_0\|_{L^1_{2,v}} \right] \\ &+ 2C_0 \|f\|_{L^\infty_{[0,T]\times\mathbb{R}^d}} \left[C(\lambda_2) \sup_{|u-v|\leqslant\lambda_2} (e(f)(t,u)) + \frac{2^{d-2}C_\gamma}{\lambda_2^{d-1}} T \|f_0\|_{L^1_{2,v}} \right], \end{aligned} \tag{6.4.9}$$

where we set

$$\begin{aligned} M_0 &= \min\{1, C_\Phi l_b \|f_0\|_{L^1_v}\}, \\ C_0 &= C_\gamma C_\Phi l_b \|f_0\|_{L^1_{2,v}}. \end{aligned}$$

We emphasize here that we slightly changed Lemma 6.4.4 since we put the integrale in time before taking the supremum in v. Which is obtain by exactly the same proof but integrating first in time.

By bounding all quantities in (6.4.9) in time and velocities, one gets

$$\begin{aligned} M_0 E(f)(t,v) \leqslant& \left[C_0 C(\lambda_1) + 2C_0 C(\lambda_2) \|f\|_{L^\infty_{[0,T]\times\mathbb{R}^d}} \right] \sup_{[0,T]\times B(v,\lambda_1+\lambda_2)} E(f) \\ &+ \left[\|f_0\|_{L^\infty_v} + C_0 T \|f\|_{L^\infty_{[0,T]\times\mathbb{R}^d}} + C_0 C_\gamma 2^{d-2} \|f_0\|_{L^1_{2,v}} T \left(\frac{1}{\lambda_1^{d-1}} + \frac{\|f\|_{L^\infty_{t,v}}}{\lambda_2^{d-1}} \right) \right]. \end{aligned}$$

To conclude, we notice that by assumption $\|f\|_{L^\infty_{[0,T]\times\mathbb{R}^d}}$ is finite and therefore we fix λ_1 and λ_2 small enough such that $C_0 C(\lambda_1) + 2C_0 C(\lambda_2) \|f\|_{L^\infty_{[0,T]\times\mathbb{R}^d}} \leqslant M_0/2$. Then we take the supremum over t in $[0,T]$ and v in $\mathbb{R}^d$ to obtain the expected result.

6.5 Creation of moments of all order

In this section we prove that moments of all order appear immediately for solutions of the Boltzmann-Nordheim equation, as long as they are in $L^\infty_{\text{loc}}\left([0,T_0), L^1_{2,v} \cap L^\infty_v\right)$.
The first part of this section is dedicated to the proof of this *a priori* result. It thoroughly follows the proof established in [77] for the Boltzmann equation which was relying on a subtle Povzner inequality. Our extension of their Povzner-type inequality, see Section 6.3, allows us to apply their methods directly to the Boltzmann-Nordheim equation.

Then, in a second part we quantify the explosion of the $(2+\gamma)^{\text{th}}$ moment as time goes to 0. This estimate will be of great importance in the proof of the uniqueness, see Section 6.6. Here again we copy the arguments of [77] thanks to the extension of Povzner inequality, Lemma 6.3.1.

All the details of the proofs are exactly the same as for the Boltzmann equation given by S. Mischler and B. Wennberg in [77]. However, we still write them down roughly in order to show that they are indeed a straight combination of their proofs and our Povzner-type inequality. Basically we show that we can apply our inequality each time they applied theirs and that the outcome is the same.

6.5.1 *A priori* estimate on the moments of a solution

The immediate appearance of moments is characterized by the following proposition.

Proposition 6.5.1 *Let $f_0(v)$ be in $L^1_{2,v} \cap L^\infty_v$.*
Let f be a non-negative solution of (6.1.3) in $L^\infty_{loc}\left([0,T_0), L^1_{2,v} \cap L^\infty_v\right)$, with initial value f_0, satisfying the conservation of mass and energy.
Then for all for all $s>0$ and for all $0<T<T_0$,

$$\int_{\mathbb{R}^d} |v|^s f(t,v)\,dv \in L^\infty_{loc}\left([T,T_0)\right)$$

The proof of that proposition is done by induction and requires two lemmas, which gives the same estimates as the ones for the Boltzmann equation in [77]. The first one is the initialisation of the induction, it controls the $L^1_{2+\gamma/2,v}$-norm, and the second lemma gives an inductive bound on moments.

We start by taking f_0, f, T_0 as in Proposition 6.5.1. We have that f_0 is positive and such that $\left(1+|v|^2\right) f_0(v)$ is in L^1_v. Proposition $A1$ in the appendix of [77] gives the existence of ψ a positive convex function on $\mathbb{R}^+$ such that there exists $C>0$ such that

$$\int_{\mathbb{R}^d} \psi\left(|v|^2\right) f_0(v)\,dv \leqslant C.$$

Moreover, ψ can be written $\psi(x) = x\phi(x)$, where ϕ is concave, increasing to infinity, and such that for any $\varepsilon > 0$ and any α in $(0,1)$, it satisfies $(\phi(x) - \phi(\alpha x))\, x^{\varepsilon} \to \infty$ as $x \to \infty$.

Lemma 6.5.2 *We have that for all T in $[0, T_0)$ there exists $c_T, C_T > 0$ such that for all $0 \leqslant t \leqslant T$,*

$$\begin{aligned}\int_{\mathbb{R}^d} f(t,v)\psi\left(|v|^2\right)\, dv + c_T \int_0^t \int_{\mathbb{R}^d} f(\tau, v)\left[M_{2+\gamma/2} + \psi\left(|v|^2\right)\right]\, dv d\tau \\ \leqslant \int_{\mathbb{R}^d} \psi\left(|v|^2\right) f_0(v)\, dv + C_T t.\end{aligned} \tag{6.5.1}$$

Proof of Lemma 6.5.2 We fix T in $[0, T_0)$ and we consider $0 \leqslant t \leqslant T$.

As proved in the proof of uniqueness in [77], we can construct an increasing sequence $(\psi_n)_{n\in\mathbb{N}}$ of convex function converging pointwise to ψ and such that $\psi_{n+1} - \psi_n$ is convex. The ψ_n are such that there exists a sequence $(p_n)_{n\in\mathbb{N}}$ of polynomial of order 1 such that $\psi_n - p_n$ is of compact support.
Moreover, for a given F satisfying the assumptions of Lemma 6.3.1, we have that H_{ψ_n} is non-negative and converges pointwise to H_ψ (see Remark 6.3.2) and $|G_{\psi_n}(v, v_*)| \leqslant C_G\, |v|\, |v_*|$ for all n.

We know that f preserves mass and energy and therefore

$$\int_{\mathbb{R}^d} [f(t,v) - f_0(v)]\, \psi_n\left(|v|^2\right)\, dv = \int_{\mathbb{R}^d} [f(t,v) - f_0(v)] \left(\psi_n\left(|v|^2\right) - p_n\left(|v|^2\right)\right)\, dv.$$

Now, $\psi_n - p_n$ is of compact support so we can use the fact that f is solution to the Boltzmann-Nordheim equation and the integral property of the operator Q, Lemma 6.1.1. This yields

$$\begin{aligned}&\int_{\mathbb{R}^d} [f(t,v) - f_0(v)]\, \psi_n\left(|v|^2\right)\, dv \\ &\qquad = \frac{C_\Phi}{2} \int_0^t \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} q(f)(\tau, v, v_*) \left[\psi'_{n*} + \psi'_n - \psi_{n*} - \psi_n\right]\, dv dv_* d\tau,\end{aligned}$$

with

$$q(f)(\tau, v, v_*) = |v - v_*|^\gamma b\left(\cos\theta\right) f(\tau) f_*(\tau) \left(1 + f'(\tau) + f'_*(\tau)\right).$$

We can decompose the left handside as in point *iii)* of Lemma 6.3.1 with $F(v, v_*, \sigma) = 1 + f'_* + f'$ which fulfils the assumptions needed since f is in $L^\infty_{[0,T]}$. We obtain

$$\int_{\mathbb{R}^d} f(t,v)\psi_n\left(|v|^2\right)\,dv + \frac{C_\phi}{2}\int_0^t\int_{\mathbb{R}^d\times\mathbb{R}^d} f(\tau)f(\tau)_*\,|v-v_*|^\gamma\,H_{\psi_n}\,dv_*dvd\tau$$
$$= \int_{\mathbb{R}^d} f_0(v)\psi_n\left(|v|^2\right)\,dv + \frac{C_\phi}{2}\int_0^t\int_{\mathbb{R}^d\times\mathbb{R}^d} f(\tau)f(\tau)_*\,|v-v_*|^\gamma\,G_{\psi_n}\,dv_*dvd\tau \quad (6.5.2)$$

Thanks to Lemma 6.3.1, our operators H_{ψ_n}, G_{ψ_n}, H_ψ and G_ψ satisfies exactly the same properties than the operators H_{ψ_n}, G_{ψ_n}, H_ψ and G_ψ in step 1 of proof of Theorem 1.1$'$ in [77], with C_H and C_G depending on T which has been fixed. Equality (6.5.2) is exactly equality (3.4) in step 1 of proof of Theorem 1.1$'$ in [77]. Thus we can compute this equality in exactly the same way as Mischler and Wennberg did.

This yields the inequality of Lemma 6.5.2. ■

We turn to the induction property.

Lemma 6.5.3 *Let T be in $(0,T_0)$.*
For all n in $\mathbb{N}$ there exists $T_n>0$ as small as we want such that

$$M_{2+(2n+1)\gamma/2}(T_n) < \infty$$

and such that for all t in $[T_n,T]$ there exists $C_T>0$ and $c_{T_n,T}>0$ such that

$$M_{2+(2n+1)\gamma/2}(t) + c_T\int_{T_n}^t \left[M_{2+(2n+1)\gamma/2}(\tau) + M_{2+(2n+3)\gamma/2}(\tau)\right]\,d\tau \leqslant C_{T_n,T}(1+t), \quad (6.5.3)$$

where $M_s(t)$ is the moment of order s at time t, see (6.2.1).

Proof of Lemma 6.5.3 The fact that there exists T_{n+1} as small as we want such that $M_{2+(2n+3)\gamma/2}(T_n) < \infty$ is given by the second term on the left-hand side of inequality (6.5.3) at rank n, and from the second term on the left-hand side of inequality (6.5.1) in Lemma 6.5.2 for $n=0$.

Then the proof amounts to using Povzner inequality of Lemma 6.3.1 in exactly the same way as in [77], this time considering the function ψ to be $\psi(x) = x^{1+(2n+3)\gamma/4}$ (so we use point i) of Lemma 6.3.1). We can therefore follow the proof of [77] (step 2 of proof of Theorem 1.1$'$) and apply our Lemma 6.3.1 with $F(v,v_*,\sigma) = 1+f'+f'_*$ and the constants C_H and C_G depending on T via $\|f\|_{[0,T],v}$. ■

We are now able to finish the proof of the main proposition of this section.

