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Abstract

This paper studies a game theoretic model where agents choose between two updating rules to predict a future endogenous variable. Agents rationally choose between these predictors based on relative performance. Conditions for evolutionary stability and stability under learning are found for the Nash solutions and corresponding parameter equilibria. Stability conditions are contingent upon parameter values and the initial distribution of heterogeneity. However, when the cost of using the more advanced updating rule is sufficiently large, all agents will asymptotically use the more parsimonious, or Minimum State Variable (MSV), updating rule.

Key Words: Adaptive Learning; Evolutionary Dynamics; Heterogeneous Expectations; Multiple Equilibria; Rational Expectations.

JEL Classification: C62, D84, E37

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1 Introduction

Expectations continue to play a key role in macroeconomic research. Since its introduction by Muth (1961), Lucas (1972, 1973), and Sargent (1973), the Rational Expectations Hypothesis (REH) has been the dominant paradigm in expectations formation. According to the REH, agents form expectations using the mathematical expectations operator conditioned upon available information. In modelling, economists usually assume that agents posses perfect knowledge of the true stochastic process of the variables they need to forecast.

Two objections to the REH come from the literature on bounded rationality. First, it may be a very strong assumption to assume that agents know the true stochastic process of the variables they need to forecast. A general suggestion from the literature is to allow agents to form expectations from less sophisticated schemes as in Bray and Savin (1986), Evans and Honkapohja (2001), and Hommes and Sorger (1998). Another objection to the REH is that under an environment with heterogeneous expectations, economic outcomes depend upon expectations of all participants.[1] Heterogeneous expectations may alter the stochastic process of aggregate variables. Thus, if agents with rational expectations are to know the form of this stochastic process, then they must be able to observe the expectations of all agents in the economy. The learning literature has also discussed expectation formation schemes with heterogeneity, e.g. in Evans and Honkapohja (1997), Evans, Honkapohja, and Marimon (2001), Honkapohja and Mitra (2005), Giannitsarou (2003), and Guse (2005).

Guse (2005) was the first to consider a model where heterogeneous expectations came from agents using different forecasting models when learning. In this model, a proportion of agents, μ, used a parsimonious forecasting model while the remaining $(1-\mu)$ agents used a forecasting model that corresponded to a "bubble solution." The stability properties under learning (E-stability) in this model were determined by the proportion of agents using each forecasting model. When this proportion of agents was allowed to vary arbitrarily, it turned out that the stability properties guaranteed that the stable solution was always stationary. Finally, the central result of the paper was that the two possible equilibria exchange stability properties at the smallest μ where the Mean Squared Errors from the two forecasting models are equal.

The limitation to Guse (2005) is that there is a restrictive assumption that the proportion of agents using each forecasting model is determined exogenously. Exogenously determined heterogeneity may produce a result where many agents are using an obviously inefficient forecasting model due to the form of the equilibrium. Agents who notice this efficiency disparity could form better expectations by using the most efficient forecasting model. This paper extends Guse (2005) by incorporating predictor choice into the model so that the agents will be able to change the forecasting model they use to form expectations. In this case, the agents are not only learning, but they are also learning

[1] There are some rational expectation models with heterogeneous information where agents try to improve upon their information using market data and the actions or expectations of other agents (if they are observable). In the learning literature, the assumption that other agents' expectations or actions are observable is not made.

the best way to learn the equilibrium.

Many papers have recently studied including predictor choice as an economic decision in models with expectation formation.[2] In Evans and Ramey (1992) agents choose whether or not to use a costly algorithm to update beliefs every period. This is later extended in Evans and Ramey (1998) where they allow agents to pay a resource cost for the privilege to use a mechanism that directly calculates expectations. Brock and Hommes (1997) use an approach they call the Adaptively Rational Equilibrium Dynamics (A.R.E.D.) to examine predictor decision. Under the setup of a cobweb model, they conclude that when the set of predictors are a stable predictor, rational expectations, and an unstable predictor, naive expectations, the dynamics of the system may not settle down to an equilibrium. However, this result may disappear when the set of predictors available increases. Branch (2002) examines Brock and Hommes' model and finds that the set of predictors available affects the local stability properties of the system. Sethi and Franke (1995) consider a model where agents have the choice between using a costless adaptive expectations rule or using rational expectations which incurs a cost. Predictor decision is then dictated via an evolutionary process. The papers in the predictor choice literature have focused on deterministic models and not stochastic models of learning. The main reason for this hole in the literature is that there were no learning models studied with multiple available predictors prior to Guse (2005).

As agents are acting like econometricians when forming their expectations, it only makes sense that the predictor choice mechanism should be set up such that agents are acting like econometricians in testing the available forecasting models. In this paper, I will follow the work of Sethi and Franke (1995) and use evolutionary learning as a selection criterion as it is similar to adaptive learning. This is shown by Marimon and McGrattan (1995) who find an isomorphism between adaptive learning and evolutionary learning. Furthermore, Kandori et. al. (1993) argue that the evolutionary approach reflects "limited ability (on the players' part) to receive, decode, and act upon information..." Predictor choice will be modelled as a game where a proportion μ agents are "programmed" to use the parsimonious forecasting model and $(1-\mu)$ are "programmed" to use the "bubble" forecasting model. Agents will then tend to switch to the forecasting model that awards the highest level of "fitness." As in Kandori et. al., there is limited information, so only some agents are able to detect the relative fitness differences and switch to the "better" forecasting model. However, as time goes by, all agents will be able to detect the difference and the population will change until there is no relative difference in the fitness levels awarded by the forecasting models.

The technique used in this paper will provide a tool for a test of robustness of learnability of equilibria under homogeneous expectations. A rational expectations equilibrium (REE) is commonly considered to be relevant if it is stable under learning, or expectationally stable (E-stable). However, some models produce multiple equilibria where many solutions may be E-stable. A natural question would be: If a model has multiple equilibria (of different forms) and agents can choose a forecasting model

[2] The works of Arthur (1992), De Grauwe, DeWachter, and Embrechts (1993), and Sethi (1996) present numerical results for this type of research.

(to form expectations) based on past performance, what solution, or solutions, would be stable under learning?[3] Furthermore, would the resulting (stable) Nash solution, under the predictor choice model, involve homogeneous or heterogeneous expectations? Conditions for stability under predictor choice and learning may be more strict than E-stability conditions for homogeneous expectations.

This paper presents a self-referential linear stochastic model with the possibility of multiple equilibria. Guse (2005) discusses the stability results under learning of such a model when agents have different perceptions of the true equilibrium. In this paper, the model in Guse (2005) is expressed as a game where agents benefit from using the most efficient predictor of the economy. I examine the stability properties of the equilibria in the game under RE and under least squares learning. When the model is expressed as a game with predictors, only some Nash equilibria are shown to be evolutionary stable when disturbed by mutant populations. Furthermore, only Nash equilibria with homogeneous expectations can be evolutionary stable with a corresponding learnable equilibrium. The central conclusion is that dynamics depend on the initial level of heterogeneity and parameter values in the model. The stability path dependence disappears when the cost of using the expensive predictor is sufficiently high and leads to all agents asymptotically using the more parsimonious, or Minimum State Variable (MSV), updating rule.

