Rethinking Risk:
Aspiration as Pure Risk

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January 2005

CWPE 0507

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Abstract

There exists no satisfactory theory of risk in current normative decision theories. Notions based on utility curvature, loss aversion and probability weighting are derivative, cannot be applied to non-numerical consequences, and are not psychologically intuitive. I develop Pure Risk theory which resolves these problems – it is consistent with existing normative theories, and both internalises and generalises the intuitive notion of risk being related to the probability of not achieving one's aspirations. The theory shows that existing models are misspecified. Effects hitherto modelled as loss aversion or utility curvature may be due instead to Pure Risk.

Keywords: Risk; Pure Risk; Aspiration Levels; Subjective Expected Utility Theory; Prospect Theory; Pure Risk Prospect Theory.

JEL Classification: D81

Rethinking Risk: Aspiration as Pure Risk

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1 Risky Goals

When faced with a choice between acts in a risky world, what goals should be rationally used to guide one's choice of act? One intuitively plausible answer, and one that is given almost axiomatic status by the financial world, is that the decision maker should seek to maximise the expected benefit and to minimise the inherent risk of the decision. However, the precise content of what constitutes appropriate concepts of "benefit" and "risk" is by no means unproblematic. The vast majority of research on rational decision making in the past half-century has focussed heavily on the first of these potential goals, characterising "benefit" as the utility of the potential outcomes of the decision taken. What then of risk? The rather surprising answer given by the axiomatisations of Expected Utility Theory (von Neumann and Morgenstern 1947) and Subjective Expected Utility (Savage 1954) is that, given any preference ordering of acts (leading to risky, uncertain, or certain outcomes) that meets the rather minimal requirements of the axioms, a utility value may be ascribed to each possible consequence in such a way that the preference ordering of acts is fully replicated by the expected utility ordering of these acts.

So, if any preference ordering may be fully reflected in the expected utilities attached to the consequences, then it must be the case that these utilities already reflect any attitudes to risk that may be held by the decision maker. In maximising expected utility, the risk concerns of the decision maker are already accounted for. Expected Utility Theory (EUT) and Subjective Expected Utility

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I would like to thank R. Duncan Luce for providing valuable comments on an earlier draft of this manuscript and the ESRC for funding (Ref: PTA–030-2002-00325).

(SEU) Theory tell us that the utility allocation required to represent a given preference ordering over acts is unique up to an affine transformation, but it does not tell us anything about the way in which the utilities required to represent this preference ordering are affected by attitudes to risk versus return. Where risk is discussed in the context of EUT it is purely in the context of the way in which utility values might be attached to numerical (usually monetary) outcomes. I shall argue in this article that these concepts of risk do not, in fact, represent true risk attitudes but rather merely the influences of unrelated psychophysical factors that have an incidental effect on risk attitude. What is missing is a theory of risk that is both psychologically intuitive and that provides an explanation for the way in which risk attitudes affect preference orderings. This paper will attempt to provide such a theory.

Risk has had precise formulations within EUT, SEU and their later variants (of which Quiggin's Rank-Dependent Utility (1982) and Tversky and Kahneman's Cumulative Prospect Theory (1992) stand out as current candidates to render EUT more descriptively valid). In the more sophisticated versions of these theories risk attitude may arise from three potentially distinct components. The first two relate to the way in which utility is attached to monetary outcomes, firstly in terms of the concavity or convexity of the value function (which may be different for gain and loss outcomes in reference dependent models), and secondly from loss aversion, the degree to which individuals react more strongly to losses rather than gains. The third component is found in Rank Dependent Utility (RDU) models and arises from the way in which decision makers transform probabilities by giving more or less weight to extreme outcomes in a lottery.

There are, however, at least three problems with these versions of risk. Firstly, they do not have an intuitive psychological interpretation: in particular they do not seem to adequately characterise what we intuitively mean when we refer to the riskiness of a decision. There is no real sense in which SEU seeks to minimise risk and, if risk in any sense is minimised by a decision maker employing SEU, it will only be insofar as the interaction of the three components result in that individual attributing to the gamble a certainty equivalent that is less than the expected value of the gamble. This may be guaranteed, for example, in the situation where the decision maker has a value function that is globally concave and does not transform probabilities. However, this concavity is not implied by SEU, but is rather a descriptive assertion which places limitations on the theory. This assertion may be justi…ed in a number of ways, the most obvious being that, if SEU is used as the theory of risky choice, then empirical observation suggests that people are risk-averse and must have a concave value function. However, there is nothing in SEU that states that this need be the case - individuals may be risk-seeking as well as risk averse, and indeed may have value functions that have segments of both convexity and concavity. I would argue, however, that if we think introspectively about what the "risk" of a decision intuitively means to us, then we …nd that it is not an ambivalent concept which one may plausibly seek to maximise or minimise depending on preferences, but rather that risk is, in a deep sense, a negative concept, something that a coherent concept of rationality should mandate that we minimise. That is, minimising risk in some sense is a fundamental goal of instrumental rationality.

The second problem with existing concepts of risk is that, aside from risk attitude due to rank dependent decision weights, they are reliant either on a value function that transforms monetary values to utilities. If the value function is concave, the individual can be shown to be risk averse with respect to the underlying monetary outcomes. Any concept of risk that relies on transformation through such a function cannot be applied in its absence. Hence, there is no way that the standard concepts of risk based on the value function (through curvature or loss aversion) can be applied directly to a preference ordering between acts if the consequences of these acts cannot be entirely described be some underlying numerical scale. To say that risk only has meaning as a concept insofar as there is a function to translate numerical monetary outcomes to utility seems extremely narrow and restricts risk only to decision problems where the consequences can all by fully described on some prior numerical scale. The consequences related to the acts do not have to be speci…ed as monetary outcomes in Savage's (1954) axiomatic treatment and one can certainly imagine that changing risk attitudes could change the structure of a preference ordering in such cases, without requiring that the consequences be fully described on a monetary scale.

A third problem relating to the historical association between risk and the curvature of the utility function is that, because some of the e¤ects of a curved value function may be given interpretations in terms of risk, this has led to a confusion between psychophysical and psychological reactions to monetary amounts on the one hand, and risk attitudes themselves on the other. In a broader context, the fact that the three components mentioned above all induce some degree of risk attitude in that the a¤ect the certainty equivalent value of a gamble does not imply that they are the sum total of what we mean by risk. These three factors certainly in‡uence risk attitude, but this does not imply that they jointly constitute risk attitude. In fact, I shall argue that some concept of risk may and should be given pride of place as a fundamental concept governing the way in which utilities are attached to outcomes, and that this concept is independent of psychophysical and psychological reactions to numerical or monetary amounts as well as to subjective probability. This novel and distinct source of risk attitude I shall term our attitudes to Pure Risk (PR). Whilst reactions to value, which will be discussed in detail below, may be said to in‡uence decisions in a manner that resembles the functioning of risk attitude, they are not themselves PR attitude. If nothing else, these concepts apply equally in situations of certainty as they do under uncertainty.

The primary problem with this oversight is that it has led to the assumption that risk may be adequately characterised through the use of a value function (and possibly, in the case of rank-dependent utility theories, through the additional e¤ects imparted by a nonlinear decision weighting function) and that attaining an adequate theory of risk is simply a matter of getting the form and parameterisation of this function correct. The failure of these value functions to adequately describe empirical or introspective risk attitudes is a result of trying to get one function to do the work of two - one governing psychological and psychophysical reactions to values, and the second governing attitudes to PR. That the three components of the standard approach in combination lead to some measure of risk aversion has resulted in the resumption of any quest for a theory of PR itself.

This paper will provide a theory of risk that may applied to any uncertainty or variability in possible future outcomes following a choice of act rather than solely to transformations of a numerical scale to utilities, or probabilities to decision weights. Since I intend this theory to represent risk in its pure sense and not be derived incidentally from other effects, I term the notion PR. An attitude to PR may be held by a decision maker regardless of her value function curvature or degree of rank-dependent probability distortion, although these concepts may have an additional effect on the Total Risk Attitude in that they influence the individual's certainty equivalence for a given gamble.

It is worth noting from the outset that SEU and its later derivative theories have been increasingly under fire as descriptive theories and, whilst RDU Theories have been successful at describing actual choices of binary prospects, they appear to be unable to predict choices accurately when the prospects become more complex, i.e., in exactly the choice contexts where one would expect to see risk becoming an important determinant of choice (see Birnbaum 1999, Luce 2000, Payne 2004).

In what follows I will first present and discuss concepts of risk that are found in standard SEU and in its more advanced derivative theories, particularly CPT and variations thereon. If SEU is valid as the normative criterion of choice, then we already have some concept of risk embedded in current theories when applied to monetary outcomes. It will be important for the development of a theory of risk as a fundamental factor in rational choice to understand what is, and is not, accomplished by these alternative concepts of risk.

It is important to emphasise that I am not suggesting that SEU is incorrect as a normative theory of risky choice, but that it does not contain a complete theory of how the appropriate SUE utility allocation is influenced by risk. For this it is necessary to describe how PR attitudes affect our preference ordering between acts in a risky environment. After discussing the SEU embedded concepts of risk I will move on to the development of the normative theory of risk itself, and its implications for the standard normative framework.

2 Traditional Notions of Risk

Through long, and frequently sloppy, use of the term "risk" in both theoretical and practical applications we are left with a plethora of alternative definitions, concepts, and measures, all masquerading under the name of Risk. Many of these do not warrant discussion here as they are either practical measures that proxy for some unspecified concept of risk (e.g., many financial risk measures such as downside volatility, VaR, etc.), or they will not be confused with the uses I will employ in our current discussion. However, there are a number of concepts of risk embedded in the standard discussions of SEU (although these do not have to follow from the framework of SEU itself), and it is essential to understand why these differ from PR and how they can be justified within a broader theory that introduces PR explicitly as a basic notion. Before discussing these, however, there is one further meaning of risk that needs to be clarified, that of the Knightian distinction between decision making under uncertainty, and decision making under risk. The latter refers to a decision environment where numerical probabilities may be assigned to the possible states of the world, the former to an environment where this is not the case. PR theory may be incorporated into either situation and when referring to standard model of expected utility maximisation I mean this to refer broadly to theories compatible with Savage's (1954) Subjective Expected Utility (SEU) where the preference ordering is formed in a situation of uncertainty.

2.1 Risk Attitude Through the Value Function

2.1.1 Perceptual Risk

Let us now examine the concepts of risk that are embedded in the standard criteria of rational decision making under risk. The ‡avour of risk that is most central to EUT is that due to the curvature of the value function - where the function is concave the decision maker is “risk averse”, whilst “risk-seeking” behaviour results from a convex function. Due to the long use of the terms risk-aversion and risk-seeking within the theory of expected utility maximisation, they have almost become synonymous with concavity and convexity respectively. It is important, therefore, for us to enquire in precisely what sense concavity justi…es the use of the term “risk aversion”, and to ask how this …ts with our intuitive concepts of “risk”. That is, what precisely is the nature of the risk to which one is averse if one possesses a concave value function.

The concavity of the value function may be associated with the notion of certainty equivalence. Given any act with risky outcomes, we may specify an act with no risk (i.e., the same outcome in every state of the world) for which the decision maker is indi¤erent between the two. If the utility function is concave the value of the certain outcome of the second act will be lower than the expected outcome of the …rst. That is, an individual with a concave function will be prepared to accept lower expected value in order to eliminate the uncertainty inherent in the risky act. A corollary to this is that a decision maker with a concave function will always prefer the act with a single certain outcome to any other act with the same expected value. The greater the concavity, the lower the certainty equivalence of any act - the more expected value the decision maker will sacri…ce to attain an equal level of expected utility. The reverse is true of a convex value function.

There are two broad ‡avours of models with di¤erent implications for how utility is assigned to money. The standard version of SEU assumes that utility is derived from absolute monetary amounts. That is, from total wealth where utility increases monotonically with wealth. The risk attitude inherent in the traditional version of SEU arises because people attribute utility to absolute monetary values in a non-linear way. The standard assumption is that people attribute utility to wealth at a decreasing rate. Thus, greater wealth leads to greater utility, but a given additional increment of wealth provides less utility to the wealthy than to the poor. This results in a concave value function for which, through the application of Jensen's inequality, $E[u(x)] \leq u(E[x])$ (which is a defining property of a concave function and states that the expected utility of a prospect is less than the utility of the expected value of the prospect), we see that an individual with concave utility will always prefer the prospect with the certain outcome equal to the expected value of the risky prospect, to the gamble itself. This notion of risk is tightly bound to the way in which values are transformed psychologically into utility, that is, the way that humans perceive values on some scale (monetary or otherwise). Because of this link I shall refer to this concept of risk as absolute perceptual risk.

Note that, whilst the reasonable assumption of diminishing marginal utility with wealth leads automatically to risk aversion, it is by no means a necessary consequence of the theory. An individual could transform monetary wealth in such a way as to have a convex utility function, in which case maximising expected utility would result in choosing increased risk in this sense. The Weber-Fechner law in the psychology of perception, however, does give us both theoretical and empirical grounds for believing that we would face diminishing sensitivity to many scales, including monetary scales. Introspection too offers clear guidance here. It would seem pathological for an individual to value an additional £10 more if he were already a millionaire than if he were living in poverty.

The other, more recent, perspective on utility transformations holds rather that it is changes in wealth that are the carriers of utility - utility is measured in terms of changes in wealth from a reference point and for any decision can be negative as well as positive[2].

[2] A further distinction often debated by economists is whether utility should be derived

This latter approach was introduced by Kahneman and Tversky's Prospect Theory of 1979. It appears to fit far better with empirical evidence of actual choices people make - faced with a risky prospect, people seem to treat the outcomes as gains and losses relative to a reference point.

This use of a reference point leads to the possibility of a difference between transforming numerical gains to utility values, and transforming numerical losses. People seem to psychologically perceive losses quite differently from gains. The extensive empirical research on gain and loss utility functions most commonly finds that people make choices as though they were using a concave function for gains, and a convex function for losses (for example Tversky and Kahneman 1992). Thus, if people attribute utility to changes in wealth rather than to absolute levels of wealth then they display risk aversion for gains, and risk seeking behaviour for losses[3]. Again, this pattern is not a consequence of the theory, but arises from other theoretical and empirical considerations. Cognitive psychology predicts this pattern by invoking diminishing sensitivity to stimuli relative to the reference point. If what people notice is gains and losses relative to some status quo, then we would expect them to be less able to distinguish a given increment the further it is from the reference point. An extra £1 added to a loss of £1 million will be less noticeable than £1 added to a loss of £5. This provides a psychophysical explanation for the common pattern we actually observe although this may not be the whole story. This concept of risk I shall term reference dependent perceptual risk.

