On Zero Stiffness

Mark Schenk and Simon D Guest [*]

Abstract

Zero stiffness structures have the remarkable ability to undergo large elastic deformations without requiring external work. Several equivalent descriptions exist, namely i) continuous equilibrium, ii) constant potential energy, iii) neutral stability and iv) zero stiffness. Each perspective on zero stiffness provides different methods of analysis and design. This paper reviews the concept of zero stiffness and categorises examples from the literature by the interpretation that best describes their working principle. Lastly, a basic spring-to-spring balancer is analysed to demonstrate the equivalence of the four different interpretations, and illustrate the different insights that each approach brings.

1 Introduction

A fascinating combination of geometry, stiffness and prestress enables *zero stiffness structures* to deform and maintain their deformed configuration without any external work — in effect, these elastic structures behave as mechanisms. This remarkable property forms the topic of this paper. Paradoxically, there exists a considerable body of work describing zero stiffness structures, while the concept of zero stiffness is at the same time generally unknown. One of the reasons is that examples of zero stiffness structures are found in disparate disciplines and go by various monikers (for example, in mechanical engineering they are often referred to as *statically balanced mechanisms*). As a result, the shared underlying mechanical principles are seldom recognised.

There are several equivalent interpretations of the concept of zero stiffness. The key property is the ability of an elastic structure to deform with *zero stiffness*, where any change in configuration requires no external work. These structures are thus said to be *neutrally stable*: at the cusp at stability and instability, they can undergo large displacements for a constant critical load or self-stress. Another interpretation is that these structures are in *continuous equilibrium* with an applied load over a finite range of motion. Lastly, a zero stiffness structure maintains a *constant potential energy*, which is continuously redistributed as the structure deforms, thus eliminating any preferred position under the applied load or self-stress. Throughout this paper the term *zero stiffness structures* will be used as the most general description; while these elastic systems behave as mechanisms, they will generally be referred to as structures.

---
[*]Department of Engineering, University of Cambridge, Cambridge, UK
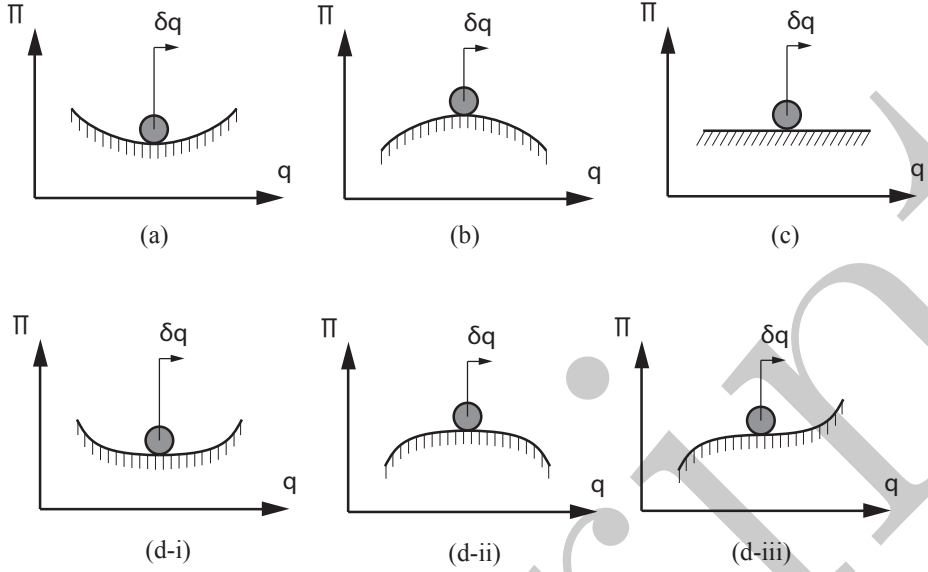Email: M.Schenk@cantab.net

Figure 1: A ball lying on a curved surface illustrates the different stability conditions of a single degree of freedom elastic structure. As the potential energy of the ball is proportional to its vertical position, the shape of the surface is analogous to a potential energy function Π for an elastic structure. Consider small variations δq around the equilibrium state. Shown are configurations which are (a) stable $\left(d^2\Pi/dq^2 > 0\right)$, (b) unstable $\left(d^2\Pi/dq^2 < 0\right)$, and (c) neutrally stable $\left(d^2\Pi/dq^2 = 0\right)$. The examples shown in (d) are all locally neutrally stable (i.e. the system has quasi-zero stiffness), but are ultimately either stable (d-i) or unstable (d-ii, d-iii).

Zero stiffness structures can be more formally defined by considering the total potential energy Π of an elastic structure under conservative loads. If the geometric configuration of the structure is defined by a vector of coordinates $\mathbf{p}$ with components p_i, then setting the derivatives of Π $(\partial\Pi/\partial p_i)$ to zero provides the equilibrium equations. The Hessian $\left(\partial^2\Pi/\partial p_i \partial p_j\right)$ subsequently yields the stiffness equations for the structure. By looking at the eigenvalues of the Hessian matrix the stability of the structure can be assessed [1,2]. A structure is stable if all eigenvalues are positive, unstable if there exists a negative eigenvalue, and has a zero stiffness deformation mode if the Hessian is singular.

In general a zero stiffness mode will be infinitesimal, and higher-order strain and prestress effects will ultimately determine the stability of the structure, as illustrated in Figure 1. Despite the local nature of the zero stiffness behaviour, such quasi-zero stiffness structures may nonetheless be of use in engineering applications such as vibration isolation. However, of primary interest to our discussion are cases where all higher-order derivatives of the potential energy are also zero, and the structure therefore has a finite zero stiffness path. Barring any imperfections, friction or material damping, these elastic structures can deform over large displacements with zero stiffness. This singular stuctural behaviour only exists for very specific combinations of material stiffness, structural geometry and prestress.

It is important to note that in order to fully describe the mechanical properties of zero stiffness structures, a geometrically non-linear analysis is necessary to distinguish between infinitesimal and large-displacement zero stiffness modes. This is analogous to the study of buckling, where only a postbuckling analysis can provide information about the structure's ability to carry further loads.

Applications In some respects zero stiffness structures are a structural curiosity, as they cannot carry any further loads. Nonetheless, zero stiffness has interesting practical applications. Zero stiffness structures can maintain their deformed state without any plastic deformation, and static balancing can be used to reduce the operating energy of mechanisms, as a quasi-static change of position will require no effort. Furthermore, from an academic point of view zero stiffness structures are singularities in structural theory, and their study advances our understanding of structural mechanics.

The concept and application of zero stiffness is certainly not new: the first known academic description dates from 1867 [3][1], and the first engineering example can be found in a 1901 patent [4]. Many applications have since been proposed, and this paper only provides a brief overview of the wide range of possibilities. An important application of zero stiffness is in the counterbalancing of a mass. The classic example is the Anglepoise desk lamp, where the lamp shade can be repositioned effortlessly [5]. Other examples of counterbalancing can be found in drafting tables [6], blackboards [7], instrument tables [8], computer screens [9] (*e.g.* the Apple G4 iMac), supports for industrial pipes [10,11], and rehabilitation aids [12]. In robotics the use of static balancing enables manipulators to become lighter and faster, as the weight of the links no longer needs to be carried by the actuators [13,14]. Constant force generators can also be found in machine design applications [4,15,16]. In recent years compliant mechanisms have gained increasing interest; static balancing can significantly reduce the required operating force [17] by compensating for the parasitic stiffness inherent in the flexible hinges. Lastly, vibration isolation [18–20] provides further applications for zero stiffness structures, with as notable examples the Steadicam [21,22] and a gravity measurement device [23].

Scope and Outline This review aims to familiarize the reader with the concept of zero stiffness structures, by drawing together examples from disparate fields and describing the mechanical principles that unite them. Accordingly, the intention of this paper is to develop an over-arching insight, rather than present an in-depth review of individual research fields.