2 The Model and E-stability

Guse (2005) considers a self-referential linear stochastic macroeconomic model with the possibility of multiple REE, as presented in Taylor (1977) and discussed in the learning literature[4], e.g. Evans and Honkapohja (2001) and Heinemann (2000). It is a linear stochastic model with real balance effects consisting of four parts: aggregate demand, aggregate supply, money demand, and a fixed money supply. The reduced form is as follows:

$$y_t = \alpha + \beta_0 E^*_{t-1} y_t + \beta_1 E^*_{t-1} y_{t+1} + v_t \tag{1}$$

where E^* denotes a not necessarily rational expectation and v_t is a linear combination of stochastic shocks where $v_t \sim N\left(0, \sigma^2\right)$. Although it may be any variable that is affected by expectations, think of the variable y_t to be prices at time t. Under a situation of homogeneous expectations, there are two REE:

$$PLM_1 \quad : \quad y_t = a_1 + v_t \tag{2}$$

$$PLM_2 \quad : \quad y_t = a_2 + b_2 y_{t-1} + v_t. \tag{3}$$

[3]These solutions could also involve sunspots, however, Guse (2005) notes that the E-stability conditions do not change when sunspots are included in the "bubble" equilibrium. I do not include sunspots in this paper as they complicate the model without adding any interesting results.

[4]Although the model is ad hoc and not derived based on microfoundations, it continues to be a workhorse to study learning in a model with the possibility of multiple equilibria.

The first equilibrium represented by equation (2) is commonly referred to as the minimum state variable (MSV) solution or the "fundamental solution." The second equilibrium is commonly referred to as the "bubble solution" as it includes an extra state variable (y_{t-1}) that is not included in equation (1). I will refer to this equilibrium as the AR(1) REE.

In this model of heterogeneous expectations, assume that agents have the choice of using one of two forecasting models, corresponding to the two REE (2) and (3), where agents recursively estimate the coefficients of their forecasting model to form expectations of y_t and y_{t+1}. If a proportion of μ agents have a perceived law of motion (PLM) of equation (2) and the remaining $(1-\mu)$ agents have a PLM of (3), then the actual law of motion (ALM) is:

$$\begin{aligned} y_t &= \alpha + \mu a_1(\beta_0 + \beta_1) + (1-\mu)a_2(\beta_0 + \beta_1(1+b_2)) \\ &\quad +[(1-\mu)b_2(\beta_0 + \beta_1 b_2)]y_{t-1} + v_t \end{aligned} \tag{4}$$

The above system defines a mapping from the PLM to the ALM as follows:

$$T(\phi) = T\begin{pmatrix} a_1 \\ a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} \frac{\alpha+\mu a_1(\beta_0+\beta_1)+(1-\mu)a_2(\beta_0+\beta_1(1+b_2))}{1-(1-\mu)b_2(\beta_0+\beta_1 b_2)} \\ \alpha + \mu a_1(\beta_0 + \beta_1) + (1-\mu)a_2(\beta_0 + \beta_1(1+b_2)) \\ (1-\mu)b_2(\beta_0 + \beta_1 b_2) \end{pmatrix} \tag{5}$$

The resulting equilibria are expressed as :

$$\begin{aligned} a_1 &= \frac{\alpha}{1-\beta_0-\beta_1} \\ a_2 &= a_1(1-b_2) \\ b_2 &= \frac{1-(1-\mu)\beta_0}{(1-\mu)\beta_1} \end{aligned} \tag{6}$$

or

$$\begin{aligned} a_1 &= \frac{\alpha}{1-\beta_0-\beta_1}, \\ a_2 &= \frac{\alpha}{1-\beta_0-\beta_1}, \text{ and} \\ b_2 &= 0 \end{aligned} \tag{7}$$

Equilibrium (6) is referred to the AR(1) mixed expectations equilibria (MEE).[5] In this equilibrium, the proportion of agents using PLM_1 are underparameterizing the model when they are forming their expectations as they are ignoring the bubble in the ALM. Therefore, forecast errors from PLM_1 will tend to be larger, on average, in this equilibrium. Equilibrium (7) is referred to as the minimum state variable (MSV) MEE.

[5] The equilibria are referred to as "mixed" because they are generated from more than one PLM. Branch and McGough (2004) refer to such an equilibrium as the Heterogeneous Expectations Equilibrium (HEE). The MEE includes the REE when $\mu = 0$ or $\mu = 1$.

In this equilibrium, forecast errors will be the same under each PLM as they produce the same forecasts.

Under the two MEE's, economic agents have a great deal of knowledge of the economy. It is common to ask whether these equilibria are robust when agents form expectations using less sophisticated schemes than RE. Suppose that the agents act like econometricians and construct forecasts using their econometric model that they update every period when new information becomes available. The condition for an equilibrium to be (locally) stable under such a learning rule is known as Expectational Stability (E-stability):

Definition 1: *E-stability is the condition of local asymptotic stability of $\bar{\phi}$ under the differential equation*[6]

$$\frac{d\phi}{d\tau} = T(\phi) - \phi, \tag{8}$$

where T is the mapping from the perceived law of motion, ϕ, to the implied actual law of motion, $T(\phi)$ and τ denotes "notional" or "artificial" time.

$\bar{\phi}$ is a fixed point of the ODE which is also a MEE. For stability, the eigenvalues of the Jacobian matrix of equation (8) must have negative real parts. It is commonly known that (e.g. Marcet and Sargent (1989) and Evans and Honkapohja (2001)) an E-stable equilibrium is learnable under ordinary least squares and other similar learning mechanisms. Learnability of an equilibrium may be regarded as a necessary condition for the relevance of that equilibrium. Guse (2005) presents the E-stability conditions for a fixed proportion of heterogeneity, μ, in the following proposition:

Proposition 1: *E-stability conditions for the above linear stochastic model with heterogeneous expectations.*

1. *All MSV MEE in the parameter set*

$$ES_1 = \left\{ (\beta_0, \beta_1) \,|\beta_0 < \left(\frac{1}{1-\mu} \right), \beta_0 + \beta_1 < 1 \right\}$$

are E-stable. All MSV MEE outside of this set are E-unstable.

2. *All AR(1) MEE in the parameter set*

$$ES_2 = \left\{ (\beta_0, \beta_1) \,| \frac{1}{1-\mu} < \beta_0 < 1 - \beta_1 \right\}$$

are E-stable. All AR(1) MEE outside of this set are E-unstable.

For the Taylor (1977) real balance model, the parameter restrictions are $\beta_1 = -\beta_0$ and $\beta_0 \neq 0$. Therefore, either solution, MSV or AR(1) may be E-stable under heterogeneous expectations, depending on the parameter values of the model. E-stability

[6] In the homogeneous expectations case, $\phi = (a_1)$ if all agents use PLM_1 and $\phi = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$ if they use PLM_2.

of the two solutions may change when the level of heterogeneity, μ, is allowed to change. When μ is allowed to arbitrarily change, Guse (2005) presents the condition for E-stability for any μ:[7]

Proposition 2: *Let*

$$A = \{(\beta_0, \beta_1)\,|\beta_0 < 1, \beta_0 + \beta_1 < 1\}$$

and

$$S = \{(\beta_0, \beta_1)\,|\beta_0 > 1, \beta_0 + \beta_1 < 1\}.$$

If $(\beta_0, \beta_1) \in A \cup S$, *then for each* $\mu \in [0, 1]$ *exactly one of the two MEE is E-stable.*

Within this set, if μ changes for some reason, the other MEE may become E-stable, but there is no μ such that both solutions are E-unstable. The set A is where only the MSV equilibrium is E-stable for all $\mu \in [0, 1]$. The set S is the set where the two equilibria exchange E-stability at $\mu = 1 - \frac{1}{\beta_0}$. When determining stability of the system with predictor choice and learning, I will assume that $(\beta_0, \beta_1) \in A \cup S$. I focus on the set S where stability of each equilibrium is determined by the level of heterogeneity, μ, however, I also discuss stability properties when $(\beta_0, \beta_1) \in A$.

