Nontrivial solutions for nonlinear discrete boundary value problems of the fourth order

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ABSTRACT

We study the existence of multiple nontrivial solutions for two nonlinear fourth order discrete boundary value problems. We first establish criteria for the existence of at least two nontrivial solutions of the problems and obtain conditions to guarantee that the two solutions are sign-changing. Under some appropriate assumptions, we further prove that the problems have at least three nontrivial solutions, which are respectively positive, negative, and sign-changing. We include two examples to illustrate the applicability of our results. Our theorems are proved by employing variational approaches, combined with the classic mountain pass lemma and a result from the theory of invariant sets of descending flow.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

It is well known that nonlinear difference equations of higher order appear naturally as discrete analogue and as numerical solutions of differential equations. The applications of such equations have been well documented in [16, 17]. In recent years, the existence of solutions of boundary value problems (BVPs) of difference equations, with various boundary conditions (BCs), has received increasing attention from many researchers. In this thesis, let $[c,d]_{\mathbb{Z}} = \{z \in \mathbb{Z} \mid c \leq z \leq d\}$, where $c,d \in \mathbb{Z}$ with $c \leq d$. We are concerned with the existence of multiple nontrivial solutions of the BVP for the discrete beam equation

$$\begin{cases} \Delta^4 u(k-2) - \alpha\Delta^2 u(k-1) + \beta u(k) = f(k,u(k)),\ k \in [a+1,b+1]_{\mathbb{Z}}, \\ u(a) = \Delta^2 u(a-1) = 0,\ u(b+2) = \Delta^2 u(b+1) = 0, \end{cases} \tag{1.1}$$

where $\alpha,\beta \in [0,\infty)$, $a,b \in \mathbb{Z}$ with $b \geq a$, $f : [a+1,b+1]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ is continuous, and Δ is the forward difference operator defined by $\Delta u(k) = u(k+1) - u(k)$ and $\Delta^n u(k) = \Delta(\Delta^{n-1} u(k))$. By a *solution* of BVP (1.1), we mean a function $u : [a-1,b+3]_{\mathbb{Z}} \to \mathbb{R}$ such that u satisfies both the equation and the BCs in (1.1). If $u(k) > 0$ for all $k \in [a+1,b+1]_{\mathbb{Z}}$, then u is called a *positive solution*; if $u(k) < 0$ for all $k \in [a+1,b+1]_{\mathbb{Z}}$, then u is said to be a *negative solution*; and if $u(k)$ changes signs on $[a+1,b+1]_{\mathbb{Z}}$, then u is called a *sign-changing solution*. We also obtain existence criteria for the BVP

$$\begin{cases} \Delta^4 u(k-2) - \alpha\Delta^2 u(k-1) + \beta u(k) = \lambda f(k,u(k)),\ k \in [a+1,b+1]_{\mathbb{Z}}, \\ u(a) = \Delta^2 u(a-1) = 0,\ u(b+2) = \Delta^2 u(b+1) = 0, \end{cases} \tag{1.2}$$

where λ is a positive parameter.

BVP (1.1) can be regarded as a discrete analogue of the beam problem

$$\begin{cases} u^{(4)}(t) - \alpha u''(t) + \beta u(t) = f(t, u(t)), \quad t \in (0,1), \\ u(0) = u''(0) = 0, \quad u(1) = u''(1) = 0, \end{cases} \tag{1.3}$$

where the BCs correspond to both ends of the beam being hinged when there is no bending moment. The equation in BVP (1.3) is often referred to as the beam equation since it describes the deflection of a beam under a certain force. BVP (1.3) and a number of its variations have been investigated by many authors. A small sample of the work can be found in, for example, [2, 5, 27, 28] and the included references.

The existence of solutions of discrete BVPs, with various BCs, of the fourth order has been extensively studied in the literature. The reader is refered to [1, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 18, 24, 25, 26, 29, 31] for some work on this subject. In particular, paper [8] studied the existence of three solutions of the BVP

$$\begin{cases} \Delta^4 u(k-2) - \alpha \Delta^2 u(k-1) + \beta u(k) = \lambda f(k, u(k)),\ k \in [1, T]_{\mathbb{Z}}, \\ u(0) = \Delta u(-1) = \Delta^2 u(T) = 0,\ \Delta^3 u(T-1) - \alpha \Delta u(T) = \mu g(u(T+1)), \end{cases} \tag{1.4}$$

where $T \geq 2$ is an integer, $\alpha, \beta, \lambda, \mu \in \mathbb{R}$ are parameters, $f \in C([1,T]_{\mathbb{Z}} \times \mathbb{R}, \mathbb{R})$, and $g \in C(\mathbb{R}, \mathbb{R})$; while the existence of infinitely many solutions of BVP (1.4) was considered in [7]. The existence theorems in [7, 8] give the existence of solutions for the parameters λ and μ in different intervals. Paper [6] investigated the existence of two solutions of the BVP

$$\begin{cases} \Delta^4 u(k-2) - \alpha \Delta^2 u(k-1) + \beta u(k) = f(k, u(k)),\ k \in [1, T]_{\mathbb{Z}}, \\ u(-1) = \Delta u(-1) = 0,\ u(T+1) = \Delta^2 u(T) = 0. \end{cases}$$

Variational methods and critical point theory were used in [6, 7, 8] to prove the existence results. By using fixed point theory, paper [1] obtained a number of criteria for the existence of positive solutions of BVP (1.2) with $\beta = 0$, i.e., the problem

$$\begin{cases} \Delta^4 u(k-2) - \alpha \Delta^2 u(k-1) = \lambda f(k, u(k)), \ k \in [a+1, b+1]_{\mathbb{Z}}, \\ u(a) = \Delta^2 u(a-1) = 0, \ u(b+2) = \Delta^2 u(b+1) = 0. \end{cases} \tag{1.5}$$

Later, paper [9] studied the following slightly more general version of BVP (1.5)

$$\begin{cases} \Delta^4 u(t-2) - \beta \Delta^2 u(t-1) = \lambda \left[f(t, u(t), u(t)) + r(t, u(t)) \right], \ t \in [a+1, b-1]_{\mathbb{Z}}, \\ u(a) = \Delta^2 u(a-1) = 0, \ u(b) = \Delta^2 u(b-1) = 0, \end{cases} \tag{1.6}$$

where $f : [a+1, b-1]_{\mathbb{Z}} \times [0, \infty) \times (0, \infty) \to [0, \infty)$ and $r : [a+1, b-1]_{\mathbb{Z}} \times [0, \infty) \to [0, \infty)$ are continuous. By applying some results from mixed monotone operator theory, not only was the existence and uniqueness of positive solutions of BVP (1.6) obtained , but also the dependence of positive solutions on the parameter λ was discussed. Moreover, two sequences are constructed in such a way so that they converge uniformly to the unique positive solution of the problem. See [9, Theorems 3.1 and 3.2] for details. Paper [10] studied the existence of solutions of BVP (1.2) by using the critical point theory and monotone operator theory. We comment that none of these papers studied the existence of sign-changing solutions. One of the goals of this work is to study sign-changing solutions of BVPs (1.1) and (1.2).

In this thesis, we prove some new existence criteria for multiple nontrivial solutions of BVPs (1.1) and (1.2). We first establish an equivalent variational structure for BVP (1.1). During the process, we derive a symmetric positive definite matrix M, defined by (2.19) below, whose eigenvalues are exactly eigenvalues of a linear BVP. See Remark 2.0.3 in Chapter 2 for details. The smallest and largest eigenvalues of the matrix M are used in the statements and proofs of our theorems. Spectral properties of several BVPs for the linear discrete beam equation have been studied by Ji and Yang in [12, 13, 14]. In our first existence result (Theorem 3.1.1) for BVP (1.1), we utilize variational approaches, combined with the classic mountain pass lemma, to show that BVP (1.1) has at least two nontrivial solutions. Then, by the positivity of the associated Green's function (see Remark 2.0.5), we further establish sufficient conditions to guarantee that the two nontrivial solutions are sign-changing. In our second existence result (Theorem 4.1.1) for BVP (1.1), we combine variational methods with the theory of invariant sets of descending flow to show that, under some suitable conditions, BVP (1.1) has at least three nontrivial solutions consisting of one positive, one negative, and one sign-changing solutions. The theory of invariant sets of descending flow was introduced by Liu and Sun in [22] in 2001 and has now become a useful tool in the study of existence theory for nonlinear problems. We refer the reader to [19, 20, 21, 23] for some recent applications of this theory. As applications of Theorems 3.1.1 and 4.1.1, we also obtain several criteria for the existence of multiple nontrivial solutions of BVP (1.2).