This paper is structured as follows. Section 2 describes three illustrative, and purposefully dissimilar, examples of zero stiffness structures. In Section 3 we discuss four interpretations of zero stiffness, and group examples from the literature accordingly. Section 4 analyses the basic spring-to-spring balancer from four perspectives; the example demonstrates the equivalence of the different zero stiffness interpretations, but also illustrates that each approach provides its own insights.

---

[1]William Thomson and Peter Guthrie Tait's *Treatise on Natural Philosophy* was first published in 1867. In this paper we shall refer to the 1883 second edition, which is more widely available. William Thomson is generally better known as Lord Kelvin.
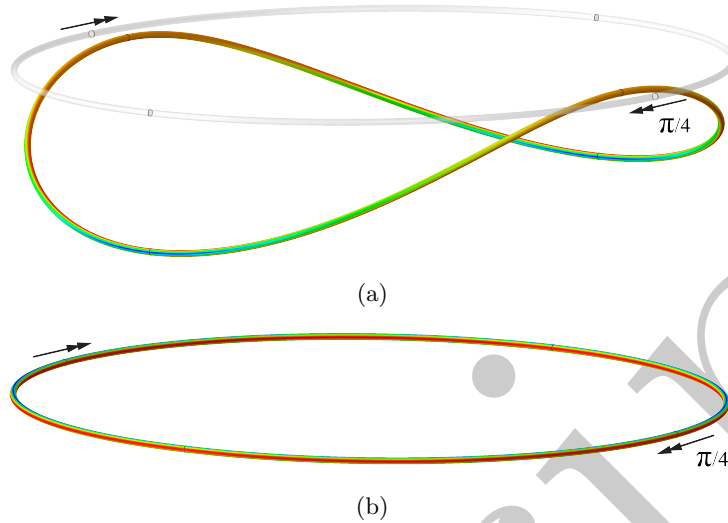
3

(a)

(b)

Figure 2: The structural response of a ring subject to an equal and opposite axial rotation at two diametrically opposed points depends crucially on the presence of prestress. An unstressed ring (a) deforms into a saddle-shaped configuration, whereas a prestressed ring (b), formed from an initally straight rod, twists effortlessly around its centroidal axis with zero stiffness. The numerical simulations were performed using the finite element software ABAQUS 6.11. The ring and rod have a diameter of 200 mm and 2 mm respectively, and the axial rotation was $\pi/4$. The colours represent the Von Mises stress, with warmer colours indicating higher values, illustrating the difference in stress distribution in the two cases.

2   Three Examples

In this section we highlight three examples from the literature. While at first glance these structures appear dissimilar, they share the common feature of deforming with zero stiffness.

2.1   Twisting Rod

The first known description of a zero stiffness elastic structure was given by Thomson and Tait [3, §621-626]. Take an initially straight rod with uniform bending stiffness, and bend it arbitrarily (but elastically) along its length. In this deformed configuration, any twisting of the rod along its centroidal axis requires no torque, and thus has zero stiffness. The explanation is immediately intuitive: there exists no preferential bending axis and thus any twisted orientation of the rod will have identical strain energy. An example of this type of structure is shown in Figure 2. When a thin straight rod is bent uniformly and its ends are rigidly joined, the resulting prestressed ring can be twisted along its axis with no stiffness (in practise there will be some, negligible, stiffness due to imperfections in the initial straightness of the rod). In contrast, when an unstressed ring is twisted along its axis, it deforms into a saddle shape. The Thomson & Tait
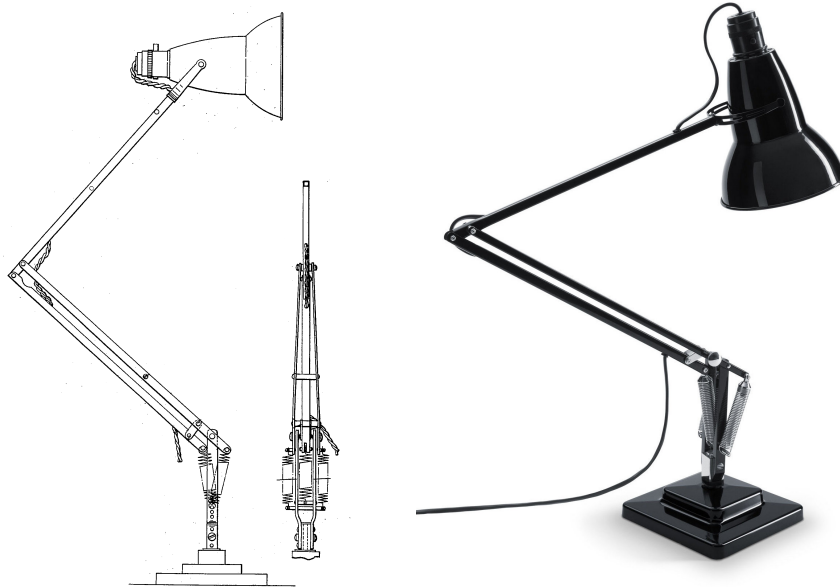
4

Figure 3: Designed in 1935 by George Carwardine, the three-spring design of the Anglepoise 1227 has become iconic. Left, the original patent drawings [5]; right, the modern reinterpretation Type 1227 as pictured on the 2009 Royal Mail stamp celebrating British Design Classics.

twisting rod is an elegant example of a structure with a constant strain energy, where the elastic stresses are continuously redistributed as it deforms in its zero stiffness mode.

2.2 Anglepoise Lamp

The classic Anglepoise lamp exemplifies a category of spring-and-linkage based statically balanced mechanisms. The spring-based balancing system enables effortless positioning of the lamp shade [24]. The original design by George Carwardine [5] dates to the 1930s, but its elegant mechanical design has remained effectively unchanged since. In this type of statically balanced mechanisms the links are regarded as freely hinged rigid bodies and the linear springs are the only elastic components. Often these are zero-free-length springs, which are pretensioned to the extent that the spring has an effective zero rest length, and the force is therefore proportional to the length of the spring. (The necessary pretension can be achieved, for instance, by twisting the wire prior to the coiling of the spring.) The Anglepoise spring mechanism illustrates two zero stiffness interpretations: by generating a constant upward force the weight of the lamp shade is continuously balanced in all positions, and a constant potential energy is maintained by redistributing energy between the strain energy in the springs and the gravitational potential energy of the lamp shade.

Figure 4: A neutrally stable cylindrical shell [25] can be twisted from fully coiled to fully extended configuration and again to fully coiled (clockwise from top-left) with zero stiffness. In all deformed configurations the shell can be wrapped around an underlying cylinder with constant radius.

2.3 Neutrally Stable Cylindrical Shells

Guest *et al.* [25] describe a neutrally stable prestressed shell, discovered serendipitously during work on bistable cylindrical shell structures. These bistable structures have two stable states, and can transition between the two through elastic deformations. The bistability can be achieved by imposing a state of prestress in the shell by an initial plastic forming process [26]. Using a simple inextensional shell deformation model [27], a particular prestress state for the shell was found where the twisting deformation requires no external energy; see Figure 4. Upon closer inspection, a hidden symmetry was revealed, with a parallel in the neutrally stable buckled state of a bi-metallic disc. When such a disc is subjected to a change in temperature it will first develop a spherical curvature due to the different thermal expansion coefficients of the two layers, before bifurcating into a cylindrical configuration [28]. As the orientation of the cylindrical axis is arbitrary, the cylindrical shell can effortlessly be twisted into other configurations with identical strain energy. The same underlying mechanism operates in the redistribution of the strain energy in the neutrally stable cylindrical shell.