3 Evolutionary Stability

Next, I will focus on the AR(1) and MSV processes and ignore learning for the time being. Suppose that the agents have the ability to change their forecasting model (PLM) if they believe that the other is doing a better job at predicting the economic variable. In this case, agents may act like econometricians and test the performance of the available forecasting models.

When testing the existence of omitted or irrelevant variables, a typical econometrician will compare how well the two forecasting models have performed. One measure of forecasting usefulness is the mean squared error (MSE). This is obtained by supposing a quadratic loss function for forecast errors:

$$MSE_i = E\left(y_{t+1} - F_i\left(y_{t+1}\right)\right)^2$$

where $F_i(y_{t+1})$ is the forecast of y_{t+1} obtained using forecasting model i. Since the agents in this model are concerned with predictability, they would like to use the forecasting model that gives the smallest MSE.[8] In order to incorporate this into a model of predictor choice, I will assume that agents will receive utility inversely related to the MSE of the forecasting model they are currently using. Also assume that agents typically prefer parsimony and thus there is some fixed cost associated with using the more advanced updating rule (if both predictors give the same MSE, then the

[7] This proposition is a combination of two propositions presented in Guse (2005).

[8] One could consider other properties as well, however, predictive performance of the estimated model (not just estimators) is very natural.

parsimonious predictor is always preferred). Assume the following utility function for an agent using predictor i:

$$U_i = \frac{1}{MSE_i} - \text{ cost of using predictor } i$$

In real-time, MSE_i would be the MSE realized in the previous period, however, I will use the MSE of each predictor that corresponds to the E-stable MEE (MSE realized under RE) of the current value of μ when evaluating for stability. I discuss the reasons for using these MSE's below. The MSE's are written as MSE_1 and MSE_2 and can be found in the appendix A. Assume that the cost of using the AR(1) process is greater or equal to the cost of using the MSV process, so, without loss of generality, the cost of using the MSV process will be normalized to zero.[9]

3.1 The Model Expressed as a Game with a Continuum of Players

Assume that there is a continuum of players so that each agent's decision does not affect the state of the economy. Let $([0,1], \mathcal{B})$ be the underlying space where $[0,1]$ is the player set and $\mathcal{B}$ is the σ-algebra of Borel subsets of $[0,1]$. Let $S_i = \{PLM_1, PLM_2\}$ be the set of strategies for each player.

At the beginning of the period, each agent will form $E_{t-1}y_t$ and $E_{t-1}y_{t+1}$ using the PLM corresponding to their strategy s_i. In this artificial game, assume that the agents have RE and thus will know the MEE values based on their given strategy. Later, learning will be included in the game.

Suppose that each player receives a payoff from choosing either strategy in the following manner:

$$\begin{aligned} v_i(s_i, \mu) &= \frac{1}{MSE_1} = U_1 \text{ if } s_i = PLM_1 \\ &= \frac{1}{MSE_2} - k = U_2 \text{ if } s_i = PLM_2 \end{aligned}$$

where $k \geq 0$ is the cost of using the AR(1) predictor, MSE_i is the MSE realized under RE using strategy s_i, and the population state at time t is:

$$x_t = (\mu_t, 1 - \mu_t).$$

In evolutionary game theory, individuals are programmed to make a single strategy and the population changes depending on relative evolutionary "fitness." In this model, the proportion of agents using the MSV PLM, μ, will adjust as individuals choose to change their "programmed strategy." If $v_i(s_i, \mu) > v_j(s_j, \mu)$, then agents will tend to choose s_i over s_j until $v_i(s_i, \mu) = v_j(s_j, \mu)$ or when the proportion of those using strategy s_j converges to zero. The replicator dynamics, discussed below, will dictate

[9] This can be done since agents will only consider differences in utility and not the differences in the estimated parameter values.

predictor choice dynamics in this game as it will favor the strategy awarding the highest utility.

3.1.1 The Nash equilibria of the game

Next, I solve for the Nash equilibria for the above game. The Nash solutions in evolutionary game theory are the population states where all individuals receive the same level of fitness regardless of their strategy. In the above game, this is where $v_1 = v_2$ if $\mu \in (0,1)$, $v_1 > v_2$ if $\mu = 1$, or $v_2 > v_1$ if $\mu = 0$. Since there are two possible solutions, MSV and AR(1), I solve for the Nash equilibria associated with each solution.

For the MSV MEE, it turns out that $\forall \mu \in [0,1]$, $MSE_1 = MSE_2 = \sigma^2$. If $k = 0$, agents are indifferent in which PLM they use, so the Nash equilibria consists of $\mathcal{B}$, the set of all possible combinations of heterogeneous expectations. If $k > 0$, the Nash equilibrium is $\mu = 1$ where all of the agents choose to use PLM_1.

There are several Nash equilibria for the AR(1) MEE. If $k = 0$, then there are two Nash equilibria which are

$$\begin{aligned} \mu &= 0 \\ \mu &= 1 - \frac{1}{\beta_0} = \bar{\mu}. \end{aligned}$$

Guse (2005) shows that $\bar{\mu}$ is where the MSV and AR(1) MEE exchange stability. If $\mu > \bar{\mu}$, then the MSV solution is E-stable and the AR(1) solution is E-unstable and if $\mu < \bar{\mu}$, the stability properties are reversed. Next, if $0 < k \leq k_1$, then the three Nash equilibria are:

$$\begin{aligned} \mu &= 0 \\ \mu &= \mu_1 \\ \mu &= \mu_2. \end{aligned}$$

where

$$\begin{aligned} \mu_1 &= 1 - \frac{1}{\beta_0 + \beta_1\sqrt{k\sigma^2}} \\ \mu_2 &= 1 - \frac{1}{\beta_0 - \beta_1\sqrt{k\sigma^2}} \\ k_1 &= \frac{(1-\beta_0)^2}{\beta_1^2\sigma^2} \end{aligned}$$

Note that $\mu_1 < \bar{\mu} < \mu_2$, so the AR(1) MEE is E-stable for μ_1 and is E-unstable for μ_2. Finally, if $k > k_1$, then there is one Nash equilibrium:

$$\mu = \mu_2.$$

3.1.2 Evolutionary Stability of the Nash Equilibria

Evolutionary game theory considers a pure or mixed strategy and determines whether this strategy is stable when the population is disturbed by some "mutant strategy." The game described above does not allow mixed strategies, but the population average of those choosing PLM_1, μ, will be considered a "mixed strategy." To determine stability in the context of the above game, consider all "mixed" ($\mu \in (0,1)$) and pure equilibria ($\mu = 0$ or $\mu = 1$) for the population and determine if it will be beneficial for a small proportion of agents to switch from using their current PLM to using the other PLM. If some "mutant" population strategy is allowed to enter and thus changing μ, will the population return to the given (equilibrium) population strategy when agents are allowed to change strategies? If so, such a population strategy is an Evolutionary stable strategy. Following Weibull (1995), Evolutionary Stability is defined as follows:

Definition 2: $x \in \Delta$ *is an evolutionary stable strategy (ESS)*[10] *if for every strategy* $y \neq x$, there exists some $\bar{\varepsilon}_y \in (0,1)$ such that

$$v(x, \varepsilon y + (1-\varepsilon)x) > v(y, \varepsilon y + (1-\varepsilon)x) \tag{9}$$

holds for all $\varepsilon \in (0, \bar{\varepsilon}_y)$.