More recent theories are attempting to combine these two perspectives by, for example, using changes in wealth as the carriers of

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from wealth or from consumption. Either choice can be reconciled with both the relative and absolute versions of EUT. Consumption may seem the more natural choice, with the relative version being deviations of consumption from some reference level. However, wealth can be interpreted as the means to a flow of consumption and whilst there are distinctions to be made here, as long as the utility from wealth is monotonically related to that from consumption, they will not be of great concern for our discussion here.

[3] I should point out that non-parametric investigations of reference dependent utility functions show a great deal of individual variation, but risk aversion for gains, and risk seeking for losses seems to describe both the representative agent, and the median individual (Abdellaoui 2000). Most work on reference dependent utility also employs rank-dependence which may result in more complex patterns of risk attitude.

utility, but allowing the assessment of these gains and losses (i.e., the transformation of the values to utilities) to change as overall wealth changes (e.g., see Schmidt 2003, Sugden 2003). Thus the function is defined over gains and losses, but the precise shape of this function will be partly wealth dependent. Both perspectives have some intuitive appeal and may both play a role in the transformation of numerical values to utility for any given person.

Regardless of which of these two perspectives is adopted, there is no sense in which either resulting concept of risk is something to be minimised. Where the functions result in risk aversion due to a concave transformation this is largely attributable to the psychophysics of value, rather than to any intuitive notion of risk. Risk as a concept in traditional SEU models is entirely derivative on value perception - it is an accidental result of the theory rather than an important concept of rationality in its own right.

2.1.2 Loss Aversion

The move to reference dependent utility led to the introduction of a further concept that captures some of what we might consider to be naturally part of a comprehensive notion of risk. It intuitively seems that risk should have more to do with the chance of things going badly, than of things going well, a notion referred to by Fishburn (1982) as "the conventional notion that risk is a chance of something bad happening". It has long been noticed that losses are psychologically more important than gains, and empirical studies have shown that people do indeed treat losses differently from gains, and that this difference is deeper than simply different curvatures on the relevant portions of the value function (though see Schmidt and Traub 2002 for recent experimental work challenging the universality of loss aversion). Decision makers will generally weight potential losses more strongly than gains of the same magnitude. This leads to a type of risk embedded in reference dependent EUT that does recognise risk as a negative and focuses on the downside: the concept of Loss Aversion.

Loss aversion can be incorporated into reference dependent utility theory by decomposing the function into a loss aversion index, and a basic perceptual transformation. This is represented as:

$$V(x) = \begin{cases} v(x) & \text{if } x \geq 0 \\ \lambda v(x) & \text{if } x < 0 \end{cases}$$

λ is a loss aversion index capturing the exchange rate between gains and losses. If people are loss averse then $\lambda > 1$ and V is steeper in the loss domain. As Köbberling and Wakker (2003) point out, Tversky and Kahneman (1992) assume that $v(x) = x^{\alpha}$ for gains and $v(x) = -(-x)^{\beta}$ for losses. With this formulation $v(1) = -v(-1) = 1$, which implies a scaling convention such that $\lambda = \frac{-V(-1)}{V(1)}$. This convention is not independent of the unit of payment, so they suggest an alternative convention of $\lambda = \frac{-V'_{\uparrow}(0)}{V'_{\downarrow}(0)}$ where $V'_{\uparrow}(0)$ is left derivative at 0, and $V'_{\downarrow}(0)$ the right. This imposes an assumption of smoothness on the basic perceptual transformation function $v(x)$ which is thus not directly affected by the shift from gains to losses. Thus, the effect of loss aversion on risk attitude is completely separated from the transformation of monetary outcomes into basic values.

To show that loss aversion can be precisely disentangled from utility transformations as separate components of risk aversion Köbberling and Wakker utilise Yaari's (1969) definition of comparative risk aversion which is framed in terms of acceptance sets. The acceptance set of sure outcome x is the set of all prospects that the agent prefers to receiving the amount x with certainty. Yaari proposed that if the acceptance set of agent 2 is contained in the acceptance set of agent 1, then agent 2 is more risk averse. In other words agent 1 will be prepared to take all the gambles that agent 2 is prepared to take, plus some gamble or gambles that agent 2 is not prepared to take. This definition is a powerful tool as it allows us to characterise an individual as more (or less) risk averse than another, without having to specify a precise definition for risk. It may be applied to decision theories to determine the characterisation of risk aversion within that theory. Applied to SEU Yaari's definition characterises concave value functions, as expected. In theories where risk aversion may result from more than one component Yaari's definition is less useful as it does not separate the components. For example an agent may have greater risk aversion than another due to perceptual risk, but less loss aversion. Examining the case where one acceptance set is contained in another requires that both components work in the same direction and thus may be difficult to separate.

Köbberling and Wakker examine loss aversion by looking at two agents who have identical acceptance sets for gains only prospects and for loss only prospects[4], but where agent 2's acceptance set is contained with agent 1's set for mixed prospects. This precisely characterises loss aversion: it is observed only in mixed prospects and relates to exchange rate of gain and loss utility. Because their acceptance sets are identical for gains only and loss only prospects, risk attitude in this case cannot be due to the way basic values are attributed to monetary outcomes. However, since agent 2 does have an acceptance set that is contained within agent 1's set, there is an element of risk aversion. This must therefore be due to the way losses are treated relative to gains.

Because loss aversion is a component of the value function, of the way in which utility is assigned to numerical values, it is subject to the same critiques as the other concepts of risk found in EUT (or its variants). In particular the risk attitude due to loss aversion is concept that is derivative on psychological reactions to losses versus gains, rather than being due to any fundamental notion of risk. In addition it is subject to the problem of non-numerical outcomes which I discuss next. Nonetheless, it is evident that Loss Aversion encapsulates an element of the notion of risk that cannot be captured by the traditional measures involving perceptual curvature.

2.1.3 The Problem of Non-Numerical Outcomes

The risk attitude that arises from standard SEU is a direct consequence of how the individual assigns utility to numerical outcomes. It resides in the transformation of monetary outcomes to utility val-

[4] And identical decision weighting functions, a further component of risk attitude that will be discussed in the next section.

ues. One problem with this is that the concept only has any traction if the outcomes themselves can be numerically described. If the outcomes in different states of the world can only be verbally described then it is not clear in what sense the transformation of these outcomes to utility values can be said to be concave or convex. This is not a problem of uncertainty rather than risk in the Knightian sense. If we can assign both probabilities and utilities to outcomes, but cannot also describe the outcomes themselves by numerical values, then there is no way in which maximising expected utility has anything meaningful to say about risk. This framework may tell us which act will be preferred, but not why this should be due in any way to risk attitude.

To illustrate this problem consider the example below which has been borrowed from Savage (1954). Assume you have just entered a room where a friend has been making an omelette, and has already broken 5 good eggs into a bowl. You offer to finish making the omelette and must now decide what to do with the sixth egg which must either be used or will be wasted. The egg is in one of two unknown states: good, or rotten. In a simple framing of this problem, which will suffice to illustrate my point, you have three possible acts to choose between: either to throw the egg away, to break in directly into the bowl of 5 good eggs, or to break it into a saucer to check whether or not it is rotten before adding it to the other eggs. The possible consequences are illustrated in the table below.

	State	
Act	Good	Bad
Break into bowl	six-egg omelette	no, omelette, and five good eggs destroyed
Break into saucer	six-egg omelette, and a saucer to wash	five-egg omelette, and a saucer to wash
Throw away	five-egg omelette, and one good egg destroyed	five-egg omelette

According to Subjective Expected Utility Theory, as long as your preferences between the three acts obey the appropriate axioms, then there exists an allocation of utilities to each of these consequences such that your subjective expected utility of the acts maintains the preference ordering.

However, there is no way in which two individuals may be said to be more, or less, risk averse than the other according to the standard description of risk as the curvature of the value function. This is simply because the consequences do not consist of values that may be translated into utilities through a function - the utilities that maintain the preference ordering over acts are allocated directly to the verbal descriptions without the need to convert these descriptions into monetary or numerical values first. The standard theories of risk cannot be used in any decision situation such as this where the decision problem cannot be completely described on a single numerical scale.

In general we may have a number of possible acts A_i, and states of the world s_j, for each combination of which there is some consequence c_{ij}, which can be any verbal (or even non-verbal in the case of emotive or affective responses to consequences) description of the outcome associated with that combination . If the preference ordering over A_i obeys the requisite axioms then we may allocate a number u_{ij} to each c_{ij} such that $A_i \succsim A_{i'}$ implies $E_j[u_{ij}] \geq E_j\left[u_{i'j}\right]$ for all $i' \neq i$, where the expectation $E_j[\cdot]$ is taken over the subjective assessment of the probabilities of the states of the world s_j. Only if all consequences c_{ij} can be completely described by a single numerical scale, say $x_{ij} \in \mathbb{R}$, can we have the situation where the utilities u_{ij} may be the result of a numerical transformation of the form $u_{ij} = v(x_{ij})$ which may have the properties of convexity or concavity.

This may appear not to be a problem for the vast range of issues to which this concept of risk aversion is used (finance, gambling, economics, etc.) because it may be argued that in these cases we have a clear numerical scale with which to describe the outcomes, that of monetary value. This may be approximately true for large subsets of these problems. However, for it to be exactly true that risk aversion in this sense is meaningful it must be the case that the outcomes are completely described by their monetary value. It is by no means certain that this can ever be the case. As an example consider that a complete description of a future outcome must necessarily incorporate the decision itself. To describe the outcome in a particular future state of the world as, for example, "The decision maker receives £X" is incomplete. Completeness requires at a minimum a description such as: "The decision maker receives £X having chosen act A". To receive £X after having chosen some other act is a di¤erent state of the world, and may well be assigned a di¤erent utility. Even if there is no other possible act that results in the decision maker receiving £X, the fact that it was act A that was chosen is still a part of the description of the resulting state.

Further evidence for the impossibility of purely numerical outcome descriptions comes from the theory of mental accounting. Thaler (1999) speaks of money as having "labels" in peoples' minds. Money is not treated as being completely fungible in decision making, but is rather labelled according to its sources (regular income, bonus, windfall, etc.) and its earmarked uses (housing, leisure, etc.). Thus, even money cannot be described solely by a numerical value in decision making, but also incorporates descriptive labels that mean that £1 from one source may be treated quite di¤erently to £1 from another source. In behavioural game theory (see Camerer 2003) too, it has been repeatedly shown that people imbue monetary amounts with descriptive components that re‡ect aspects of the "fairness" of the situation that resulted in the monetary outcome, of the previous strategic choices of the other player, or of many other aspects of social preference, all of which may be seen as giving descriptive labels to purely monetary outcomes. For example, for a child to receive £x as Christmas gift from a relative may be evaluated quite di¤erently depending on whether her siblings received the same as her, or received more than her. It thus seems that it is not possible to have purely numerical descriptions - it could still be the case that an identical utility value is assigned to another outcome in which the decision maker receives $\$Y \neq \$X$ after choosing a different act. The risk aversion generally associated with SEU then, is, at best, a partial concept.

I do not wish by means of this argument to imply that risk aversion due to concave utility is a useless concept, just that it can only apply to that portion of the utility assignments that is derived from a numerical value attached to each outcome (and these numerical values need not necessarily be monetary outcomes, other numerical scales, e.g., environmental temperature, quantity of food, or acoustic volume, might also lend themselves to this concept of risk aversion). This portion is never 100%, although in certain decision tasks the extent to which the link between numerically valued outcomes and utility outcomes is influenced by additional verbal (or perhaps non-verbal but nonetheless non-numerical) descriptions may be slight.

What are we to make of a concept of risk that only exists when the utilities assigned to outcomes are transformations of some other numerical scale? It would seem that, when faced with a choice in which every possible end-state (in the complete sense) is associated with a utility value, we should be able to say something about the riskiness of the acts available to us without having to examine the numerical provenance of those utilities. Consider a simple decision problem with three equally probable states of the world

	s_1	s_2	s_3	Expected Utility
A_1	100	100	100	100
A_2	80	100	120	100
A_3	60	100	140	100

but imagine that these final utility figures are all we have available and have not arisen by transforming monetary values through a utility function - the outcomes in the different states of the world may be described but, like the outcomes in Savage's omelette example, are not amenable to direct numerical comparisons. SEU tells us that there exists an assignment of utilities for any complete preference ordering that obeys the continuity and independence axioms, regardless of the whether the consequences may be described numerically. Using a concept of risk that is dependent on the curvature of the utility function there is nothing we can say about the risk inherent in these options.

Economics often implicitly side-steps this problem by assuming that some monetary value can be attributed to virtually any descriptive component of an outcome. In the example where the decision taken is itself a component of the outcome description this would involve adjusting any actual monetary outcomes by some monetisation of the "act utility". This seems to be going backwards though: why should we require both a utility scale and a monetary assignment to have a coherent theory of risk. Indeed if we know the utility allocations to individual outcomes that maintain the preference ordering, and we assume a specific shape for the value function it is possible to go backwards to the money amounts that would need to be assigned to the non-monetary outcomes. In doing this however, we are inferring an additional monetary description that is unnecessary to ensure an expected utility representation. Worse still, we have assumed that risk attitudes are solely of the type embedded in the form we have assumed for the value function, and that nothing is lost in the move from the full descriptions c_{ij} to the necessary monetary descriptions x_{ij}. I claim that all we have achieved by this move is to stipulate what the monetary equivalent of the outcomes would be after taking into account our psychophysical attitudes to money. That these attitudes can affect our certainty equivalents (and thus overall risk attitude) does not imply that the monetary equivalents incorporate all there is to say about risk attitude.

The concept of risk inherent in utility curvature is merely a consequence of the way utility relates to the numerically measurable component of the descriptions of outcomes, and is in no way a universally applicable definition of what "risk" actually is. It may be applied with varying degrees of success to any decisions that have some numerically measurable component of all of their outcomes (such as gambling). However, even here there are difficulties. There may be more than one component of the outcome descriptions that can be numerically measurable. Think of an environmental decision problem where different investment packages (acts) lead to different possible levels of global wealth and overall levels of global warming. Avoiding the all too easy economic assumption that the global wealth levels can simply be adjusted by some amount to reflect the effect of the global temperature levels, we are now left with two numerical components in this problem. There are thus separate measures of risk aversion that deal with wealth and temperature, and therefore two different concepts of risk. And yet, since by SEU we are theoretically justified in attributing a single utility level to these consequences, could we not also hope for a single, universal concept of risk that is related to the outcome descriptions themselves?