3 Design and Analysis Methods

The concept of zero stiffness can be described in various manners: equivalent formulations are *continuous equilibrium*, *constant potential energy*, *neutral stability* and *zero stiffness*. However, different interpretations will lead to different methods for the design and analysis of zero stiffness structures. In this section examples are drawn from a wide range of disparate disciplines, and categorised by an interpretation that best clarifies their working principle. Inevitably, the boundaries between the categories are blurred.
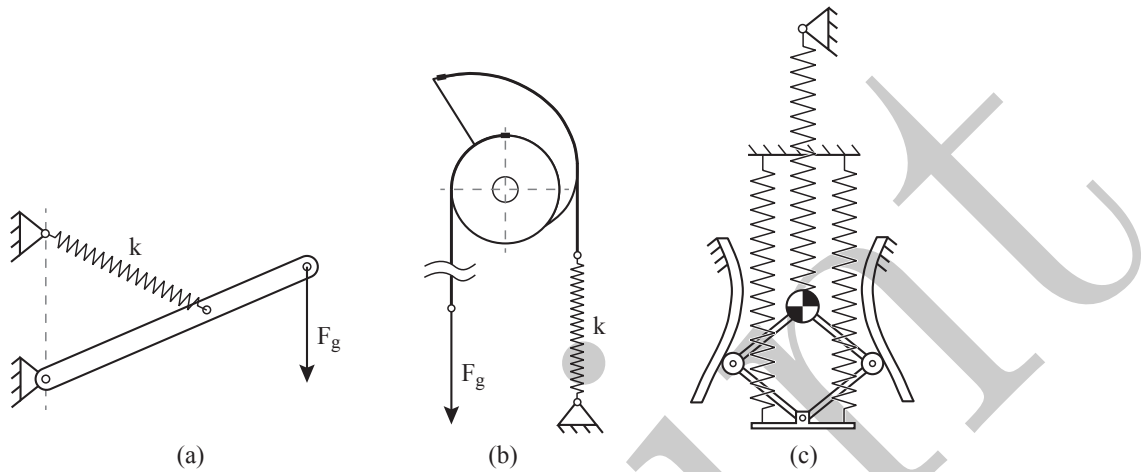
Figure 5: Three gravity equilibrators: (a) basic linkage with a zero-free-length spring; (b) wrapping cam with a regular spring [30]; (c) curved guides with two sets of springs [31].

3.1 Continuous Equilibrium

A key approach to the design of zero stiffness structures is to focus on the force and moment equilibrium equations, and maintain continuous equilibrium throughout the structure's range of motion.

The driving application is the design of gravity equilibrators, where the zero stiffness structure provides a constant vertical force to counterbalance a mass. A wealth of gravity balancing mechanisms can be found in the patent literature; a full review is beyond the scope of this paper, and we shall highlight only key examples. Static balancing of a mass can naturally be achieved through the use of counterweights. However, spring-based mechanisms are often preferred for minimal increase in inertia, and compactness of design. As the force of a regular coiled spring is proportional to its elongation, a mechanical conversion is necessary to produce a constant force at the end effector. This can be achieved through appropriate mechanical linkages, or cams and other curved surfaces [29].

In linkage based statically balanced mechanisms, only the springs are considered to deform elastically and the links are regarded as rigid. (A classic example is the Anglepoise desk lamp discussed in Section 2.2.) Often a key component is the use of zero-free-length springs [32, 33]. The spring is prestressed such that the spring force is proportional to the length of the spring, as opposed to its elongation, and thus has an effective zero rest length. These springs greatly simplify the design of statically balanced linkages, but are seldom used in practise as accurate manufacture is difficult. Fortunately, their properties can straightforwardly be emulated (or closely approximated) in practical implementations, for example by using multiple pulleys [32]. The basic spring balancer, as shown in Figure 5(a), forms the core of many statically balanced mechanisms. Its design has been extended by many authors, notably [5, 15, 32, 34, 35], and it can be recognized in many engineering applications.

7

Another means of converting a linear spring to a constant balancing force, is the use of wrapping cams or curved guides. Ostler and Zwick [30] describe cam based balancers where moment equilibrium is preserved between the constant load and the varying spring force; see Figure 5(b). More generally, curved guides can be used to control the deformation of a linear or torsional spring, and produce a desired force output [9, 31, 36]. The curve geometry can be generated using equilibrium [36] or energy [31] equations. Either approach produces a differential equation for the curve, which can be solved for a desired payload, geometry and spring stiffness [37]. Recent development have resulted in the design of curve-based balancers where adjustment to different payloads is possible without redesign of the curve geometry [31]. This is achieved by using two sets of springs: the first is coupled directly to the payload, and the vertical force component of the second set is controlled by the curved surface; together they produce a constant upward force over the balancer's range of motion; see Figure 5(c). Adjusting the balancer to different payloads is done by increasing the initial tension in the first set of springs. An important advantage of the curve and cam based balancers is their compactness, compared to linkage based gravity equilibrators.

Instead of counterbalancing a constant (gravity) load, mechanisms may also be designed to counterbalance elastic forces. For instance, to counteract undesired forces, such as the elasticity in cosmetic gloves for prosthetics or the parasitic stiffness in compliant mechanisms. Another application is in spring-to-spring balancers that enable energy-free adjustability of gravity equilibrators to multiple payloads [8, 38]. A classic spring-to-spring balancer is discussed in detail in Section 4.

3.2   Constant Potential Energy

An intuitive interpretation of zero stiffness structures is the realisation that throughout their deformation a constant total energy is maintained, and no preferred configuration thus exists. In the zero stiffness deformation mode the elastic strain energy is either redistributed within the structure, and/or exchanged with potential energy in an external field (*e.g.* gravity or magnetic [39]). The constant energy interpretation also clarifies the necessity of a prestress or preload: energy must first be introduced to the structure before it can be redistributed in the zero stiffness deformation mode.

The constant energy approach has proven to be an effective method to analyse and design zero stiffness structures. For example, Walsh *et al.* [40] described the perfect balancing of a two degree of freedom balancer with spatially arranged springs using potential energy functions; approximately balanced spring-linkage mechanisms can be designed by exploring energy landscapes [32, 41]; and the study of strain energy contours for cylindrical shell deformations led to the discovery of zero stiffness shell structures [25, 42]. The constant energy approach is also used for spring-to-spring balancers, where mechanisms transfer energy from one spring to the next. Barents *et al.* [38] proposed several designs for spring-to-spring balancers, where one spring is used to counterbalance a payload, and the second is available to store/release energy when the payload mass is varied; this enables energy-free adjustment of a gravity balancer.
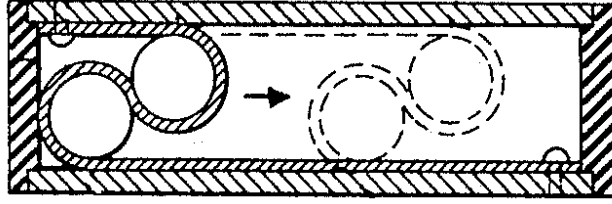
Figure 6: In a *Rolamite* linear bearing a flat spring is wrapped around two cylinders. For any displaced configuration of the two rollers the total stored strain energy will remain constant; the structure therefore has no preferred position and deforms with zero stiffness. Image from [43].

In other structures, a constant potential energy is maintained by ensuring a constant amount of material deformation within the zero stiffness mode. A classic example is the *Rolamite* linear bearing [43]; see Figure 6. A flat blade spring is wrapped around two cylinders; for any displaced configuration of the two rollers, the deformed spring length (and thus stored strain energy) remains constant, and the mechanism therefore has no preferred position. The concept was extended to a rolling hinge [44], and elegantly employed in a preloading mechanism for a knife edge pivot [16, §10.2]. A similar configuration was also proposed as a mechanical model for lotus receptacles [45]. Vehar *et al.* [46] described closed-loop tape-spring mechanisms with multiple travelling elastic hinges. Despite the non-linear and complex mechanics of the bending of a tape spring [47], the constant strain energy in the mechanisms can be recognized by the constant total material deformation. Lastly, the Thomson & Tait [3] zero stiffness twisting ring, where any twisted configuration has identical strain energy, was previously described in Section 2.1.