Under an evolutionary stable strategy, if a small proportion of agents "mutate" from using one predictor to the other predictor, then they will not receive more utility than before the mutation. Furthermore, no other agents will wish to follow the "mutants." When there exists a selection criterion for the population, the population will tend to return to the evolutionary stable strategy.

A best response function can be drawn to present evolutionary stability. $x \in \Delta$ is evolutionary stable if:[11]

$$v(s_1, \varepsilon y + (1-\varepsilon)x) - v(s_2, \varepsilon y + (1-\varepsilon)x) \lesseqgtr 0 \text{ if } y \gtreqless x$$

The only potential strategies that can be evolutionary stable are the Nash equilibria. The set of ESS will thus be a subset of the Nash equilibria. Formally, $\Delta^{ESS} \subset \Delta^{NE}$.

Consider the Nash equilibria for the MSV MEE. For the MSV solution where $k = 0$, if μ_0 is allowed to change, there is equality for equation (9) for any $\mu_0 \in [0,1]$. Therefore, all the Nash equilibria in this case fail to be evolutionary stable strategies. This results from the fact that utility from each updating rule is the same for all $\mu \in [0,1]$. I will assume that $k > 0$ for the MSV MEE to ignore this uninteresting result. Next, consider the Nash equilibrium for the MSV solution when $k > 0$. When a small proportion of agents use PLM_2, inequality (9) always holds. Therefore, the Nash equilibrium of $\mu = 1$ for the MSV solution where $k > 0$ is an evolutionary stable strategy.

[10] Δ denotes the set of potential strategies. In this particular continuous framework, $x = (\mu_t, 1-\mu_t) \in [0,1]^2$.

[11] Note that $\mu \in [0,1]$, so for pure strategies, we only have to increase or decrease μ depending on which strategy we are considering.

Next, consider the Nash equilibria for the AR(1) solution. Figures 1 and 2 depict the best response functions used to determine evolutionary stability. Figure 1 shows the best response function when $0 \leq k \leq k_1$. The three Nash equilibria are $\mu = 0$, $\mu = \mu_1$, and $\mu = \mu_2$ where $\mu_1 \leq \bar{\mu} \leq \mu_2$. The first and third of these Nash equilibria are ESS's, but the second solution is not an ESS.[12] Figure 2 presents the best response function when $k > k_1$. The Nash equilibrium, $\mu = \mu_2$, is ESS.

(Figures 1 and 2 about Here)

These results bring forward a natural question, "Are there any evolutionary stable Nash equilibria with E-stable MEE's?" The most interesting candidates are the Nash MEE defined with $0 < \mu < 1$. The following proposition states that these candidates can never be evolutionary stable with E-stable solutions.

Proposition 3: *For any $k \geq 0$, there does not exist a Nash equilibrium with $\mu \in (0,1)$ that is evolutionary stable with an E-stable MEE.*

The proof for this proposition is given in appendix B. As we will be interested in learning in the next section, we can conclude that in the above model with learning, heterogeneity can only be a short run phenomenon. When learning is included in the game, the ESS $\mu_2 \in (0,1)$ can not be considered a relevant equilibrium as its associated MEE is not E-stable.

4 The Replicator Dynamics and Evolutionary E-stability

Next, assume that agents do not have RE in the game and must form expectations using least squares learning from their given strategy. Now, their are two possible problems for stability: stability of the Nash equilibria and E-stability of the MEE given the Nash population. Before moving to stability analysis, however, I will first define the replicator dynamics which will dictate predictor choice.

There are two elements of evolutionary game theory: a mutant mechanism which provides variety and a selection criterion that favors one variety over another. The replicator dynamics provides the role of selection. Following Weibull (1995), a discrete version of the replicator dynamics can be defined as follows:

$$\mu_t = \left(\frac{\zeta + U_1}{\zeta + \mu_{t-1} * U_1 + (1 - \mu_{t-1}) * U_2} \right) * \mu_{t-1} \tag{10}$$

where ζ is a non-negative constant[13] and U_i is a measure of utility for using forecasting

[12] When $k = 0$, there are only two Nash solutions, $\mu = 0$ and $\mu = \mu_1 = \mu_2 = \bar{\mu}$. In this case, $\mu = 0$ is the ESS.

[13] ζ is included in the replicator dynamics for two reasons. First, it determines the speed of convergence for real time dynamics. Second, it can be used to guarantee that both the numerator and denominator of the replicator dynamics are positive.

model i in period $t-1$. The replicator dynamics directs the population to use the forecasting model that awards a higher utility, or fitness level, at time $t-1$.

For testing the econometric model, the replicator dynamics can be attained. Following Kandori et. al. (1993) I will assume agents may not react instantaneously to their environment and they are myopic when they do react. As agents may not react instantaneously, assume that agents tend to imitate others in the use of their forecasting model which is seen in the μ_{t-1} term outside of the brackets on the right side of equation (10). This means that agents will decide not to test their econometric model if most agents are using the same one. Next, as agents are testing the two econometric models, they will compare the MSE's which is shown inside the brackets of equation (10). According to the replicator dynamics (and our assumption), not all agents will choose to change forecasting models all at once.[14] This can be explained by assuming heterogeneity within the testing procedure. For instance, some agents may require a larger sample size to do the test, or there may be heterogeneity in the critical test statistic to reject one's current forecasting model. The type of learning procedures may be heterogeneous as in Giannitsarou (2003) such that some agents may not learn very quickly causing a heterogeneous arrival time for testing. Finally, one could assume an environment of imperfect knowledge. Since agents are only using one forecasting rule, they must get the information of the MSE of the other process from an outside source. Assume that the quality of this information is dependent upon the distance (on the $[0,1]$ line) between an agent and the last agent who uses the other forecasting model. The agents located at the poles of the $[0,1]$ player set may have very poor information on the other MSE and thus would not wish to run a specification test unless the majority of the agents are using the other forecasting model. As heterogeneity is expressed as a proportion of agents in this paper, it is only natural to use evolutionary learning as the mechanism for econometric testing in the dynamic system.

Brock and Hommes (1997) and others assumed that the role of selection was dictated by a multinomial logit law of motion. With a multinomial logit, convergence to a single predictor is not necessarily attainable unless the expected value of utility from each predictor, except one, is equal to zero. The replicator dynamics will provide a tool to produce the possibility of convergence to homogeneous expectations due to the asymptotic nature of the replicator dynamics. However, this does not guarantee convergence to a single predictor as it may be that $U_1 = U_2$ for some $\mu_{t-1} \in (0,1)$.