I have argued that, although existing concepts of risk attitude have no traction in decision problems with non-numerical outcomes, it is nonetheless meaningful to speak of risk attitude in some more general sense in analysis of these problems. It may be argued here that perhaps the numerical transformations are indeed all there is to risk attitude and that risk attitude has no bearing on non-numerically described problems. To this argument I would counter that it certainly seems to be meaningful to speak of acts that are more or less risky, even where the outcomes cannot be reduced to a numerical scale. Since each of the acts results in outcomes that are different depending on the realised state, it would seem that some notion of risk must play a role in the rational decision making process. In Savage's omelette, for example, it seems to be meaningful to claim that breaking the egg straight into the bowl is in some sense riskier that first breaking it into a saucer to see if it is good. If such comparisons of possible acts are meaningful, then there must exist some notion of risk attitude that does not rely on the psychophysics of value.

The existence of this attitude to PR may be made more formally by revisiting Yaari's (1969) definition of risk attitude. His acceptance sets do not rely on any particular theory of risk and may be applied to acts with non-numerically described outcomes. It is perfectly intelligible in the omelette example to define some certain outcome and ask what, for a given individual, is the set of acts that would be preferred to this outcome. The chosen outcome need not be one of the set of available outcomes in the decision problem, and could be numerical or non-numerical. Intuitively it would appear that breaking the egg straight into the bowl is in some sense more risky than the other two alternatives. If this is the case it should be possible to pick some certain outcome such that this outcome will be preferred to breaking the egg straight into the bowl, but not preferred to the other two available acts (which thus constitute the acceptance set of this individual in this circumstance). Another individual may be said to be less risk averse if her acceptance set, relative to the same certain outcome, contained all three risky acts. Hence, since we can coherently describe what it means to have different degrees of risk aversion in a situation where the standard theories of risk attitude are useless to us, it must be the case that there is more to risk attitude than is contained within these traditional approaches. That is, there exists a concept of PR that is distinct from the standard theories.

2.2 Probabilistic Risk

There is one further source of risk in SEU derived models. The failure of SEU to adequately describe observed choice behaviour led to the development of rank dependent utility (RDU) models (Quiggin 1982). Tversky and Kahneman's (1992) Cumulative Prospect Theory (CPT) incorporates RDU into a reference dependent context. These models allow for decision makers to distort the probabilities associated with outcomes. In particular, the weight associated with a particular outcome is no longer determined purely by the (objective or subjective) probability of it occurring, but also by the ranking of that outcome amongst all possible outcomes. The psychological intuition behind this probability distortion is the Principle of Attention - the attention of the decision maker is naturally drawn to the best outcomes and the worst outcomes, leading to these extreme outcomes being overweighted in the decision process. Lopes (1987) discussed the psychological bases for these e¤ects in terms of hope and fear: hope for good outcomes leads to increased attention to the best possible outcomes, whereas fear of a bad outcome leads to overemphasis of particularly bad outcomes.

Incorporating these decision weights into SEU requires a transformation of the cumulative probability distributions for losses, and the decumulative distribution for gains. For this transformation to meet the conditions imposed by the principal of attention, the function needs to be inverse-S shaped (that is, concave near 0 and convex near 1). Empirical evidence suggests that there is a great deal of variation between individuals, but that the median decision weighting function is indeed inverse-S shaped (Abdellaoui 2000, Luce 2000). If people do distort probabilities in this way the distortion will affect the certainty equivalent value, and it forms a further source of observed risk attitude.

The transformation of probabilities results in the decision being made as if the decider were using the undistorted probabilities of an alternative distribution, the Imputed Distribution (Davies and Satchell 2004a) and can result in a change in the certainty equivalent value and hence in the risk premium of the prospect (see Davies and Satchell 2004b for a discussion of the e¤ects of risk types on the risk premium in CPT). With an inverse-S shaped weighting function, the e¤ect of rank-dependent probability weightings on the overall certainty equivalent is indeterminate. The tendency to focus on the worst outcomes will lead to an increase in risk aversion and hence in the risk premium, whereas the distortion due to hope will reduce risk aversion. In CPT the relative magnitude of these two factors will also be a¤ected by the position of the reference point, relative to the ranked outcomes.

Although this component of risk aversion does not require a transformation from consequences that can be fully described by a single numerical scale as for the perceptual risk and loss aversion components, like them it is also completely derivative on a psychological intuition that is not intended to be a theory of risk itself. The principle of attention that leads to inverse-S shaped decision weighting functions draws attention to both the extreme gains and extreme losses of a distribution. In addition, a loss of a given amount may have its weighting increased in one prospect, whilst decreased in another prospect. This concept appears to relate far more to the perception of individual outcomes relative to the set of outcomes with which it is presented, and to the salience of extreme members of a set of consequences than it does to any fundamental notion of risk. Again, we have a concept that may incidentally influence the risk attitude of a subject for a particular decision, but is not itself part of the PR attitude of the subject.

CPT and RDU models have in recent years taken an empirical pounding from the experimental evidence amassed by Michael Birnbaum (e.g., 1998, 1999, 2004) that unearths a number of new choice paradoxes that violate CPT. As alternatives he offers a more general class of “configural weight models” of which RDU and CPT are special cases, but which all cause decision weights to be distorted from the underlying subjective probabilities. Other members of the class cause distortions which, though different from those predicted by RDU, would also function as a source of probabilistic risk. Thus, whilst these models might provide descriptively better predictions of choice behaviour, they do not get us closer to a true theory of risk.

3 The Need for a Theory of Pure Risk

To summarise the classical position then, within the SEU or RDU framework risk attitude arises from a number of different effects. Firstly from the Perceptual Risk associated with attaching basic utility to numerical outcomes. This can be either absolute perceptual risk, or reference dependent perceptual risk for which the curvature of the basic utility function may affect risk attitude separately for gains and losses. Secondly, from Loss Aversion resulting from the disparity in psychological attitudes to gains and losses. The third component is due to the distortion of probabilities implied by attention being subconsciously drawn to extreme outcomes.

Of more importance here, however, is the fact that all three of these classes of risk attitude are entirely derivative concepts.

The first two only have traction insofar as there is an underlying numerical scale which can be transformed into utility and they do not encapsulate our intuition of what is meant by risk. They arise entirely as a side effect of human psychophysical or psychological perceptions of value and gains versus losses, rather than from any notion of risk as a specific entity to be minimised. Although these concepts have been at the centre of vast amounts of research on risk aversion, their connection with PR is incidental. The existence of these derivative risk measures has had the deleterious effect that we have not noticed that the maximisation of expected utility is insufficient to characterise precisely how utility is allocated to particular outcomes, and that what is lacking is, in fact, a theory of PR.

Before moving on I should first like to digress briefly on why we should want a normative theory of risk that fulfills this role. After all, given that we can only ever observe the preference ordering of the true individual imbued with an attitude towards risk which therefore masks the role played by PR attitudes themselves, the notion is purely conceptual. The first response is that it would seem desirable to have some theory of what it is that we mean by the term "risk", an understanding of which might allow us to approach the problem of rational choice under uncertainty more thoroughly. That the theory I offer is satisfactory in such a sense is an evaluation that will have to follow the exposition of the theory below.

As a second answer, we should ask ourselves what it is about the structure of the acts in the decision problem above that makes the theory of expected utility such an important concept. In a risk free decision problem we only have need of a simple criterion to ensure our goals are best accommodated in the act chosen: chose the act with the best (certain) consequence. This is simply a matter of ranking certain outcomes by their preference. Once we introduce risk to the structure of the problem our analogue of that simple goal fails to adequately resolve the choice problem - the outcomes are no longer certain and we require a theory that indicates how we should weigh up the various outcomes and the uncertainty associated with each.

We should therefore suspect that, since what we have added to the problem is risk, some fundamental concept of risk attitude might be required to adequately describe the rational choice. Indeed, as Lola Lopes says of EUT in her 1987 article on the psychology of risk, “after all the study and all the clever theorizing, we are left with a theory of risk taking that fails to mention risk.”

Re‡ect for a moment on the superordinate goal of the decision problem. Given our structure of beliefs, desires, needs, wants, etc., we ascribe utility to the outcomes associated with various possible future states of the world. Our overall goal with relation to the choice we face is to achieve maximum ex post utility. In a risky environment achieving that goal is not directly accessible to us because part of the determination of the …nal outcome is out of our control. However, even though we cannot act to achieve maximal ex post utility directly, it still remains the optimal ex post resolution of the problem. The translation of this goal in our risky environment is “to maximise achieved (i.e., ex post) utility given the acts that are available”. Von Neumann and Morgenstern demonstrated that the utilities can be allocated in such a way that picking the act, or one of the acts, with the maximal expected utility is the best way of choosing in the sense that it maintains our preference ordering over acts, given our beliefs and desires. However, the unsatisfactory part of this theory is that, although it provides the solution to the problem of choosing the optimal act when we move to a choice environment that embodies risk, there is no explanation within the theory of the role played by risk attitude in the allocation of appropriate utility values.

For a third reason recall that SEU says that there exists a utility allocation to all consequences in a decision framework such that the preference ordering over acts is maintained by the (subjective) expected utilities of the acts. If the axioms leading to RDU hold then there exists both a utility allocation to consequences and a mapping from subjective probabilities to decision weights, and the preference ordering in this case is maintained by the expected utility where the subjective probabilities are replaced by the rank-dependent decision weights. Because the two models are di¤erent we should expect the utility allocation to be different depending on whether preferences obey the axiomatic structure leading to SEU, or to RDU. In neither case, however, does the model tell us how to allocate these utilities to consequences, just that a suitable allocation exists. Traditional value functions are attempts to fill this gap, to provide a method for allocating utilities to consequences where the consequences may be completely described numerically. This method does not start from the preference ordering and ask what utilities need to be assigned to maintain the preference ordering through the criterion of expected utility. Instead it begins directly with monetary consequences and postulates that the utility function is of a specific form, the choice of which is guided by theoretical and empirical considerations. This approach is essential if SEU is to be at all useful: if one cannot somehow assign utilities directly to consequences in such a way as to be confident that they are the "correct" utilities in the sense that they will maintain the preference ordering over acts, then one is left with a theory with no predictive content. Assigning utilities directly to consequences enables one to examine choices without having to observe the entire preference ordering of the decision maker.

How, though, are we to be sure that the value functions that are standardly used are, in fact, assigning the correct utilities to consequences? This is at least partially an empirical question. If using a given function enables us to perfectly predict choices in every case, then we can be reasonably sure that this method of attaching utility numbers to consequences is arriving at the "correct" utilities that maintain the preference ordering over acts. Even if this is case, however, we can never be entirely sure of the correctness of the value function. In any observed choice we can only observe the top ranked act. The preference ordering of non-chosen options is never revealed. SEU shows that there exists a utility allocation that maintains the entire preference ordering but we can never be entirely sure that our chosen allocation methodology achieves this. Since the usefulness of the theory relies on us not having to observe the entire preference ordering, it becomes important how we choose to assign utilities to outcomes.

In the more common scenario where the observed choices are not entirely predicted by the SEU or RDU model when employing a given value function then there is a real problem of determining whether or not it is a misspeci…ed function that is to blame. Failure to predict choices accurately could arise from a number of factors. Firstly, as previously discussed the value function is only useful when transforming numerical scales to utilities - the failure to predict choices using such a utility function could re‡ect the failure of the monetary inputs into the utility function to provide a comprehensive description of the consequences. There may be aspects of the decision outcomes that are lost in the reduction of the descriptions to purely monetary outcomes. Secondly, there may be aspects of the decision environment that cause the decision maker to violate some of the axioms underlying SEU or RDU. That is, the decision maker may not be "rational" in the sense required by the theory being applied. Thirdly the decision maker may be subject to random errors when making decisions such that observed choices can only ever be modelled to a certain degree of accuracy. Lastly, the value function used may itself be a poor model of how consequences may be assigned utilities in the manner required by the theory, either because some other functional form is more accurate, or because the some component is missing entirely.

For many years it was the case that value functions were assumed to relate to absolute wealth levels. With the advent of Prospect Theory it was observed that choices seem to be modelled better by using reference dependent utility which incorporates loss aversion. It may be the case that some combination of these two perspectives is correct, or that further re…nements will be proposed that improve the …t of choice models to actual choices. Given this (probably not exhaustive) list of possible ways for observed choices to deviate from predicted choices we have little hope of determining empirically whether any particular function for attaching utility numbers to consequences is, in fact, taking us the full distance to the correct utilities that the theories tell us exist. We thus rely on other means to motivate the type of value function we choose. The psychological principle that we appear to react to changes from the status quo provides a rationale for reference dependent functions. The principle of diminishing sensitivity to changes from this reference point may be used to postulate that this function should be concave for gains, and convex for losses. Observed attitudes to gains and losses motivates the use of a parameter governing loss aversion. But once we have applied all these intuitions through some function $v(x)$, we still have no way of really knowing whether the transformed values v are the same as the utilities u which maintain the complete preference ordering over acts. We have transformed the monetary values using a mix of psychological intuitions and empirical research in an attempt to arrive at the utilities required by SEU, but have no reason to believe that we have arrived at our destination other than by the brute force strategy of trying to parameterise the chosen utility function to fit available choice data, which is itself only a glimpse at the peak of the preference ordering as a whole.