The rest of this thesis is organized as follows. Chapter 2 contains some preliminaries. Chapter 3 studies the existence of at least two nontrivial solutions of BVPs (1.1) and (1.2) and the existence of at least three nontrivial solutions of the problems are investigated in Chapter 4.

CHAPTER 2

PRELIMINARY RESULTS

In this chapter, we first obtain the equivalent variational structure for BVP (1.1). Define a set X of functions by

$$X = \left\{u : [a-1,b+3]_{\mathbb{Z}} \to \mathbb{R} \mid u(a) = \Delta^2 u(a-1) = 0,\ u(b+2) = \Delta^2 u(b+1) = 0\right\}. \quad (2.1)$$

Then, X is a vector space with $au+bv = \{au(k)+bv(k)\}$ for any $u, v \in X$ and $a, b \in \mathbb{R}$. Moreover, X is a $b-a+1$ dimensional Banach space equipped with the norm

$$\|u\| = \left(\sum_{k=a+1}^{b+1} (u(k))^2\right)^{1/2} \quad \text{for any } u \in X.$$

Define the functionals $\Phi, \Psi, I : X \to \mathbb{R}$ by

$$\Phi(u) = \frac{1}{2}\sum_{k=a+2}^{b+2} (\Delta^2 u(k-2))^2 + \frac{1}{2}\alpha \sum_{k=a+1}^{b+2} (\Delta u(k-1))^2 + \frac{1}{2}\beta \sum_{k=a+1}^{b+1} u^2(k), \quad (2.2)$$

$$\Psi(u) = \sum_{k=a+1}^{b+1} F(k, u(k)),$$

and

$$I(u) = \Phi(u) - \Psi(u), \quad (2.3)$$

where $u \in X$ and

$$F(t,x) = \int_0^x f(k,s)ds. \quad (2.4)$$

Then, Φ, Ψ, I are well defined and continuously differentiable whose derivatives are the linear functionals $\Phi'(u)$, $\Psi'(u)$, and $I'(u)$ given by

$$\Phi'(u)(v) = \sum_{k=a+2}^{b+2} \Delta^2 u(k-2)\Delta^2 v(k-2) + \alpha \sum_{k=a+1}^{b+2} \Delta u(k-1)\Delta v(k-1) + \beta \sum_{k=a+1}^{b+1} u(k)v(k),$$

$$\Psi'(u)(v) = \sum_{k=a+1}^{b+1} f(k,u(k))v(k),$$

and

$$\begin{aligned} I'(u)(v) &= \sum_{k=a+2}^{b+2} \Delta^2 u(k-2)\Delta^2 v(k-2) + \alpha \sum_{k=a+1}^{b+2} \Delta u(k-1)\Delta v(k-1) \\ &\quad + \beta \sum_{k=a+1}^{b+1} u(k)v(k) - \sum_{k=a+1}^{b+1} f(k,u(k))v(k) \end{aligned} \tag{2.5}$$

for any $u, v \in X$.

Using the summation by parts formula

$$\sum_{k=m}^{n} f_k \Delta g_k = f_{n+1}g_{n+1} - f_m g_m - \sum_{k=m}^{n} g_{k+1}\Delta f_k, \tag{2.6}$$

we can prove the following lemma.

Lemma 2.0.1For any $u, v \in X$, we have

$$\sum_{k=a+2}^{b+2} \Delta^2 u(k-2)\Delta^2 v(k-2) = \sum_{k=a+1}^{b+1} \Delta^4 u(k-2)v(k) \tag{2.7}$$

and

$$\sum_{k=a+1}^{b+2} \Delta u(k-1)\Delta v(k-1) = -\sum_{k=a+1}^{b+1} \Delta^2 u(k-1)v(k). \tag{2.8}$$

Proof. For any $u, v \in X$, we have

$$\Delta^2 u(a-1) = \Delta^2 u(b+1) = v(a) = v(b+2) = 0. \tag{2.9}$$

We now prove (2.7). Using (2.6) and (2.9), we obtain that

$$\begin{aligned}
& \sum_{k=a+2}^{b+2} \Delta^2 u(k-2)\Delta^2 v(k-2) \\
= \;& \Delta^2 u(b+1)\Delta v(b+1) - \Delta^2 u(a)\Delta v(a) - \sum_{k=a+2}^{b+2} \Delta^3 u(k-2)\Delta v(k-1) \\
= \;& -\Delta^2 u(a)v(a+1) - \sum_{k=a+2}^{b+2} \Delta^3 u(k-2)\Delta v(k-1) \\
= \;& -\Delta^2 u(a)v(a+1) - \Delta^3 u(b+1)v(b+2) + \Delta^3 u(a)v(a+1) + \sum_{k=a+2}^{b+2} \Delta^4 u(k-2)v(k) \\
= \;& \left(-\Delta^2 u(a) + \Delta^3 u(a)\right) v(a+1) + \sum_{k=a+2}^{b+2} \Delta^4 u(k-2)v(k) \\
= \;& \Delta^4 u(a-1)v(a+1) + \sum_{k=a+2}^{b+2} \Delta^4 u(k-2)v(k) \\
= \;& \sum_{k=a+1}^{b+1} \Delta^4 u(k-2)v(k).
\end{aligned} \tag{2.10}$$

Thus, (2.7) holds. Similarly, we can show (2.8). This completes the proof of the lemma. ■

Lemma 2.0.2A function $u \in X$ is a critical point of the functional I if and only u is a solution of BVP (1.1).

Proof. In view of (2.5) and Lemma 2.0.1, we have the following equivalence

$$\begin{aligned}
& u \in X \text{ is a critical point of } I \\
\iff \;& I'(u)(v) = 0 \text{ for any } v \in X \\
\iff \;& \sum_{k=a+1}^{b+1} \left[\Delta^4 u(k-2) - \alpha\Delta^2 u(k-1) + \beta u(k) - f(k,u(k))\right] v(k) = 0 \text{ for any } v \in X \\
\iff \;& \Delta^4 u(k-2) - \alpha\Delta^2 u(k-1) + \beta u(k) = f(k,u(k)) \quad \text{for all } k \in [a+1,b+1]_{\mathbb{Z}}.
\end{aligned}$$

Note that the BCs in (1.1) are obviously satisfied since $u \in X$. Then, the conclusion of the lemma is true. This completes the proof of the lemma. ■

Below, we present an equivalent form of the functional $\Phi(u)$. Let

$$u = (u(a-1), u(a), u(a+1), \cdots, u(b+1), u(b+2), u(b+3)) \in X.$$

Since X is isomorphic to $\mathbb{R}^{b-a+1}$ and u satisfies the BCs in (1.1), in the sequel, we always identify u with the vector

$$u = (u(a+1), \cdots, u(b+1)) \in \mathbb{R}^{b-a+1}.$$

For the first term in $\Phi(u)$, we have

$$\begin{aligned} \frac{1}{2}\sum_{k=a+2}^{b+2}(\Delta^2 u(k-2))^2 &= \frac{1}{2}\sum_{k=a+2}^{b+2}\big[u^2(k)+4u^2(k-1)+u^2(k-2)+2u(k)u(k-2) \\ &\quad -4u(k)u(k-1)-4u(k-1)u(k-2)\big]. \end{aligned} \tag{2.11}$$