4.1 Fast-Slow Dynamics

Econometric tests for goodness of fit tend to require a large set of observations before the test can produce meaningful results. Furthermore, an econometrician may not be too keen to changing to another forecasting model if the MSE of that model has a large variability. It makes more sense for the econometricians to test the two models when they have a large information set and a pair of relatively stable MSE's. Therefore,

[14] Agents simultaneously choosing the same forecasting model can only occur if ζ can take on negative values. Agents will all at once choose to use the MSV predictor if $\zeta = -U_1$ and will all use the AR(1) predictor if $\zeta = -U_2$ for all $\mu_{t-1} \in [0,1]$.

the speed of learning will be much faster than the population dynamics as a result of econometric testing. For mathematical purposes, I will assume that the MSE that agents observe will be the MSE corresponding to the MEE for the current value of μ. This will be from a process of fast learning dynamics with slow replicator dynamics, or "fast-slow" learning. The agents will fully learn the MEE corresponding to the current value of μ before each period when μ is updated by the replicator dynamics. Therefore, the speed of the learning is infinitely faster than the speed of the replicator dynamics. Under real-time, when agents compute the MSE as

$$MSE_{i,t} = MSE_{i,t-1} + t^{-1}((y_t - z'_{t-1}\phi_{i,t})^2 - MSE_{i,t-1})$$

(no assumption of "fast-slow" dynamics), simulations show that the results are comparable to the results discussed below.[15] The plausible assumption of fast-slow dynamics is used in order to theoretically evaluate for evolutionary E-stability defined below.

4.2 Evolutionary E-stability

Now, I will examine when a Nash solution is stable given β_0, β_1, α, and a cost parameter, k. Here, I introduce a concept I will call evolutionary E-stability:

Definition 3: *Assume that the model is updated using fast parameter learning dynamics with slow replicator dynamics. An MEE or REE, $\phi(\mu^*)$, is Evolutionary E-stable, under the defined game above, if for all $\mu \in [0,1]$* sufficiently close to μ^* (1) $\mu_t \to \mu^*$ *under the replicator dynamics and* (2) $\phi(\mu_t)$ *is E-stable for all* μ_t.

Here, $\phi(\mu)$ refers to an E-stable MEE that is determined by the level of heterogeneity, μ, and therefore, $\phi(\mu^*)$ is the MEE determined by the Nash solution of μ^*. Under evolutionary E-stability, if a mutation occurs to the level of heterogeneity to slightly change μ, then the system will return to the evolutionary E-stable MEE or REE, $\phi(\mu^*)$. Furthermore, at each μ in the neighborhood of μ^*, the corresponding MEE is E-stable.

(Figure 3 about here)

4.3 AR(1) Evolutionary E-stability

First, consider the AR(1) REE where all agents use the AR(1) predictor. For the AR(1) REE to be E-stable, it must be that

$$(\beta_0, \beta_1) \in S$$

Figure 3 presents the replicator dynamics for the AR(1) REE[16] when there is a deviation to $\mu_0 < \bar{\mu}$. If the population begins at a μ to the left of the intersection point of $\mu_t = \mu_1$

[15] Simulations are not included as they do not provide any additional conclusions.

[16] Recall that $\bar{\mu} = 1 - \beta_0^{-1}$ is the point where the two MEE exchange stability.

of the replicator dynamics, then the replicator dynamics will direct the entire population to using the AR(1) predictor. If μ is to the right of this intersection point, then the replicator dynamics will direct the agents away from using the AR(1) predictor. This result will be explained later with MSV Dominance. The following proposition presents the conditions for stability for the AR(1) REE under the replicator dynamics:

Proposition 4: *Assume that* $(\beta_0, \beta_1) \in S$. *Under fast-slow dynamics, the AR(1) REE is stable under the replicator dynamics for all*

$$0 \leq \mu_0 < \mu_1$$

if

$$0 \leq k < k_1.$$

The proof is given in Appendix C. Note that this stability result is path dependent. It must be that the initial level of heterogeneity must be contained in the above limits stated in the proposition. Evolutionary E-stability conditions for the AR(1) REE (the E-stable REE when $\mu = 0$) come from the previous proposition.

Corollary 1: *If* $(\beta_0, \beta_1) \in S$. *The AR(1) REE is Evolutionary E-stable for*

$$0 \leq k < k_1.$$

The proof is given in Appendix D.

4.4 MSV Evolutionary E-stability

(Figure 4 about Here)

Next, I examine when the MSV REE is evolutionary E-stable. It turns out that it can not be Evolutionary E-stable when the cost of using the AR(1) forecasting model is zero since both of the MSE's are equal to σ^2. The replicator dynamics in this case would be $\mu_{t+1} = \mu_t$. Since the MSV predictor is easier to use, suppose that there is a preference of using the MSV predictor so that $k > 0$. Figure 4 shows the replicator dynamics for MSV evolutionary E-stability which gives us the following proposition:

Proposition 5: *Under Fast-slow dynamics, the MSV REE is Evolutionary E-stable if* $k > 0$. *Furthermore, the solution is always stable under the replicator dynamics for all*

$$\begin{aligned} 1 - \frac{1}{\beta_0} &< \mu_0 \leq 1 \text{ if } \beta_0 > 1 \\ 0 &< \mu_0 \leq 1 \text{ if } \beta_0 < 1 \end{aligned}$$

if $k > 0$.

The proof of this proposition is given in Appendix E. Figure 4 shows the replicator dynamics for the case where $\beta_0 < 1$, i.e. where $(\beta_0, \beta_1) \in A$. For $(\beta_0, \beta_1) \in S$, one can look at figure 3 to the right of $\mu = \bar{\mu} = 1 - \frac{1}{\beta_0}$. When the MSV MEE is E-stable, both updating rules provide the same MSE. Therefore, as long as $k > 0$, the replicator dynamics will direct the population to all use the MSV predictor. As the cost of using the AR(1) predictor, k, increases, the replicator dynamics becomes more bowed out from the line $\mu_t = \mu_{t-1}$. This will create a result of MSV dominance which is discussed below.

4.5 MSV Dominance

There is one more question to answer. From proposition 4, if k is large enough, then the AR(1) REE, is not evolutionary E-stable and there is no evolutionary E-stable AR(1) MEE. What happens in this case? It turns out that the solution converges to the MSV REE as all agents switch to using the MSV predictor. I refer to this phenomenon as minimum state variable dominance.

Definition 4: *Minimum state variable (MSV) dominance is said to occur if a model begins at an AR(1) E-stable MEE and converges to an MSV E-stable REE with homogenous expectations under the replicator dynamics.*

MSV dominance is shown in figure 3. If $\mu_1 < \mu_0 < \bar{\mu}$, the AR(1) MEE is E-stable and the MSV MEE is E-unstable. Due to the cost, k, of using the AR(1) forecasting model, the MSV forecasting model awards more utility even though it has a larger MSE. The replicator dynamics, therefore, directs the population away from using the AR(1) forecasting model. As more agents use the MSV updating rule, the AR(1) MEE solution becomes more like the MSV MEE. In fact, when $\mu = \bar{\mu}$, both solutions are the same where[17]

$$\begin{aligned} a_1 &= a_2 = \frac{\alpha}{1 - \beta_0 - \beta_1} \\ b_2 &= 0. \end{aligned}$$

At $\mu > \bar{\mu}$, the MSV MEE replaces the AR(1) MEE as the E-stable solution. Now the population is in the area of MSV evolutionary E-stability and the replicator dynamics will continue to direct all agents to use the MSV predictor. Therefore, the relevant branch for the replicator dynamics in figure 3 is the one corresponding to the MSV solution. The E-stable MEE was initially the AR(1) solution, but due to the replicator dynamics, all agents asymptotically switched to the MSV updating rule which corresponds to the new E-stable MEE. The following proposition gives the conditions for MSV dominance:

Proposition 6: *Assume* $(\beta_0, \beta_1) \in S$. *If*

$$0 < k \leq k_1$$

[17] The solution is, however, not E-stable. Simulations suggest this does not present a problem as long as b_2 is sufficiently near zero when $\mu = \bar{\mu}$.

and

$$\mu_1 < \mu_0 < 1 - \frac{1}{\beta_0} = \bar{\mu},$$

and the MEE is an E-stable AR(1) solution, then MSV dominance will occur. If

$$k > k_1$$

and

$$0 < \mu_0 < 1 - \frac{1}{\beta_0}$$

and the MEE is an E-stable AR(1) solution, then MSV dominance will always occur.