Based on existing concepts of risk that are grounded within SEU or RDU models, there are, I believe, sound theoretical reasons for believing that traditionally utilised value functions cannot be sufficient to take us from consequences to utilities. If we believe that decisions are (and should be) influenced by attitudes to some form of PR, then existing concepts of risk are severely lacking: they apply only to numerical descriptions of outcomes and they are not about fundamental attitudes to risk. Thus if attitudes to some primary description of PR are meaningful, they are not captured through existing methods of allocating utilities to outcomes. These traditional functions (which I shall denote by $v(x)$ and may be reference dependent or not, and may display loss aversion or not) may tell us how individuals react psychologically to consequences when described numerically, but ignore PR attitudes and thus, as long as attitudes to PR influence preference orderings, $v(x)$ cannot be taken to fully describe the utilities required by the preference ordering, u. This is shown schematically in Figure 1. In moving from c_{ij} to u_{ij} there remain two gaps. One between c_{ij} and x_{ij} which reflects that portion of the consequence description that is lost in requiring numerical scales to which $v(x)$ can be applied. As discussed above this gap could be formed in part through the application of mental accounting labels" to monetary outcomes, to strategic components of the decision, or to social preferences which are not reflected in the straight monetary outcomes. There may also may many other aspects of the decision of which we are unaware that require further descriptive labels, due to the effects of framing or of general affective or emotional responses to the decision environment. The second gap is between $v(x_{ij})$ and u_{ij} which partially reflects the fact that PR attitudes are not in any way incorporated into the function $v(x)$, and partially the fact that there may be additional components that influence preference orderings of which we are currently unaware, or that there may be psychophysical components missing in our best current theory of how individuals transform value, i.e., missing elements of $v(x)$.

In trying to model choices between acts using $v(x)$, all previous attempts to fit value functions to choice data have implicitly tried to force a particular specification for $v(x)$ to do all the work of moving from c_{ij} to u_{ij}. If a theory of PR does have a place in decision making then previous parameterisations of specific forms for $v(x)$ have been misspecified as they have been trying to fit attitudes to PR as well as loss aversion and diminishing sensitivity. Given this, it is perhaps more surprising how well such models have performed in fitting observed choices. Does this not cast doubt on the importance of some primary concept of PR attitudes in decision making? For instance in RDU models it is the combination of the value function $v(x)$ and the decision weighting function that are together involved in getting from c_{ij} to u_{ij}, whilst also transforming the probability distribution. The weighting function may be replaced by some other of the class of generic configural weight models, which are combined with the value function. These models have proved far more able to describe observed choice behaviour than SEU models, and yet they too ignore risk attitudes and cannot be applied to non-numerical consequence descriptions. Is an intuitive sense of PR then only an illusion?

That RDU …ts choice data reasonably well, however, does not eliminate the space for a theory of PR. Because the many components built into these models have an in‡uence on total risk attitude (they may all a¤ect the certainty equivalent value of a gamble, for instance) it is not too surprising that, when forced to, they might pick up a considerable amount of that portion of risk attitude that should properly be attributed to PR. As I will show later once I have fully speci…ed the theory of PR, it is possible that by ignoring it we have been forcing loss aversion, value function curvature and probability distortion to take on values that do not re‡ect the actual role of these factors in decision making. Thus, concavity for gains and loss aversion may be exaggerated, whilst convexity of money attitudes for losses appears lower than it actually would be were PR attitude taken into account.

If there is a theory of PR attitude that …lls the role I have described, then any attribution of utilities to outcomes, monetary or otherwise, that ignores this role of risk is inherently misspeci…ed and any predictions or models based on this incomplete model will be incorrect. Since the existing utility functions that are commonly used do not refer to PR attitude they will not enable us to determine the preference ordering for any individual except for that idealised individual that is neutral with respect to PR. It is important, therefore, that we specify our theory of PR attitude and test its validity in actual choices.

I have indicated that PR is something to be avoided. Whilst I do not wish to exclude *a priori* the possibility that decision makers might prefer PR, we might ask why risk should *a priori* be something that we wish to minimise. After all the standard concepts of risk in SEU allow individuals to be risk-averse or risk-seeking. The discussion above provides an answer to this question: the risk we wish to include as our rational criterion of choice is the risk of not maximising achieved utility given the options available to us - the probabilistic risk of not getting as close to our superordinate goal as possible. Seen in this context risk is uncontroversially something that a rational being would wish to minimise. As I shall argue in a later section this, I believe, corresponds to our intuitive, psychological conception of risk as a negative concept.

Further anecdotal evidence of why a concept of risk, in particular, should be an important consideration when analysing choices comes from the …nancial world where theories of rational decision making are applied to problems as an everyday tool. As far back as 1952 A. D. Roy writing on asset selection in Econometrica acknowledged that practical decision making was as much concerned with minimising risk as with maximising utility:

> Decisions taken in practice are less concerned with whether a little more of this or that will yield the largest net increase in satisfaction than with avoiding known rocks of uncertain position, or with deploying forces so that, if there is an ambush round the next corner, total disaster is avoided.

The analogue to maximising expected utility in the …nancial world is generally the maximisation of expected return (the "value function" of a …rm is frequently deemed to be linear in monetary return). However, expected return is normally maximised on a risk adjusted basis, or is maximised per unit of risk. If the concept of risk inherent in SEU was all that was relevant to decision making, then there would be no need for further adjustment. Using expected return divided by the standard deviation of returns (the commonly used Sharpe ratio introduced in 1966) can be justi…ed in SEU with certain restrictive assumptions on the shape of the utility curve or on the outcome distributions. However, even in this case there has been a recognition that standard deviation is an unsatisfactory proxy for our intuitive conception of risk and there has been a move towards downside measures of risk such as downside volatility and Value at Risk (VaR). As will become clear, the concept of VaR has a similar, though more simplistic, structure to my proposed normative measure of PR.

4 Pure Risk

So how do we …nd a place for a theory of Pure Risk within the powerful framework provided by some version of expected utility maximisation? If an attitude to PR exists, then it must be incorporated into any given rational preference ordering over possible acts, or gambles, **%**. Given a coherent theory of PR it is also intelligible to examine what it means to be una¤ected by an attitude to PR, or to be PR neutral. My approach here is to postulate a hypothetical, but related preference ordering, $\boldsymbol{\%}^{N}$, that is neutral with regard to PR. That is $\boldsymbol{\%}^{N}$ would be the ordering over the same acts that would be held by the same individual were she absolutely indi¤erent to risk in this new, and yet to be speci…ed, pure sense. This separates **%** into a PR neutral component and a second component that embodies only attitude the individual's attitudes to PR. This second component may be seen as an ordering over acts, $\boldsymbol{\%}^{PR}$, that describes preferences that are informed solely by attitudes to PR. We assume that the original preference ordering obeys the axioms required by SEU (or indeed RDU or CPT as desired) and thus develop a version of PR theory consistent with which ever of these theories one prefers. In what follows I shall assume that the original preference ordering **%** admits an SEU representation for simplicity. That is, the preference ordering over acts is such that there exists a mapping from consequences c_{ij} (which need not be numerical descriptions) to utilities u_{ij} 2 R such that the preference ordering is maintained by the subjective expectation of utility for each act. The restriction to SUE, however, is not necessary for the development of PR theory and I shall comment throughout on the e¤ects that would be expected by the move to RDU or CPT. In a rank dependent model maintaining the preference ordering would also require a non-linear transformation of the subjective probabilities and reference dependence would require that the utilities were allocated relative to some status quo outcome which would be accorded a utility value of zero. This preference decomposition is shown schematically in Figure 2.

As discussed above PR attitude does not, by de…nition, re‡ect responses to monetary amounts, attitudes to gain vs. losses, or subjective probabilities. Thus, when the overall preference order is decomposed into $\%^{PR}$ and $\%^{N}$ all these behavioural components of risk attitude are accounted for by the PR neutral preferences $\%^{N}$. To the extent that these preferences are reference dependent, or rank dependent then, this will be preserved in the PR neutral preference ordering. Although I do not assume here that the PR preferences are rational in the sense that they don't violate any of the basic axioms of SEU, it will later be clear that this must be the case given the structure of PR attitude. However, it will be necessary for the development of the theory that $\%^{N}$ admits a utility representation. For reason reason it must be the case that the PR neutral preference ordering of an individual whose natural preference ordering is fully rational is also rational. $\%^{N}$ must obey the same axioms as $\%$.

If this is the case there thus exists an attribution of utilities to consequences that is unique up to an a¢ne transformation that preserves this PR neutral preference ordering. That is, denoting the PR free utilities as u^{N}(¢) for any gambles f and g, it is the case that f $\%^{N}$ g if and only if $u^{N}(f)$ ¸ $u^{N}(g)$. This contrasts with the original utilities, u(¢) which re‡ect the full preference order such that f $\%$ g if and only if $u(f)$ ¸ $u(g)$. Unless the decision maker is originally neutral with regard to PR, the ordering $\%^{N}$ will be di¤erent from $\%$. It is in the gap between these two preference orderings that we …nd a place for our theory of PR attitude. A PR theory is thus required to explain, in a plausible way, both theoretically and intuitively, how $\%^{N}$ is derived from $\%$, and vice versa, or equivalently, how the utilities u are related to the PR neutral utilities u^{N}. Note that nowhere will this theory of risk be required to have recourse to transformations from monetary values into utilities. Both PR neutral preferences and the original preferences are de…ned solely on a preference ordering over acts. The consequences that will be realised on the resolution of uncertainty following these acts may, or may not, be fully described in terms of numerical values.

To give a simple example of the two utilities assume that the utilities for an individual of a set of acts A_i, given preference ordering **%**, are attached to consequences in equally possible states of the world. s_j, by the following values:

	s_1	s_2	s_3	$E[u_i]$
A_1	100	100	100	100
A_2	89	100	110	99:7
A_3	78	100	118	98:6

When the PR neutral preferences of the same individual are considered the attributions of utility will be required to maintain the preference ordering $\mathbf{\%}^N$ will be different from those which maintain **%**. An illustrative allocation might be:

	s_1	s_2	s_3	$E\left[u_i^N\right]$
A_1	100	100	100	100
A_2	80	100	120	100
A_3	60	100	140	100

Here, where the individual has a clear preference between the three acts, the PR neutral preferences are indifferent to which act is chosen. The presence of PR attitude in an individual will alter the preference ordering that would have been observed in an individual who did not care in any sense about PR. In this case the upper utility table has been derived from the first by using a simple concave transformation of the lower table. This reflects the fact that the individual is assumed to be averse to PR and will thus weight the downward deviation from the middle outcome more than the upward deviation. This transformation has been chosen arbitrarily at this point to illustrate the difference between the two utility attributions. Without a theory of PR we can at this point say nothing yet about how the two utilities are related. PR theory will provide such a link between the utilities required to render the preference ordering for a PR neutral individual and those for a normal individual who displays PR aversion. It will show how risk may be considered to have meaning as a natural concept within the framework of expected utility maximisation without resorting to consequences described in purely monetary terms. In doing so it will also discern PR attitude from existing concepts of risk attitude which are entirely derivative on psychological or psychophysical responses to monetary values and to subjective probability distributions.

To specify a theory of PR we need to ask how the addition of PR attitude to $\%^{N}$ will transform the PR neutral utility allocations u^{N} to the comprehensive utility allocations u. That is, were an "individual" embodying only PR attitudes to face a decision between the possible acts, where the consequences already represent all PR neutral preferences, on what basis would this PR decision be made? We thus imagine a situation where the preference ordering $\%^{PR}$ is required to make a decision between the A_i where the consequences c_{ij} have been amended to incorporate all non-PR aspects of the choice. The consequences are thus given by u_{ij}^{N}. PR attitude resides in the translation from u_{ij}^{N} to u_{ij}.

Describing the place for a theory of PR as I have above has made use of a few assumptions that I feel it best to make explicit before attempting to outline the theory itself, which will be developed within the framework provided by these assumptions. Firstly, I have assumed that there is a natural, intuitively plausible concept of PR that is entirely di¤erent from derived concepts of risk that arise from psychological or psychophysical reactions to money (or other numerical scales) or subjective probability. These concepts of risk are assumed not to capture any element of PR so that the components of Total Risk Attitude that arise from PR aversion may be entirely separated from those that arise through derivative e¤ects. This assumption of the existence of PR is an obvious requirement to formulate a theory of PR - if a PR attitude does not exist then $\%$ and $\%^{N}$ will be identical.

Secondly, I have assumed that it is a meaningful concept to describe an individual that is neutral with regard to PR. Thus, the preference ordering $\%^{N}$ embodies no PR attitude, regardless of what other psychological attitudes it may incorporate. These two assumptions may be combined to assume that a PR neutral preference ordering may be formed from any original preference ordering **%** so that the two di¤er precisely and only with regard to PR attitude. That is, no other aspects that a¤ect preference have been tampered with in moving from **%** to **%**^{N}.

Lastly comes a set of assumptions such that the **%**, and **%**^{N} admit a utility representation in the sense that they all obey at least some minimal set of choice axioms su¢ cient to ensure a representation. In the case of the overall preference ordering this is uncontroversial as I wish this to be a normative theory that applies within the rational framework as speci…ed by some pre-existing normative model of decision making (EUT, SEU, RDU, CPT, GDU, etc.). It will turn out that the structure of PR ensures that **%**^{PR} also satis…es the axioms of SEU, which is perhaps a desideratum of a normative theory of risk preference, but this need not be assumed.

Assumption: There exists a normative and psychologically plausible attitude towards risk that is not derivative on the curvature of perceptual transformations of numerical outcomes, loss aversion, or rank-dependent distortions of subjective probability. This risk attitude is termed Pure Risk attitude.

Assumption: Given a complete, well-behaved preference ordering between acts, **%**, that admits a utility representation (possibly with the addition of con…gural decision weights), we may decompose **%** into two components **%**^{PR} and **%**^{N}. **%**^{PR} embodies solely and completely the PR attitude of the individual. **%**^{N} embodies all other aspects of the rational preference ordering and is neutral with regard to PR.

Assumption: **%**, and **%**^{N} admit utility representations. **%**^{N} thus permits an allocation of PR neutral utilities u^{N} to all consequences which are di¤erent to the overall utilities u that preserve **%**, unless **%** is also PR neutral. If **%** also has a rank-dependent representation, then this aspect of the decision is carried by **%**^{N}.