In certain types of zero stiffness structures the stored elastic energy is not constant, but the working principle is nonetheless best understood by noting a constant elastic deformation zone. Constant force springs (also known as *negator* springs [48, pp. 152]) are unstressed in their coiled configuration. The length of the transition zone between coiled and straightened state is always constant, and any further extension will therefore require no increase in force. Similarly, propagating instabilities [49] can be regarded as zero stiffness deformations. A classic example is the inflation of a long thin balloon [50]: after the initial inflation stage there exists a constant deformation zone between the flat and inflated section, and further inflation takes place at a nominally constant pressure.

3.3   Neutral Stability

In the stability of structures, buckling is said to occur when a structure can no longer support a load and undergoes a sudden large displacement. In effect, the structure has zero stiffness in the buckling mode, and one or more of the eigenvalues of the stiffness matrix becomes zero [1]. A classic small-displacement buckling analysis, however, cannot determine whether the structure is ultimately stable, unstable or neutrally stable; therefore, an analysis of the *postbuckling* bifurcation paths is necessary [1,2]. The neutrally stable postbuckling approach
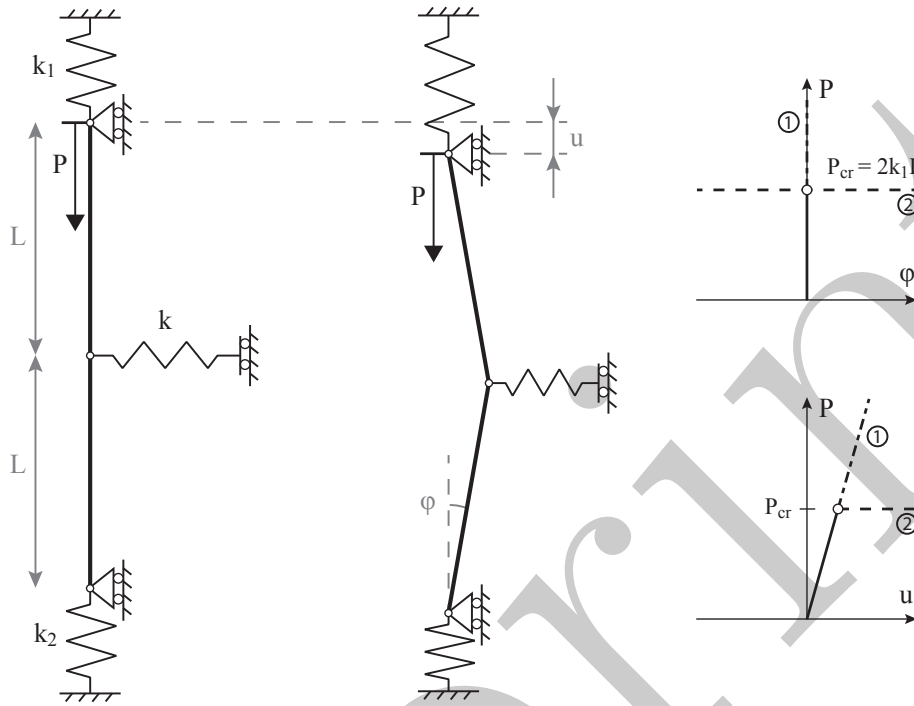
9

Figure 7: The zero stiffness elastic structure studied by Tarnai [45]; it will exhibit a neutrally stable post-buckling path (**2**) if the balancing condition $k_1 k - 4k_1 k_2 + k_2 k = 0$ is satisfied. For $k_1 = k_2 = \frac{k}{2}$, the critical buckling load is $P_{\text{cr}} = 2k_1 L$.

to zero stiffness structures has primarily been studied within the context of structural mechanics. However, the connection is increasingly recognised and utilised in the field of mechanical engineering [51].

Tarnai [45] provides an elegant analysis of the neutral stability of a rigid-linked pin-jointed structure under an external load, as shown in Figure 7. By studying analytical expressions for the first differential and the Hessian of the potential energy function, it was shown that there exists a transition from an unstable to a stable symmetric bifurcation as the ratio k_2/k_1 is increased. In between, for a specific combination of spring stiffnesses ($k_1 k - 4k_1 k_2 + k_2 k = 0$), the structure is neutrally stable over a finite range of motion. Tarnai includes a brief history of the analysis of this zero stiffness structure, which suggests it to be the earliest known example of a neutrally stable postbuckled structure; Gáspár [52] and Hegedüs [53] analysed simplified versions of Tarnai's structure, and provide further examples. Steinboeck *et al.* [54] studied the imperfection sensitivity of postbuckling behaviour in terms of symmetric, anti-symmetric and zero-stiffness bifurcations, within the framework of Koiter's initial postbuckling analysis. An example zero stiffness structure was described, which showed that zero stiffness does not necessarily lie at the boundary of imperfection sensitivity and insensitivity. Continued work [55–58] revealed further examples, and provides necessary and sufficient conditions for neutrally stable postbuckling behaviour.

While neutral stability in postbuckling has become a subject of study in its own right, it may also elucidate the underlying mechanics of other zero stiffness structures. For example, the neutrally stable shell structure described by Guest *et al.* [25] was best understood by means of the parallel with the post-buckling behaviour of a bimetallic disc subject to a change in temperature [28] (see Section 2.3). This insight explained the critical level of prestress for the neutrally stable behaviour, which had previously been found empirically by studying strain energy contours.

As a closing example, Van Eijk [59] described a zero stiffness platform supported on buckled blade springs. The postbuckling behaviour, however, is stable and the zero stiffness displacements would thus be infinitesimal. Examples of such quasi-zero stiffness structures can be found in precision engineering applications.

3.4 Zero Stiffness

Lastly, the stiffness equations of an elastic structure can be analysed directly, in search of zero stiffness conditions. In essence this is identical to the buckling analysis discussed previously, where zero stiffness identifies the point of buckling of the structure. Nevertheless, many problems in structural mechanics are formulated directly in terms of stiffness equations, and a link to buckling is often non-obvious.

In general, stiffness formulations are in the form of a linearised tangent stiffness matrix. Any singularities in the stiffness matrix will therefore only reveal infinitesimal zero stiffness modes, and further effort is required to find conditions for a finite zero stiffness path. For example, this was done for zero stiffness tensegrity structures (*i.e.* prestressed pin-jointed frameworks), by exploiting the specific nature of the zero stiffness deformations [60]. For the zero stiffness tensegrity structures, the connection with stability and critical load was revealed by Guest [61]. When the level of prestress in a tensegrity structure is increased, the overall stiffness first increases, before dropping off in certain deformation modes, reaching zero (when making use of tension members with zero rest length) and eventually becoming negative. This illustrates that the zero stiffness tensegrities are at the cusp of stability and instability for a critical level of self-stress; a small change in prestress would result in either a stable or unstable structure. In practise, however, insensitivity to the precise level of prestress may be provided by friction in the structure [62]. It is important to note that the stiffness matrices must correctly take into account any prestressing or preloading of the structure, to be able to correctly identify zero-stiffness modes. While stiffness matrices are traditionally the tool of structural mechanics, they have been adapted for use in the design of statically balanced mechanisms [63].