The proof is given in Appendix F.

4.6 Global Stability, MSV Dominance, and Path Dependence

(Figure 5 about Here)

Above, convergence to a Nash equilibrium was dependent upon the initial population level, μ_0. Figure 5 presents the Nash solution for every corresponding μ_0 and k for $\beta_0 > 1$. The curve in the figure represents μ_1 for the corresponding cost, k. Recall that these Nash solutions did not exist for some (β_0, β_1, k) . This curve will shift to the left as β_0 decreases and disappear when $\beta_0 < 1$, the case where the MSV REE is evolutionary E-stable for all $\mu_0 \in [0, 1]$.

When (μ_0, k) is below this curve, the resulting Nash solution of the system is $\mu = 0$ for $0 \leq k \leq k_1$. Therefore, the corresponding Evolutionary E-stable equilibrium is the AR(1) REE. When (μ_0, k) is above this curve, the resulting Nash solution is $\mu = 1$ with the corresponding MSV parameter equilibrium. The resulting Nash solutions are path dependent when the cost of using the AR(1) predictor is below k_1. However, when $k > k_1$, for all $\mu \in (0, 1]$, the model is MSV dominant. Therefore, path dependence of the Nash solution no longer exists when the cost of the AR(1) predictor is sufficiently high.

With MSV dominance, even the existence of a single agent who believes that the law of motion is MSV, will provide a result of asymptotic homogeneity of the MSV predictor provided the cost parameter is large enough. This result provides a reasonable situation where the MSV solution may be the relevant solution even when it is not initially learnable due to heterogeneous expectations.

5 Conclusion

This paper introduces the use of evolutionary dynamics to further evaluate REE under learning. Furthermore, it investigates the possibility of more equilibria defined under heterogeneous expectations. Evolutionary and adaptive learning are combined so agents not only learn the parameter values of a perceived equilibrium, they also learn which forecasting model will be the "best" to learn the equilibrium.

The paper investigates a well discussed model with the possibility of multiple equilibria and shows that each solution may be stable under the combined evolutionary-adaptive learning dynamics. In these equilibria, one of the two predictors is always superior to the other, so the superior predictor is used by all agents. It turns out that if the cost of using the AR(1) forecasting model is sufficiently large, then the parsimonious forecasting model becomes the unambiguously preferred forecasting model. This results in a global convergence of the minimum state variable (MSV) REE as long as at least one individual initially believes the equilibrium to be of this form. This result suggests that the MSV solution, of the above model, may be the universally relevant solution even when it is not initially learnable.

Appendix A. Calculation of the MSE for both of the PLM's

MSE for the first PLM

PLM1:

$$\begin{aligned} MSE_1 &= E(y-a_1)^2 \\ &= E\left(y-E\left(y\right)\right)^2 \\ &= var\left(y\right) \\ &= \frac{\sigma^2}{1-b^2} \end{aligned}$$

If b=0 then the MSE from the first predictor becomes:

$$MSE_1 = \sigma^2 \tag{11}$$

When we enter the AR(1) MEE values in for the MSE_1 we get the following solution:

$$MSE_1 = \frac{(1-\mu)^2\sigma^2\beta_1^2}{(1-\mu)^2\beta_1^2 - (1-(1-\mu)\beta_0)^2} \tag{12}$$

MSE for the second PLM

PLM2:

$$\begin{aligned} MSE_2 &= E(y-a_2-b_2y_{t-1})^2 \\ &= E(T_{a2}+T_{b2}y_{t-1}+v_t-a_2-b_2y_{t-1})^2 \\ &= \sigma^2 \end{aligned} \tag{13}$$

The mean square error for the second predictor will always be σ^2 as long as y follows a stationary process. This means that the $MSE_1 \geq MSE_2$ for all E-stable stationary values of α, β_0, and β_1. This intuitively makes sense because the AR(1) predictor is always unbiased while the MSV predictor is unbiased only when $b_2 = 0$.

Appendix B. Proof of Proposition 3

For $k \geq 0$ the only evolutionary stable Nash MEE with $\mu \in (0,1)$ is the AR(1) Nash solution $\mu = \mu_2$. For this value of μ, we find that $\beta_0 < \frac{1}{1-\mu}$. Proposition 1 shows that The AR(1) solution is not E-stable at this value. ■

Appendix C. Proof of Proposition 4

When $0 \leq k < k_1$, solutions to the replicator dynamics are:

$$\begin{aligned} \mu_t &= 0 \\ \mu_t &= \mu_1 \\ \mu_t &= \mu_2 \\ \mu_t &= 1 \end{aligned}$$

For μ_1 and μ_2, we must see when these solutions are between zero and one. For $k = 0$, it turns out that

$$\mu_1 = \mu_2 = 1 - \frac{1}{\beta_0},$$

so here both solutions are between zero and one. For $k = k_1$, we see that

$$\begin{aligned} \mu_1 &= 0 \\ 1 - \frac{1}{\beta_0} &< \mu_2 < 1 \end{aligned}$$

Since μ_1 is strictly decreasing for $k \in [0, k]$, it turns out that $\mu_1 \in \left(0, 1 - \frac{1}{\beta_0}\right)$ and *therefore, $\mu_2 > \mu_1$ for all $k \in (0, k_1)$. For any $k \in [0, k_1)$,the slope of the replicator dynamics evaluated at $\mu = 0$* and $\mu = \mu_1$ are:

$$0 < \frac{\partial \mu_t}{\partial \mu_{t-1}}|_{\mu=0} < 1$$

and

$$\frac{\partial \mu_t}{\partial \mu_{t-1}}|_{\mu=\mu_1} > 1.$$

So if $\mu_0 < \mu_1$, then the system will converge to $\mu = 0$ and if $\mu_0 > \mu_1$, the system will diverge away from $\mu = 0$. Therefore the replicator dynamics are stable under the above conditions. ■

Appendix D. Proof of Corollary 1

Proposition 1 shows the E-stability properties for AR(1) Equilibrium. For the AR(1) *solution to be E-stable, it must be that:*

$$\begin{aligned} \frac{1}{1-\mu} &< \beta_0 < 1 - \beta_1 \\ \beta_1 &< 0. \end{aligned}$$