Proof of Proposition 6.5.1 First of all we notice that f is in $L^1_{2,v}$ and therefore the Proposition is true for all s in $[0,2]$.

For $s > 2$ we just have to notice that the first term on the left-hand side of inequality (6.5.3) in Lemma 6.5.3 gives that $M_{2+(2n+1)\gamma/2}(t) \leqslant C_{T_n,T}(1+T)$ for all t in $[T_n, T]$, T_n given but being as small as we want. This being true for all n gives that M_s is bounded on $[T_1, T_2]$ for all $0 < T_1 \leqslant T_2 < T_0$. ∎

Remark 6.5.4 *We can emphasize here that this result is slightly different from the one in the Boltzmann equation. Indeed, in that case $T_0 = +\infty$ and the bounds on the moments on $[T, T_0)$ only depend on T. For the Boltzmann-Nordheim equation in our settings we can only reach locally bounded moments since we require the boundedness of the solution f in order to apply Povzner inequality. This boundedness property is only local (as shown by the explosion at T_0^- of the L^∞-norm) and so we cannot expect the moments to be in $L^\infty_{[0,T_0]}$ even if $T_0 = +\infty$.*

6.5.2 Control of the explosion of the $L^1_{2+\gamma,v}$-norm at time 0

In this section we show that $M_{2+\gamma}$ explodes at most like $1/t$ when t goes to zero. This is the purpose of the next proposition.

Proposition 6.5.5 *Let $f_0(v)$ be in $L^1_{2,v} \cap L^\infty_v$.*
Let f be a non-negative solution of (6.1.3) in $L^\infty_{loc}\left([0,T_0), L^1_{2,v} \cap L^\infty_v\right)$, with initial value f_0, satisfying the conservation of mass and energy.
Then there exists $0 < \tau < T_0$ and there exists $C_\tau > 0$ such that

$$\forall t \in (0, \tau], \quad M_{2+\gamma}(t) \leqslant \frac{C_\tau}{t}.$$

Proof of Proposition 6.5.5 We take $0 < t \leqslant T < T_0$. Thanks to Proposition 6.5.1 we know that $M_s(t)$ is bounded by a constant $C_T > 0$ depending on T and t.

The technical Lemma 6.1.1 yields

$$\frac{d}{dt} M_{2+\gamma}(t) = \frac{C_\phi}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} |v - v_*|^\gamma \, f f_* K_{1+\gamma/2}(v, v_*) \, dv_* dv, \tag{6.5.4}$$

where $K_{1+\gamma/2}(v, v_*)$ is given in Lemma 6.3.1 for $\psi(x) = x^{1+\gamma/2}$. We can use point $i)$ of Lemma 6.3.1 since f is bounded on $[0, T]$. Hence, (6.5.4) becomes

$$\frac{d}{dt} M_{2+\gamma}(t) \leqslant \frac{C_\phi}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} |v - v_*|^\gamma \, f f_* \left[C'_G \, |v|^{1+\gamma/2} \, |v_*|^{1+\gamma/2} - C'_H \, |v|^{2+\gamma} \right] \, dv_* dv,$$

where C_G and C_H are given in Lemma 6.3.1 (up to a multiplicative constant only depending on γ).

Since f preserves the mass and the energy and since $0 \leqslant \gamma \leqslant 1$, a mere triangular inequality in the first term in the integral yields

$$\frac{d}{dt}M_{2+\gamma}(t) \leqslant C_\phi C_\gamma \left\|f_0\right\|_{L^1_{2,v}} M_{1+3\gamma/2} - \frac{C_\phi C'_H}{2}\left\|f_0\right\|_{L^1_v} M_{2+\gamma}, \tag{6.5.5}$$

where C_γ has been defined in (6.4.2). Besides,

$$\forall \varepsilon > 0, \quad |v|^{1+3\gamma/2} \leqslant \varepsilon\, |v|^{2+\gamma} + C_\varepsilon\, |v|^{2\gamma}.$$

Then, because $2\gamma \leqslant 2$ and taking ε small enough, (6.5.5) shows that there exists c_T and C_T positive constants depending on T and independent of t such that

$$\frac{d}{dt}M_{2+\gamma}(t) \leqslant c_T - C_T M_{2+2\gamma}(t).$$

We have the following Holder's inequality

$$M_{2+\gamma} \leqslant M_2^{1/2} M_{2+2\gamma}^{1/2}$$

and therefore

$$\frac{d}{dt}M_{2+\gamma}(t) \leqslant c_T - C_T M^2_{2+\gamma}(t).$$

So we have two cases to consider. Either $M_{2+\gamma}(t)$ is bounded when t goes to 0 and then Proposition 6.5.5 is proven. Or there exists τ such that $M_{2+\gamma}(\tau) \geqslant \sqrt{2c_T/C_T}$ and then for all $t \leqslant \tau$, $M_{2+\gamma}(t)$ is decreasing and

$$\forall t \in (0,\tau], \quad \frac{d}{dt}M_{2+\gamma}(t) \leqslant -\frac{C_T}{2} M^2_{2+\gamma}(t),$$

which gives the expected bound on $M_{2+\gamma}(t)$. ∎

6.6 Uniqueness of solution for the Boltzmann-Nordheim equation

In this section we prove that there exists at most one solution to the Boltzmann-Nordheim equation for bosons (6.1.3) in the space $L^\infty_{\text{loc}}\left([0,T_0), L^1_{2,v}\cap L^\infty_v\right)$ for $T_0 > 0$.

The proof relies on precise estimates on the L^1_v, the $L^1_{2,v}$ and the L^∞_v-norms of the difference of two solutions. The main problem arises when, in order to control the $L^1_{2,v}$-norm, one has to deal with terms of the form $|v|^{2+\gamma}$. A careful study allows us to control this weight thanks to the $2+\gamma$ moment of the solution (which has been studied in Section 6.5.2) and the fact that if f in $L^\infty_{\text{loc}}\left([0,T_0), L^1_{2,v}\cap L^\infty_v\right)$ is a solution to the Boltzmann-

Nordheim equation then, by Proposition 6.4.1,

$$\forall T \in [0, T_0),\ \exists N_T > 0, \quad \sup_{[0,T]\times\mathbb{R}^d} E[f](t,v) \leqslant N_T, \tag{6.6.1}$$

where

$$E[f](t,v) = f(t,v) + \int_0^t (1+|v|^\gamma)\, f(s,v)\, ds.$$

These estimates lead to a system of three differential and non-differential inequalities that we solve thanks to an extended Nagumo's uniqueness criterion for small times and an extended Gronwall lemma for larger times.

In the end, we prove the following theorem.

Theorem 6.6.1 *Let $h_0(v)$ be in $L^1_{2,v} \cap L^\infty_v$.*
Let f and g be two non-negative solutions of (6.1.3) in $L^\infty_{loc}\left([0,T_0), L^1_{2,v} \cap L^\infty_v\right)$ satisfying the conservation of mass and energy.
If f and g have the same initial data h_0 then $f = g$ on $[0, T_0)$.

For now on we take f and g satisfying the assumptions of Theorem 6.6.1. In order to shorten notations we still denote by N_T the maximum of N_T for f and for g, defined in (6.6.1).

6.6.1 Evolution of $\|f - g\|_{L^1_v}$

First of all we can write the following algebraic identity which we are going to use throughout this section.

$$abc - def = \frac{1}{2}(a-d)(bc+ef) + \frac{a+d}{4}\left[(b-e)(c+f) + (c-f)(b+e)\right]. \tag{6.6.2}$$

We have the following differential inequality,

Lemma 6.6.2 *For all T in $[0,T_0)$, there exists $C_T > 0$ such that for all t in $[0,T]$,*

$$\frac{d}{dt}\|f-g\|_{L^1_v} \leqslant C_T\left[\|f-g\|_{L^1_v} + \|f-g\|_{L^1_{2,v}}\right].$$

C_T only depends on C_Φ, l_b, γ, $\|h_0\|_{L^1_{2,v}}$ and N_T (see (6.6.1)).

Proof of Lemma 6.6.2 We fix T in $[0,T_0)$ and we consider t in $[0,T]$. Thanks to the integral property of Q, Lemma 6.1.1,

$$\begin{aligned}\frac{d}{dt}\|f-g\|_{L^1_v} &= \int_{\mathbb{R}^d} \mathrm{sgn}(f-g)\partial_t(f-g)\,dv = \int_{\mathbb{R}^d} \mathrm{sgn}(f-g)\left(Q(f)-Q(g)\right)\,dv \\ &= \frac{C_\Phi}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} b(\cos\theta)\,|v-v_*|^\gamma P(f,g)\left[\Psi'_*+\Psi'-\Psi_*-\Psi\right]\,d\sigma dv_* dv,\end{aligned} \tag{6.6.3}$$

where we wrote $\Psi(t,v)=\mathrm{sgn}(f-g)(t,v)$ and

$$P(f,g)=ff_*(1+f'+f'_*)-gg_*(1+g'+g'_*) \tag{6.6.4}$$

We easily have that $|[\Psi'_*+\Psi'-\Psi_*-\Psi|\leqslant 4$.
Furthermore, using the arithmetic identity (6.6.2) we compute

$$\begin{aligned}|P(f,g)|\leqslant &\frac{1+2N_T}{2}\,|f-g|\,(f_*+g_*)+\frac{1+N_T}{2}(f+g)\,|f_*-g_*| \\ &+\frac{1}{4}(f+g)(f_*+g_*)\Big[\,|f'-g'|+|f'_*-g'_*|\,\Big].\end{aligned}$$

We plug these two inequalities inside (6.6.3) which we cut into for integrals. The change of variable $(v,v_*)\to(v_*,v)$ shows that the first two terms are equal as well as the last two. Thus,

$$\begin{aligned}\frac{d}{dt}\|f-g\|_{L^1_v} \leqslant\, &2C_\Phi l_b\,(1+2N_T)\int_{\mathbb{R}^d\times\mathbb{R}^d}|v-v_*|^\gamma\,|f-g|\,(f_*+g_*)\,dv_*dv \\ &+C_\Phi\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} b(\cos\theta)\,|v-v_*|^\gamma(f+g)(f_*+g_*)\,|f'-g'|\,d\sigma dv_* dv.\end{aligned}$$

The first integral is easily dealt with by a mere triangular inequality together with (6.4.2). In the second integral we use the change of variable $(v,v_*)\to(v',v'_*)$ and the terms in v' and v'_* are dealt with Lemma 6.4.4, inequality (6.4.4) with $\lambda=1$. Therefore,

$$\begin{aligned}\frac{d}{dt}\|f-g\|_{L^1_v} \leqslant\, &2C_\Phi l_b\,(1+2N_T)\,C_\gamma\left(\|h_0\|_{L^1_{2,v}}\|f-g\|_{L^1_v}+\|h_0\|_{L^1_v}\|f-g\|_{L^1_{2,v}}\right) \\ &+C_\gamma C_\Phi l_b\,\|h_0\|_{L^1_{2,v}}\left[2C(1)N_T+2^{d-1}C_\gamma\,\|h_0\|_{L^1_{2,v}}\right]\int_{\mathbb{R}^d}(1+|v|^\gamma)\,|f-g|\,dv.\end{aligned}$$

By setting C_T the maximum among the multiplicative constants above, we reach the inequality of Lemma 6.6.2. ∎

6.6.2 Evolution of $\|f-g\|_{L^1_{2,v}}$

The differential inequality satisfies by $\|f-g\|_{L^1_{2,v}}$ is more intricate and requires to control the $(2+\gamma)^{\text{th}}$ moments of g and f by the L^1_v-norm of the difference. This is achieve thanks to the next lemma.