By simple calculations, we see that

$$\sum_{k=a+2}^{b+2}\big[u^2(k)+4u^2(k-1)+u^2(k-2)+2u(k)u(k-2)\big] = uAu^T$$

and

$$\sum_{k=a+2}^{b+2}\big[-4u(k)u(k-1)-4u(k-1)u(k-2)\big] = uBu^T,$$

where A and B are two $(b-a+1)\times(b-a+1)$ matrices given by

$$A=\begin{cases}\begin{pmatrix}4\end{pmatrix} & \text{if } b=a,\\ \begin{pmatrix}5&0\\0&5\end{pmatrix} & \text{if } b=a+1,\\ \begin{pmatrix}5&0&1\\0&6&0\\1&0&5\end{pmatrix} & \text{if } b=a+2,\\ \begin{pmatrix}5&0&1&0&0&\cdots&0&0&0&0\\0&6&0&1&0&\cdots&0&0&0&0\\1&0&6&0&1&\cdots&0&0&0&0\\0&1&0&6&0&\cdots&0&0&0&0\\0&0&1&0&6&\cdots&0&0&0&0\\\vdots&\vdots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\vdots\\0&0&0&0&0&\cdots&6&0&1&0\\0&0&0&0&0&\cdots&0&6&0&1\\0&0&0&0&0&\cdots&1&0&6&0\\0&0&0&6&0&\cdots&0&1&0&5\end{pmatrix} & \text{if } b\geq a+3,\end{cases}$$

and

$$B = \begin{cases} \begin{pmatrix} 0 \end{pmatrix} & \text{if } b = a, \\ \begin{pmatrix} 0 & -4 \\ -4 & 0 \end{pmatrix} & \text{if } b = a+1, \\ \begin{pmatrix} 0 & -4 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -4 & 0 & -4 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & -4 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & -4 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & -4 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -4 & 0 & -4 \\ 0 & 0 & 0 & 6 & 0 & \cdots & 0 & 0 & -4 & 0 \end{pmatrix} & \text{if } b \geq a+2. \end{cases}$$

Thus, from (2.11), it follows that

$$\frac{1}{2}\sum_{k=a+1}^{b+2}(\Delta^2 u(k-2))^2 = \frac{1}{2}uCu^T, \tag{2.12}$$

where $C = A + B$ is the $(b-a+1)\times(b-a+1)$ matrix given by

$$C = \begin{cases} \begin{pmatrix} 4 \end{pmatrix} & \text{if } b = a, \\ \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix} & \text{if } b = a+1, \\ \begin{pmatrix} 5 & -4 & 1 \\ -4 & 6 & -4 \\ 1 & -4 & 5 \end{pmatrix} & \text{if } b = a+2, \\ \begin{pmatrix} 5 & -4 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -4 & 6 & -4 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & -4 & 6 & -4 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 6 & -4 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 & 6 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 6 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & -4 & 6 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -4 & 6 & -4 \\ 0 & 0 & 0 & 6 & 0 & \cdots & 0 & 1 & -4 & 5 \end{pmatrix} & \text{if } b \geq a+3. \end{cases} \tag{2.13}$$

For the second term in $\Phi(u)$, we have

$$\begin{aligned} \frac{1}{2}\alpha \sum_{k=a+1}^{b+2} (\Delta u(k-1))^2 &= \frac{1}{2}\alpha \sum_{k=a+1}^{b+2} \left[u^2(k) + u^2(k-1) - 2u(k)u(k-1)\right] \\ &= \frac{1}{2}\alpha \sum_{k=a+1}^{b+1} 2u^2(k) - \frac{1}{2}\alpha \sum_{k=a+2}^{b+1} 2u(k)u(k-1) \\ &= \frac{1}{2} uDu^T, \end{aligned} \tag{2.14}$$

where D is a $(b-a+1)\times(b-a+1)$ matrix defined by

$$D = \begin{cases} \begin{pmatrix} 2\alpha \end{pmatrix} & \text{if } b = a, \\ \begin{pmatrix} 2\alpha & -\alpha \\ -\alpha & 2\alpha \end{pmatrix} & \text{if } b = a+1, \\ \begin{pmatrix} 2\alpha & -\alpha & 0 & \cdots & 0 & 0 \\ -\alpha & 2\alpha & -\alpha & \cdots & 0 & 0 \\ 0 & -\alpha & 2\alpha & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2\alpha & -\alpha \\ 0 & 0 & 0 & \cdots & -\alpha & 2\alpha \end{pmatrix} & \text{if } b \geq a+2. \end{cases} \tag{2.15}$$

For the last term in $\Phi(u)$, we have

$$\frac{1}{2}\beta \sum_{k=a+1}^{b+1} u^2(k) = \frac{1}{2} u E u^T, \tag{2.16}$$

where E is a $(b-a+1)\times(b-a+1)$ matrix given by

$$E = \begin{pmatrix} \beta & 0 & 0 & \cdots & 0 & 0 \\ 0 & \beta & 0 & \cdots & 0 & 0 \\ 0 & 0 & \beta & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \beta & 0 \\ 0 & 0 & 0 & \cdots & 0 & \beta \end{pmatrix}. \tag{2.17}$$

Now, from (2.2), (2.12), (2.14), (2.16), it follows that

$$\Phi(u) = \frac{1}{2} u M u^T \qquad \text{for all } u \in X, \tag{2.18}$$

where

$$M = C + D + E. \tag{2.19}$$

It is obvious that M is a symmetric positive definite matrix. Let λ_i, $i = 1,\dots,b-a+1$, be the eigenvalues of M ordered as follows

$$0 < \lambda_1 \le \lambda_2 \le \dots \le \lambda_{b-a+1} < \infty,$$

and let ξ_i be a normalized eigenvector of M associated with λ_i such that

$$(\xi_i, \xi_j) = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$$

Then, it is easy to verify that

$$\frac{1}{2}\lambda_1 \|u\|^2 \le \Phi(u) \le \frac{1}{2}\lambda_{b-a+1}\|u\|^2 \quad \text{for all } u \in X. \tag{2.20}$$

Remark 2.0.3Consider the following associated linear version of BVP (1.2)

$$\begin{cases} \Delta^4 u(k-2) - \alpha\Delta^2 u(k-1) + \beta u(k) = \lambda u(k),\ k \in [a+1, b+1]_{\mathbb{Z}}, \\ u(a) = \Delta^2 u(a-1) = 0,\ u(b+2) = \Delta^2 u(b+1) = 0. \end{cases} \tag{2.21}$$

For a fixed $\lambda \in \mathbb{C}$, if BVP (2.21) has a nontrivial solution $u \in X$, then λ is called an eigenvalue of BVP (2.21) and the corresponding nontrivial solution u is said to be an eigenfunction of BVP (2.21) corresponding to λ. It is easy to verify that λ is an eigenvalue of problem (2.21) if and only if λ is an eigenvalue of the matrix M defined by (2.19). Thus, all the eigenvalues of BVP (2.21) are positive and are given by $\lambda_1,\dots,\lambda_{b-a+1}$. As mentioned in Chapter 1, the properties of eigenvalues for several BVPs, consisting of linear discrete beam equations and different BCs, have been investigated by Ji and Yang in [12, 13, 14].

Next, we recall how to rewrite the solution of BVP (1.1) as a fixed point of some appropriate operator. To this end, we assume that α and β satisfy $\alpha^2 \ge 4\beta$ and let r_1 and r_2 be the roots of the polynomial $P(r) = r^2 - \alpha r + \beta$, i.e.,

$$r_1 = \frac{\alpha + \sqrt{\alpha^2 - 4\beta}}{2} \quad \text{and} \quad r_2 = \frac{\alpha - \sqrt{\alpha^2 - 4\beta}}{2}.$$

Then, $r_1 \geq r_2 \geq 0$. For $i = 1, 2$, let

$$G_i(t,k) = \frac{1}{\rho_i(1,0)\rho_i(b+2,a)} \begin{cases} \rho_i(t,a)\rho_i(b+2,k), & a \leq t \leq k \leq b+1, \\ \rho_i(k,a)\rho_i(b+2,t), & a+1 \leq k \leq t \leq b+2, \end{cases}$$

where

$$\rho_i(t,k) = \begin{cases} t-k & \text{if } r_i = 0, \\ \gamma_i^{t-k} - \gamma_i^{k-t} & \text{if } r_i > 0, \end{cases}$$

with

$$\gamma_i = \frac{r_i + 2 + \sqrt{r_i(r_i+4)}}{2}.$$

The following lemma follows from [10, Lemma 2.2].