Alternatively, rather than looking for zero eigenvalues in the stiffness matrices, a zero stiffness structure can be designed by explicitly combining structures with a positive and negative stiffness [64, 65]. The necessary negative stiffness is often derived from the postbuckling solution of a (multi-stable) elastic structure [66]. This technique is increasingly used in the design of compliant mechanisms. Such mechanisms rely on the elastic deformation of their structural members for motion, thereby eliminating friction and backlash. However, the elastic deformations introduce a parasitic stiffness which impacts the operating

11

energy of the mechanism. By coupling the compliant mechanism with a negative stiffness component, the parasitic stiffness can be counterbalanced and the mechanism's operating force is reduced. The resulting zero stiffness structures are referred to as Statically Balanced Compliant Mechanisms (SBCM) [17,67]. The structural behaviour of compliant mechanisms is often strongly non-linear and matching the opposing stiffness of two non-linear structures is challenging; as a result the zero stiffness mode will generally be imperfect, and restricted to a limited range of motion. In order to avoid problems with stress relaxation in prestressed structures, an initial loading step can be used to introduce the negative stiffness necessary for the zero stiffness regime. The results of the analysis of a simplified quasi-zero stiffness compliant mechanism are shown in Figure 8. For designs with compliant hinges where beam bending dominates, finite element analysis is necessary to produce the force-displacement profiles and optimise the mechanism geometry [65,67].

In designing for zero stiffness, one can also study the dynamic behaviour of the structure, and design for a low (or zero) natural frequency. This approach is used in the design of passive vibration isolation systems. An extensive body of research is available, and a detailed review by Ibrahim [18] covers a wide range of non-linear passive vibration isolators, including mechanical structures such as postbuckled beams, as well as magnetic quasi-zero stiffness systems [20]. For most vibration isolation applications, the use of a quasi-zero stiffness structure will suffice, as often only small amplitude vibrations are encountered. In fact, quasi-zero stiffness structures may even be preferable for vibration isolation, as they can combine a high overall static stiffness with a low dynamic stiffness around the quasi-zero stiffness configuration [19,39]. Nonetheless, applications for large-displacement vibration isolation systems exist, including the Steadicam [21], and accurate gravity measurement devices [23].
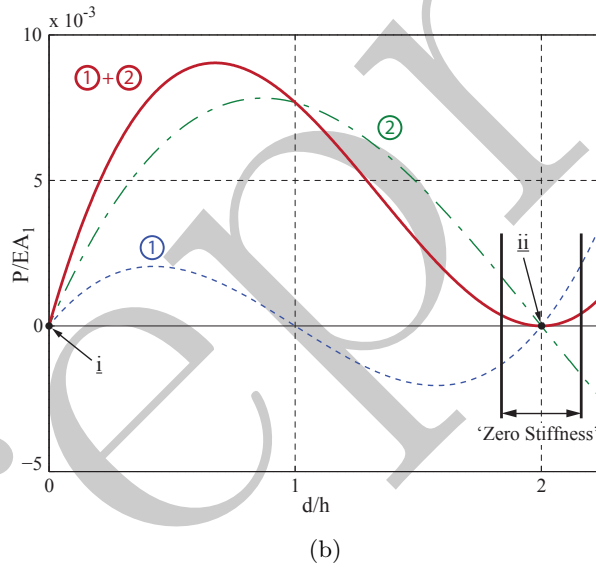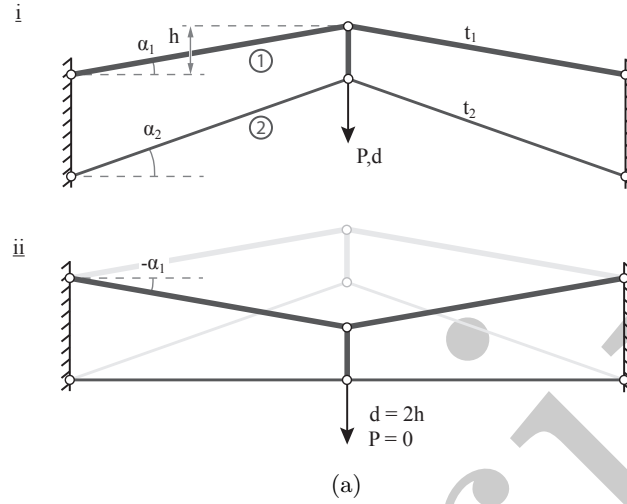
(a)

(b)

Figure 8: A compliant mechanism with zero stiffness can be obtained by combining the positive and negative stiffness of two elastic structures. In (a-i) are shown two coupled Von Mises trusses [1] in their unstressed configuration; the bars are elastic and the Euler load is assumed sufficiently large to avoid buckling. Due to their differing rise α, the two arches will snap through at different displacements d during the preloading. The configuration considered here has $\alpha_1 = 10°$, $\alpha_2 = 19.4°$ and beam thickness $t_2 = 0.52 \cdot t_1$ (to ensure that the opposing stiffness of the two structures is equal at the zero stiffness point). In (b) is shown the force-displacement profile of the two arches. Around $d/h = 2$ the positive stiffness of arch 1 and negative stiffness of the post-buckled arch 2 add up to produce zero stiffness over a limited working range; see (a-ii) for the zero stiffness configuration. Note that this quasi-zero stiffness point is ultimately unstable.
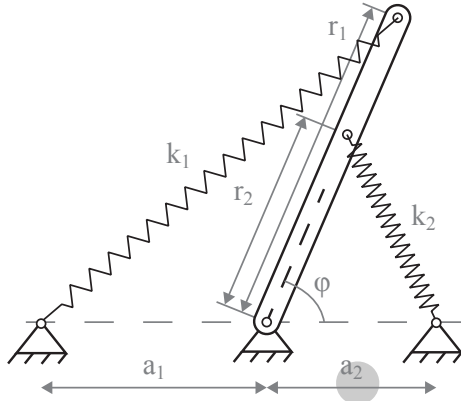
13

Figure 9: The basic spring-to-spring balancer consists of a rigid link connected to two zero-free-length springs [32]. The geometric parameters are a_1, a_2, r_1, r_2, with spring stiffnesses k_1, k_2; note that the attachment points at the base are collinear. The zero-stiffness condition $a_1 k_1 r_1 = a_2 k_2 r_2$ will be derived in this section.

4   Example: Zero Stiffness Four Ways

To illustrate the equivalence of the different interpretations of zero stiffness, this section details the mechanics of the classic spring-to-spring balancer shown in Figure 9. We extend an earlier analysis by Herder [32] and describe the spring-to-spring balancer in terms of i) continuous equilibrium, ii) constant potential energy, iii) neutral stability and iv) zero stiffness. Each method will be used to derive the combined geometric and stiffness conditions for zero stiffness (the level of prestress is implied by the use of zero-free-length springs), and each method provides different insights into the mechanics of the structure.

4.1   Continuous Equilibrium

For a spring-to-spring balancer with zero-free-length springs, the spring forces acting on the connecting link are straightforwardly resolved into a horizontal component and a component parallel to the direction of the link; see Figure 10.

$$F_\| = k_1 r_1 + k_2 r_2 \tag{4.1}$$

$$F_{1,\text{hor}} = a_1 k_1 \tag{4.2}$$

$$F_{2,\text{hor}} = a_2 k_2 \tag{4.3}$$

By virtue of using zero-free-length springs, these forces are independent of the link angle φ. The moment equilibrium around the bottom pivot is given by

$$a_1 k_1 r_1 \sin \varphi = a_2 k_2 r_2 \sin \varphi \tag{4.4}$$

and will hold, for any value of φ, if the relationship

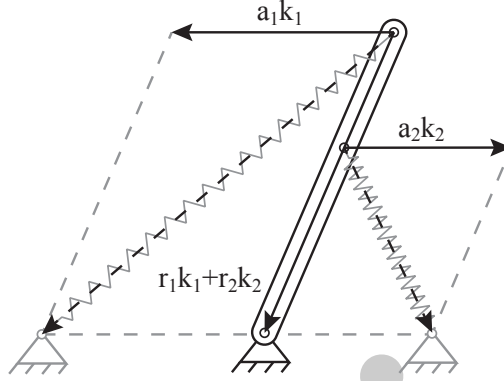$$a_1 k_1 r_1 = a_2 k_2 r_2 \tag{4.5}$$

14

Figure 10: The forces of the zero-free-length springs can be decomposed into their horizontal components and components parallel to the rigid link. Note that these forces are independent of the link angle φ.

is satisfied. This is the zero stiffness condition for the spring-to-spring balancer which describes the required relationship between the geometry and stiffness parameters of the structure; the level of prestress is implicit in the use of zero-free-length springs.