4.1 Aspirations as Pure Risk

Lola Lopes posited a dual criterion theory of decision making under risk (1987) which she called SP/A theory. This theory was not intended to be a normative theory of rational decision making, but rather a descriptive theory incorporating a psychological perspective on how individuals assess risk. However, being concerned with the psychology of risk it contains insights into an approach to risk that is intuitively appealing, as well as providing data to support these insights. The first criterion is Security vs. Potential (SP) which is concerned with how people focus differentially on the extreme good outcomes, or the extreme bad outcomes depending on the degree to which they are motivated by achieving security (fear) or achieving potential (hope). This component of the model is essentially an exposition of the Principle of Attention, leading the decision maker to overweight certain outcomes relative to other depending of the ranking of the particular outcome. It is a form of RDU which was given a later mathematical formulation (Lopes and Oden 1999) which involves a linear utility function and a decision weighting function that (whilst unaxiomatised and reliant on some unsubstantiated assumptions (see Davies 2004)) gave a similar inverse-S shape to those proposed by Kahneman and Tversky (1992) or Prelec (1998). In requiring no utility transformation of numerical values and simultaneously transforming probabilities, the SP criterion effectively removes one component of an EUT based theory such as CPT (that of utility curvature), whilst retaining the descriptive psychological component of decision weighting. For the purposes of SP/A this was a simplification that seemed justifiable in terms of the theory's ability to fit the data, but will not lead us to a normative measure of risk. It is interesting, however, that in discussing the psychology of risk, Lopes feels no need to discuss the forms of risk that are derivative on the shape of the utility function except in a negative context to point out that these do not seem to be adequate descriptions of risk as a psychological notion.

The second criterion of Aspiration (A), however, contains the kernel of a rational theory of risk. Lopes postulates that individuals have an aspiration level (which may differ for each decision being made) and that, in addition to maximising the SP criterion, they also wish to maximise the probability that they achieve the appropriate aspiration level. These aspirations will be affected by what opportunities are available and by constraints imposed by the environment. Even a risk averse person may still be inclined to take large risks if playing it safe in a particular context will not provide a high enough outcome to ensure survival, whereas taking a risk at least provides this possibility. For example, an impoverished farmer who realises that he will be unable to meet subsistence levels even by planting entirely with low risk subsistence crops, may instead quite rationally choose to take the risky alternative of planting cash crops which might at least provide a possibility of achieving the minimum survival level (Lopes 1987; Shefrin and Statman 2000). Aspiration will be determined by what the subject considers "reasonable or safe to hope for", by the influence of the other alternatives available, and by the influence of the external environment. These will all serve to provide the "frame" in which the choice is made and will all change the way in which the individual approaches the decision. Thus, regardless of the desire for security or potential on the part of the decision maker, the aspiration factor means that the person will also seek to maximise the chance that the aspiration level is reached.

Technically, in Lopes' model the decision maker seeks not only to maximise the score on the SP criterion, but also to minimise the probability of not achieving the aspiration level. Letting α represent the aspiration level, this means maximising $A = Pr(x > \alpha)$. This conception of risk is entirely absent from all the traditional measures reviewed above. That the probability of achieving some aspiration level is a psychologically plausible consideration when choosing among risky options is supported by Lopes' 1987 protocol analyses, as well as descriptive data (Lopes 1987, Lopes and Oden 1999, Payne 2004) and simple introspection. This notion has also been widely mentioned and discussed, but rarely formalised as a theoretical basis for risk (Dubins and Savage 1976; March and Shapira 1992; Payne 2004; Roy 1952; Sokolowska and Pohorille 2000; Sokolowska 2003). I propose to use this concept as the cornerstone of Pure Risk Theory.

The first way in which Lopes criterion does not meet our requirements is easily amended. Her theory applies only to initial consequences described in monetary values. We wish, however, to ask how this criterion would be applied by a hypothetical individual whose preference ordering embodies only PR attitude, $\succsim^N$. By assuming that this hypothetical individual is making a decision between acts where the consequences already encompass all choice preferences but his own, we can adapt Lopes' criterion to $A = \Pr\left(u^N > \alpha\right)$, where α is a PR neutral utility level. That is, the PR attitude is concerned solely with the probability of achieving a PR neutral utility (which therefore incorporates a full consideration of all non-PR choice concerns from $\succsim^N$) that is greater than some level utility given by α. Put another way, the PR neutral utilities u^N reflect the preference ordering of the individual in the absence of the desire to achieve some minimum aspiration level. $\succsim^N$ incorporates all the same factors that might influence the decision as $\succsim$ (including possibly loss aversion, attitudes to money and attitudes to probability) with the exception of the resorting of the options implied by the attitude to PR, which is precisely that the individual wishes to maximise $A = \Pr\left(u^N > \alpha\right)$.

It is important to see here that $\succsim^N$ is hypothetical - the overall preference ordering $\succsim$ includes the individual's attitudes to PR and, in attempting to define what that attitude consists of we have gone backwards to a hypothetical preference ordering that excludes these risk attitudes. PR attitude is thus acting on the hypothetical entities u^N which represent the utilities that would be allocated to the preference ordering $\succsim^N$ were the individual not to have any attitude to PR. This method may seem somewhat circular in that I postulate a theory of PR that acts on the utilities that would obtain were we to remove any attitude to this notion of PR. However, working this way has a number of advantages in that in enables us to work directly with preference orderings rather than with consequences, and it enables us, through construction, to eliminate all other aspects of the decision that might influence the preference ordering in ways that given the appearance of relating to risk attitude but are, in fact, completely unrelated. It may help to view the problem in terms of breaking the individual's preferences between gambles down into two components: the overall preference ordering $\succsim$ is decomposed (hypothetically) into those preferences that would be observed in the absence of any attitude to PR **%**^{N}, and those preferences that consist solely of these attitudes to PR. The overall preferences may thus be retrieved from **%**^{N} by imagining that a decision maker that embodies only your PR attitudes is given a decision that consists of a series of acts for which all the consequences are completely described by the utilities u^N. The theory of PR needs to provide a convincing story for how this hypothetical decision maker, who embodies solely PR attitudes, would make this decision. If this can be done, and our version of Lopes' aspiration criterion is the …rst step, we have described what it is to have an attitude to PR as well as how to switch between **%**^{N} and **%**.

Claim: Pure Risk is related to the concept of achieving an aspiration level; the probability of attaining an outcome above some aspiration level. Since Pure Risk is not the only component of the decision process, the aspiration level must re‡ect non-Pure Risk aspects of the decision and must therefore be evaluated on Pure Risk neutral utility levels.

4.2 Problems with Simple Aspiration as Pure Risk

This particular version of what it is to have an attitude to PR is extremely simplistic and poses a number of problems. However, there are already bene…cial comments that can be made about the general approach that incorporates aspiration levels. Firstly, it encapsulates a measure of risk that is congruous with the intuitive idea that avoiding risk is somehow about avoiding the worst that can happen. In minimising the probability that your realised PR neutral utility is below some important level, one has reduced the chance of the worst happening which is surely very close to the way in which many people think about risk. The aspiration level criterion is analogous to the Value at Risk (VaR) measure used in practical …nance where ® is a measure of risk determined by the value at a constant percentile of the distribution. For example, VaR at 5% is the loss (i.e., value of ®) that would only be exceeded of 5% of trading days. The fact that an aspiration level type criterion is used as the most in‡uential risk measure in the finance industry is indicative of the degree to which the folk psychology of risk is captured by examining the most extreme downside of an outcome distribution.

The major problem with the Aspiration criterion, however, is its essential arbitrariness: how precisely do we choose the aspiration point α? And how do we defend this choice against other possible contenders? Lopes gives little guidance on this question beyond suggesting that it may be influenced by many factors, both psychological and environmental. A survival level of outcome certainly seems an appropriate choice in many development economic contexts, but has little resonance where the decision maker is not living at subsistence levels. In any case, it may be reasonable to assume that utility levels cannot go below that associated with the survival level so this level is a lower bound on utility (i.e., if α is the survival level, then $\Pr\left(u^N > \alpha\right) = 1$ by definition[5]) and thus this aspiration level will have no effect on the PR neutral choice.

In many (or indeed most) cases considered in the applied decision making, however, the survival level with not be of great concern, particularly with regard to much financial decision making. A number of other contenders can be envisaged for the aspiration level though. One is the status quo reference level (i.e., the point of zero utility in a reference dependent context). This makes the aspiration level and the reference point identical, but it is certainly a level that could be argued to have some special significance in that the individual might care disproportionately about avoiding a loss and thus seek to minimising the probability of this occurring (independently of the utilities assigned to these outcomes) (Payne 2004). However, I believe that this is a descriptive point that doesn't

[5] This point is perhaps somewhat contentious as it is easy to image situations where decision makers ascribe utility levels above the lower bound to outcomes that involve death. These may vary according to the manner in which death came about, such as the case where honourable death in battle is assigned a high utility at the time that choices are being made, whereas a cowardly death is not. Certainly these attitudes may influence risk attitude during life. It is also true that people both live with and, in some cases (e.g., polar explorers) intentionally face, a small probability of death on a daily basis. An alternative approach may be to postulate some utility associated with non-survival whilst retaining the assumption that there is some lower bound on utility - this leads us well out of the world of financial decision making, but certainly is an assumption employed to great effect by organised religion over the centuries.

have any particular normative justification, particularly when we consider that the distinction between gains and losses is already incorporated directly into the perceptual transformation and loss aversion. Another possible natural aspiration level might be some form of peer group benchmark: one might plausibly wish to minimise the probability of not achieving some benchmark utility level relative to others. A financial example of this “keeping up with the Joneses” effect can be found in fund management where fund performance is measured less on the absolute returns, and more on returns relative to some industry benchmark. No doubt plausible justifications can be found for numerous other possible aspiration points in specific contexts. In addition, since we do not require that outcomes need to take descriptions in terms of a numerical scale it will not be clear how specific aspiration points might be identified. The PR free utilities u^N are numerical, but only unique up to an affine transformation[6] and it will not necessarily be clear how to select given values of u^N for special consideration points without perhaps singling out particular outcome descriptions for special attention.

Since minimising the risk of not achieving a single aspiration point does not completely determine a theory of PR, and since the description of the aspiration point is essentially arbitrary and may differ from context to context, could we not argue that each of the possible contenders has a role to play and that a true notion of risk requires that we simultaneously minimise the probability of not achieving multiple utility aspiration points α_i? If, for example we were to postulate two aspiration points this would imply that we would be concerned with minimising both the probability of not achieving the lower aspiration level, $A_1 = \mathrm{pr}\left(u^N < \alpha_1\right)$, as well as the further criterion of minimising the probability mass between the lower and the higher aspirations, $A_2 = \mathrm{pr}\left(\alpha_1 < u^N < \alpha_2\right)$.

This would be a step forward, but does not go far enough, and would result in a theory of rational decision making that required an unspecified number of risk minimisation goals to be added to the maximisation goal, each of which is still arbitrary in nature. In

[6] With SEU as the underlying basis for **%**^{N}. The axioms of CPT are more restrictive.

addition, there now exists a problem of how these multiple criteria are to be combined. They surely cannot all be given equal weight? As a first approximation it seems reasonable that the aspiration of surviving is more important than that of not taking a loss, which is in turn more important than reaching a positive benchmark. Perhaps the aspiration points should take a lower weighting as the threshold associated with each increases. In addition, the weight would depend crucially on the distance between the aspiration points - an aspiration that was only just higher than another point would need to have a commensurately smaller weight if the incremental probability mass between the two was not to get a disproportionately large weighting.

4.3 Aspiration Weighting Function

Instead of multiple aspiration points, I wish to argue that our rational notion of risk minimisation can be incorporated into a single goal by stipulating that every possible PR neutral utility point is an aspiration point. This enables us to arrive at a preference condition for PR. Since the hypothetical PR neutral utility allocations u^N reflect all aspects of the preference ordering $\succsim$ except PR attitude it must be the case that if one possible act f has a higher probability of satisfying an aspiration level than another act g, for every possible aspiration level, then this preference ordering $\succsim^{PR}$concerned only with PR attitude, must have $f \succsim^{PR} g$. A decision maker who chose f would have a higher probability of achieving any possible aspiration level than one who chose g.

Claim: (PR Neutral Dominance) If, for any two acts f and g we have $\Pr_f\left(u^N \geq \alpha\right) \geq \Pr_g\left(u^N \geq \alpha\right)$ for all α, then $f \succsim^{PR} g$.

This preference condition effectively tells us that, if we equate PR with the chance of not achieving a desired outcome (where the outcomes are in terms of PR neutral utility, and are thus fully adjusted for any other aspects of the overall preference ordering that do not reflect PR) then we must never choose a gamble that is dominated for all possible aspiration levels.

At first sight this proposition is not that much use to us as it deals only with pairs of acts of which one is dominated by the other at every possible aspiration level. The criterion also requires that we compare any two acts for an infinity of aspiration levels which are themselves on hypothetical PR neutral utilities. However, notice that the criterion given in the proposition is precisely that of first-order stochastic dominance (sd) applied to PR neutral utility.

For a distribution $F\left(u^N\right)$ to first-order stochastically dominate another $G\left(u^N\right)$ requires that

$$F\left(u^N\right) \leq G\left(u^N\right) \text{ for all } u^N \tag{1}$$

which is precisely the criterion in the proposition above.

Applying standard stochastic dominance results this means that there exists a function of u^N, which I term the Aspiration Weighting Function $\alpha\left(u^N\right)$ such that:

Proposition: (Bawa) If, for any two acts f and g we have $F\left(u^N\right) \leq G\left(u^N\right)$ for all u^N (so $f \succsim^{PR} g$) then

$$\int \alpha\left(u^N\right) dF\left(u^N\right) \geq \int \alpha\left(u^N\right) dG\left(u^N\right) \tag{2}$$

if and only if $\alpha\left(u^N\right)$ is a nondecreasing function.

If stochastic dominance is to be satisfied for any two distributions, then the Aspiration Weighting Function must be nondecreasing. Bawa (1975) provides a proof of this for the case where the function is bounded from below. That is, if $\alpha\left(u^N\right)$ only needs to be defined for utility above a certain level (presumably analogous to the survival level, below which the concept of aspiration ceases to have meaning) then stochastic dominance over PR neutral utilities is equivalent to an upward sloping aspiration weighting function. This use of dominance enables us to represent PR attitude using a single transformation of PR neutral utilities using an Aspiration Weighting Function.

The act chosen by an individual whose preference ordering embodies only PR attitude would be the act with the highest expected value of aspiration weighted PR neutral utility. Thus, the full intuition of the need to avoid PR can be expressed as the need to maximise the total expected utility

$$U = \int \alpha\left(u^N\right) dF\left(u^N\right) \quad \text{or} \tag{3}$$

$$U = E\left[\alpha\left(u^N\right)\right] \tag{4}$$

Conclusion: Pure Risk attitude is related to the probability of not achieving levels of utility that account for all aspects of choice except Pure Risk. All possible aspiration levels (and thus the entire distribution of PR neutral utility) are important for Pure Risk attitude, and an act that is PR neutral dominant will be as least as good as the alternative act. The decision maker will therefore choose the act that maximises

$$U = E\left[\alpha\left(u^N\right)\right]$$

where $\alpha\left(u^N\right)$ is a nondecreasing aspiration weighting function that transforms PR neutral utility, U^N, to final utility U.