Lemma 2.0.4Assume that $\alpha^2 \geq 4\beta$. Then, a function $u \in X$ is a solution of BVP (1.1) if and only if u is a fixed point of the completely continuous operator $T : X \to X$ defined by

$$T(u(k)) = \sum_{l=a+1}^{b+1} G(k,l) f(l, u(l)), \quad k \in [a+1, b+1]_{\mathbb{Z}}, \tag{2.22}$$

where

$$G(k,l) = \sum_{s=a+1}^{b+1} G_2(k,s) G_1(s,l). \tag{2.23}$$

Remark 2.0.5From (2.23), we see that $G(k,l) > 0$ for all $k, l \in [a+1, b+1]_{\mathbb{Z}}$.

Finally in this chapter, we introduce the following notations that will used in the remainder of this thesis:

$$\begin{cases} F^{\infty} = \limsup_{|x| \to \infty} \dfrac{\max_{k \in [a+1,b+1]_{\mathbb{Z}}} F(k,x)}{|x|^2}, \\ F^{0} = \limsup_{|x| \to 0} \dfrac{\max_{k \in [a+1,b+1]_{\mathbb{Z}}} F(k,x)}{|x|^2}, \\ F_{\infty} = \liminf_{|x| \to \infty} \dfrac{\min_{k \in [a+1,b+1]_{\mathbb{Z}}} F(k,x)}{|x|^2}, \\ f^{0} = \limsup_{|x| \to 0} \dfrac{\max_{k \in [a+1,b+1]_{\mathbb{Z}}} |f(k,x)|}{|x|}, \\ f^{\infty}_{\vartheta} = \limsup_{|x| \to \infty} \dfrac{\max_{k \in [a+1,b+1]_{\mathbb{Z}}} |f(k,x)|}{|x|^{\vartheta}}, \quad \vartheta > 1, \end{cases} \tag{2.24}$$

where f is the nonlinear function given in problem (1.1) and F is defined by (2.4).

CHAPTER 3

EXISTENCE OF TWO NONTRIVIAL SOLUTIONS

3.1 MAIN RESULTS

In this chapter, we study the existence of at least two nontrivial solutions of BVPs (1.1) and (1.2). Below, we first state our existence criteria and then provide one example to illustrate the results. Recall that λ_1 and λ_{b-a+1} are the smallest and largest eigenvalues of the matrix M defined by (2.19).

Theorem 3.1.1Assume that

(H1) $F^\infty < \frac{1}{2}\lambda_1$ and $F^0 < \frac{1}{2}\lambda_1$;

(H2) there exists $w \in X$ such that $\sum_{k=a+1}^{b+1} F(k,w(k)) > \frac{1}{2}\lambda_{b-a+1}\|w\|^2$.

Then, BVP (1.1) has at least two nontrivial solutions.

If, in addition to (H1) and (H2), we further assume that

(H3) $\alpha^2 \geq 4\beta$ and $xf(k,x) < 0$ for all $k \in [a+1,b+1]_{\mathbb{Z}}$ and $x \neq 0$,

then the two nontrivial solutions are sign-changing solutions.

The following corollaries are direct consequences of Theorem 3.1.1.

Corollary 3.1.2Assume that there exists $w \in X$ such that

$$\frac{\lambda_{b-a+1}\|w\|^2}{\sum_{k=a+1}^{b+1} F(k,w(k))} < \min\left\{\frac{\lambda_1}{F^\infty}, \frac{\lambda_1}{F^0}\right\}, \tag{3.1}$$

Then, for each

$$\lambda \in \left(\frac{\lambda_{b-a+1}\|w\|^2}{2\sum_{k=a+1}^{b+1} F(k,w(k))}, \min\left\{\frac{\lambda_1}{2F^\infty}, \frac{\lambda_1}{2F^0}\right\}\right), \tag{3.2}$$

BVP (1.2) has at least two nontrivial solutions. Moreover, if (H3) holds, the two nontrivial solutions are sign-changing solutions.

Corollary 3.1.3Assume that $F^{\infty} = F^0 = 0$ and there exists $w \in X$ such that $\sum_{k=a+1}^{b+1} F(k, w(k)) > 0$, Then, for each

$$\lambda \in \left(\frac{\lambda_{b-a+1} \|w\|^2}{2 \sum_{k=a+1}^{b+1} F(k, w(k))}, \infty \right),$$

BVP (1.2) has at least two nontrivial solutions. Moreover, if (H3) holds, the two nontrivial solutions are sign-changing solutions.

Example 3.1.4Consider the BVP

$$\begin{cases} \Delta^4 u(k-2) - 3\Delta^2 u(t-1) + 2u(t) = \lambda f(k, u(k)), \ k \in [1,5]_{\mathbb{Z}}, \\ u(0) = \Delta^2 u(-1) = 0, \ u(6) = \Delta^2 u(5) = 0, \end{cases} \tag{3.3}$$

where $\lambda > 0$ is a parameter and

$$f(k,x) = \begin{cases} -4k & \text{if} \quad x > 1, \\ -4kx^3 & \text{if} \quad |x| \le 1, \\ 4k & \text{if} \quad x < -1, \end{cases} \qquad \text{for all } (k,x) \in [1,5]_{\mathbb{Z}} \times \mathbb{R}. \tag{3.4}$$

We claim that, for each $\lambda \in (4.5207, \infty)$, BVP (3.3) *has at least two nontrivial sign-changing solutions.*

In fact, first note that BVP (3.3) is of the form of BVP (1.1) with $a = 0$, $b = 4$, $\alpha = 3$, and $\beta = 2$. Obviously, $\alpha^2 > 4\beta$. From (3.4), we obtain that $xf(k,x) < 0$ for all $k \in [1,5]_{\mathbb{Z}}$ and $x \neq 0$, and

$$F(k,x) = \begin{cases} -k(4x-3) & \text{if} \quad x > 1, \\ -kx^4 & \text{if} \quad |x| \le 1, \\ k(4x+3) & \text{if} \quad x < -1, \end{cases} \qquad \text{for all } (k,x) \in [1,5]_{\mathbb{Z}} \times \mathbb{R}.$$

Then, in view of (2.24), we have $F^\infty = F^0 = 0$. From (2.13), (2.15), and (2.17), we see that the matrix M, defined by (2.19), is given by

$$M = \begin{pmatrix} 13 & -7 & 1 & 0 & 0 \\ -7 & 14 & -7 & 1 & 0 \\ 1 & -7 & 14 & -7 & 1 \\ 0 & 1 & -7 & 14 & -7 \\ 0 & 0 & 1 & -7 & 13 \end{pmatrix}.$$

Using MATLAB, we find that the smallest and largest eigenvalues λ_1 and λ_5 of M are given by $\lambda_1 \approx 2.8756$ and $\lambda_5 \approx 27.1244$. Choose $w \in X$ so that $w(k) = -1$ for all $k \in [1,5]_{\mathbb{Z}}$. Then, we have $\sum_{k=1}^{5} F(k,w(k)) = 15 > 0$. Thus, all the conditions of Corollary 3.1.3 are satisfied. Note that

$$\frac{\lambda_5 \|w\|^2}{2\sum_{k=1}^{5} F(k,w(k))} \approx 4.5207.$$

The claim then follows from Corollary 3.1.3.

3.2 PROOFS

We now prove our results. Recall that the functional I is said to satisfy the Palais–Smale (PS) condition if every sequence $\{u_n\} \subset H_\mu$, such that $I(u_n)$ is bounded and $I'(u_n) \to 0$ as $n \to \infty$, has a convergent subsequence. Here, the sequence $\{u_n\}$ is called a PS sequence of I.

Lemma 3.2.1Assume that $F^\infty < \frac{1}{2}\lambda_1$. Then, the functional I, defined by (2.3), is coercive and satisfies the PS condition.