4.2   Constant Potential Energy

The total potential energy U in the spring-to-spring balancer depends solely on the elongation of the springs. Using the cosine rule to calculate the spring lengths, the energy stored in the zero-free-length springs is given by

$$
\begin{aligned}
U & = \frac{1}{2} k_1 \left[a_1^2 + r_1^2 - 2 a_1 r_1 \cos\left(\pi - \varphi\right) \right] + \frac{1}{2} k_2 \left[a_2^2 + r_2^2 - 2 a_2 r_2 \cos\varphi \right] \\
& = \frac{1}{2} k_1 \left(a_1^2 + r_1^2 \right) + \frac{1}{2} k_2 \left(a_2^2 + r_2^2 \right) - a_1 r_1 \cos\left(\pi - \varphi\right) - a_2 r_2 \cos\varphi \\
& = C + a_1 r_1 k_1 \cos\varphi - a_2 r_2 k_2 \cos\varphi
\end{aligned}
$$

which will be constant for any value of φ if the following equality holds:

$$
a_1 k_1 r_1 = a_2 k_2 r_2
$$

For the special case where $k_1 = k_2 = k$ and $a_1 = a_2 = r_1 = r_2 = R$, the constant energy solution is elegantly demonstrated geometrically using Thales' theorem, as shown in Figure 11.

4.3   Neutral Stability

Here we consider the basic balancer as a column loaded under a compressive load P, with a zero-free-length spring providing a restoring force; see Figure 12. For the axially loaded column with zero-free-length spring, the potential energy
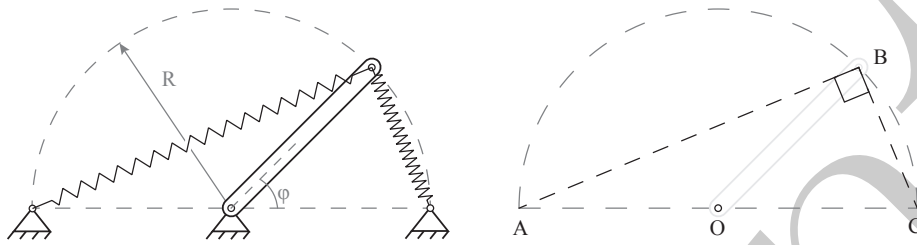
15

Figure 11: For a spring-to-spring balancer with $k_1 = k_2 = k$ and $a_1 = a_2 = r_1 = r_2 = R$ the constant potential energy can be illustrated elegantly with Thales' theorem: if A, B and C are points on a circle where the line AC is a diameter of the circle, then the angle $\angle ABC$ is a right angle. Using Pythagoras' theorem it is then evident that the total spring energy in the system will be constant for any angle φ of the rigid link.
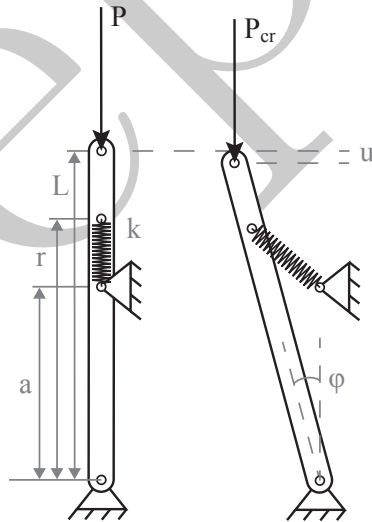

Figure 12: A column is loaded under a compressive axial load P, with a zero-free-length spring providing a restoring force. This will result in a neutrally stable post-buckling configuration at the critical buckling load P_{cr} (right).

16

function Π is given by:

$$\begin{aligned}
\Pi &= \frac{1}{2}k\left(r^2 + a^2 - 2ar\cos\varphi\right) - Pu \\
&= \frac{1}{2}k\left(r^2 + a^2 - 2ar\cos\varphi\right) - PL\left(1 - \cos\varphi\right)
\end{aligned}$$ (4.6)

For equilibrium $\partial\Pi/\partial\varphi = 0$, and thus:

$$\frac{\partial\Pi}{\partial\varphi} = \sin\varphi\left(akr - PL\right) = 0$$ (4.7)

which will hold for $\varphi = 0$, as well as for the critical buckling load:

$$P_{\mathrm{cr}} = \frac{akr}{L}$$ (4.8)

The second derivative determines the stability of the solution:

$$\frac{\partial^2\Pi}{\partial\varphi^2} = \cos\varphi\left(P_{\mathrm{cr}} - P\right)L$$ (4.9)

and will be zero for any angle φ at the critical buckling load. It is easy to verify all higher order derivatives of the potential function Π will be zero at the buckling load, and thus the structure is neutrally stable for any position under the critical buckling load P_{cr}. In effect, this becomes a constant force generator, and can be used to counterbalance a mass [38]. Two such structures can be coupled to form the statically balanced spring-to-spring balancer, as long as they share an equal critical load. Using Equation 4.8 this requirement again yields the balancing condition $a_1 k_1 r_1 = a_2 k_2 r_2$ for the system shown in Figure 9.

4.4 Zero Stiffness

Here the spring-to-spring balancer is modelled as a prestressed pin-jointed truss, where $r_1 = r_2 = r$; see Figure 13. The tangent stiffness matrix $\mathbf{K}_t$ can then be constructed according to the derivation by Guest [68], which takes into account prestressing of the members. The first component is the equilibrium matrix $\mathbf{A}$, which relates the internal tensions $\mathbf{t}$ to the external forces $\mathbf{f}$:

$$\mathbf{A}\mathbf{t} = \mathbf{f}$$ (4.10)

and is here given by:

$$\mathbf{A} = \left[\begin{array}{ccc} \frac{a_1 + r\cos\varphi}{l_1} & -\frac{a_2 - r\cos\varphi}{l_2} & \cos\varphi \\ \frac{r\sin\varphi}{l_1} & \frac{r\sin\varphi}{l_2} & \sin\varphi \end{array}\right]$$ (4.11)

with l_1 and l_2 the lengths of the respective springs. As the structure is statically indeterminate, the nullspace of $\mathbf{A}$ provides the state of self-stress $\mathbf{t}_0$:

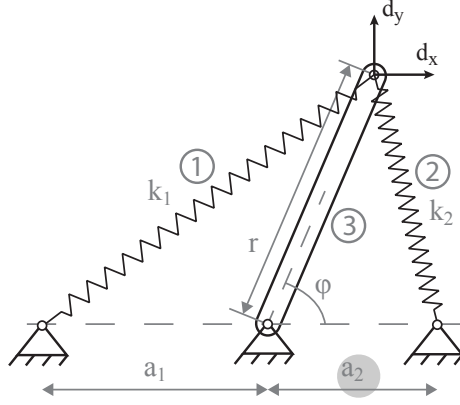$$\mathbf{t}_0 = \left[\begin{array}{c} a_2 l_1 \\ a_1 l_2 \\ -r(a_1 + a_2) \end{array}\right]$$ (4.12)

17

Figure 13: The spring-to-spring balancer with $r_1 = r_2 = r$ can be modelled as a pin-jointed truss with two degrees of freedom, d_x and d_y.