Lemma 6.6.3 *For all T in $[0,T_0)$, there exists $C_T>0$ such that for all t in $[0,T]$,*

$$\frac{d}{dt}\|f-g\|_{L^1_{2,v}} \leqslant C_T\left[M_{2+\gamma}(t)\|f-g\|_{L^1_v}+\|f-g\|_{L^1_{2,v}}+\|f-g\|_{L^\infty_{[0,T],v}}\right].$$

$M_{2+\gamma}$ is the $(2+\gamma)^{th}$ moment of $f+g$ (see (6.2.1)) and C_T only depends on C_Φ, l_b, γ, $\|h_0\|_{L^1_{2,v}}$ and N_T (see (6.6.1)).

Proof of Lemma 6.6.3 We fix T in $[0,T_0)$ and we consider t in $[0,T]$. As in the proof of Lemma 6.6.2 we have

$$\frac{d}{dt}\|f-g\|_{L^1_v} = \frac{C_\Phi}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} b|v-v_*|^\gamma P(f,g)\left[\Psi'_*+\Psi'-\Psi_*-\Psi\right]\,d\sigma dv_* dv, \quad (6.6.5)$$

where this time we wrote $\Psi(t,v)=\mathrm{sgn}(f-g)(t,v)\left(1+|v|^2\right)$ and $P(f,g)$ is still given by (6.6.4).

Using the algebraic inequality (6.6.2) and using the change of variable $(v,v_*)\to(v_*,v)$ we obtain

$$\frac{d}{dt}\|f-g\|_{L^1_v} = C_\Phi\left(\frac{1}{2}I_1+\frac{1}{4}I_2+\frac{1}{8}I_3+\frac{1}{4}I_4\right) \quad (6.6.6)$$

with

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} b|v-v_*|^\gamma\left[G(\Psi)-\Psi\right](f-g)(f_*+g_*)\,d\sigma dv_* dv,\\
I_2 &= \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} b|v-v_*|^\gamma\left[G(\Psi)-\Psi\right](f-g)(f_*(f'+f'_*)+g_*(g'+g'_*))\,d\sigma dv_* dv,\\
I_3 &= \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} b|v-v_*|^\gamma\left[G(\Psi)-\Psi\right](f+g)(f_*-g_*)(f'+f'_*+g'+g'_*)\,d\sigma dv_* dv,\\
I_4 &= \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} b|v-v_*|^\gamma\left[G(\Psi)-\Psi\right](f+g)(f_*+g_*)(f'_*-g'_*)\,d\sigma dv_* dv,
\end{aligned}$$

where we defined $G(\Psi)=\Psi'_*+\Psi'-\Psi_*$ and we have straightforwardly

$$|G(\Psi)|\leqslant 3+\left|v'\right|^2+\left|v'_*\right|^2+|v_*|^2 = 2\left(1+|v_*|^2\right)+\left(1+|v|^2\right). \quad (6.6.7)$$

Thanks to the latter bound on $G(\Psi)$ and the fact $\Psi.(f-g)=\left(1+|v|^2\right)|f-g|$ we find

$$\begin{aligned}
|I_1| &\leqslant 2l_b\int_{\mathbb{R}^d\times\mathbb{R}^d}\left(1+|v_*|^2\right)\left(|v|^\gamma+|v_*|^\gamma\right)|f-g|\,(f_*+g_*)\,dv dv_*\\
&\leqslant 4l_b C_\gamma^2\|h_0\|_{L^1_{2,v}}\|f-g\|_{L^1_{2,v}}+2l_b C_\gamma M_{2+\gamma}\|f-g\|_{L^1_v}, \quad (6.6.8)
\end{aligned}$$

where C_γ has been defined in (6.4.2).

The term I_2 is dealt exactly the same way, remembering the f and g are bounded by N_T.

$$|I_2| \leqslant 8N_T l_b C_\gamma^2 \|h_0\|_{L^1_{2,v}} \|f-g\|_{L^1_{2,v}} + 4N_T l_b C_\gamma M_{2+\gamma} \|f-g\|_{L^1_v}. \tag{6.6.9}$$

In the term I_3 we make the change of variable $(v,v_*) \to (v_*,v)$ and after bounding the terms in v' and v'_* by N_T we recover $|I_1|$. Therefore,

$$|I_3| \leqslant 8N_T l_b C_\gamma^2 \|h_0\|_{L^1_{2,v}} \|f-g\|_{L^1_{2,v}} + 4N_T l_b C_\gamma M_{2+\gamma} \|f-g\|_{L^1_v}. \tag{6.6.10}$$

The last term, I_4, is more intricate and we have to deal with it carefully so that the terms of order $2+\gamma$ in v only appear in front of the $L^1_{2,v}$-norm of $f-g$.

First of all, thanks to (6.6.7), we have

$$|G(\Psi)-\Psi| \leqslant 2\left[\left(1+|v|^2\right)+\left(1+|v_*|^2\right)\right],$$

so that

$$|I_4| \leqslant 2\int_{\mathbb{R}^{2d}\times\mathbb{S}^{d-1}} b|v-v_*|^\gamma \left[\left(1+|v|^2\right)+\left(1+|v_*|^2\right)\right](f+g)(f_*+g_*)\left|f'_*-g'_*\right| \, d\sigma dv_* dv.$$

Then, the change of variable $(v,v_*) \to (v_*,v)$ followed by the change of variable $\sigma \to -\sigma$, which brings v' to v'_* and reciprocally, gives

$$\begin{aligned}&\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} b(\cos\theta)|v-v_*|^\gamma\left(1+|v_*|^2\right)(f+g)(f_*+g_*)\left|f'_*-g'_*\right| \, d\sigma dv_* dv \\ &\quad = \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} b(-\cos\theta)|v-v_*|^\gamma\left(1+|v|^2\right)(f+g)(f_*+g_*)\left|f'_*-g'_*\right| \, d\sigma dv_* dv.\end{aligned}$$

Therefore, if we denote $\tilde{b}(x) = b(x)+b(-x)$ we obtain

$$\begin{aligned}|I_4| &\leqslant 2\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} \tilde{b}|v-v_*|^\gamma\left(1+|v|^2\right)(f+g)(f_*+g_*)\left|f'_*-g'_*\right| \, d\sigma dv_* dv \\ &\leqslant 16 l_b C_\gamma \|h_0\|^2_{L^1_{2,v}} \|f-g\|_{L^\infty_{[0,T],v}} \\ &\quad +2\int_{\mathbb{R}^d} dv\, \left(1+|v|^2\right)|v|^\gamma(f+g)\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \tilde{b}(f_*+g_*)\left|f'_*-g'_*\right| \, d\sigma dv_*.\end{aligned} \tag{6.6.11}$$

The last integral is dealt with in the same way as in the proof of Lemma 6.4.4, (6.4.6), by studying the cases v' is close to v and when not. We use the Carleman representation of this integral, which reads, with $v_* = v'_* + v' - v$,

$$\begin{aligned}
\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \tilde{b}(f_* + g_*)\left|f'_* - g'_*\right| \, d\sigma dv_* &= \int_{\mathbb{R}^d\times E_{vv'}} \frac{\tilde{b}}{|v-v'|^{d-1}}(f_*+g_*)\left|f'_*-g'_*\right| \, dv' dv'_* \\
&\leqslant \int_{\mathbb{R}^d} \left|f'_* - g'_*\right| \left(\int_{\mathbb{R}^d} \frac{\tilde{b}}{|v-v'|^{d-1}}(f_*+g_*)\,dv'\right) dv'_* \\
&\leqslant l_b \int_{\mathbb{R}^d} \left|f'_* - g'_*\right| \left(2C(1)N_T + 2^{d-2}\times 2\left\|h_0\right\|_{L^1_v}\right) dv'_*,
\end{aligned}$$

where $C(1)$ is defined in (6.4.7).

We plug the latter inequality into (6.6.11) to obtain the following control from above

$$|I_4| \leqslant 16 l_b C_\gamma \left\|h_0\right\|^2_{L^1_{2,v}} \left\|f-g\right\|_{L^\infty_{[0,T],v}} + 4 l_b C_\gamma \left(C(1)N_T + 2^{d-2}\left\|h_0\right\|_{L^1_v}\right) M_{2+\gamma}\left\|f-g\right\|_{L^1_v}. \tag{6.6.12}$$

To conclude we gather (6.6.8), (6.6.9), (6.6.10) and (6.6.12) into (6.6.6) ■

6.6.3 Control of $\|f-g\|_{L^\infty_{[0,T],v}}$

For the L^∞-norm of $f-g$, we derive the following inequality

Lemma 6.6.4 *There exists τ in $[0, T_0)$, there exists $C_\tau > 0$ such that for all t in $[0,\tau]$ and for all m in $\mathbb{N}$,*

$$\|f-g\|_{L^\infty_{[0,t],v}} \leqslant C_\tau \sup_{[0,t],v} \|f-g\|_{L^1_{2,v}}.$$

C_τ only depends on C_Φ, l_b, γ, $\|h_0\|_{L^1_{2,v}}$, τ and N_τ (see (6.6.1)).

Proof of Lemma 6.6.2 We fix T in $[0, T_0)$ and we consider t in $[0, T]$. We have the following decomposition

$$\begin{aligned}
f(t,v) - g(t,v) &= C_\Phi \int_0^t \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} b(\cos\theta)\,|v-v_*|^\gamma\, P(f',g')\,d\sigma dv_* ds \\
&\qquad - C_\Phi \int_0^t \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} b(\cos\theta)\,|v-v_*|^\gamma\, P(f,g)\,d\sigma dv_* ds \\
&= \int_0^t J_1(s,v)\,ds + \int_0^t J_2(s,v)\,ds,
\end{aligned}$$

where P is given by (6.6.4).