Proof. We first show that I is coercive, i.e.,

$$\lim_{\|u\| \to \infty} I(u) = \infty \quad \text{for all } u \in X. \tag{3.5}$$

Since $F^\infty < \frac{1}{2}\lambda_1$, for a fixed $c_1 \in (F^\infty, \frac{1}{2}\lambda_1)$, there exists a constant $c_2 > 0$ such that

$$F(k,x) \le c_1|x|^2 + c_2 \quad \text{for all } (k,x) \in [a+1,b+1]_{\mathbb{Z}} \times \mathbb{R}. \tag{3.6}$$

Then, from (2.3), (2.20), and (3.6), we have

$$\begin{aligned} I(u) &\geq \frac{1}{2}\lambda_1\|u\|^2 - \sum_{k=a+1}^{b+1}\left(c_1|u(k)|^2 + c_2\right) \\ &= \left(\frac{1}{2}\lambda_1 - c_1\right)\|u\|^2 - c_2(b-a+1). \end{aligned}$$

Note that $c_1 < \frac{1}{2}\lambda_1$. Then, (3.5) holds, i.e., I is coercive. Now, from the fact that X is a finite dimensional Banach space, we see that I satisfies the PS condition. This completes the proof of the lemma. ■

Next, we recall the following classic mountain pass lemma of Ambrosetti and Rabinowitz (see, for example, [15, Theorem 7.1]). Below, we denote by $B_r(u)$ the open ball centered at $u \in X$ with radius $r > 0$, $\overline{B}_r(u)$ its closure, and $\partial B_r(u)$ its boundary.

Lemma 3.2.2Let $(X, \|\cdot\|)$ be a real Banach space and $I \in C^1(X, \mathbb{R})$. Assume that I satisfies the PS condition and there exist $u_0, u_1 \in X$ and $\rho > 0$ such that

(A1) $u_1 \notin \overline{B}_\rho(u_0)$;

(A2) $\max\{I(u_0), I(u_1)\} < \inf_{u\in\partial B_\rho(u_0)} I(u)$.

Then, I possesses a critical value which can be characterized as

$$c = \inf_{\gamma\in\Gamma}\max_{s\in[0,1]} I(\gamma(s)) \geq \inf_{u\in\partial B_\rho(u_0)} I(u),$$

where

$$\Gamma = \{\gamma \in C([0,1], X) \ : \ \gamma(0) = u_0, \ \gamma(1) = u_1\}.$$

We now prove Theorem 3.1.1.

Proof of Theorem 3.1.1. We first show that 0 is a strict local minimizer of I. First, we have

$$I(0) = \Phi(0) - \Psi(0) = 0. \tag{3.7}$$

Since $F^0 < \frac{1}{2}\lambda_1$, for a fixed $c_3 \in \left(F^\infty, \frac{1}{2}\lambda_1\right)$, there exists $\rho > 0$ such that

$$F(t,x) \le c_3|x|^2 \quad \text{for all } (t,x) \in [a+1,b+1]_{\mathbb{Z}} \times \mathbb{R} \text{ with } |x| \le \rho. \tag{3.8}$$

Let $u \in B_\rho(0) \setminus \{0\}$. Then, $u(k) < \rho$ on $[a+1,b+1]_{\mathbb{Z}}$. From (2.3), (2.20), and (3.8),

$$\begin{aligned} I(u) &\ge \frac{1}{2}\lambda_1 \|u\|^2 - c_3 \sum_{k=a+1}^{b+1} |u(k)|^2 \\ &= \left(\frac{1}{2}\lambda_1 - c_3\right)\|u\|^2 > 0. \end{aligned}$$

This shows that 0 is a strict local minimizer of I.

Let w be given in (H2). From (2.3), (2.20), and (H2), it follows that

$$I(w) \le \frac{1}{2}\lambda_{b-a+1}\|w\|^2 - \sum_{k=a+1}^{b+1} F(k,w(k)) < 0.$$

Then, 0 is not a global minimizer of I.

Next, we show that I has a global minimizer. Choose a constant $I_0 \in (I(w),0)$. Define a set S by

$$S = \{u \in X \mid I(u) \le I_0\}.$$

Then, $S \neq \emptyset$ since $w \in S$. In view of Lemma 3.2.1, I is coervive, which in turn implies that S is bounded. Hence, by [30, Corollary 38.10], I has a minimum I_1 on S, which is also the minimum of I on X. Thus,

$$0 > I_1 = \min_{u\in S} I(u) = \min_{u \in H_\mu} I(u) > -\infty,$$

and there exists $u_1 \in X$ such that

$$I(u_1) = I_1 < 0. \tag{3.9}$$

Thus, u_1 is a critical point of I and $u_1 \not\equiv 0$. From Lemma 2.0.2, it follows that u_1 is a nontrivial solution of BVP (1.1).

Below, we show the existence of a second critical point of I. By Lemma 3.2.1, I satisfies the PS condition. We have shown that $u_0 := 0$ is a strict local minimizer of I. Then, there exists $0 < \rho < \|u_1\|$ such that $r := \inf_{u\in\partial B_\rho(u_0)} I(u) > 0$. In view of (3.7) and (3.9), we see that all the conditions of Lemma 3.2.2 are satisfied. Then, from Lemma 3.2.2, there exists a critical point u_2 of I such that

$$I(u_2) \geq r > 0. \tag{3.10}$$

From(3.9) and (3.10), we have $u_1 \neq u_2$ and $u_2 \not\equiv 0$. Thus, Lemma 2.0.1 implies that u_2 is a second nontrivial solution of BVP (1.1).

Finally, we show that if (H3) holds, then u_1 and u_2 are sign-changing solutions. Suppose by the contradiction that u_1 is not sign-changing. Then, we have either $u_1(k) \geq 0$ or $u_1(k) \leq 0$ for all $k \in [a+1,b+1]_{\mathbb{Z}}$. Without loss of generality, we may assume that $u_1(k) \geq 0$ on $[a+1,b+1]_{\mathbb{Z}}$. Then, in view of (H3), $f(k,u_1(k)) \leq 0$ for all $k \in [a+1,b+1]_{\mathbb{Z}}$. By Lemma 2.0.4, we obtain that

$$u_1(k) = \sum_{l=a+1}^{b+1} G(k,l) f(l,u_1(l)), \quad k \in [a+1,b+1]_{\mathbb{Z}}.$$

Then, from Remark 2.0.5, $u_1(k) \leq 0$ on $[a+1,b+1]_{\mathbb{Z}}$. Hence, $u_1(k) \equiv 0$ on $[a+1,b+1]_{\mathbb{Z}}$. This contradicts with the fact that $u_1(t)$ is nontrivial. Hence, u_1 is a sign-changing solution. By a similar argument, we can show that u_2 is also a sign-changing solution. This completes the proof of the theorem. ■

Proof of Corollary 3.1.2. For any λ satisfying (3.2), we have

$$\lambda F^\infty < \frac{1}{2}\lambda_1, \quad \lambda F^0 < \frac{1}{2}\lambda_1,$$

and

$$\sum_{k=a+1}^{b+1} \lambda F(k,w(k)) > \frac{1}{2}\lambda_{b-a+1}\|w\|^2.$$

Thus, with F replaced by λF and f replaced by λf, the assumptions (H1), (H2), and (H3) of Theorem 3.1.1 are satisfied. The conclusion then follows directly from Theorem 3.1.1. ■

Proof of Corollary 3.1.3. Under the assumptions of the corollary, we see that

$$\frac{\lambda_{b-a+1}\|w\|^2}{\sum_{k=a+1}^{b+1} F(k,w(k))} < \infty \quad \text{and} \quad \min\left\{\frac{\lambda_1}{F^\infty}, \frac{\lambda_1}{F^0}\right\} = \infty.$$

Then, (3.1) holds. Hence, the conclusion follows directly from Corollary 3.1.2. ■

CHAPTER 4

EXISTENCE OF THREE NONTRIVIAL SOLUTIONS

4.1 MAIN RESULTS

In this chapter, we study the existence of at least three nontrivial solutions of BVPs (1.1) and (1.2). As in Chapter 3, we first state the existence results and then present one example to illustrate the applicability of the results.

Theorem 4.1.1Assume that

(A1) $F_\infty > \frac{1}{2}\lambda_{b-a+1}$;

(A2) $f^0 = 0$;

(A3) there exist $\vartheta > 1$ and $K > 0$ such that $|f(k,x)| \leq K(1+|x|^\vartheta)$ for all $(k,x) \in [a+1,b+1]_\mathbb{Z} \times \mathbb{R}$;

(A4) $\alpha^2 \geq 4\beta$ and $xf(k,x) > 0$ for all $k \in [a+1,b+1]_\mathbb{Z}$ and $x \neq 0$.

Then, BVP (1.1) has at least three nontrivial solutions, one of which is positive, one is negative, and one is sign-changing.