We have assumed that $(\beta_0, \beta_1) \in S$*, so that*

$$\beta_0 + \beta_1 < 1$$

and

$$\beta_0 > 1.$$

Also note that $0 < \mu_1 \leq 1 - \beta_0^{-1}$*, so for any* μ_0 *sufficiently close to* $\mu = 0$*, it must be that* $\beta_0 > \frac{1}{1-\mu_0}$*. Therefore, the AR(1) REE is evolutionary E-stable for*

$$0 \leq k < k_1.$$

■

Appendix E. Proof of Proposition 5

The only solution to the replicator dynamics under the MSV MEE is

$$\mu = 1.$$

It can be shown that

$$\frac{\partial \mu_t}{\partial \mu_{t-1}}|_{\mu=1} < 1,$$

so the replicator dynamics are stable here. Also, as long as

$$\mu_0 > 1 - \frac{1}{\beta_0},$$

the MEE, for all μ_t, is E-stable. Therefore, the MSV REE is Evolutionary E-stable and the replicator dynamics are always stable for all

$$\begin{aligned} 1 - \frac{1}{\beta_0} &< \mu_0 \leq 1 \text{ if } \beta_0 > 1 \\ 0 &< \mu_0 \leq 1 \text{ if } \beta_0 < 1 \end{aligned}$$

if $k > 0$. ■

Appendix F. Proof of Proposition 6

There is only one Nash solution in this case, $\mu = \mu_2$. The derivative of the replicator dynamics is

$$\frac{\partial \mu_t}{\partial \mu_{t-1}}|_{\mu=\mu_2} < 1.$$

In this case the replicator dynamics move μ *toward* $\mu = \mu_2 > 1 - \frac{1}{\beta_0}$. *We also see that the MEE solutions as* $\mu \to 1 - \beta_0^{-1}$ *are:*

$$\begin{aligned} \lim_{\mu \to (1-\beta_0^{-1})^-} \frac{1-(1-\mu)\beta_0}{(1-\mu)\beta_1} &= 0 \\ \lim_{\mu \to (1-\beta_0^{-1})^-} \frac{\alpha}{1-\beta_0-\beta_1}(1-b_2) &= \frac{\alpha}{1-\beta_0-\beta_1} \end{aligned}$$

This means that as $\mu \to 1 - \beta_0^{-1}$, *the MEE goes from the AR(1) solution to the MSV solution. Since we assumed fast-slow dynamics, the replicator dynamics move* μ *slow enough, and* b_2 *and* a_2 *will be such that* $b_2 \in nbhd(b_2 = 0)$ *and* $a_2 \in nbhd(a_2 = \frac{\alpha}{1-\beta_0-\beta_1})$. *When the dynamics move us to* $\mu > 1 - \beta_0^{-1}$, *we are in the area of MSV E-stability. The fast-slow dynamic assumption leads us to know that* $b_2 \in nbhd(b_2 = 0)$ *and* $a_2 \in nbhd(a_2 = \frac{\alpha}{1-\beta_0-\beta_1})$, *so the MSV solution is E-stable. The inequality above implies that* $k > 0$, *so the MSV REE is stable under the replicator dynamics. Therefore, MSV dominance has occurred.* ■

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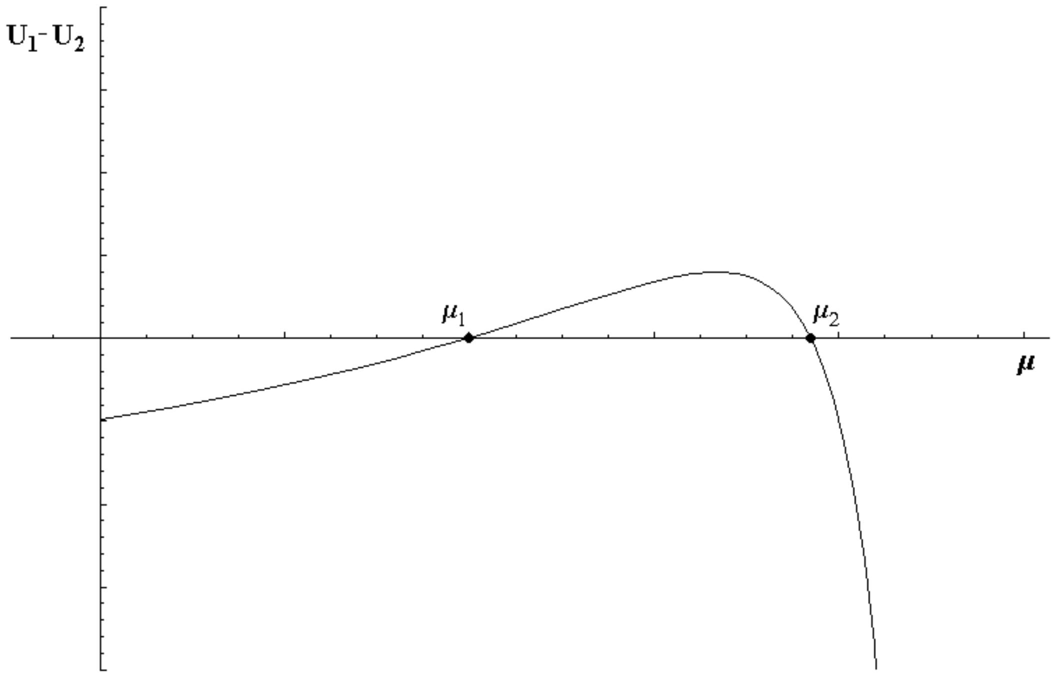

FIGURE 1. Best Response Function for AR(1) Solution when $0 \leq k \leq k_1$.

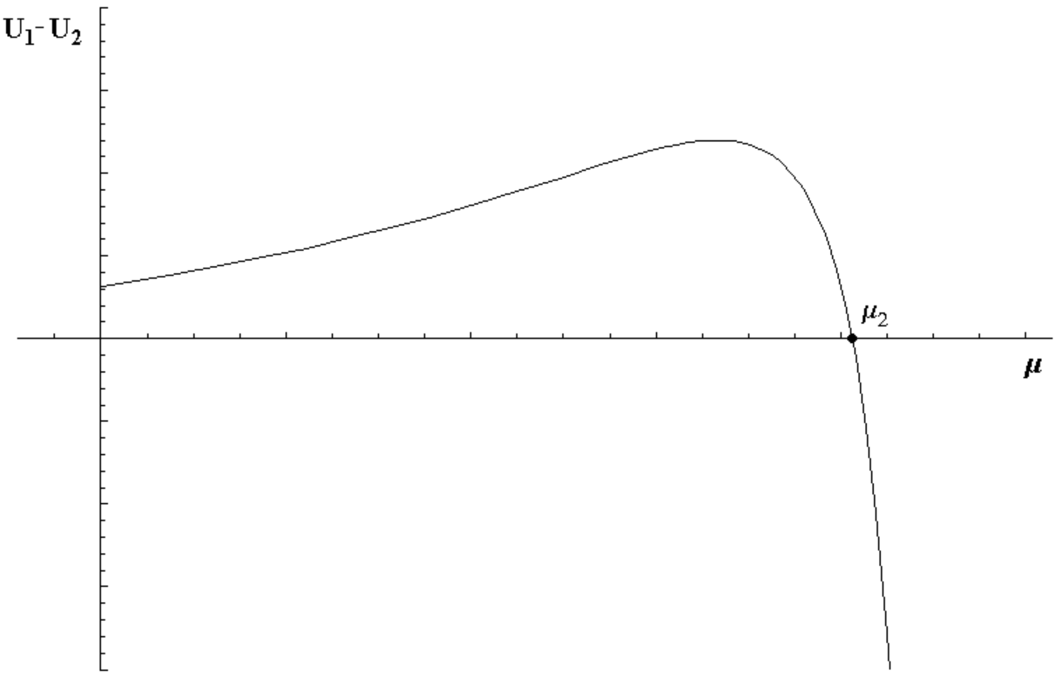

FIGURE 2. Best Response Function for AR(1) Solution when $k > k_1$.