We have that at $t=0$, $f(0,v)-g(0,v)=0$ for all v and therefore

$$\begin{aligned} |f-g|(t,v) &= \int_0^t \partial_t |f-g|(s,v)\,ds = \int_0^t \mathrm{sgn}(f-g)(s,v)\partial_t(f(s,v)-g(s,v))\,ds \\ &\leqslant \int_0^t |J_1|(s,v)\,ds + \int_0^t \mathrm{sgn}(f-g)(s,v)J_2(s,v)\,ds. \end{aligned} \tag{6.6.13}$$

We start by the first term J_1. Using the algebraic equality (6.6.2) and the definition of P one can bound $P(f',g')$ by

$$\begin{aligned} \left|P(f',g')\right| \leqslant &(1+N_T)\left|f'-g'\right|(f'_*+g'_*) + \frac{(1+2N_T)}{2}\left|f'_*-g'_*\right|(f'+g') \\ &+\frac{1}{2}\left\|f-g\right\|_{L^\infty_{[0,t],v}}(f'+g')(f'_*+g'_*). \end{aligned}$$

The change of variable $\sigma\to-\sigma$ sends v' to v'_* and reciprocally. Therefore we have

$$\begin{aligned} \int_0^t |J_1|\;ds \leqslant &2C_\Phi(1+2N_T)\int_0^t\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \tilde{b}\,(\cos\theta)\,|v-v_*|^\gamma\left|f'_*-g'_*\right|(f'+g')\,d\sigma dv_* ds \\ &+\frac{C_\Phi}{2}\left\|f-g\right\|_{L^\infty_{[0,t],v}}\int_0^t\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} b\,(\cos\theta)\,|v-v_*|^\gamma\left|f'_*+g'_*\right|(f'+g')\,d\sigma dv_* ds, \end{aligned}$$

where we defined $\tilde{b}(x)=b(x)+b(-x)$.

For both integrals, we use Lemma 6.4.4 with $\lambda>0$ to be chosen later. This yields

$$\begin{aligned} \int_0^t |J_1|\;ds \leqslant &4C_\gamma C_\Phi l_b(1+2N_T)\sup_{[0,t]}\left(\left\|f-g\right\|_{L^1_{2,v}}\right)\left[2C(1)N_T + 2^{d-2}t\left\|h_0\right\|_{L^1_{2,v}}\right] \\ &+\frac{1}{2}C_\gamma C_\Phi l_b\left\|f-g\right\|_{L^\infty_{[0,t],v}} 2\left\|h_0\right\|_{L^1_{2,v}}\left[2C(\lambda)N_T + \frac{2^{d-2}}{\lambda^{d-1}}2t\left\|h_0\right\|_{L^1_v}\right]. \end{aligned}$$

We choose λ small enough such that

$$2C_\gamma C_\Phi l_b\left\|h_0\right\|_{L^1_{2,v}} C(\lambda)N_T \leqslant \frac{1}{4}, \tag{6.6.14}$$

and we define $\tau<T$ such that

$$C_\gamma C_\Phi l_b\left\|h_0\right\|_{L^1_{2,v}}\frac{2^{d-1}}{\lambda^{d-1}}\left\|h_0\right\|_{L^1_v}\tau \leqslant \frac{1}{4}. \tag{6.6.15}$$

These choices of constants lead to

$$\forall t\in[0,\tau],\quad \int_0^t |J_1|\;ds \leqslant \frac{1}{2}\left\|f-g\right\|_{L^\infty_{[0,t],v}} + C_\tau\sup_{[0,t]}\left\|f-g\right\|_{L^1_{2,v}}, \tag{6.6.16}$$

with C_τ a constant depending on τ.

We now turn to the last term J_2 in (6.6.13). By using (6.6.2) and the change of variable $\sigma \to -\sigma$ we get

$$\begin{aligned}
&\int_0^t \operatorname{sgn}(f-g)(s,v)J_2(s,v)\,ds \\
&\quad\leqslant \frac{C_\Phi}{2}\int_0^t\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} b\,|v-v_*|^\gamma \operatorname{sgn}(f-g)(g-f) \\
&\qquad\qquad \times \left(f_*(1+f'+f'_*)+g_*(1+g'+g'_*)\right)dv_*ds \\
&\quad + \frac{C_\Phi}{4}(2+4N_T)l_b\int_0^t\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} |v-v_*|^\gamma\,|f_*-g_*|\,(f+g)\,dv_*ds \\
&\quad + \frac{C_\Phi}{4}\int_0^t\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \tilde{b}\,|v-v_*|^\gamma\,(f+g)(f_*+g_*)\left|f'_*-g'_*\right|\,d\sigma dv_*ds,
\end{aligned} \tag{6.6.17}$$

where we wrote $\tilde{b}(x)=b(x)+b(-x)$.

The second integral is easily dealt with and we have

$$\int_0^t\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} |v-v_*|^\gamma\,|f_*-g_*|\,(f+g)\,dv_*ds \leqslant 2N_TC_\gamma \sup_{[0,t],v}\|f-g\|_{L^1_{2,v}} \tag{6.6.18}$$

The third and last integral is a bit more intricate and we use the Carleman representation of the integral against (σ,v_*). We emphasize that in the integral against (v',v'_*) we denote $v_*=v'_*+v'-v$. This yields

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \tilde{b}\,|v-v_*|^\gamma\,(f+g)(f_*+g_*)\left|f'_*-g'_*\right|\,d\sigma dv_*ds \\
&\quad\leqslant \int_0^t (g+f)\left(\int_{\mathbb{R}^d}\left|f'_*-g'_*\right|\left(1+\left|v'_*\right|^\gamma\right)\left[\int_{\mathbb{R}^d}\frac{(1+|v'|^\gamma)}{|v-v'|^{d-1}}(f_*+g_*)\,dv'\right]dv'_*\right)ds
\end{aligned}$$

and we follow the idea developed in Lemma 6.4.4 with $\lambda=1$

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \tilde{b}\,|v-v_*|^\gamma\,(f+g)(f_*+g_*)\left|f'_*-g'_*\right|\,d\sigma dv_*ds \\
&\quad\leqslant \int_0^t \left(1+|v|^\gamma\right)(g+f) \\
&\qquad\left(\int_{\mathbb{R}^d}\left|f'_*-g'_*\right|\left(1+\left|v'_*\right|^\gamma\right)\left[2N_TC(1)+2^{d-1}\|h_0\|_{L^1_{2,v}}\left(1+\left|v'_*\right|^\gamma\right)\right]dv'_*\right)ds.
\end{aligned}$$

γ is in $[0,1]$ and thus we have

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \tilde{b}\,|v-v_*|^\gamma\,(f+g)(f_*+g_*)\left|f'_*-g'_*\right|\,d\sigma dv_*ds \\
&\quad\leqslant\quad 2C_\gamma N_T\left[2N_TC(1)+2^{d-1}\|h_0\|_{L^1_{2,v}}\right]\sup_{[0,t],v}\|f-g\|_{L^1_{2,v}}
\end{aligned} \tag{6.6.19}$$

The first integral on the left-hand side of (6.6.17) is negative because f and g are both non-negative solutions. Thus we plug (6.6.18) and (6.6.19) in (6.6.17), which gives

$$\forall t \in [0,T], \quad \int_0^t \operatorname{sgn}(f-g)(s,v) J_2(s,v)\, ds \leqslant C_T \sup_{[0,t],v} \|f-g\|_{L^1_{2,v}}, \tag{6.6.20}$$

where C_T is a constant depending on T.

Therefore, if we take τ given by (6.6.15), we can use (6.6.16) and (6.6.20) inside (6.6.13). This yields the result given in the statement of Lemma 6.6.4. ■

6.6.4 Uniqueness result: proof of Theorem 6.6.1

In this section we prove the uniqueness result stated in Theorem 6.6.1.
We set τ to be the minimum between τ in Proposition 6.5.5 and τ in Lemma 6.6.4. Throughout this section, C will stand for a positive generic constant depending only on τ, N_τ, on the parameters of the operator Q and on $\|h_0\|_{L^1_{2,v}}$.

We use Lemma 6.6.2, Lemma 6.6.3 and Lemma 6.6.4 together to see that there exists τ such that if t belongs to $[0,\tau]$ then

$$\begin{cases} \dfrac{d}{dt}\|f-g\|_{L^1_v} \leqslant C\left[\|f-g\|_{L^1_v} + \|f-g\|_{L^1_{2,v}}\right] \\ \dfrac{d}{dt}\|f-g\|_{L^1_{2,v}} \leqslant C\left[M_{2+\gamma}(t)\,\|f-g\|_{L^1_v} + \|f-g\|_{L^1_{2,v}} + \|f-g\|_{L^\infty_{[0,T],v}}\right] \\ \|f-g\|_{L^\infty_{[0,t],v}} \leqslant C \sup\limits_{[0,t],v} \|f-g\|_{L^1_{2,v}}. \end{cases} \tag{6.6.21}$$

The L^1_v, $L^1_{2,v}$ and $L^\infty_{[0,t],v}$-norms of f and g are bounded by assumption. Therefore, the first inequality in (6.6.21) gives $\|f-g\|_{L^1_v} \leqslant Ct$.
Moreover, Proposition 6.5.5 says that for t in $(0,\tau]$, $M_{2+\gamma}(t) \leqslant \frac{C_\tau}{t}$, where C_τ has been defined in Proposition 6.5.5 and therefore the second inequality in (6.6.21) gives $\|f-g\|_{L^1_{2,v}} \leqslant Ct$.
We can use these results to get $\|f-g\|_{L^\infty_{[0,t],v}} \leqslant Ct$ in the third inequality in (6.6.21).

We can use this argument again to obtain that in fact $\|f-g\|_{L^1_v} \leqslant Ct^2$, $\|f-g\|_{L^1_{2,v}} \leqslant Ct^2$ and $\|f-g\|_{L^\infty_{[0,t],v}} \leqslant Ct^2$. By induction we obtain that for all n in $\mathbb{N}$, $\|f-g\|_{L^1_v} \leqslant Ct^n$, $\|f-g\|_{L^1_{2,v}} \leqslant Ct^n$ and $\|f-g\|_{L^\infty_{[0,t],v}} \leqslant Ct^n$.

Remark 6.6.5 *We emphasize here that one would like to take the limit as n goes to $+\infty$ to obtain the uniqueness on short times. Unfortunately, C is a generic constant and we do not explicitely mentionned that this constant is increasing with n.*

Therefore the three norms are time-differentiable at 0 with their time-derivatives being 0 at $t = 0$. Therefore we can use (6.6.21) for all t in $[0, \tau]$ and combining the second with the third one we get

$$\forall T \in [0, \tau], \quad \frac{d}{dt} \|f - g\|_{L^1_{2,v}} \leqslant \frac{K_1}{t} \|f - g\|_{L^1_{2,v}} + K_2 \sup_{[0,t],v} \|f - g\|_{L^1_{2,v}}, \tag{6.6.22}$$

where $K_1, K_2 > 0$ only depend on τ, h_0 and the operator Q.

We fix $n \geqslant K_1$ and we defined $X(t) = \|f - g\|_{L^1_{2,v}} / t^n$. We have that $X(t) \leqslant C_\tau t^2$ which means that $X(t)$ is continuous at 0 and also right-differentiable at 0 with $X'(0) = 0$.