$\alpha\left(u^N\right)$ is a consequence of extending Lopes' Aspiration criterion to an infinite number of aspiration levels over the whole space of possible PR neutral utility outcomes. Lopes justifies her aspiration criterion by an appeal to the psychology of risk and to empirical data that observed choices over distributions of outcomes show patterns that cannot be accommodated within a traditional model that solely maximises some version of expected utility (Lopes 1987, 1996, and Lopes and Oden 1999). These justifications are still important within my normative model of PR as I am arguing that the notion of risk embedded in the extension of aspiration level to an aspiration function is inherently intuitive from a deep psychological point of view.

With the addition of an additional assumption we can put some further structure on the shape of the function. $\alpha\left(u^N\right)$ governs the degree to which the decision maker is more or less concerned with achieving lower values of u^N when faced with a choice. An intuition for the slope of $\alpha\left(u^N\right)$ can be provided when we see that the slope governs the importance given to each increment of the cumulative distribution of u^N. By following through on our intuition that fulfilling a given Aspiration point should become less important, the higher the level of utility attached to that point (i.e., the importance of each incremental gain in u^N is decreasing in u^N) it must be the case that the aspiration weighting function is concave as well as nondecreasing. In this case the choice between distributions over PR neutral utilities in addition satisfies second-order stochastic dominance (Bawa 1975) This notion of PR also satisfies our intuitions about risk as being an inherently downside concept.

In addition, the concavity of the aspiration weighting function may be derived if we apply to PR theory Yaari's (1969) general definition of risk aversion. The fact the one individual's acceptance set is contained in another's implies that the first decision maker is more risk averse. For PR theory, as for SEU, Yaari's definition characterises a concave function. Like the value function, the aspiration weighting function will be unique only up to an affine transformation.

Proposition: If the importance attached to achieving any given aspiration level of u^N is greater the lower the level, then the aspiration function is concave as well as nondecreasing, and Pure Risk attitude satisfies second-order stochastic dominance. This also precisely characterises Pure Risk aversion.

The value of $U = E\left[\alpha\left(u^N\right)\right]$ given to each act is necessarily equal to the final expected utility of that act when the original consequences of the act, c_{ij}, are evaluated by the total preference function ϱ so we have:

$$\int \alpha\left(u^N\right) dF\left(u^N\right) = \int u\,dF(u) \tag{5}$$

The PR neutral utilities u^N embody all aspects of preference except PR attitude. PR attitude is thus all that is required to apply to U^N in order to transform the PR neutral utility to the final utility U.

4.4 An Alternative Derivation of Pure Risk

A striking aspect of this criterion is its similarity to the primary criterion of expected utility maximisation which, when applied to monetary consequences, is given by

$$U = \int u(x)\, dF(x) \tag{6}$$

There is, however, a crucial difference: expected utility maximisation thus expressed is the expectation of the utility of monetary outcomes, whereas PR is an expectation on a transformation of PR neutral utility values themselves.

Nonetheless because (3) is an expected utility representation over acts with PR neutral outcomes, it must be the case that $\succsim^{PR}$ is a rational preference ordering that obeys some set of axioms that permit such a representation. Certainly, since we are after a normative theory of PR, it would not seem desirable for $\succsim^{PR}$ to violate the axiomatic structure of SEU. This also suggests a possible way of axiomatising the theory of PR. If we are prepared to assume in advance that $\succsim^{PR}$ obeys the axioms of SEU when applied to u^N, then any axiomatisation of SEU can be adapted to produce a criterion of PR. So, constraining $\succsim^{PR}$ to satisfy continuity and independence, it must be the case that there exists a function $\phi\left(u^N\right)$, such that

$$\{\succsim^{PR}\} \Leftrightarrow \int \phi\left(u^N\right) dF\left(u^N\right) \geq \int \phi\left(u^N\right) dG\left(u^N\right)$$

In addition, given the monotonicity of u^N (in the absence of uncertainty the individual will always prefer greater certain PR Neutral utility to less) we may say that $\phi\left(u^N\right)$ is increasing. Because it is tautologically true that more PR neutral utility is preferred to less if it is obtained with certainty, the aspiration weighting function must be positively related to final utility. However, the fact that $\succsim^{PR}$ obeys the axioms of SEU is not sufficient to guarantee the concavity of $\phi\left(u^N\right)$ which, as before, must come from additional assumptions about the declining importance of aspiration as the level increases.

4.5 Restrictions on the Aspiration Weighting Function

For two prospects with identical valuations from a PR neutral perspective (so $E\left[u^N\right]$ is the same for both), the prospect that is less Purely Risky will necessarily have the higher utility U. We may thus use $-E\left[\alpha\left(u^N\right)\right]$ to obtain a measure of PR of a distribution over u^N for a given decision maker:

$$PR = -E\left[\alpha\left(u^N\right)\right] \tag{7}$$

The fact that PR is always concerned with minimising the cumulative probabilities (i.e., is inherently concerned with the downside), means that, given a consistent set of PR neutral utilities, in maximising $E\left[\alpha\left(u^N\right)\right]$ we are naturally minimising PR.

To put further structure on the aspiration function we need to delve deeper into the existing literature on risk measures. This is a surprisingly sparse literature, probably because without a notion of PR, SEU leaves little wriggle room for introducing additional measures of risk. The literature attempts either to add a risk measure as a secondary variable to be considered in addition to SEU (Coombs 1975), to derive measures of perceived risk that are incidental to the actual preference structure over outcomes (Pollatsek and Tversky 1970; Luce 1980; Sarin 1987), or to examine risk measures for practical application in finance with no necessary link to the normative theories of SEU (Szegö 2002).

All of these measures have been concerned primarily with the measurement of risk inherent in monetary outcomes and not with separating a concept of PR from risk effects derived from other sources. However, the axiomatic approaches to risk measurement used by Pollatsek and Tversky, Luce, and Sarin may be easily applied to gambles over PR neutral utilities and, since they intended to axiomatise a measurement of risk alone, some of the axioms may have better traction when applied to that component of choice which by definition focusses solely on PR. Sarin (1987) uses two assumptions to derive a model of risk, both of which may be placed within the context of PR. I present the assumptions here, adapted somewhat to fit the purposes of the current framework. Our risk measure for a density function f on u^N is the value of $PR(f) = -\int \alpha\left(u^N\right) dF\left(u^N\right)$ as required in our framework. This PR measure is defined relative to the density function f on random variable U^N. If a constant amount β is added to every PR neutral utility in a gamble we define the density of the modified gamble as f^{β}. Sarin's first assumption is that the PR of the modified density $PR\left(f^{\beta}\right)$ is a multiplicative function of $PR(f)$ and β. This assumption is justified by both intuition and evidence that the PR of an option should decrease when a constant is added to all outcomes of the gamble (Pollatsek and Tversky 1970; Coombs and Lehner 1981; Keller, Sarin, and Weber 1986; Jia, Dyer, and Butler 1999).

Assumption 1: (Risk Multiplicitivity) There is a strictly monotonic function S such that for all density functions f and all real $\beta > 0$.

$$PR\left(f^{\beta}\right) = PR(f)S(\beta) \tag{8}$$

$S(\beta)$ is strictly decreasing if $PR : f \to R^{+}$ and strictly increasing if $PR : f \to R^{-}$.

The second assumption used by Sarin (and by Luce 1980) is that the densities can be aggregated into a single number using a form of expectation.

Assumption 2: There is a function T such that for all densities f

$$PR(f) = \int T\left(u^N\right) dF\left(u^N\right) = E\left[T\left(u^N\right)\right] \tag{9}$$

This assumption is already satisfied given the structure required for PR theory.

Pure Risk Theory and Risk Multiplicitivity together ensure that the aspiration weighting function takes the form

$$\alpha\left(u^N\right) = \frac{1 - e^{-\rho u^N}}{\rho}$$

with $\rho > 0$.

Sarin (1987) proves that given these two assumptions, (which amount to the sole additional assumption of multiplicitivity for

PRT), that, for some constants K and c,

$$PR(f) = \int K e^{cu^N} dF\left(u^N\right) \tag{10}$$

where $K > 0$, $c < 0$, or $K < 0$, $c > 0$.
Given our definition of PR, this implies

$$\alpha\left(u^N\right) = -Ke^{cu^N}$$

and since $\alpha\left(u^N\right)$ is required to be nondecreasing and concave, we can restrict the constants to $K > 0$, $c < 0$. In addition, given that $\alpha\left(u^N\right)$ is unique only to an affine transformation, we can define $\rho = -c$, and $K = \frac{1}{\rho}$, and add the constant $\frac{1}{\rho}$, to obtain

$$\alpha\left(u^N\right) = \frac{1 - e^{-\rho u^N}}{\rho} \tag{11}$$

with $\rho > 0$ as a single parameter that governs the curvature of the aspiration weighting function, and thereby the degree of risk aversion.

4.6 A Measure of Pure Risk Aversion

Analogous to traditional risk concepts we can also define the degree of PR aversion implied by the aspiration function for any given distribution over PR utility. Standard risk aversion over monetary may be characterised by whether the certainty equivalent of the evaluation is above or below the expected value of the distribution. A risk averse individual will always prefer the expected value of any distribution received with certainty, to the gamble of the distribution itself. The certainty equivalent is the certain amount for which the decision maker is indifferent between receiving that amount or playing the gamble. For our PR criterion with a concave aspiration weighting function, PR aversion is an inherent property of the theory. Any rational person will thus at least prefer to attain the expect utility to playing the prospect. We define the Pure Certainty Equivalent (u^N_{PCE}) as that utility level which satisfies:

$$\alpha\left(u^N_{PCE}\right) = \int \alpha\left(u^N\right) dF\left(u^N\right) \tag{12}$$

With a concave aspiration weighting function it will always be the case that $\alpha\left(u^N_{PCE}\right) \le E\left(u^N\right)$.

Because of the tight analogy between the criterion of PR, and EUT applied to monetary outcomes, it is possible to adapt many of the results applicable to this latter theory. For example, the degree of PR aversion is dependent on the degree of relative concavity of the aspiration weighting function - the greater the curvature, the greater the PR aversion. We may thus employ an analogy of the Arrow-Pratt coefficients of risk aversion: the absolute coefficient of PR aversion at utility level u^N may be defined as

$$PR_A\left(u^N\right) = -\alpha''\left(u^N\right)/\alpha'\left(u^N\right) \tag{13}$$

and the relative coefficient of PR at u^N is

$$PR_R\left(u^N\right) = -u^N\alpha''\left(u^N\right)/\alpha'\left(u^N\right) \tag{14}$$

Using these concepts it becomes possible to investigate how PR attitude varies with overall utility levels. Note though, that with (11) as the aspiration weighting function we have

$$\begin{aligned} PR_A\left(u^N\right) &= -\alpha''\left(u^N\right)/\alpha'\left(u^N\right) \\ &= -\frac{e^{-\rho u^N}}{-\rho e^{-\rho u^N}} \\ &= \frac{1}{\rho} \end{aligned} \tag{15}$$

and

$$PR_R\left(u^N\right) = \frac{u^N}{\rho}$$

Thus, an individual will display constant absolute PR aversion, and increasing relative PR aversion, unless the PR parameter ρ itself varies with the PR-neutral utility level. This last observation appears very likely to hold and, whilst, this is not the place to explore these effects in detail, there are a number of examples in the literature which tend to indicate that risk taking increases as the survival level is approached. Kunreuther and Wright (1979) describe this tendency for subsistence farmers (as discussed later by Lopes 1987). It has been observed too in troubled firms (Bowman 1982; Bromiley and Wiseman 1989), as well as in the field of animal behaviour, where foraging choices get more risky as the calorific survival threshold is approached (Kacelnik and Bateson 1996). If the animal is unlikely to survive even with the yield of the sure food source, it becomes optimal to exhibit risk seeking behaviour as this provides the only chance of survival. On the other hand PR aversion also seems to decrease where the current levels of utility are extremely high because the notion of PR loses its bite for the extremely wealthy (March and Shapira 1992). This might indicate that, although from any given status quo point we might expect constant absolute PR aversion, this is still governed by the PR parameter ρ which might be responsive to shifts in the reference point.

5 Pure Risk Prospect Theory

In the exposition above I have outlined a theory of PR that forms a single component of any preference structure that admits a utility representation (or a utility and probability weighting function representation). PRT can thus be made fully compatible with SEU or RDU, or variations thereon. It relates to a component of the manner in which utilities are allocated to consequences within these theories, and not to any aspect of the final utility allocation itself. This structure, however, is highly theoretical and not of a great deal of use to us until we specify fully the remainder of the influences of choice within which it operates. We have derived the aspiration weighting function that describes the effect of PR attitude on PR neutral utilities. However, these utilities u^N are hypothetical and unobservable entities. To use the theory we need to be able to apply it to actual choices, not to hypothetical transformations of these choices. This requires a specification of the content of the PR neutral preference ordering $\succsim^N$.

Examining Figure 3 we now see that the advent of PR theory allows us to partially (or perhaps completely) plug the gap that was left by the fact the standard value function, which I denote as above by v (x) did not include any such considerations. The move from u^N to u is now fully accounted for by $\boldsymbol{\%}^{PR}$ and thus by PR. As can be seen from the diagram, this still leaves two potential gaps in how we allocate the correct utilities to the original preference ordering. This depends on how completely we can take v (x) to be a complete proxy for the PR neutral preference ordering $\boldsymbol{\%}^{N}$. If we ignore for the time being the problem of converting descriptions of the consequences to numerical valuations x to which we can apply the value function v (x), then our problem reduces to ensuring that v (x) accounts for all aspects of choice that relate monetary values to utility, and that are not accounted for by PR attitude. I shall take CPT to be the basis for this as it allows the most comprehensive set of e¤ects to be brought to bear on this evaluation. Thus, v (x) may be taken to include reference dependent utilities, diminishing sensitivity from the reference point, as well as loss aversion. In addition, following Schmidt (2003) and Sugden (2003) we may allow these e¤ects to vary as the reference point is varied. That is, whilst the utility of a potential outcome is measured in terms of its deviation from the reference point, the form of the function that performs this translation may change as the reference point itself shifts. If it is the case that these e¤ects capture exhaustively the rational and PR free attitudes to monetary outcomes, then the gap between v (x) and u^N has been closed completely. It may well be the case, however, that there are some (or even many) psychological or psychophysical e¤ects that in‡uence our attitudes to money of which we are currently unaware. If this is the case then v (x) will at best be a good proxy, limited by our current knowledge for the translation of monetary values to PR neutral utilities.