When using zero-free-length springs with k_1 and k_2, the zero stiffness condition $a_1 k_1 = a_2 k_2$ follows directly from this state of self-stress. The specific self-stress $\mathbf{t}$ thus becomes

$$\mathbf{t} = \begin{bmatrix} l_1 k_1 \\ l_2 k_2 \\ -r(k_1 + k_2) \end{bmatrix} \tag{4.13}$$

A modified axial stiffness $\hat{g}$ is formulated for each of the members, taking into account the prestress:

$$\hat{g} = g - \frac{t}{l} \tag{4.14}$$

where g is the axial stiffness of the member. For zero-free-length springs this modified axial stiffness will be zero [60], and the diagonal matrix $\hat{\mathbf{G}}$ becomes:

$$\hat{\mathbf{G}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (k_1 + k_2 + \frac{EA}{r}) \end{bmatrix} \tag{4.15}$$

The modified material stiffness matrix $\hat{\mathbf{K}}$ is given as:

$$\hat{\mathbf{K}} = \mathbf{A} \hat{\mathbf{G}} \mathbf{A}^T \tag{4.16}$$

which results in

$$\hat{\mathbf{K}} = (k_1 + k_2 + \frac{EA}{r}) \begin{bmatrix} \cos^2 \varphi & \cos \varphi \sin \varphi \\ \cos \varphi \sin \varphi & \sin^2 \varphi \end{bmatrix} \tag{4.17}$$

The tangent stiffness matrix $\mathbf{K}_t$ requires a further component, the stress matrix $\mathbf{S}$, to represent the stiffness due to the reorientation of the prestressed members.

$$\mathbf{K}_t = \hat{\mathbf{K}} + \mathbf{S} \tag{4.18}$$

The total stress matrix $\mathbf{S}$ is composed of the stress matrices $\mathbf{S}_i$ for the individual members,

$$\mathbf{S}_i = \frac{t_i}{l_i} \begin{bmatrix} \mathbf{I} & -\mathbf{I} \\ -\mathbf{I} & \mathbf{I} \end{bmatrix} \tag{4.19}$$

18

where $\mathbf{I}$ is the identity matrix. After removing the appropriate rows and columns for constrained nodes, the combined stress matrix for the spring-to-spring balancer becomes:

$$\mathbf{S} = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right] \tag{4.20}$$

and the tangent stiffness matrix $\mathbf{K}_t$ is thus equal to Equation 4.17. The tangent stiffness matrix has a nullity of one, and the nullspace has a basis vector:

$$\mathbf{v} = \left[\begin{array}{c} \tan\varphi \\ -1 \end{array} \right] \tag{4.21}$$

This basis describes the infinitesimal zero-stiffness displacement mode. It exists for every configuration φ, and is always orthogonal to the rigid link (with unit vector $\mathbf{r}$):

$$\mathbf{r} \cdot \mathbf{v} = \left[\begin{array}{cc} \cos\varphi & \sin\varphi \end{array} \right] \left[\begin{array}{c} \tan\varphi \\ -1 \end{array} \right] = 0 \tag{4.22}$$

The stiffness analysis shows that for any configuration φ of the spring-to-spring balancer there exists an infinitesimal zero-stiffness mode which is orthogonal to the rigid link; the infinitesimal modes are thus connected in a finite zero-stiffness path. The geometric-stiffness balancing condition for the spring-to-spring balancer was here derived from the state of self-stress of the structure. While more elaborate than the previous methods in describing this example structure, the stiffness matrix approach has proven to be powerful in analysing more complex zero-stiffness structures [60].

It is interesting to consider the parallel of this example to a cable-stayed mast, where the tension in the guy-ropes is increased. Intuitively, an increase in pretension would serve to stabilise the mast, and while it is true that this will help prevent cable slackening under load, actually the horizontal stiffness of the system decreases as tension is increased. To reach the critical level of prestress required for zero stiffness will, however, require specially constructed stays and mast.

5 Discussion and Conclusions

This paper has focused on the surprising and fascinating phenomenon that certain structures can deform elastically without external work. These singular structures are said to have zero stiffness, and their remarkable behaviour relies on a specific combination of geometry, stiffness and prestress.

Several equivalent descriptions for these structures exist, and examples from the literature have been discussed in terms of the following interpretations: i) continuous equilibrium, ii) constant potential energy, iii) neutral stability, and iv) zero stiffness. Each interpretation can yield alternative insights, and provide different methods of design and analysis. By rephrasing examples from the literature in different interpretations, disparate examples can be united by the same underlying principle. In cases where this is not straightforward — for

19

example, the Thomson and Tait twisting ring is not easily interpreted as a neutrally stable buckling solution — the most general description is that of zero stiffness, which covers all examples presented in this review.

The design of zero stiffness structures, *i.e.* finding the appropriate combination of geometry, stiffness and prestress, remains a specialised field. We hope that the present review will provide new insights and serve as a source of inspiration for the development of new and exciting examples of zero stiffness structures.

References

[1] Bazant ZP, Cedolin L. Stability of Structures: Elastic, Inelastic, Fracture and Damage Theories. Oxford University Press; 1991.

[2] Thompson JMT, Hunt GW. General Theory of Elastic Stability. John Wiley & Sons; 1973.

[3] Thomson W, Tait PG. A Treatise on Natural Philosophy. vol. 2. 2nd ed. Cambridge University Press; 1883.

[4] Lynen W. Federwerk; 1901. DE Patent 125645.

[5] Carwardine G. Equipoising mechanism; 1937. US Patent 2076446.

[6] Laporte JL. Counterweight system; 1982. US Patent 4351245.

[7] Bong F. Spring balanced adjustable blackboard; 1966. US Patent 3269035.

[8] Jackson EV, Matthew MP. Instrument table; 1965. US Patent 3179364.

[9] Sweere HC, Fluhrer RW, Al-zebdeh K, Eliason J, Lindblad SC, Showalter MJ, et al.. Monitor support system; 2006. US Patent 6994306.

[10] Debbage L. Constant load supports; 1961. US Patent 2989276.

[11] Hardtke H. Suspension device with a compensatory spring system; 1986. US Patent 4613119.

[12] van Dorsser WD, Barents R, Wisse BM, Herder JL. Gravity-Balanced Arm Support With Energy-Free Adjustment. ASME Journal of Medical Devices. 2007;1(2):151–158.

[13] Gosselin C, Wang J, Laliberté T, Ebert-Uphof I. On the design of a statically balanced 6-dof parallel manipulator. In: Proceedings of the IFToMM Tenth World Congress on the Theory of Machines and Mechanisms. Oulu, Finland; 1999. p. 1045–1050.

[14] Carricato M, Gosselin C. A statically balanced Gough/Stewart-type platform: conception, design and simulation. ASME Journal of Mechanisms and Robotics. 2009;1(3):031005.

[15] Hain K. Der Federausgleich von Lasten. Grundlagen der Landtechnik. 1952;p. 37 – 50.

20

[16] Koster MP. Constructieprincipes voor het nauwkeurig bewegen en positioneren. Twente University Press; 2000.

[17] Hoetmer K, Herder JL, Kim CJ. A Building Block Approach for the Design of Statically Balanced Compliant Mechanisms. In: ASME 2009 International Design Engineering Technical Conferences and Computers and Information in Engineering Conference (IDETC/CIE2009). DETC2009-87451. San Diego, California, USA; 2009. p. 313–323.

[18] Ibrahim RA. Recent advances in nonlinear passive vibration isolators. Journal of Sound and Vibration. 2008;314(3–5):371–452.

[19] Carrella A, Brennan MJ, Waters TP. Static analysis of a passive vibration isolator with quasi-zero-stiffness characteristic. Journal of Sound and Vibration. 2007;301(3–5):678–689.

[20] Robertson WS, Kidner MRF, Cazzolato BS, Zander AC. Theoretical design parameters for a quasi-zero stiffness magnetic spring for vibration isolation. Journal of Sound and Vibration. 2009;326(1–2):88–103.

[21] Jurgens J. Steadicam as a Design Problem. SMPTE Motion Imaging Journal. 1978;87(9):587–591.

[22] Brown G, Diguilio AO. Support apparatus; 1980. US Patent 4208028.