We differentiate $X(t)$ in the same spirit that Nagumo's fixed point theorem. The main difference relies on the fact that we shall have to deal with terms of the form $\sup X$ in the differential inequality. Thanks to (6.6.22) we have

$$\begin{aligned} \frac{d}{dt} X(t) &= \frac{1}{t^n} \left(\frac{d}{dt} \|f - g\|_{L^1_{2,v}} - \frac{n}{t} \|f - g\|_{L^1_{2,v}} \right) \\ &\leqslant \frac{K_2}{t^n} \sup_{[0,t],v} \|f - g\|_{L^1_{2,v}} \\ &\leqslant K_2 \sup_{[0,t],v} X(s). \end{aligned}$$

We integrate in time between 0 and t and because $X(t)$ is positive we obtain

$$X(t) \leqslant t K_2 \sup_{[0,t],v} X(s)$$

and by induction we obtain

$$\forall t \in [0, \tau], \, \forall m \in \mathbb{N}, \quad X(t) \leqslant (t K_2)^m \sup_{[0,t],v} X(s).$$

Hence, we can take the limit as m goes to $+\infty$ for all $t < 1/K_2$. Which means that

$$\forall t \in [0, \min(\tau, 1/K_2)], \quad X(t) = 0$$

and as a result, if we denote $\tau_1 = \min(\tau, 1/K_2)$,

$$\forall t \in [0, \tau_1], \quad \|f - g\|_{L^1_{2,v}} = 0. \tag{6.6.23}$$

To conclude for all time in $[0, \tau]$ we know that for $t \geqslant \tau_1$, $M_{2+\gamma}(t)$ is bounded by C_τ / τ_1 (see Proposition 6.5.5) and therefore (6.6.22) becomes

$$\forall T \in [\tau_1, \tau], \quad \frac{d}{dt} \|f - g\|_{L^1_{2,v}} \leqslant K_{\tau_1} \|f - g\|_{L^1_{2,v}} + K_2 \sup_{[0,t],v} \|f - g\|_{L^1_{2,v}},$$

which we can multiply by $e^{-K_{\tau_1}t}$, which is decreasing in t, to use an extended Gronwall lemma:

$$\forall T \in [\tau_1, \tau], \quad \frac{d}{dt}\left(e^{-K_{\tau_1}t}\|f-g\|_{L^1_{2,v}}\right)(t) \leqslant K_2 \sup_{[0,t],v}\left(e^{-K_{\tau_1}s}\|f-g\|_{L^1_{2,v}}(s)\right),$$

which once again gives uniqueness between $[\tau_1, 2\tau_1]$ and by induction we obtain that

$$\|f-g\|_{L^1_{2,v}} = 0$$

on $[0,\tau]$.

Finally, the uniqueness on $[0,T]$ is obtain by interating the latter proof starting from τ to go up to 2τ. Indeed, τ only depends on the operator Q, N_T and on $\|h_0\|_{L^1_{2,v}}$ which is equal to $\|g\|_{L^1_{2,v}}$ and $\|f\|_{L^1_{2,v}}$ at time τ since these two solutions preserves mass and energy. Therefore starting the proof at τ will give us the uniqueness between τ and 2τ. By induction we have that $f=g$ on $[0,T]$ for all $0 \leqslant T < T_0$.

6.7 Local existence of solutions

This section is dedicated to proving the following theorem

Theorem 6.7.1 *Let $f_0(v)$ be in $L^1_{2,v} \cap L^\infty_v$.*
Then there exists $T_0 > 0$, depending only on C_Φ, l_b, γ, $\|f_0\|_{L^1_{2,v}}$ and $\|f_0\|_{L^\infty_v}$, such that there exists f in $L^\infty_{loc}\left([0,T_0), L^1_{2,v} \cap L^\infty_v\right)$ solution on (6.1.3) on $[0,T_0) \times \mathbb{R}^d$ such that

- $T_0 = +\infty$ *or* $\lim_{T \to T_0^-} \|f\|_{L^\infty_{[0,T]\times\mathbb{R}^d}} = +\infty$,
- *f preserves the mass, momentum and energy of f_0,*
- *for all $T < T_0$,*

$$\sup_{[0,T]\times\mathbb{R}^d}\left(f(t,v) + \int_0^t (1+|v|^\gamma)\, f(s,v)\, ds\right) < \infty.$$

For now on, we take f_0, non identically 0, in $L^1_{2,v} \cap L^\infty_v$.

The proof of this theorem relies on a time discretisation of equation (6.1.3) together with an approximation of the Boltzmann-Nordheim operator Q. This raises a sequence of functions $(f_n)_{n\in\mathbb{N}}$ that will be proven to be approximations of a solution of the Boltzmann-Nordheim equation. This step is done by establishing the weak convergence of the sequence $(f_n)_{n\in\mathbb{N}}$ to the unique solution of (6.1.3) (see Theorem 6.6.1).

We shall first derive some properties for truncated operators approximating the Boltzmann-Nordheim operator. Then we define some constants and construct a sequence of functions and finally show that this sequence convergences to a solution of equation (6.1.3).

6.7.1 Some properties of truncated operators

This idea of approximating the collision kernel in the case of hard potentials is a common one in the Boltzmann equation litterature (see for instance [2] or [77]). We consider now the following truncated operators, where n is a positive integer,

$$Q_n(f) = C_\Phi \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} (|v-v_*|\wedge n)^\gamma\, b\left[f'f'_*(1+f+f_*) - ff_*(1+f'+f'_*)\right] dv_* d\sigma.$$

where $a\wedge b = \min(a,b)$.

We associate to these operator the natural decomposition (6.1.4) – (6.1.5):

$$Q_n(f) = -fQ_n^-(f) + Q_n^+(f).$$

The truncated operators are much easier to handle because they are easily bounded in $L^1_{2,v}\cap L^\infty_v$, which is not the case for the full Boltzmann-Nordheim operator Q. Indeed, we have the following controls on the negative part

Lemma 6.7.2 *Let f be in $L^1_{2,v}\cap L^\infty_v$. Then we have the following inequalities*

- $\left\| fQ_n^-(f)\right\|_{L^1_{2,v}} \leqslant C_\Phi l_b n^\gamma \left(1+2\left\|f\right\|_{L^\infty_v}\right)\left\|f\right\|^2_{L^1_{2,v}}$,
- $\left\| Q_n^-(f)\right\|_{L^\infty_v} \leqslant C_\Phi l_b n^\gamma \left(1+2\left\|f\right\|_{L^\infty_v}\right)\left\|f\right\|_{L^1_v}$,
- *if, moreover, $f\geqslant 0$, then there exists $C_\gamma>0$ (defined by (6.4.2)) such that*

$$\forall v\in\mathbb{R}^d,\quad Q_n^-(f)(v) \geqslant C_\Phi l_b\left(n^\gamma\wedge(1+|v|^\gamma)\right)\left\|f\right\|_{L^1_v} - C_\Phi C_\gamma l_b \left\|f\right\|_{L^1_{2,v}}.$$

Proof of Lemma 6.7.2 We have that

$$Q_n^-(f)(v) = C_\Phi \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} (n\wedge|v-v_*|)^\gamma\, b(\cos\theta) f_*\left[1+f'_*+f'\right] dv_* d\sigma.$$

Therefore the first two inequalities are trivially obtained by bounding f'_*+f' by $2\left\|f\right\|_{L^\infty_v}$ and the kernel by $n^\gamma b(\cos\theta)$ and then taking the supremum in v or integrating against $(1+|v|^2)\,dv$.

Now we suppose that f is positive and we copy the proof of Lemma 6.4.2, thus

$$\begin{aligned} Q_n^-(f)(v) &\geqslant C_\Phi \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} (n \wedge |v-v_*|)^\gamma \, b(\cos\theta) f_* \, dv_* d\sigma \\ &\geqslant C_\Phi l_b \left[\int_{|v-v_*|\leqslant n} |v-v_*|^\gamma \, f_* \, dv_* + \int_{|v-v_*|\geqslant n} n^\gamma f_* \, dv_* \right]. \end{aligned}$$

Since γ is in $[0,1]$, we know that the following triangular inequality holds

$$(|v|^\gamma - |v_*|^\gamma) \leqslant |v-v_*|^\gamma \leqslant (|v|^\gamma + |v_*|^\gamma).$$

This yields

$$\begin{aligned} Q_n^-(f)(v) &\geqslant C_\Phi l_b \left[\int_{|v-v_*|\leqslant n} \left((1+|v|^\gamma) - (1+|v_*|^\gamma)\right) f_* \, dv_* + \int_{|v-v_*|\geqslant n} n^\gamma f_* \, dv_* \right] \\ &\geqslant C_\Phi l_b \left[(n^\gamma \wedge (1+|v|^\gamma)) \, \|f\|_{L^1_v} - C_\gamma \int_{|v-v_*|\leqslant n} (1+|v_*|^2) f_* \, dv_* \right], \end{aligned}$$

by definition of C_γ, see (6.4.2). We obtained the expected lower bound. ∎

Moreover we also have the following bounds on the positive part

Lemma 6.7.3 *Let f be in $L^1_{2,v} \cap L^\infty_v$. Then we have the following inequalities*

- $\|Q_n^+(f)\|_{L^1_{2,v}} \leqslant C_\Phi l_b n^\gamma \left(1 + 2\|f\|_{L^\infty_v}\right) \|f\|^2_{L^1_{2,v}}$,
- *for all $\lambda > 0$,*

$$\left\|Q_n^+(f)\right\|_{L^\infty_v} \leqslant C_\Phi l_b C_\gamma \|f\|_{L^1_{2,v}} \left(1 + 2\|f\|_{L^\infty_v}\right) \left[\int_{|u-v|\leqslant\lambda} \frac{(n^\gamma \wedge (1+|u|^\gamma)) \, f(u)}{|u-v|^{d-1}} \, du + \frac{C_\gamma}{\lambda^{d-1}} \|f\|_{L^1_{2,v}} \right].$$

Proof of Lemma 6.7.3 For the first inequality, we just have to notice that, after the change of variable $(v', v'_*) \to (v, v_*)$ one obtains

$$\int_{\mathbb{R}^d} (1+|v|^\gamma) \, Q_n^+(f) \, dv = \int_{\mathbb{R}^d} (1+|v|^\gamma) \, f Q_n^-(f) \, dv,$$

and we deal with the $L^1_{2,v}$-norm the same way we did in proof of Lemma 6.7.2.

For the L^∞_v-norm, we use exactly the same approach than in Lemma 6.4.4, using Carleman representation. This yields

$$\left|Q_n^+(f)(v)\right| \leqslant 2^{d-2} C_\Phi l_b \left[1 + 2\left\|f\right\|_{L^\infty_v}\right] \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{(|v'-v'_*|\wedge n)^\gamma}{|v-v'|^{d-1}} f' f'_*\, dv' dv'_*. \qquad (6.7.1)$$

We just have to deal with the integral whether $(1+|v|^\gamma)$ is smaller than n^γ or not.