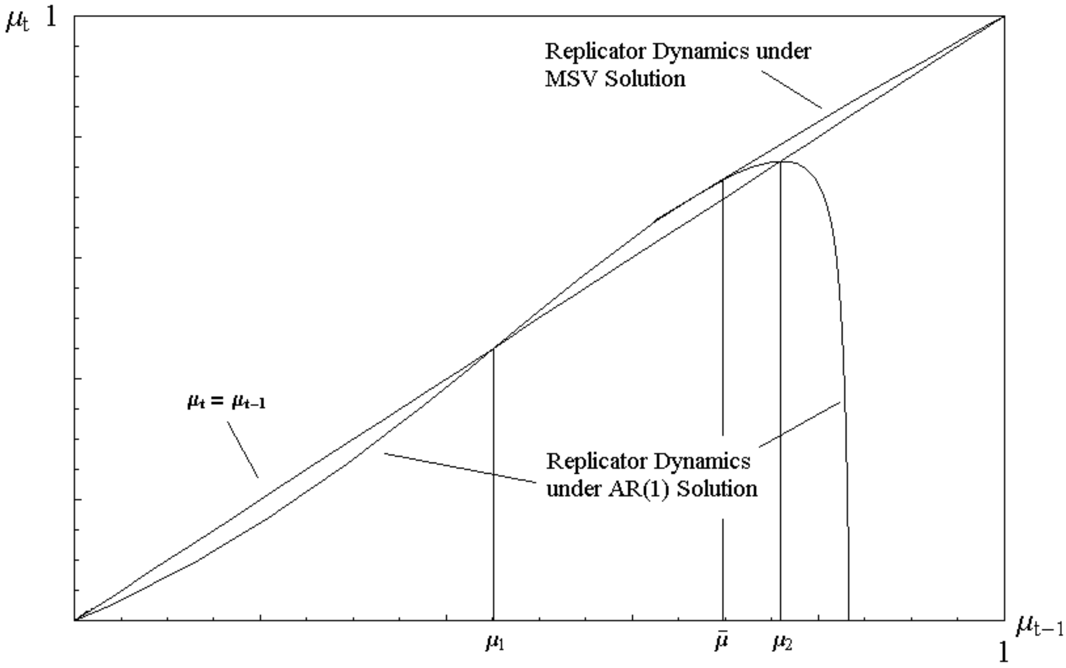

FIGURE 3. The Replicator Dynamics for an AR(1) Evolutionary E-stable REE.

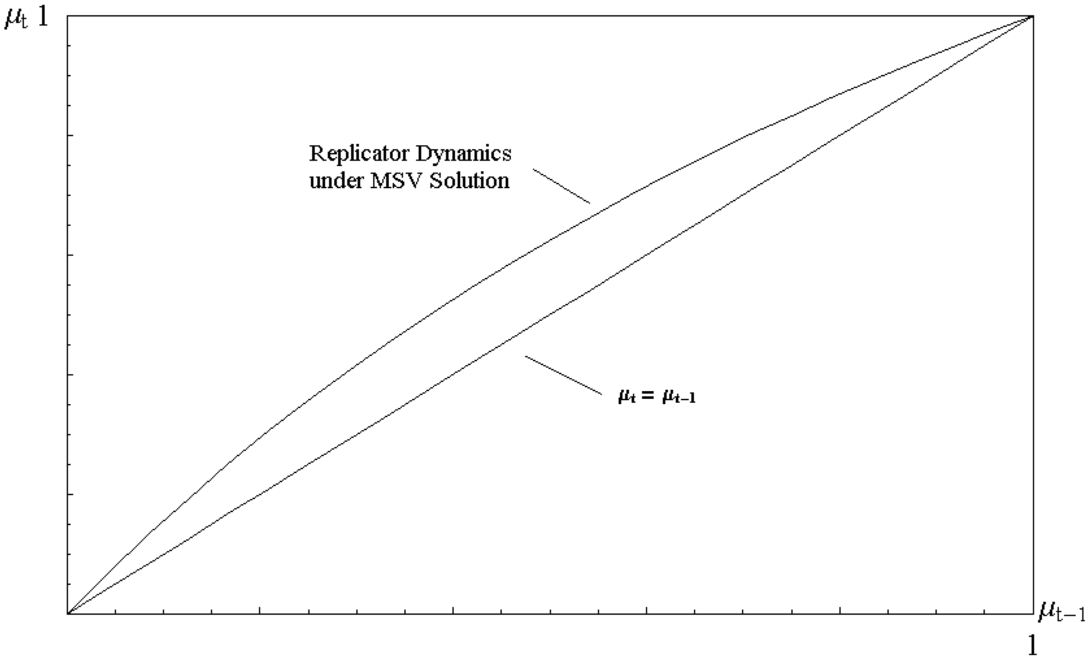

FIGURE 4. The Replicator Dynamics for an MSV Evolutionary E-stable REE.

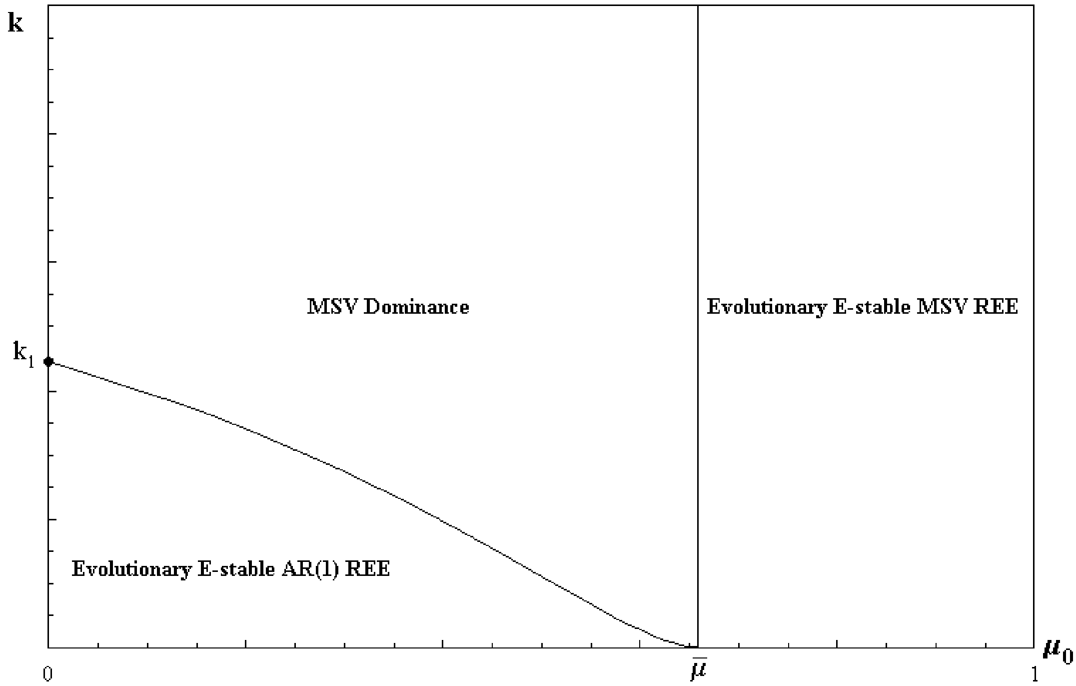

FIGURE 5. Evolutionary E-stability Conditions