Corollaries 4.1.2 and 4.1.3 below follow directly from Theorem 4.1.1.

Corollary 4.1.2Assume that (A2) and (A4) hold and there exist $\vartheta > 1$ and $L > 0$ such that

$$\frac{\lambda_{b-a+1}}{2F_\infty} < \frac{L}{f_\vartheta^\infty}. \tag{4.1}$$

Then, for each

$$\lambda \in \left(\frac{\lambda_{b-a+1}}{2F_\infty}, \frac{L}{f_\vartheta^\infty}\right), \tag{4.2}$$

BVP (1.2) has at least three nontrivial solutions, one of which is positive, one is negative, and one is sign-changing.

Corollary 4.1.3Assume that (A2) and (A4) hold, $F_\infty > 0$, and $f_\vartheta^\infty = 0$, where $\vartheta > 1$. Then, for each

$$\lambda \in \left(\frac{\lambda_{b-a+1}}{2F_\infty}, \infty \right),$$

BVP (1.2) has at least three nontrivial solutions, one of which is positive, one is negative, and one is sign-changing.

Example 4.1.4Consider the BVP

$$\begin{cases} \Delta^4 u(k-2) - 4\Delta^2 u(t-1) + 3u(t) = \lambda f(k,u(k)),\ k \in [1,6]_{\mathbb{Z}}, \\ u(0) = \Delta^2 u(-1) = 0,\ u(7) = \Delta^2 u(6) = 0, \end{cases} \tag{4.3}$$

where $\lambda > 0$ is a parameter and

$$f(k,x) = \begin{cases} 4kx^3 & \text{if} \quad |x| \le 1, \\ 4kx & \text{if} \quad |x| > 1, \end{cases} \qquad \text{for all } (k,x) \in [1,6]_{\mathbb{Z}} \times \mathbb{R}. \tag{4.4}$$

We claim that, for each $\lambda \in (8.1656, \infty)$*, BVP* (4.3) *has at least three nontrivial solutions, one of which is positive, one is negative, and one is sign-changing.*

In fact, first note that BVP (4.3) is of the form of BVP (1.1) with $a = 0$, $b = 5$, $\alpha = 4$, and $\beta = 3$. Obviously, $\alpha^2 > 4\beta$. From (4.3), it follows that $xf(k,x) > 0$ for all $k \in [1,6]_{\mathbb{Z}}$ and $x \neq 0$, and

$$F(k,x) = \begin{cases} kx^4 & \text{if} \quad |x| \le 1, \\ 2kx^2 - k & \text{if} \quad |x| > 1, \end{cases} \qquad \text{for all } (k,x) \in [1,6]_{\mathbb{Z}} \times \mathbb{R}.$$

Then, in view of (2.24), we obtain that $F_\infty = 2 > 0$ and $f^0 = f_\vartheta^\infty = 0$ for any $\vartheta > 1$. Hence, all the conditions of Corollary 4.1.3 are satisfied. From (2.13), (2.15), and (2.17), we see that the matrix M, defined by (2.19), is given by

$$M = \begin{pmatrix} 16 & -8 & 1 & 0 & 0 & 0 \\ -8 & 17 & -8 & 1 & 0 & 0 \\ 1 & -8 & 17 & -8 & 1 & 0 \\ 0 & 1 & -8 & 17 & -8 & 1 \\ 0 & 0 & 1 & -8 & 17 & -8 \\ 0 & 0 & 0 & 8 & -8 & 16 \end{pmatrix}.$$

Using MATLAB, we find that the smallest and largest eigenvalues λ_1 and λ_6 of M are given by $\lambda_1 \approx 3.8315$ and $\lambda_6 \approx 32.6625$. Note that

$$\frac{\lambda_6}{2F_\infty} \approx 8.1656.$$

Then, the claim follows from Corollary 4.1.3.

4.2 PROOFS

We now prove our results. Let X be defined by (2.1). We equip X with the inner product

$$\langle u, v\rangle = \sum_{k=a+2}^{b+2} \Delta^2 u(k-2)\Delta^2 v(k-2) + \alpha \sum_{k=a+1}^{b+2} \Delta u(k-1)\Delta v(k-1) + \beta \sum_{k=a+1}^{b+1} u(k)v(k). \quad (4.5)$$

The induced norm $\|\cdot\|_1$ is given by

$$\|u\|_1 = \left(\sum_{k=a+2}^{b+2} (\Delta^2 u(k-2))^2 + \alpha \sum_{k=a+1}^{b+2} (\Delta u(k-1))^2 + \beta \sum_{k=a+1}^{b+1} u^2(k), \right)^{1/2}, \ u \in X. \quad (4.6)$$

Then, X is an $b-a+1$ dimensional Hilbert space and the norms $\|\cdot\|_1$ and $\|\cdot\|$ are equivalent.

Lemma 4.2.1Assume that $\alpha^2 \geq 4\beta$. Then, for any $u, v \in X$. we have

$$\langle Tu, v\rangle = \sum_{k=a+1}^{b+1} f(k, u(k))v(k) \quad (4.7)$$

and

$$I'(u) = u - Tu, \tag{4.8}$$

where I and T are defined by (2.3) and (2.22), respectively.

Proof. For any $u, v \in X$, from (2.5) and (4.5), it follows that

$$I'(u)(v) = \langle u, v\rangle - \sum_{k=a+1}^{b+1} f(k, u(k))v(k) \tag{4.9}$$

and

$$\begin{aligned}\langle Tu, v\rangle &= \sum_{k=a+2}^{b+2} \Delta^2(Tu)(k-2)\Delta^2 v(k-2) + \alpha \sum_{k=a+1}^{b+2} \Delta(Tu)(k-1)\Delta v(k-1)\\ &\quad +\beta \sum_{k=a+1}^{b+1} (Tu)(k)v(k). \end{aligned}\tag{4.10}$$

Using the summation by parts formula, similar as in (2.10), we obtain that

$$\begin{aligned}&\sum_{k=a+2}^{b+2} \Delta^2(Tu)(k-2)\Delta^2 v(k-2)\\ =\ & \Delta^2(Tu)(b+1)\Delta v(b+1) - \Delta^2(Tu)(a)\Delta v(a) - \sum_{k=a+2}^{b+2} \Delta^3(Tu)(k-2)\Delta v(k-1)\\ =\ & -\Delta^2(Tu)(a)v(a+1) - \sum_{k=a+2}^{b+2} \Delta^3(Tu)(k-2)\Delta v(k-1)\\ =\ & -\Delta^2(Tu)(a)v(a+1) - \Delta^3(Tu)(b+1)v(b+2) + \Delta^3(Tu)(a)v(a+1)\\ & + \sum_{k=a+2}^{b+2} \Delta^4(Tu)(k-2)v(k)\\ =\ & \left(-\Delta^2(Tu)(a) + \Delta^3(Tu)(a)\right) v(a+1) + \sum_{k=a+2}^{b+2} \Delta^4(Tu)(k-2)v(k)\\ =\ & \Delta^4(Tu)(a-1)v(a+1) + \sum_{k=a+2}^{b+2} \Delta^4(Tu)(k-2)v(k)\\ =\ & \sum_{k=a+1}^{b+1} \Delta^4(Tu)(k-2)v(k).\end{aligned}$$

By a similar argument, we have

$$\sum_{k=a+1}^{b+2} \Delta(Tu)(k-1)\Delta v(k-1) = -\sum_{k=a+1}^{b+1} \Delta^2(Tu)(k-1)v(k).$$

Then, from (4.10), we see that

$$\begin{aligned}\langle Tu, v\rangle &= \sum_{k=a+1}^{b+1} \left[\Delta^4(Tu)(k-2) - \alpha\Delta^2(Tu)(k-1) + \beta(Tu)(k)\right] v(k) \\ &= \sum_{k=a+1}^{b+1} f(k, u(k))v(k),\end{aligned}$$

i.e., (4.7) holds. Now, (4.7) and (4.9) imply that

$$I'(u)(v) = \langle u, v\rangle - \langle Tu, v\rangle.$$

Thus, (4.8) holds. This completes the proof of the lemma. ■

Lemma 4.2.2Assume that (A1) holds. Then, the functional I satisfies the PS condition.