In the case where $(1+|v|^\gamma) \leqslant n^\gamma$ we bound the integral from above in the following way,

$$\begin{aligned} \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{(|v'-v'_*|\wedge n)^\gamma}{|v-v'|^{d-1}} f' f'_*\, dv' dv'_* &\leqslant \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|v'-v'_*|^\gamma}{|v-v'|^{d-1}} f' f'_*\, dv' dv'_* \\ &\leqslant C_\gamma \left\|f\right\|_{L^1_{2,v}} \int_{\mathbb{R}^d} \frac{1+|v'|^\gamma}{|v-v'|^{d-1}} f'\, dv', \qquad (6.7.2) \end{aligned}$$

since γ is in $[0,1]$ and therefore, thanks to (6.4.2),

$$|v'-v'_*|^\gamma \leqslant \left(|v'|^\gamma + |v'_*|^\gamma\right) \leqslant C_\gamma \left(1+|v'_*|^2\right)\left(1+|v'|^\gamma\right).$$

We obtain the expected result in the case $(1+|v|^\gamma) \leqslant n^\gamma$ by splitting the integral in (6.7.2) into small and large relative velocities $v - v_*$.

The case $(1+|v|^\gamma) \geqslant n^\gamma$ is dealt with in exactly the same way but we bound $(|v'-v'_*|\wedge n)$ by n in (6.7.2). This gives us the expected result. ■

6.7.2 Construction of a sequence of approximations

We now fix a positive integer n and we want to discretise in time the Boltzmann-Nordheim equation associated to the truncated operators Q_n. Thus we need to work on a closed interval. More precisely, we shall solve the truncated Boltzmann-Nordheim equation

$$\partial_t f_n = Q_n(f_n)$$

by a implicit Euler scheme on an interval $[0, T_0]$, T_0 not depending on n.

To this end, we require to fix some constants (like the ones appearing in Lemma 6.7.3) that we are going to define below.
First of all, in order to shorten notations, we define

$$C_L = C_\Phi l_b \left\|f_0\right\|_{L^1_v}, \qquad (6.7.3)$$

$$C_0 = \frac{C_\Phi l_b C_\gamma \left\|f_0\right\|_{L^1_{2,v}}}{\min(1, C_L)} \qquad (6.7.4)$$

where C_γ has been defined in (6.4.2). Then, we define

$$K_0 = \frac{2\left\|f_0\right\|_{L^\infty_v}}{\min\left(1, C_L\right)}. \tag{6.7.5}$$

We have that $|u|^{1-d}$ is integrable near 0 in $\mathbb{R}^d$. Therefore we can consider λ strictly positive such that

$$C(\lambda) = \int_{|u|\leqslant\lambda} \frac{1}{|u|^{d-1}}\, du \leqslant \frac{1}{2C_0\left(1+2K_0\right)}. \tag{6.7.6}$$

Now we are able to define the time interval we shall work on,

$$T_0 = \frac{K_0}{4C_0}\left[1 + \left(1+2K_0\right)\frac{C_\gamma}{\lambda^{d-1}}\left\|f_0\right\|_{L^1_{2,v}}\right]^{-1}. \tag{6.7.7}$$

We emphasize here that all the constants are independent of the integer n.

We consider the following explicit Euler scheme on $[0, T_0]$,

$$\left\{\begin{array}{l} f_n^{(0)}(v) = f_0(v) \\ f_n^{(k+1)}(v) = f_n^{(k)}(v)\left(1 - \Delta_n Q_n^-\left(f_n^{(k)}\right)\right) + \Delta_n Q_n^+\left(f_n^{(k)}\right), \quad \text{for } k \in \left\{0, \dots, \dfrac{T_0}{\Delta_n}\right\}, \end{array}\right. \tag{6.7.8}$$

where Q_n^- and Q_n^+ have been defined in (6.1.4) – (6.1.5). Δ_n is the time step such that

$$\Delta_n = \min\left(1, \frac{1}{2C_\Phi l_b n^\gamma \left\|f_0\right\|_{L^1_v}\left[1+2K_0\right]}\right). \tag{6.7.9}$$

We first need to prove that the sequence $\left(f_n^{(k)}\right)_{k\in\left\{0,\dots,\frac{T_0}{\Delta_n}\right\}}$ is well-defined. This is the purpose of the next proposition.

Proposition 6.7.4 *For all k in $\{0, \dots, T_0/\Delta_n\}$, we have that $f_n^{(k)}$, see (6.7.8), is well-defined and*

i) $f_n^{(k)} \geqslant 0$,

ii) $\left\|f_n^{(k)}\right\|_{L^1_v} = \left\|f_0\right\|_{L^1_v}$, $\left\||v|^2 f_n^{(k)}\right\|_{L^1_v} = \left\||v|^2 f_0\right\|_{L^1_v}$ *and* $\int_{\mathbb{R}^d} v f_n^{(k)}\, dv = \int_{\mathbb{R}^d} v f_0\, dv$,

iii)

$$\begin{aligned} f_n^{(k)}(v) \;\; &\leqslant \;\; f_0(v) - C_L \sum_{i=0}^{k-1} \Delta_n \left(n^\gamma \wedge (1+|v|^\gamma)\right) f_n^{(j)} \\ &\quad + k\Delta_n C_\Phi C_\gamma l_b \left\|f_0\right\|_{L^1_{2,v}} \left[1 + (1+2K_0)\,\frac{C_\gamma}{\lambda_1^{d-1}} \left\|f_0\right\|_{L^1_{2,v}}\right] \\ &\quad + C_\Phi l_b C_\gamma \left\|f_0\right\|_{L^1_{2,v}} (1+2K_0) \int_{|u-v|\leqslant\lambda} \frac{\sum_{i=0}^{k-1} \Delta_n \left(n^\gamma \wedge (1+|u|^\gamma)\right) f_n^{(i)}(u)}{|u-v|^{d-1}}\, du. \end{aligned}$$

iv) if we define

$$E_n(f_n^{(k)}) = \sup_{v\in\mathbb{R}^d} \left[f_n^{(k)}(v) + \Delta_n \sum_{j=0}^{k-1} \left(n^\gamma \wedge (1+|v|^\gamma)\right) f_n^{(j)}(v) \right]$$

then

$$E_n(f_n^{(k)}) \leqslant K_0,$$

where C_γ has been defined in (6.4.2) and C_L, λ, K_0, T_0 and Δ_n have been defined in (6.7.3) – (6.7.9).

Remark 6.7.5 *In particular, point iv) gives a uniform control on the L^∞_v-norm of $f_n^{(k)}$ which is bounded by K_0.*

Proof of Proposition 6.7.4 The proof of the proposition above will be done by induction. The step $k = 0$ is direct since $K_0 \geqslant \|f_0\|_{L^\infty_v}$ (see (6.7.5)). So let us assume that this is true at rank k with $k < T_0/\Delta_n$.

Combining Lemma 6.7.2 and points $ii)$ and $iv)$ of Proposition 6.7.4 at rank k we have that

$$\Delta_n \left\| Q_n^- \left(f_n^{(k)}\right)\right\|_{L^\infty_v} \leqslant \Delta_n C_\Phi l_b n^\gamma \left\|f_0\right\|_{L^1_v} (1+2K_0) \leqslant \frac{1}{2}.$$

Therefore we have that, by definition of $f_n^{(k+1)}$, (6.7.8),

$$f_n^{(k+1)}(v) \geqslant \frac{1}{2} f_n^{(k)}(v) + \Delta_n Q_n^+ \left(f_n^{(k)}\right)$$

and because $f_n^{(k)} \geqslant 0$ we obtain that $f_n^{(k+1)} \geqslant 0$, which is $i)$.

Furthermore, $f_n^{(k+1)} \geqslant 0$ implies that, thanks to Definition (6.7.8),

$$\begin{aligned}\left\| \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} f_n^{(k+1)} \right\|_{L^1_v} &= \int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} f_n^{(k+1)}(v)\, dv \\ &= \int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} f_n^{(k)}(v)\, dv + \Delta_n \int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} Q_n(f_n^{(k)})(v)\, dv.\end{aligned}$$

The last term on the right hand side is zero since Q_n satisfies the same integral property than the non-truncated Boltzmann-Nordheim operator, Lemma 6.1.1. Hence, $f_n^{(k+1)}$ satisfies point ii).

In order to prove iii) we use Lemma 6.7.2 ($f_n^{(k)}$ being positive), Lemma 6.7.3 and the fact that $\left\| f_n^{(k)} \right\|_{L^\infty_v} \leqslant K_0$ in the definition of $f_n^{(k+1)}$, (6.7.8). This yields

$$\begin{aligned} f_n^{(k+1)}(v) \;\leqslant\; & f_n^{(k)}(v) - C_L \Delta_n \left(n^\gamma \wedge (1+|v|^\gamma)\right) f_n^{(k)} \\ & + \Delta_n C_\Phi C_\gamma l_b \left\| f_0 \right\|_{L^1_{2,v}} \left[1 + (1+2K_0)\frac{C_\gamma}{\lambda^{d-1}} \left\| f_0 \right\|_{L^1_{2,v}}\right] \\ & + \Delta_n C_\Phi l_b C_\gamma \left\| f_0 \right\|_{L^1_{2,v}} (1+2K_0) \int_{|u-v|\leqslant \lambda} \frac{\left(n^\gamma \wedge (1+|u|^\gamma)\right) f_n^{(k)}(u)}{|u-v|^{d-1}}\, du. \end{aligned}$$

Then, by applying iii) for $f_n^{(k)}$ we obtain iii) for $f_n^{(k+1)}$.

Thanks to iv) at rank k we have that for all v in $\mathbb{R}^d$

$$\Delta_n \sum_{j=0}^{k-1} \left(n^\gamma \wedge (1+|v|^\gamma)\right) f_n^{(j)}(v) \leqslant k \Delta_n n^\gamma K_0.$$

Thus, $\sup_{v\in\mathbb{R}^d} \Delta_n \sum_{j=0}^{k-1} \left(n^\gamma \wedge (1+|v|^\gamma)\right) f_n^{(j)}(v)$ exists and is finite.