[23] LaCoste LJB. A New Type Long Period Vertical Seismograph. Journal of Applied Physics. 1934;5(7):178–180.

[24] French MJ, Widden MB. The spring-and-lever balancing mechanism, George Carwardine and the Anglepoise lamp. Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science. 2000;214(3):501–508.

[25] Guest SD, Kebadze E, Pellegrino S. A zero-stiffness elastic shell structure. Journal of Mechanics of Materials and Structures. 2011;6(1–4):203–212.

[26] Kebadze E, Guest SD, Pellegrino S. Bistable prestressed shell structures. International Journal of Solids and Structures. 2004;41(11–12):2801–2820.

[27] Seffen KA, Guest SD. Prestressed Morphing Bistable and Neutrally Stable Shells. Journal of Applied Mechanics. 2011;78(1):011002–011008.

[28] Seffen KA, McMahon RA. Heating of a uniform wafer disk. International Journal of Mechanical Sciences. 2007;49(2):230–238.

[29] Baradat C, Arakelian V, Briot S, Guegan S. Design and Prototyping of a New Balancing Mechanism for Spatial Parallel Manipulators. Journal of Mechanical Design. 2008;130(7):072305.

[30] Ostler J, Zwick K. Power equalizing device; 1939. US Patent 2178122.

[31] Wisse BM, Barents R, van Dorsser WD, Herder JL. Apparatus for exercising a force on a load; 2012. US Patent 8152144.

[32] Herder JL. Energy-Free Systems. Theory, conception and design of statically balanced spring mechanisms. Delft University of Technology, The Netherlands; 2001.

[33] Lynen W. Federwerk; 1904. DE Patent 157513.

[34] Carwardine G. Improvements in and relating to elastic force and equipoising mechanisms; 1938. GB Patent 489547.

[35] Nathan RH. A Constant Force Generation Mechanism. Journal of Mechanisms Transmissions and Automation in Design. 1985;107(4):508–512.

[36] Aberle H. Device for Generating a Constant Force Along a Working Path; 1998. WO Patent WO9855789.

[37] Tidwell PH, Bandukwala N, Dhande SG, Reinholtz CF, Webb G. Synthesis of Wrapping Cams. Journal of Mechanical Design. 1994;116(2):634–638.

[38] Barents R, Schenk M, van Dorsser WD, Wisse BM, Herder JL. Spring-to-Spring Balancing as Energy-Free Adjustment Method in Gravity Equilibrators. ASME Journal of Mechanical Design. 2011;133(6):061010.

[39] Carrella A, Brennan MJ, Waters TP, Shin K. On the design of a high-static-low-dynamic stiffness isolator using linear mechanical springs and magnets. Journal of Sound and Vibration. 2008;315(3):712–720.

[40] Walsh GJ, Streit DA, Gilmore BJ. Spatial spring equilibrator theory. Mechanism and Machine Theory. 1991;26(2):155–170.

[41] Radaelli G, Gallego JA, Herder JL. An Energy Approach to Static Balancing of Systems With Torsion Stiffness. Journal of Mechanical Design. 2011;133(9):091006–091014.

[42] Lachenal X, Weaver PM, Daynes S. Multi-stable composite twisting structure for morphing applications. Proceedings of Royal Society A. 2012;468(2141):1230–1251.

[43] Wilkes DF. Roller-Band Devices; 1969. US Patent 3452175.

[44] Hillberry BM, Hall AS. Rolling contact prosthetic knee joint; 1976. US Patent 3945053.

[45] Tarnai T. Zero stiffness elastic structures. International Journal of Mechanical Sciences. 2003;45(3):425 – 431.

[46] Vehar C, Kota S, Denis R. Closed-Loop Tape Springs as fully compliant mechanisms - preliminary investigations. In: Proceedings of 2004 DETC: 28th Biennial Mechanisms and Robotics Conference. Salt Lake City, Utah; 2004. .

[47] Seffen KA, Pellegrino S. Deployment dynamics of tape springs. Proceedings of teh Royal Society A. 1999;455(1983):1003–1048.

[48] Wahl AM. Mechanical Springs. 2nd ed. McGraw-Hill; 1963.

22

[49] Kyriakides S. Propagating Instabilities in Structures. Advances in Applied Mechanics. 1993;30:67–189.

[50] Chater E, Hutchinson JW. On the Propagation of Bulges and Buckles. Journal of Applied Mechanics. 1984;51(2):269–277.

[51] Gallego JA, Herder JL. Buckling as a new perspective on static balancing of mechanisms. In: 13th World Congress in Mechanism and Machine Science. Guanajuato, Mexico; 2011. .

[52] Gáspár Z. Buckling model for a degenerated case. News Letter, Technical University of Budapest. 1984;2(4):5–8.

[53] Hegedüs I. Contribution to Gáspár's paper: Buckling Model for a Degenerated Case. Newsletter Technical University of Budapest. 1986;4(1):8–9.

[54] Steinboeck A, Jia X, Höfinger G, Mang HA. Conditions for symmetric, antisymmetric, and zero-stiffness bifurcation in view of imperfection sensitivity and insensitivity. Computer Methods in Applied Mechanics and Engineering. 2008;197(45–48):3623–3636.

[55] Steinboeck A, Jia X, Höfinger G, Rubin H, Mang HA. Remarkable postbuckling paths analyzed by means of the consistently linearized eigenproblem. International Journal for Numerical Methods in Engineering. 2008;76(2):156–182.

[56] Jia X, Höfinger G, Mang HA. Imperfection Sensitivity or Insensitivity of Zero-stiffness Postbuckling that is the Question. Proceedings in Applied Mathematics and Mechanics. 2009;9(1):241–242.

[57] Jia X, Höfinger G, Mang HA. Necessary and Sufficient Conditions for Zero-stiffness Postbuckling. Proceedings in Applied Mathematics and Mechanics. 2010;10(1):173–174.

[58] Mang HA, Höfinger G, Jia X. On the predictability of zero-stiffness postbuckling. ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik. 2010;90(10–11):837–846.

[59] van Eijk J. On the design of plate-spring mechanisms. Delft University of Technology; 1985.

[60] Schenk M, Guest SD, Herder JL. Zero Stiffness Tensegrity Structures. International Journal of Solids and Structures. 2007;44(20):6569–6583.

[61] Guest SD. The stiffness of tensegrity structures. IMA Journal of Applied Mathematics. 2011;76(1):57–66.

[62] Schenk M, Herder JL, Guest SD. Design of a Statically Balanced Tensegrity Mechanism. In: ASME 2006 International Design Engineering Technical Conferences and Computers and Information in Engineering Conference (IDETC/CIE). vol. Volume 2: 30th Annual Mechanisms and Robotics Conference, Parts A and B. Philadelphia, Pennsylvania, USA; 2006. p. 501–511.

[63] Lin PY, Shieh WB, Chen DZ. A stiffness matrix approach for the design of statically balanced planar articulated manipulators. Mechanism and Machine Theory. 2010;45(12):1877–1891.

[64] Park ST, Luu TT. Techniques for optimizing parameters of negative stiffness. Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science. 2007;221(5):505–510.

[65] Chen G, Zhang S. Fully-compliant statically-balanced mechanisms without prestressing assembly: concepts and case studies. Mechanical Sciences. 2011;2:169–174.

[66] van Eijk J, Dijksman JF. Plate spring mechanism with constant negative stiffness. Mechanism and Machine Theory. 1979;14(1):1–9.

[67] Tolou N, Henneken VA, Herder JL. Statically Balanced Compliant Micro Mechanisms (SB-MEMS): Concepts and Simulation. In: Proceedings of the ASME IDETC/CIE 2010 Conference. DETC2010-28406. August 15 - 18, 2010, Montreal, Quebec, Canada; 2010. .

[68] Guest SD. The stiffness of prestressed frameworks: A unifying approach. International Journal of Solids and Structures. 2006;43(3–4):842–854.