Proof. From (A1), we see that, for a fixed $c_4 \in \left(\frac{1}{2}\lambda_{b-a+1}, F_\infty\right)$, there exists a constant $c_5 > 0$ such that

$$F(k,x) \geq c_4|x|^2 - c_5 \quad \text{for all } (k,x) \in [a+1, b+1]_{\mathbb{Z}} \times \mathbb{R}. \tag{4.11}$$

Assume that $\{u_n\} \subset X$ is a sequence such that $|I(u_n)| \leq N$ for some $N > 0$. From (2.3), (2.22), and (4.11), it follows that

$$\begin{aligned}-N \leq I(u_n) &\leq \frac{1}{2}\lambda_{b-a+1}\|u_n\|^2 - \sum_{k=a+1}^{b+1}\left(c_4|u_n(k)|^2 - c_5\right) \\ &= \left(\frac{1}{2}\lambda_{b-a+1} - c_4\right)\|u_n\|^2 + c_5(b-a+1).\end{aligned} \tag{4.12}$$

Thus,

$$\left(c_4 - \frac{1}{2}\lambda_{b-a+1}\right)\|u_n\|^2 \leq c_5(b-a+1) + N.$$

Note that $c_4 > \frac{1}{2}\lambda_{b-a+1}$. Then, $\{u_n\}$ is bounded in X. Since the dimension of X is finite, $\{u_n\}$ has a convergent subsequence. This completes the proof of the lemma. ∎

We now introduce several sets. Let

$$\Lambda^+ = \{u \in X \mid u(k) \geq 0 \text{ for all } k \in [a+1, b+1]_{\mathbb{Z}}\}$$

and

$$\Lambda^- = \{u \in X \mid u(k) \leq 0 \text{ for all } k \in [a+1, b+1]_{\mathbb{Z}}\}.$$

For any $\varepsilon > 0$, let the open convex subsets D_ε^+ and D_ε^- be defined by

$$D_\varepsilon^+ = \left\{u \in X \mid \operatorname{dist}(u, \Lambda^+) < \varepsilon\right\} \quad \text{and} \quad D_\varepsilon^- = \left\{u \in X \mid \operatorname{dist}(u, \Lambda^-) < \varepsilon\right\},$$

where $\operatorname{dist}(u, \Lambda^\pm) = \inf_{v \in \Lambda^\pm} \|u - v\|_1$. Then, $D_\varepsilon^+ \cap D_\varepsilon^- \neq \emptyset$ and $X \setminus (\overline{D_\varepsilon^+} \cup \overline{D_\varepsilon^-})$ only contains sign-changing functions.

Lemma 4.2.3Assume that (A2)–(A4) hold. Then, there exists $\bar{\varepsilon} > 0$ such that

$$T(\partial D_\varepsilon^+) \subset D_\varepsilon^+ \quad \text{and} \quad T(\partial D_\varepsilon^-) \subset D_\varepsilon^- \quad \text{for any } \varepsilon \in (0, \bar{\varepsilon}].$$

Moreover, any nontrivial critical points of the functional I in D_ε^+ (D_ε^-) are positive (negative) solutions of BVP (1.1).

Proof. We first prove the conclusion involving D_ε^-. For any $u \in X$, from (2.2) and (4.6), we have $\|u\|_1^2 = 2\Phi(u)$. This, together with from (2.20), implies that

$$\lambda_1 \|u\|^2 \leq \|u\|_1^2 \leq \lambda_{b-a+1} \|u\|^2.$$

Then,

$$\sqrt{\lambda_1}\, \|u\| \leq \|u\|_1 \leq \sqrt{\lambda_{b-a+1}}\, \|u\|. \tag{4.13}$$

For any $u \in X$, define $u^+(k) = \max\{u(k), 0\}$ and $u^-(k) = \min\{u(k), 0\}$ for all $k \in [a+1, b+1]_{\mathbb{Z}}$, and let $y = (Tu)(k) \in X$. Here, it is understood that u^+ and u^- are extended to the interval

$[a-1,b+3]_{\mathbb{Z}}$ in the way so that they satisfy the BCs in (1.1). Then, $u(k) = u^+(k) + u^-(k)$, and in view of (4.13), we have

$$\|u^+\| = \inf_{v\in\Lambda^-} \|u-v\| \le \frac{1}{\sqrt{\lambda_1}} \inf_{v\in\Lambda^-} \|u-v\|_1 = \frac{1}{\sqrt{\lambda_1}} \operatorname{dist}(u,\Lambda^-). \tag{4.14}$$

We now show the claim:

Claim: There exists a constant $c_6 > 0$ such that $\|u\|_1 \le c_6\|v\|_1$ for any $u, v \in X$ with $0 \le u(k) \le v(k)$ on $[a+1,b+1]_{\mathbb{Z}}$.

To prove the claim, suppose by the contradiction that the conclusion is not true. Then, there exist $u_n, v_n \in X$ with $0 \le u_n(k) \le v_n(k)$ on $[a+1,b+1]_{\mathbb{Z}}$ such that $\|u_n\|_1 > n^2\|v\|_1$. Let $z_n(k) = \frac{u_n(k)}{n\|v_n\|_1}$. Then, $\|z_n\|_1 > n$. This shows that $z_n(k) \not\to 0$ for all $k \in [a+1,b+1]_{\mathbb{Z}}$. On the other hand, we have

$$0 \le z_n(k) \le w_n(k) := \frac{v_n(k)}{n\|v\|_1}.$$

Note that $\|w_n\|_1 = \frac{1}{n} \to 0$. Then, $z_n(k) \to 0$ on $[a+1,b+1]_{\mathbb{Z}}$. We have reached a contradiction. Thus, the claim is true.

From (2.22), Remark 2.0.5, and (A4), we see that $0 \le (Tu)^+(k) \le T(u^+)(k)$ for all $k \in [a+1,b+1]_{\mathbb{Z}}$. Then, by the above claim, we have

$$\|y^+\|_1 = \|(Tu)^+\| \le c_6\|Tu^+\|_1.$$

Hence,

$$\operatorname{dist}(y,\Lambda^-) = \inf_{v\in\Lambda^-} \|y-v\|_1 \le \|y-y^-\|_1 = \|y^+\|_1 \le c_6\|Tu^+\|_1,$$

which in turn implies that

$$\operatorname{dist}(y,\Lambda^-)\|Tu^+\|_1 \le c_6\|Tu^+\|_1^2 = c_6\langle Tu^+, Tu^+\rangle. \tag{4.15}$$

By (A2) and (A3), there exists $c_7 \in \left(0, \frac{\lambda_1}{c_6(b-a+1)}\right)$ and $c_8 > 0$ such that

$$|f(k,x)| \le c_7|x| + c_8|x|^{\vartheta} \quad \text{for all } (k,x) \in [a+1,b+1]_{\mathbb{Z}} \times \mathbb{R}. \tag{4.16}$$

Now, from (4.7) in Lemma 4.2.2 and (4.13)–(4.16), we obtain that

$$\begin{aligned}
\operatorname{dist}(y,\Lambda^-)\|Tu^+\|_1 &\le c_6\langle Tu^+,Tu^+\rangle \\
&= c_6\sum_{k=a+1}^{b+1} f(k,u^+(k))\,T(u^+(k)) \\
&\le c_6\sum_{k=a+1}^{b+1}\left(c_7|u^+(k)|+c_8|u^+(k)|^{\vartheta}\right)T(u^+(k)) \\
&\le \left(c_6c_7(b-a+1)\|u^+\|+c_6c_8(b-a+1)\|u^+\|^{\vartheta}\right)\|Tu^+\| \\
&\le \left(\frac{c_9}{\lambda_1}\operatorname{dist}(u,\Lambda^-)+c_{10}(\operatorname{dist}(u,\Lambda^-))^{\vartheta}\right)\|Tu^+\|_1,
\end{aligned}$$

where $c_9=c_6c_7(b-a+1)$ and $c_{10}=c_6c_8(b-a+1)(\lambda_1)^{-\frac{\vartheta+1}{2}}$. Thus, we have

$$\operatorname{dist}(y,\Lambda^-)\le\frac{c_9}{\lambda_1}\operatorname{dist}(u,\Lambda^-)+c_{10}(\operatorname{dist}(u,\Lambda^-))^{\vartheta}.$$