Hence, we can consider property iii) at rank $k+1$ and take the essential supremum over v in $\mathbb{R}^d$, noticing that $k+1 \leqslant T_0/\Delta_n$,

$$\begin{aligned} E_n(f_n^{(k+1)}) \;&\leqslant\; \frac{K_0}{4} + T_0 C_0 \left[1 + (1+2K_0)\frac{C_\gamma}{\lambda^{d-1}} \left\| f_0 \right\|_{L^1_{2,v}}\right] \\ &\qquad C_0 C(\lambda)(1+2K_0) E(f_n^{(k+1)}), \\ &\leqslant\; \frac{K_0}{2} + \frac{1}{2} E_n(f_n^{(k+1)}), \end{aligned}$$

by definition of T_0, see (6.7.7) and of λ, see (6.7.6). This gives us the expected result iv)

for $f_n^{(k+1)}$. ■

6.7.3 Convergence towards a mass and momentum preserving solution of the Boltzmann-Nordheim equation

For each n in $\mathbb{N}$ we have built a sequence of functions $\left(f_n^{(k)}\right)_{k\in\{0,\dots,T_0/\Delta_n\}}$ in $L^1_{2,v}\cap L^\infty_v$. We shall see these functions as piecewise constant functions of time. Therefore we define

$$\forall n\in\mathbb{N},\ \forall (t,v)\in[k\Delta_n,(k+1)\Delta_n)\times\mathbb{R}^d,\quad f_n(t,v)=f_n^{(k)}(v). \tag{6.7.10}$$

We are about to prove that $(f_n)_{n\in\mathbb{N}}$ converges weakly in $L^1([0,T_0]\times\mathbb{R}^d)$ towards f in $L^1([0,T_0],L^1_{2,v})\cap L^\infty([0,T_0]\times\mathbb{R}^d)$, the mass and energy preserving solution of the Boltzmann-Nordheim equation (6.1.3) with initial data f_0 (which is unique thanks to Theorem 6.6.1). This is the purpose of the following proposition.

Proposition 6.7.6 *There exists f in $L^1([0,T_0]\times\mathbb{R}^d)\cap L^\infty([0,T_0]\times\mathbb{R}^d)$ such that a subsequence of $(f_n)_{n\in\mathbb{N}}$, see (6.7.10) and (6.7.8), converges towards f weakly in $L^1([0,T_0]\times\mathbb{R}^d)$ and weakly-* in $L^\infty([0,T_0]\times\mathbb{R}^d)$. Moreover, f satisfies*

- *f is a solution of the Boltzmann-Nordheim equation (6.1.3) with initial data f_0,*
- *f is positive and for all t in $[0,T_0]$, $\|f(t,\cdot)\|_{L^1_v}=\|f_0\|_{L^1_v}$ and $\int_{\mathbb{R}^d} vf(t,v)\,dv=\int_{\mathbb{R}^d} vf_0(v)\,dv$,*
- *recalling the definition of $K_0>0$ (see (6.7.5)), f satisfies*

$$\sup_{[0,T_0]\times\mathbb{R}^d}\left(f(t,v)+\int_0^t(1+|v|^\gamma)\,f(s,v)\,ds\right)\leqslant 2K_0.$$

Proof of Proposition 6.7.6 Thanks to point *ii)* of Proposition 6.7.4 we have that $(f_n)_{n\in\mathbb{N}}$ is bounded in $L^1([0,T_0]\times\mathbb{R}^d)$ but also have a uniform bound on its second moment. Therefore it is a tight sequence. Moreover, point *iv)* of Proposition 6.7.4 gives us that $(f_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty([0,T_0]\times\mathbb{R}^d)$ and thus it is equi-integrable. The Dunford-Pettis theorem concludes that $(f_n)_{n\in\mathbb{N}}$ is weakly compact in $L^1([0,T_0]\times\mathbb{R}^d)$.
Therefore, there exists f in $L^1([0,T_0]\times\mathbb{R}^d)$ such that there exists a subsequence of $(f_n)_{N\in\mathbb{N}}$, that we will keep denoting by f_n, which converges weakly in $L^1([0,T_0]\times\mathbb{R}^d)$ towards f.

Point *i)* of Proposition 6.7.4 tells us that $f\geqslant 0$. The sequence $(f_n(t,\cdot))_{N\in\mathbb{N}}$ is tight and its tightness property is independent of the time t (see the uniform control of the second moment, point *ii*) of Proposition 6.7.4). Therefore, for all $\epsilon > 0$, it exists $R_\varepsilon > 0$ such that for all n and t we have

$$\int_{\mathbb{R}^d} \mathbf{1}_{\{|v|\leqslant R_\varepsilon\}} f_n(t,v)\,dv \geqslant \|f_0\|_{L^1_v} - \varepsilon.$$

Since $\mathbf{1}_{\{|v|\leqslant R_\varepsilon\}}$ is in L^∞_v we can take the weak limit as n tends to $+\infty$ in the inequality above to obtain

$$\|f(t,\cdot)\|_{L^1_v} \geqslant \int_{\mathbb{R}^d} \mathbf{1}_{\{|v|\leqslant R_\varepsilon\}} f(t,v)\,dv \geqslant \|f_0\|_{L^1_v} - \varepsilon,$$

this being true for all ϵ. Thus, $\|f(t,\cdot)\|_{L^1_v} \geqslant \|f_0\|_{L^1_v}$. But Fatou's Lemma offers us straightforwardly the opposite inequality.
This indicates that for all t in $[0,T_0]$, $\|f(t,\cdot)\|_{L^1_v} = \|f_0\|_{L^1_v}$. A similar argument proves that $f(t,\cdot)$ has the same momentum as f_0 for all $t \geqslant 0$.

The last point of Proposition 6.7.4, shows that $(f_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty([0,T_0]\times\mathbb{R}^d)$ and therefore is weakly-* compact in this space. We can extract a subsequence of f_n, still denoted by f_n, which converges weakly-* in $L^\infty([0,T_0]\times\mathbb{R}^d)$. But since f_n converges weakly in $L^1([0,T_0]\times\mathbb{R}^d)$ to f and therefore the weak-* limit in $L^\infty([0,T_0]\times\mathbb{R}^d)$ can only be f.
Thus f belongs to $L^\infty([0,T_0]\times\mathbb{R}^d)$.

Thanks to point *iv*) of Proposition 6.7.4, we have, for all n in $\mathbb{N}$ and k in $\{0,\dots,T_0/\Delta_n - 1\}$, for all (t,v) in $[k\Delta_n,(k+1)\Delta_n)\times\mathbb{R}^d$,

$$\begin{aligned} f_n(t,v) + \int_0^t \left(n^\gamma \wedge (1+|v|^\gamma)\right) f_n(s,v)\,ds &= f_n^{(k)}(v) + \sum_{j=0}^{k-1} \Delta_n \left(n^\gamma \wedge (1+|v|^\gamma)\right) f_n^{(j)}(v) \\ &\quad + \Delta_n\,(t-k\Delta_n)\left(n^\gamma \wedge (1+|v|^\gamma)\right) f_n^{(k)}(v) \\ &\leqslant K_0 + \Delta_n^2 n^\gamma K_0 \\ &\leqslant 2K_0. \end{aligned} \tag{6.7.11}$$

Therefore, if we define

$$\forall n\in\mathbb{N},\ \forall (t,v)\in[0,T_0]\times\mathbb{R}^d, \quad g_n(t,v) = \int_0^t \left(n^\gamma \wedge (1+|v|^\gamma)\right) f_n(s,v)\,ds, \tag{6.7.12}$$

we have that the sequence $(g_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty([0,T_0]\times\mathbb{R}^d)$ and therefore is weakly-* compact. There exists a subsequence, still denoted by g_n, that converges weakly-* in $L^\infty([0,T_0]\times\mathbb{R}^d)$ to, say, g.

Besides, since $0\leqslant\gamma\leqslant 1$, we have that $(g_n)_{n\in\mathbb{N}}$ is bounded in $L^1([0,T_0]\times\mathbb{R}^d)$ and such that for all t in $[0,T_0]$,

$$\int_{\mathbb{R}^d} |v|^{2-\gamma}\, g_n(t,v)\, dv \leqslant 2C_\gamma T_0 \, \|f_0\|_{L^1_{2,v}} \, ,$$

where C_γ has been defined in (6.4.2). Therefore $(g_n)_{n\in\mathbb{N}}$ is tight and equi-integrable and therefore is weakly compact in $L^1([0,T_0]\times\mathbb{R}^d)$. As we did for f_n we obtain that g_n converges (up to a subsequence) weakly to g in $L^1([0,T_0]\times\mathbb{R}^d)$.

We are going to prove that

$$g(t,v) = \int_0^t (1+|v|^\gamma) f(s,v)\, ds. \tag{6.7.13}$$

As we emphasised before, the compactness properties of f_n and g_n are the same for $f_n(t,\cdot)$ and $g_n(t,\cdot)$ for a given t, because our bounds are independent of t. Therefore we fix a t in $[0,T_0]$ and we take ϕ in $C^\infty_c(\mathbb{R}^d)$. By weak convergence of f_n in $L^1([0,T_0]\times\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d}\int_0^t (1+|v|^\gamma)\, \phi(v) f(s,v)\, dsdv = \lim_{n\to\infty} \int_{\mathbb{R}^d}\int_0^t (1+|v|^\gamma)\, \phi(v) f_n(t,v)\, dsdv = \lim_{n\to\infty} I_n.$$

But we have the following

$$I_n = \int_{(1+|v|^\gamma)\leqslant n^\gamma} \phi(v) g_n(t,v)\, dv + \int_0^t \int_{(1+|v|^\gamma)>n^\gamma} (n^\gamma \wedge (1+|v|^\gamma))\, \phi(v) f_n(s,v)\, dvds.$$

ϕ is of compact support so for n big enough we have that

$$\forall s\in[0,t], \quad \int_{(1+|v|^\gamma)>n^\gamma} (n^\gamma \wedge (1+|v|^\gamma))\, \phi(v) f_n(s,v)\, dv = 0.$$

Finally, by the weak convergence of g_n in L^1_v we obtain

$$\lim_{n\to\infty} I_n = \int_{\mathbb{R}^d} g(t,v)\phi(v)\, dv.$$

Thus,

$$\forall \phi\in C^\infty_c(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} \phi(v) \left[g(t,v) - \int_0^t (1+|v|^\gamma)\, f(s,v)\, ds \right] dv = 0.$$

This gives us the expected equality (6.7.13), since both functions are in L^1_v.