A further component of $\boldsymbol{\%}^{N}$ that warrants discussion, though not a gap in the sense used above is the existence of non-linear decision weights. Since rank-dependence is clearly not a part of PR theory and arises from probability distortions rather than an attitude to

PR, rank dependence is not taken to be part of the preference ordering $\%^{PR}$. Thus, if overall preferences are to be a¤ected by probability distortions, then these must be part of the PR neutral preference ordering. The expectation in (3) which arrives at the …nal evaluation of the act is thus taken with respect to subjective probabilities, including any distortion due to decision weights, if appropriate. There is no second probability distortion through the agency of PR neutral preferences. Also, since what occurs in $\%^{N}$ is independent of the preference ordering $\%^{PR}$ (which acts only on the resulting allocations of $\%^{N}$) it should be possible to postulate any desired form of decision weighting, without an in‡uence on PR attitudes. As Birnbaum has demonstrated (2004) empirical choice appears to violate stochastic dominance in numerous cases. Since both CPT type decision weights and PR preserve stochastic dominance, these violations will not be explained by the addition of PR attitudes to a CPT based model. However, the concept of PR is equally consistent (descriptively at least) with other con…gural weight models that predict the appropriate violations of stochastic dominance. This hybrid would be a model that combines the descriptive power of con…gural weight models, the psychological power of reference dependent value functions, and the normative power of PR theory. It is unlikely that a purely normative theory will ever enable us to explain all empirically observed data, even where error models are included. If however, as PR theory suggests, we have so far been omitting a signi…cant normative component, then including this may be expected to increase the accuracy of future descriptive models as well.

Up to this point in CPT a decision weighting function and a value function have been used to go directly from monetary outcomes to …nal utility values. This has been based on the assumption that a combination of rank dependence, reference dependence and loss aversion are the sum total of e¤ects that in‡uence decision making under risk. In essence, the values $v(x)$ have been used as proxies for the …nal utilities u that are required to actually preserve the preference ordering through their expected values. As we have seen, the existence of PR attitude would mean that the value function has been stretched too far - we have tried to incorporate effects that are due to PR into parameterisations of utility curvature and loss aversion. In addition, because decision weights also influence overall risk attitude, estimates of the weighting functions may also have been registering some of the effects of PR attitude, even though it does not govern the translation from monetary outcomes to utility. However, if these components do cover all or the great majority of effects in evaluating acts over monetary outcomes, then $v(x)$ may be a good or even complete proxy for u^N. Using this insight we can adapt CPT to form PR Prospect Theory (PRPT), making the usual simplifying assumption of CPT that all consequences can be fully described by monetary outcomes.

5.1 The Structure of PRPT

Using the PR formulations of (4) and (11) we have:

$$U = \int \alpha\left(u^N\right) d\psi\left(u^N\right) = E_\psi\left[\alpha\left(u^N\right)\right] = E_\psi\left[\frac{1 - e^{-\rho u^N}}{\rho}\right] \tag{16}$$

where $E_\psi[\cdot]$ is the subjective expectation using the imputed distribution ψ given by the non-linear decision weights of CPT. I will not discuss here the decision weighting function in further detail as it is unchanged by the addition of PR and has been thoroughly discussed elsewhere (e.g. Davies and Satchell 2004a).

Given the assumption that the reference dependent value function and loss aversion encompass the entirety of non-PR responses to monetary amounts, we have in addition

$$u^N(x) = \begin{cases} v_+(x) & \text{if } x \geq 0 \\ \lambda v_-(x) & \text{if } x < 0 \end{cases} \tag{17}$$

where the monetary outcomes are measured relative to the reference point such that $v(0) = 0$. We also have that the value function is increasing $v'(x) > 0$. Maintaining the assumption of diminishing sensitivity which is supported by much of the empirical data, particularly for median or representative individuals, implies that the value function is convex for losses ($v''_-(x) > 0$) and concave for gains ($v''_+(x) < 0$)[7]. Combining these two gives PRPT:

$$U = E\left[\frac{1 - e^{-\rho v_+(x)}}{\rho} \,\middle|\, x \geq 0\right] + E\left[\frac{1 - e^{-\rho \lambda v_-(x)}}{\rho} \,\middle|\, x < 0\right] \tag{18}$$

Köbberling and Wakker (2003) propose an exponential value function which, given their definition of the loss aversion index, enables us to exactly separate loss aversion from utility curvature whilst retaining an index of loss aversion that is invariant to the unit of payment. Were we to employ the same function here, the overall transformation from monetary outcomes to utility would require a double exponential transformation: once of the reference dependent values through an exponential that is concave above $x = 0$ and convex below, and the second time of the resulting u^N through a globally concave exponential that reflects PR attitude. Thus, with curvature of gains governed by $g > 0$, losses by $l > 0$ and loss aversion by $\lambda > 1$, PR neutral utilities are:

$$u^N(x) = \begin{cases} \frac{1 - e^{-gx}}{g} & \text{if } x \geq 0 \\ \lambda\left(\frac{e^{lx} - 1}{l}\right) & \text{if } x < 0 \end{cases} \tag{19}$$

And the final utility allocations are:

$$u(x) = \begin{cases} \frac{1 - e^{\frac{\rho}{g}(e^{-gx} - 1)}}{\rho} & \text{if } x \geq 0 \\ \frac{1 - e^{\frac{\rho\lambda}{l}\left(1 - e^{lx}\right)}}{\rho} & \text{if } x < 0 \end{cases} \tag{20}$$

PRPT incorporates the standard CPT value function into a utility function that also accounts for attitudes to PR. Money outcomes are, in this specification, evaluated by an “expo-expo” function - an exponential of an exponential. Using the more common power function for the CPT value function would instead result in a “power-expo” curve

$$u(x) = \begin{cases} \frac{1 - e^{-\rho x^g}}{\rho} & \text{if } x \geq 0 \\ \frac{1 - e^{\rho\lambda(-x)^l}}{\rho} & \text{if } x < 0 \end{cases} \tag{21}$$

[7] Although these data have ignored the potential effects of PR and, as such have been confounding multiple effects.

Both of these permit a far richer set of behaviour than than CPT and, although an additional parameter has been introduced, this PR parameter plays a normative role and may, in fact take on some or all of the roles currently played by the descriptive parameters of CPT. Indeed, when we examine closely the behaviour of the expo-expo function over both gains and losses some very interesting results emerge. Empirical data show that, for the CPT framework, the most common pattern for individual choices has loss aversion ($\lambda > 1$), and a value function that is concave for gains ($g < 1$), convex for losses ($l < 1$), and is more linear for losses than for gains ($g < l$).

Figure 4 shows a two parameter exponential CPT function that cannot fit these patterns: $\lambda = 1$ so there is no loss aversion, and the curvature for gains and losses is equal ($g = l < 1$). The lower line shows the effect of using these values as the CPT input into PRPT. There are now only two parameters in the function, $g = l = 0.5$ governs the curvature of both gains and losses in the value function $v(x)$, and $\rho = 1$ governs PR attitude. The values have been chosen for illustrative purposes only. The resulting function is steeper for losses than for gains in a manner that is consistent with many definitions of loss aversion, but that does not require the slopes to be different at the reference point from above and below. It is also more linear for losses than for gains, without requiring either effect as assumptions of the model. PRPT can reproduce the effects of CPT with fewer parameters, and with greater normative justification. This is not to say that these psychophysical effects do not exist in reality and that a value function with all four parameters would not do a significantly better descriptive job. However, we can now provide a normative basis for some effects that had hitherto been explained purely through descriptive patches to the model.

The expo-expo PRPT function also has the additional property that it is concave for small losses before turning convex for larger losses, whilst being everywhere concave for gains. The loss part of this pattern fits that referred to by Tversky and Kahneman (1992) as the "fourfold pattern of risk aversion". Whilst this pattern can also be produced through the functioning of decision weights (which might also imply such a reversal for gains), it could be an interesting non-parametric test of PRPT to examine whether this asymmetry could be produced after accounting for probability distortions.

Holt and Laury (2002) examine the degree of risk aversion exhibited across a wide range of real payo¤s (in the gain domain) and present data to show that a "power-expo" function captures the dual e¤ects of increasing relative risk aversion which occurs as individuals become more risk averse as the stakes rise, but decreasing absolute risk aversion for large stakes which avoids the "Rabin critique" (Rabin 2000) that the degree of risk aversion required to explain choices for low stakes implies absurd levels of risk aversion for high stakes. The expo-expo function has the same properties and both functions have the desirable property that they permit risk aversion to vary for gambles of di¤erent stakes in more complex ways than traditional CPT which better …t empirical data of choices for real payo¤s[8].

A considerable part of the appeal of EUT (or derivative theories) is our ability to apply it in a vast range of situations, particularly those in economics and …nance where the outcomes can be described by a monetary scale and then transformed into utility values through the application of a parametric value function. Previous attempts to describe observed choice data without PR have been inherently misspeci…ed and, in the light of this it is perhaps unsurprising that even the most sophisticated forms of EUT (such as CPT) have failed to adequately describe observed choices for prospects other than truly simple binary prospects (Luce 2000, Lopes and Oden 1999, Birnbaum, Patten and Lott 1999). How much of the pattern of choices currently attributed to loss aversion, curved utility, or non-linear decision weights is actually a re‡ection of a rational attitude to PR? A great deal of further work will need to be done to test these speculations, and all these concepts may have a place in describing the overall risk attitude to uncertainty over monetary outcomes. However, adding PR adds a great deal of explanatory power to the

[8] Holt and Laury report that hypothetical payo¤s do not show increasing relative risk aversion with higher stakes, which may explain why the CARA power function has often provided a successful …t in experiments with large, but hypothetical stakes.

model and may begin to resolve such issues as why the curvature of the utility function required to adequately express reasonable risk aversion over small amounts implies absurd risk aversion over larger amounts (Rabin 2000); or why no single combination of parametric forms for the utility function and the decision weighting function seems to be able to account for data across numerous data sets (Burness and Neilson 2000); or why rank-dependent utility models fail to explain existing empirical data as soon as prospects are more complex than binary alternatives (Luce 2000).

5.2 The Description Gap in PRPT

PR Prospect Theory builds the concept of PR into a framework that can be used practically used to assess choices between monetary gambles. Nonetheless, it requires the assumption that there is no non-monetary descriptive element to the outcomes of these gambles - an assumption that is not required of PRT itself. Neither, however, can PRT be used to fill this gap. This needs to come from future theoretical developments. In my opinion this gap may never be filled as it incorporates a number of effects which rely on the unique circumstances of each decision, the decision environment, the individual concerned and temporal and social aspects of the decision. Nonetheless, a great deal of work is now starting to provide the basis for how we might begin to adjust a theory like PRPT for observable, generalisable and measurable aspects of the non individual, and non-"rational" components of the decision. Amongst many others, the gap between complete descriptions of the outcomes, and their restricted monetary tags may arise from: mental accounting which labels money distinctly according to its origin or planned purpose; issues of framing which cause identical numerical outcomes to appear different; social preferences which include considerations of fairness, equality, and reciprocity; reason-based thinking which requires that choices be justified either internally or externally with adequate reasons for that behaviour; strategic issues and planning which locate the outcomes of that choice within a larger framework about how the act of making that choice will influence other choices of the decision makers, and of others around her; and issues of self-control and symbolic utility (Nozick 1993) where the choice of an act comes to stand for an entire class of acts and thus acquires a symbolic utility that transcends the monetary values in the numerically described outcomes.

Notwithstanding the complexity of these issues, and the lack of any over-arching theory to tie the strands together PR Theory now provides a notion of risk that does not rely entirely on the translation of numerical values to utility values, and thus can be included in the same conceptual framework as any of these descriptive components. In addition, in those many examples in economics and …nance where the descriptive gap is small, PRT can enhance our attempts to understand and model decision making by reintroducing risk as a fundamental concept that drives our behaviour.

6 Speculations on Finance Theory

If Pure Risk plays a role in rational choice, then it has signi…cant implications for the foundations of …nance theory. Indeed, by introducing an entirely new normative concept of risk it provides an entirely new grounding for the development of rational …nance theory. These speculations will be carried further in a further set of papers, but I would like to return brie‡y to the example of the risk to return ratio commonly employed in …nance as a performance measure or investment criterion. The Sharpe ratio (1966) which divides expected (excess) returns by the standard deviation of the returns was the …rst attempt to incorporate the insight that expected return should be adjusted by the risk associated with that return - choosing the highest expected return may not be a good investment decision if it exposes the investor to considerable risks at the same time. The ratio uses standard deviation of returns as the measure of risk and, in employing returns directly, makes the assumption that we care about returns directly, rather than the utility associated with them. The ranking of options by the Sharpe ratio may be completely captured by a mean-variance frontier, which assumes that either the distributions of returns can be completely described by the mean and variance (e.g., normally distributed returns), or that the investor cares only about these two aspects of the distribution (which can be shown to be the case with a quadratic utility function) (Pedersen and Rudholm-Alfvin 2003). In practice, however, neither of these two conditions appear to hold and quadratic utility functions have the undesirable property that absolute risk aversion increases as individuals become more wealthy.

Furthermore, the concept of variance does not intuitively seem to characterise that notion of risk about which we should be concerned where choosing an investment - the fact that an investment has a widely dispersed distribution above the mean would seem to most people to be a good thing. It is losing money that they are concerned about. Given these concerns a number of alternative measures were developed that focussed on downside measures of risk, such as the Lower Partial Moments (LPM) where only returns below a general target are counted as contributing to risk. These approaches do not generally arise out of any coherent theoretical model of preferences, but rather re‡ect the desire to re‡ect the risk and return trade-o¤ in a way that appeals to the intuitions of what is meant by risk in an investment context. These measures are to a greater or lesser degree analogous to the original Aspiration criterion used by Lopes, and completely analogous in the probability of shortfall (PS) measure which simply measures the probability of a return below some target level. All su¤er from the same problem of Lopes' measure though: the essential arbitrariness of choosing a single target level that therefore ignores the e¤ect of the shape of most of the distribution.