Let $\bar{\varepsilon}=\left(\frac{\lambda_1-c_9}{2\lambda_1c_{10}}\right)^{\frac{1}{\vartheta-1}}$. Then, from the fact that $c_9=c_6c_7(b-a+1)<\lambda_1$, we know that $\bar{\varepsilon}$ is well defined and $\bar{\varepsilon}>0$. Moreover, for any $\varepsilon\in(0,\bar{\varepsilon}]$, if $\operatorname{dist}(u,\Lambda^-)\le\varepsilon$, we have

$$\begin{aligned}
\operatorname{dist}(y,\Lambda^-) &\le \frac{c_9}{\lambda_1}\operatorname{dist}(u,\Lambda^-)+c_{10}(\operatorname{dist}(u,\Lambda^-))^{\vartheta-1}\operatorname{dist}(u,\Lambda^-) \\
&\le \frac{c_9}{\lambda_1}\operatorname{dist}(u,\Lambda^-)+\left(\frac{\lambda_1-c_9}{2\lambda_1}\right)\operatorname{dist}(u,\Lambda^-) \\
&= \frac{\lambda_1+c_9}{2\lambda_1}\operatorname{dist}(u,\Lambda^-) \\
&< \operatorname{dist}(u,\Lambda^-) \\
&\le \varepsilon. \qquad (4.17)
\end{aligned}$$

Hence, $T(\partial D_{\varepsilon}^-)\subset D_{\varepsilon}^-$.

Now, let $u\in D_{\varepsilon}^-$ be a nontrivial critical point of I. Then, in view of (4.8) in Lemma 4.2.1, we see that $Tu(k)=u(k)$. By (4.17), we have $\operatorname{dist}(u,\Lambda^-)=0$. Thus, $u\in\Lambda^-\setminus\{0\}$. From Remark 2.0.5 and (A4), it follows that $u(k)<0$ for all $k\in[a+1,b+1]_{\mathbb{Z}}$. Thus, $u(k)$ is a negative solution of BVP (1.1). The proof for the conclusion involving D_{ε}^+ is similar and hence is omitted. This completes the proof of the lemma. ■

The following lemma is taken from [22, Theorem 3.2].

Lemma 4.2.4Let H be a Hilbert space. Assume that the functional $I \in C^1(H,\mathbb{R})$ satisfies the PS condition and $I'(u)$ has the expression $I'(u) = u - T(u)$ for all $u \in H$. Assume that there exist two open convex subsets D_1 and D_2 of H satisfying $D_1 \cap D_2 \neq \emptyset$, $T(\partial D_1) \subset D_1$, and $T(\partial D_2) \subset D_2$. If there exists a path $h : [0,1] \to H$ such that

$$h(0) \in D_1 \setminus D_2, \quad h(1) \in D_2 \setminus D_1,$$

and

$$\inf_{u \in \overline{D}_1 \cap \overline{D}_2} I(u) > \sup_{t \in [0,1]} I(h(t)),$$

then I has at least four distinct critical point, one in $D_1 \cap D_2$, one in $D_1 \setminus \overline{D}_2$, one in $D_2 \setminus \overline{D}_1$, and one in $H \setminus (\overline{D}_1 \cup \overline{D}_2)$.

Now, we are in a position to prove Theorem 4.1.1

Proof of Theorem 4.1.1. By (A2), we see that there exist $\nu \in \left(0, \frac{1}{2}\lambda_1\right)$ and $\delta > 0$ such that

$$|F(k,x)| \leq \nu |x|^2 \quad \text{for all } (k,|x|) \in [a+1,b+1]_{\mathbb{Z}} \times [0,\delta]. \tag{4.18}$$

Fix $\varepsilon \in \left(0, \min\left\{\bar{\varepsilon}, \delta\sqrt{\lambda_1}\right\}\right)$, where $\bar{\varepsilon}$ is given in Lemma 4.2.3. Then, for any $u \in \overline{D_\varepsilon^+} \cap \overline{D_\varepsilon^-}$, as in (4.14), we obtain that

$$\|u^\pm\| = \inf_{v \in \Lambda^\mp} \|u - v\| \leq \frac{1}{\sqrt{\lambda_1}} \operatorname{dist}(u, \Lambda^\mp) \leq \frac{1}{\sqrt{\lambda_1}} \varepsilon < \delta.$$

Thus, $|u(k)| < \delta$ on $[a+1,b+1]_{\mathbb{Z}}$. Then, from (2.3), (2.20), and (4.18), it follows that

$$I(u) \geq \frac{1}{2}\lambda_1 \|u\|^2 - \nu \sum_{k=a+1}^{b+1} |u(k)|^2 = \left(\frac{1}{2}\lambda_1 - \nu\right) \|u\|^2.$$

In view of the fact that $\nu < \frac{1}{2}\lambda_1$, there exists $I^* \geq 0$ such that $\inf_{u \in \overline{D_\varepsilon^+} \cap \overline{D_\varepsilon^-}} I(u) = I^*$. Recall that ξ_1 is the positive normalized eigenvavector of the matrix M corresponding to λ_1. Define

$Y = \text{span}\{\xi_1\}$. Then, for any $u \in Y$, as in deriving (4.12), we see that

$$I(u) \leq \left(\frac{1}{2}\lambda_{b-a+1} - c_4\right)\|u\|^2 + c_5(b-a+1),$$

where $c_4 > \frac{1}{2}\lambda_{b-a+1}$ and $c_5 > 0$. Thus, $I(u) \to -\infty$ as $\|u\| \to \infty$. Hence, there exists sufficiently large $c_{11} > 0$ such that $I(u) < I^* - 1$ for all $u \in Y$ with $\|u\| = c_{11}$. Define a path $h : [0,1] \to Y$ by

$$h(t) = c_{11}\frac{[\cos(\pi t) + \sin(\pi t)]\xi_1}{\|[\cos(\pi t) + \sin(\pi t)]\xi_1\|}.$$

Then, $\|h\| = c_{11}$ and $h(t) \in Y$ for any $t \in [0,1]$. Thus, we obtain that $I(h(t)) < I^* - 1$. Moreover, we have

$$h(0) = c_{11}\frac{\xi_1}{\|\xi_1\|} \in D_\varepsilon^+ \setminus D_\varepsilon^-, \quad h(1) = -c_{11}\frac{\xi_1}{\|\xi_1\|} \in D_\varepsilon^- \setminus D_\varepsilon^+,$$

and

$$\inf_{u \in \overline{D_\varepsilon^+} \cap \overline{D_\varepsilon^-}} I(u) = I^* > I^+ - 1 \geq \sup_{t \in [0,1]} I(h(t)).$$

This, together with Lemmas 4.2.1–4.2.3, implies that all the conditions of Lemma 4.2.4, with $H = X$, $D_1 = D_\varepsilon^+$, and $D_2 = D_\varepsilon^-$, are satisfied. Therefore, from Lemma 4.2.4, I has four critical points: $u_1 \in D_\varepsilon^+ \cap D_\varepsilon^-$, $u_2 \in D_\varepsilon^+ \setminus \overline{D_\varepsilon^-}$, $u_3 \in D_\varepsilon^- \setminus \overline{D_\varepsilon^+}$, $u_4 \in H \setminus (\overline{D_\varepsilon^+} \cup \overline{D_\varepsilon^-})$. By Lemma 2.0.2, these four critical points correspond to a trivial solutions, a positive solution, a negative solution, and a sign-changing solution of BVP (1.1). This completes the proof of the theorem. ■

Proof of Corollary 4.1.2. For any λ satisfying (4.2), we have

$$\lambda F_\infty > \frac{1}{2}\lambda_{b-a+1} \quad \text{and} \quad \lambda f_\vartheta^\infty < L.$$

Thus, with F replaced by λF and f replaced by λf, the conditions (A1)–(A4) of Theorem 4.1.1 are satisfied. The conclusion then follows directly from Theorem 4.1.1. ■

Proof of Corollary 4.1.3. Under the assumptions of the corollary, we see that

$$\frac{\lambda_{b-a+1}}{2F_\infty} < \infty \quad \text{and} \quad \frac{L}{f_\vartheta^\infty} = \infty.$$

Then, (4.1) holds. Hence, the conclusion follows directly from Corollary 4.1.2. ■

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