As a result, we have that $\int_0^t (1+|v|^\gamma)\, f(s,v)\, ds$ is in $L^\infty([0,T_0]\times\mathbb{R}^d)$. Thanks to (6.7.11) we also find

$$\sup_{[0,T_0]\times\mathbb{R}^d} \left(f(t,v) + \int_0^t (1+|v|^\gamma)\, f(s,v)\, ds \right) \leqslant 2K_0. \tag{6.7.14}$$

To conclude the proof of Proposition 6.7.6 it remains to show that f is a solution of the Boltzmann-Nordheim equation (6.1.3). However, this is now pretty straightforward since $g_n(t,\cdot)$ converges weakly-* in L^∞_v and $f_n(t,\cdot)$ converges weakly in L^1_v, for all t. Therefore, thanks to the definition of $f_n^{(k)}$, see (6.7.8), we can take the limit in

$$\int_{\mathbb{R}^d} f_n(t,v)\phi(v)\,dv = \int_{\mathbb{R}^d} \phi(v)\left[f_0(v) + \int_0^t Q_n(f_n(s,\cdot))(v)\right]\,dsdv,$$

for all test functions ϕ in $C^\infty_c(\mathbb{R}^d)$. Indeed, Q^n is basically a convolution operator with a kernel not growing faster than $(n^\gamma \wedge (1+|v|^\gamma))$. Then, since we have the equality (6.7.13), we obtain that for all test function ϕ

$$\int_{\mathbb{R}^d} \phi(v)\left[f(t,v) - \left(f_0(v) + \int_0^t Q(f(s,\cdot))(v)\,ds\right)\right] = 0,$$

and thanks to (6.7.14) we have that $f(t,v) - \left(f_0(v) + \int_0^t Q(f(s,\cdot))(v)\,ds\right)$ belongs to $L^1_v \cap L^\infty_v$ and therefore we obtain

$$f(t,v) = f_0(v) + \int_0^t Q(f(s,\cdot))(v)\,ds,$$

which means that f is a solution of the Boltzmann-Nordheim equation (6.1.3). ■

6.7.4 Preservation of the energy

This section is devoted to the proof of the following result, which is the fact that f preserves the energy of the initial data.

Proposition 6.7.7 *Let f be the function obtained in Proposition 6.7.6.*
Then for all t in $[0,T_0]$,

$$\left\| |v|^2 f(t,v) \right\|_{L^1_v} = \left\| |v|^2 f_0(v) \right\|_{L^1_v}.$$

Proof of Proposition 6.7.7 We have, thanks to Proposition 6.7.4, that for all t in $[0,T_0]$ the sequence $\left(|v|^2 f_n(t,v)\right)_{n\in\mathbb{N}}$ is bounded in L^1_v with the following preservation of the L^1_v-norm,

$$\forall n\in\mathbb{N}, \forall t\in[0,T_0], \quad \int_{\mathbb{R}^d} |v|^2 f_n(t,v)\,dv = \left\| |v|^2 f_0 \right\|_{L^1_v}.$$

Therefore, we fix t and notice that for all $R>0$ we have that, since $f_n \geqslant 0$,

$$\int_{\mathbb{R}^d} \mathbf{1}_{\{|v|\leqslant R\}} |v|^2 f_n(t,v)\,dv \leqslant \left\| |v|^2 f_0 \right\|_{L^1_v}. \tag{6.7.15}$$

As we saw in the proof of Proposition 6.7.6, $(f_n(t,\cdot))_{n\in\mathbb{N}}$ converges weakly (up to a subsequence) in L^1_v towards f. We can then take the limit as n in (6.7.15),

$$\int_{\mathbb{R}^d} \mathbf{1}_{\{|v|\leqslant R\}} |v|^2 f(t,v)\,dv \leqslant \left\| |v|^2 f_0 \right\|_{L^1_v},$$

which is true for all R. Thus, the positivity of f yields

$$\int_{\mathbb{R}^d} |v|^2 f(t,v)\,dv \leqslant \|f_0\|_{L^1_v}. \tag{6.7.16}$$

It remains to prove the opposite inequality.
To this end we shall show that $\left(|v|^2 f_n(t,v)\right)_{n\in\mathbb{N}}$ is tight in L^1_v, uniformly in t. Indeed, such a tighness property will yiels

$$\forall\varepsilon>0,\ \exists R_\varepsilon>0,\ \forall n,t,\quad \int_{\mathbb{R}^d} \mathbf{1}_{\{|v|\leqslant R_\varepsilon\}} |v|^2 f_n(t,v)\,dv \geqslant \left\| |v|^2 f_0 \right\|_{L^1_v} - \varepsilon$$

and since $\mathbf{1}_{\{|v|\leqslant R_\varepsilon\}}$ is in L^∞_v we can take the weak limit as n tends to $+\infty$ in the inequality above to obtain (remember that f is positive)

$$\left\| |v|^2 f(t,v) \right\|_{L^1_v} \geqslant \int_{\mathbb{R}^d} \mathbf{1}_{\{|v|\leqslant R_\varepsilon\}} |v|^2 f(t,v)\,dv \geqslant \left\| |v|^2 f_0 \right\|_{L^1_v} - \varepsilon,$$

this being true for all ϵ.

The tightness of our sequence of approximation is dealt with below. ∎

We have that f_0 is positive and such that $\left(1+|v|^2\right) f_0(v)$ is in L^1_v. Proposition $A1$ in the appendix of [77] gives the existence of ψ a positive convex function on $\mathbb{R}^+$ such that there exists $C>0$ such that

$$\int_{\mathbb{R}^d} \psi\left(|v|^2\right) f_0(v)\,dv \leqslant C.$$

Moreover, ψ can be written $\psi(x) = x\phi(x)$, where ϕ is concave, increasing to infinity, and such that for any $\varepsilon>0$ and any α in $(0,1)$, it satisfies $(\phi(x)-\phi(\alpha x))\,x^\varepsilon \to \infty$ as $x\to\infty$.

To prove that $\left(|v|^2 f_n(t,v)\right)_{n\in\mathbb{N}}$ is tight in L^1_v we are going to need the technical lemma about Povner-type inequality, Lemma 6.3.1.

The tightness of $\left(|v|^2 f_n(t,v)\right)_{n\in\mathbb{N}}$ directly follows from the following control of the tail of the distribution $f_n(t,v)$.

Proposition 6.7.8 *For all n in $\mathbb{N}$, for all k in $\{0,\dots,T_0/\Delta_n\}$,*

$$\int_{\mathbb{R}^d} f_n^{(k)}(v)\psi\left(|v|^2\right)\,dv \leqslant \int_{\mathbb{R}^d} \psi\left(|v|^2\right) f_0(v) + \frac{1}{2}(k+1)\Delta_n C_G C_\gamma^2 \|f_0\|^2_{L^1_{2,v}},$$

where C_G has been defined in Lemma 6.3.1.

Remark 6.7.9 *We would like to emphasize here that C_G depends on b, ψ (which depends only on f_0) and an upper bound for $\left\| f_n^{(k)} \right\|_{L^\infty_v}$ which is bounded by K_0 (see point iv) of Proposition 6.7.4). Therefore, C_G is a constant of our problem, independent of k and n.*

Proof of Proposition 6.7.8 The proof will be done by induction. The case $k = 0$ is obvious so let us assume that this is true up to rank $k < T_0/\Delta_n$.
To shorten computation we set

$$M_n^{(k)} = \int_{\mathbb{R}^d} f_n^{(k)}(v)\psi\left(|v|^2\right)\,dv.$$

By definition of $f_n^{(k+1)}$, see (6.7.8), we obtain, after integrating in v and the use of the usual changes of variables $(v, v_*) \to (v_*, v)$ and $(v, v_*) \to (v', v'_*)$,

$$\begin{aligned}
M_n^{(k+1)} &= M_n^{(k)} + \Delta_n \int_{\mathbb{R}^d} Q_n(f_n^{(k)})(v)\,dv \\
&= M_n^{(k)} + \frac{\Delta_n}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} (n \wedge |v - v_*|)^\gamma\, f_n^{(k)}(v) f_n^{(k)}(v_*) \\
&\qquad\qquad \times \left[\int_{\mathbb{S}^{d-1}} \left[1 + f_n^{(k)}(v') + f_n^{(k)}(v'_*)\right] b(\cos\theta)\left(\psi'_* + \psi' - \psi_* - \psi\right) d\sigma\right] dv_* dv \\
&= M_n^{(k)} + \frac{\Delta_n}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} (n \wedge |v - v_*|)^\gamma\, f_n^{(k)}(v) f_n^{(k)}(v_*)\left[G(v, v_*) - H(v, v_*)\right] dv_* dv.
\end{aligned} \tag{6.7.17}$$

We can use Lemma 6.3.1 with

$$\begin{aligned}
G(v, v_*) &\leqslant C_G\,|v|\,|v_*|\,, \\
H(v, v_*) &\geqslant 0.
\end{aligned}$$

This yields, applied to (6.7.17) because $f_n^{(k)}$ is positive (see Proposition 6.7.4),

$$\begin{aligned}
M_n^{(k+1)} &\leqslant M_n^{(k)} + \frac{\Delta_n}{2} C_G \int_{\mathbb{R}^d\times\mathbb{R}^d} |v - v_*|^\gamma\,|v|\,|v_*|\, f_n^{(k)}(v) f_n^{(k)}(v_*)\,dv_* dv \\
&\leqslant M_n^{(k)} + \frac{\Delta_n}{2} C_G \left[\int_{\mathbb{R}^d} (1 + |v|^\gamma)\,|v|\, f_n^{(k)}(v)\,dv\right]^2.
\end{aligned}$$

Because γ belongs to $[0, 1]$ we also have (up to a change of constant C_γ in (6.4.2))

$$(1 + |v|^\gamma)\,|v| \leqslant C_\gamma\left(1 + |v|^2\right),$$

which yields, thanks to the preservation of the $L^1_{2,v}$-norm (see Proposition 6.7.4),

$$M_n^{(k+1)} \leqslant M_n^{(k)} + \frac{\Delta_n}{2} C_G C_\gamma^2 \left\| f_0 \right\|^2_{L^1_{2,v}}.$$

Applying the induction hypothesis at rank k gives us the expected result. ■

The tightness of $(f_n(t,\cdot))_{n\in\mathbb{N}}$ follows straightforwardly from the growing property of ψ and Proposition 6.7.8 which states the following control, uniform in n and t,

$$\int_{\mathbb{R}^d} f_n(t,v)\psi\left(|v|^2\right)\,dv \leqslant \int_{\mathbb{R}^d} f_0(v)\psi\left(|v|^2\right)\,dv + \frac{T_0}{2} C_G C_\gamma^2 \left\| f_0 \right\|^2_{L^1_{2,v}} < \infty,$$

since $k \leqslant T_0/\Delta_n$ and C_G is a constant (see Remark 6.7.9).

This concludes the proof of Theorem 6.7.1 since f is a positive solution of (6.1.3), mass and energy preserving and in $L^\infty_{\text{loc}}\left([0,T_0), L^1_{2,v} \cap L^\infty_v\right)$. Therefore, see Theorem 6.6.1, f is the unique solution satisfying those properties and f preserves the momentum. Thus the sequence $(f_n)_{n\in\mathbb{N}}$ converges weakly in Proposition 6.7.6, not only just a subsequence, towards f.
If at T_0 we have that $\|f\|_{L^\infty_{t,v}} \leqslant M$ then we can apply our proof starting at T_0 and construct a solution up to T_1 (depending only on M, which depends only on f_0, and $\|f(T_0,\cdot)\|_{L^1_{2,v}} = \|f_0\|_{L^1_{2,v}}$) and by uniqueness we have in fact extended f to T_1. We can inductively build a solution on $[0,T)$ up to the point where

$$\lim_{T\to T^-} \|f\|_{L^\infty_{[0,T]\times\mathbb{R}^d}} = +\infty.$$

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