Note that in order to get practical measures that cohere with an investor's concept of risk, the industry has either had to turn to ad hoc deviations from the supposed rational criterion of choice under uncertainty, or has had to make highly restrictive and unrealistic assumptions about the shape of return distributions or the shape of the value function. Nonetheless, the …nance industry has increasingly found it useful to make investment and portfolio decisions on the basis of risk adjusted return, or return per unit of risk. Pure Risk overcomes these difficulties at the same time as providing a sound rationale for the normative status of a risk/return trade-off. It offers an intuitive measure of risk that is similar in spirit to the ad hoc measures of the finance industry and focuses naturally on the true downside nature of risk, whilst removing the arbitrary nature of choosing a particular target point.

Indeed, we can eliminate the need for a perceptual value function altogether as it is possible to simultaneously have an investor that does not experience diminishing marginal utility with respect to money (an assumption that is often desirable for corporate entities and which simply results in a linear utility function and thus no risk aversion in the traditional sense) but that still holds a rational desire to reduce risk. Specifically, we may postulate that the firm (or investor) should have no rational concern for the psychophysics of money or a concern with loss aversion, so that $u^N = x$, but still allow this entity to have a rational concern for Pure Risk, so $u = \alpha(x)$. This leads to a potential rational criterion for performance evaluation, analogous to the Sharpe ratio. The rational choice of investment to maximise $U = \int \alpha(x)\, dF(x)$. To characterise this as a performance measure in the spirit of the Sharpe ratio requires that we define a measure of risk β, such that

$$\int \alpha(x)\, dF(x) = \frac{\int x dF(x)}{\beta}$$

$$\beta = \frac{\int x dF(x)}{\int \alpha(x)\, dF(x)}$$

The investor could vary the desired level of aversion to Pure Risk by varying the parameter ρ in $\alpha(x)$ for the appropriate context. With this ratio any investment can be evaluated in terms of its risk versus return as long as ρ remains constant for the comparison. For a PR neutral individual, $\beta = 1$ and thus the decision is based on expected return only. For any level of Pure Risk attitude, however, $\beta > 1$ and increasing in the degree of Pure Risk. Thus the performance of different funds could be evaluated according to their stated degree of risk taking, rather than against the ex post benchmark o¤ered by the performance of their peers. This ratio could be applied consistently to any distribution including skewed distributions and those with fat tails, and would re‡ect the shape of the entire distribution, rather than just a few moments or partial moments. The parameter(s) in the aspiration weighting would re‡ect the degree to which the potentially large losses in the downside tail should be emphasised in the investment decision. The advantages of splitting the overall evaluation into a return component E [x] and a risk component ¯ are largely practical: the division …ts with current standard practice in …nance, and it allows the portrayal of investments on a risk/return frontier. However, unlike the mean-variance frontier, the mean-¯ frontier does not require unrealistic assumptions about either the shape of the value function (which is here entirely linear, although one could easily incorporate diminishing sensitivity and loss aversion if desired) or the nature of return distributions. And yet, the mean-¯ frontier has a …rm theoretical basis.

This is just one example of how Pure Risk seems to naturally plug practical gaps that cannot be accommodated within standard …nance theory. Its inherently downside ‡avour also holds out hope that the theory may help to resolve many of the other extant puzzles of …nance, such as the equity premium puzzle, whilst simultaneously getting closer to the tenets of rationality, rather than further away as has been the case with most recent attempts to resolve such problems.

7 Discussion

In this paper I have argued that conventional theories of risk used within the tradition of expected utility maximisation do not, in fact, characterised attitudes to PR itself. Where risk has been introduced as a concept within EUT its place is entirely subordinate to the way in which utility is attached to monetary outcomes, or to the way in which decision makers distort probabilities to arrive at decision weights. There are two separable notions of risk inherent in the attribution of utility values to monetary outcomes: Loss Aversion and the curvature of the function connecting outcomes to Basic Utility values. However, neither of these satisfactorily incorporate our intuitive notions of risk. Firstly they are entirely derived from other primitive concepts. For the former this is the way in which we psychologically evaluate gains relative to losses. For the latter the risk is entirely dependent on the way in which our marginal utility of money increases or decreases away from the reference point. Neither of these concepts expresses an intuitive and primitive concept of risk as something to be avoided, and in the case of utility curvature even the directionality is unclear. Secondly, the notions of risk thus derived are only relevant to situations where the utility of an outcome is directly and completely related to an underlying monetary scale. That is, if there is no utility function translating monetary outcomes to utility, then these concepts of loss aversion and utility curvature do not exist. In short, the traditional model of rational choice in a risky environment has hitherto completely lacked a model to explain precisely that element which makes the environment complex: it lacks a theory of risk.

I propose a theory of PR precisely fills this gap. I have shown how the concept of PR arises naturally from the extension of intuitive psychological notions of risk to continuous distributions. In particular, a natural, but basic, notion of risk is found in Lopes' concept of an Aspiration level: risk is the probability of not achieving a particular level of utility. This embodies the inherently negative flavour of risk - we should rationally wish to minimise the probability of not achieving our goals. It is instructive that this approach to risk is commonplace in the practical approaches of the finance industry. To overcome the problem that the choice of any particular aspiration level is essentially arbitrary I extend this concept to a continuum of aspiration levels and show that the second criterion of rational choice requires minimising the probability of not achieving the aspiration level for all possible choices of aspiration level simultaneously, where the importance of reaching each particular level of utility is expressed through a weighting function. By assuming the decomposition of a rational preference ordering into a component embodying only PR, and one which is PR neutral we may employ PR neutral utilities u^N as the substrate for the theory. Final utilities in this view are obtained from PR neutral utilities by the application of the Aspiration Weighting Function

$$U = \int \alpha\left(u^N\right) dF\left(u^N\right)$$

I have further shown that this criterion can also be derived by applying the axioms of SEU to lotteries of PR-neutral utility values if PR preferences are rational. Given a preference structure which obeys the axioms required to derive SEU over lotteries where the outcomes are fully expressed in utility values, we may derive the criterion of PR minimisation.

In addition, the assumption that the probability of not achieving each aspiration level decreases in importance as the level increases implies that second order stochastic dominance must be satisfied by the preference structure over the lotteries. This allows us to further stipulate that the Aspiration Weighting Function, $\alpha(u)$. must be concave and nondecreasing. The further assumption that the risk of a prospect must increase if a constant positive value is added to each outcome leads us to an exponential shape for this function.

Thus, PR may be supported by both an axiomatic derivation and the generalisation of an intuitive psychological notion of risk. Furthermore the notion of risk embodied in this criterion may be applied to choices where utility cannot be attributed entirely by transforming monetary outcomes through a value function, and thus promises far greater universality then previous notions of risk.

There are many potential ways the complete theory might be practically applied to economic and financial problems. Specifically, PRT offers the potential to investigate individual's risk attitude separately to their attitudes to value and probability. In an institutional context the theory allows an entity to have linear reactions to value (thus displaying no diminishing marginal returns to money, an assumption often thought to be irrational for corporate entities) whilst still having a rational measure of risk aversion.

The theory also allows a normative risk/return trade-off without requiring the restrictive assumptions on the utility function or returns distributions that are required for rational mean-variance analysis. Indeed, PRT offers the foundations for risk-adjusted financial performance measurement whilst maintaining an entirely linear utility function.

The inescapable implication of PRT is that the edifice of economic theory is based on a misspecified notion of rationality that ignores a fundamental component of the choice environment. That this is so has become increasingly obvious in recent years as greater complexity of descriptive modelling has failed to provide a good description of human choice, or to resolve convincingly outstanding paradoxes and problems. PR may be adapted to fit into any extant rational theory of choice, and I develop PR Prospect Theory in this paper and demonstrate that much of the work that has been performed by descriptive patches to EUT in the past may now have a normative explanation in terms of PR. There remains much empirical and theoretical work to develop and test this theory but, if true it implies some interesting amendments to the existing structure of finance theory.

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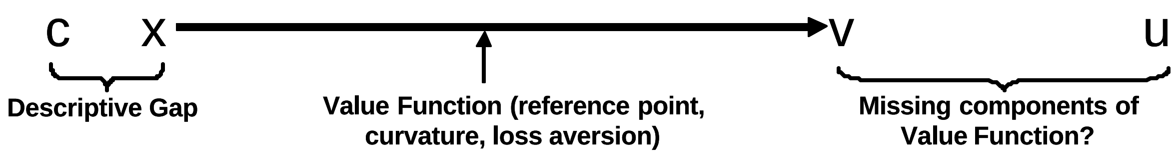

Figure 1: The standard value functions leave two potential gaps between the consequences and the utility allocations required by EUT

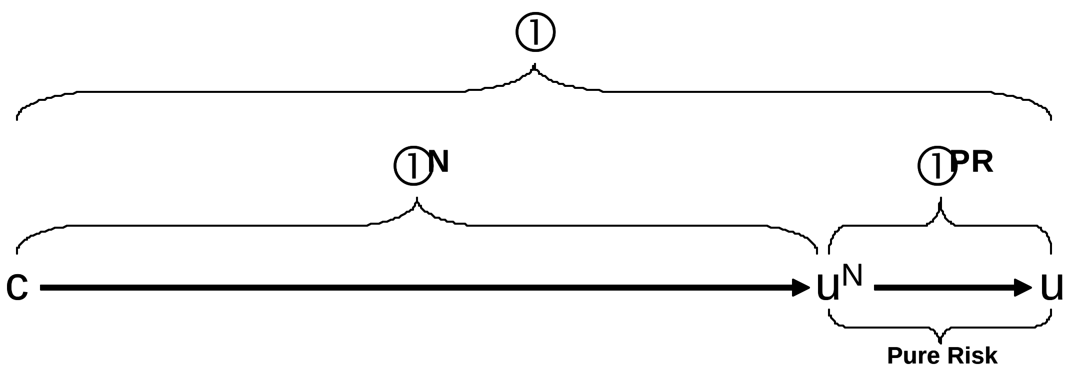

Figure 2: The overall preference ordering o is decomposed into a component that re‡ects only Pure Risk attitude o^{PR} and one which re‡ects all non-PR aspects of the decision maker and which is neutral with respect to Pure Risk, o^{N}.

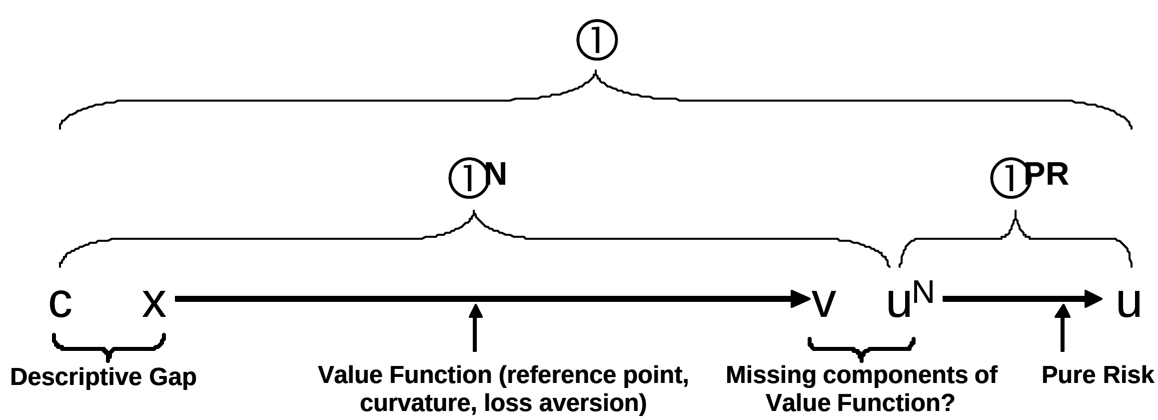

Figure 3: Gaps in utility allocations after the inclusion of Pure Risk Theory.

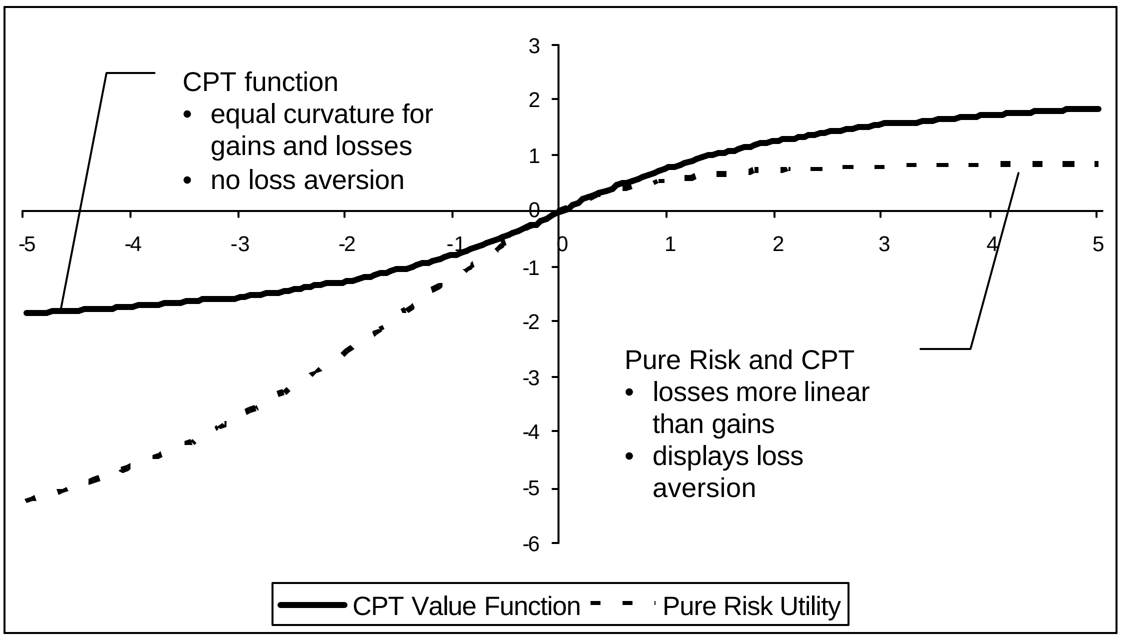

Figure 4: Transformations of monetary gains and losses for CPT and PRPT (dotted line). CPT is exponential and restricted to two parameters and thus does not display loss aversion (¸ = 1) or di¤erential values of gain and loss curvature (g = l = 0:5). The Aspiration Weighting Function is governed by ½ = 1.