SPECTRAL AND STOCHASTIC SOLUTIONS TO BOUNDARY VALUE PROBLEMS ON MAGNETIC GRAPHS

SAWYER JACK ROBERTSON

Abstract. A magnetic graph is a graph G equipped with an orientation structure σ on its edges. The discrete magnetic Laplace operator $\mathcal{L}_G^\sigma$, a second-order difference operator for complex-valued functions on the vertices of G, has been an interesting and useful tool in discrete analysis for over twenty years. Its role in the study of quantum mechanics has been examined closely since its debut in a classic paper by Lieb and Loss in 1993. In this paper, we pose some boundary value problems associated to this operator, and adapt two classic techniques to the setting of magnetic graphs to solve them. The first technique uses the spectral properties of the operator, and the second technique utilizes random walks adjusted to this particular setting. Throughout, we will prove some useful results including a Green's identity, mean value characterization of harmonic functions, and extensions of the solution techniques to Kronecker product graphs.

Contents

1. Introduction and Notation

1.1. **Introduction.** The study of discrete Laplace operators on graphs is rich and active. [12] describes some interesting general results related to this area and some surprising limitations on the felixibility of discrete Laplace operators to model real-world phenomena. Applications are manifold in the world of geometry processing and computer graphics, as well as in the mathematical models of molecular and atomic structures [9]. In the classic papers by Chung [5] [4], one finds an accessible introduction to the subject, detailing some theory and results related to the classical combinatorial Laplacian, the rich spectral theory it produces, and many interesting stochastic and geometric problems. In the spirit of both [4] and [9], we state and solve some boundary value problems associated to the magnetic Laplace operator for graphs which possess a magnetic structure; for an introduction, see [9]. We will first solve a Poisson-type problem adapting spectral theoretic methods due to Chung, then solve a Dirichlet-type problem using random walks, that is, discrete-time Markov chains. The probabilistic interpretation of the Dirichlet

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problem associated to the classical combinatorial Laplace operator on connected graphs is well-known; for detailed descriptions, see [8], [10]. The first section will cover preliminaries for the rest of the paper. In the second section, we cover the Poisson problem as well as Green's functions and their extensions to graph products. The third section will cover the Dirichlet problem in detail, and will also introduce the notion of a magnetic lift and extensions of the results to graph products.

1.2. **Some graph theory.** In this subsection, we will focus on some graph theoretic preliminaries.

Let $G = (V(G), E(G))$ be an undirected graph on $n < \infty$ vertices, without loops or multiple edges, and let $V(G)$ and $E(G)$ denote its vertex and edge sets, respectively. Henceforth, we will call such a graph *simple.* If $v \in V(G)$, we denote by d_v^G as the degree of the vertex in the graph G. If two vertices $u, v \in V(G)$ are adjacent, we write $u \sim v$.

If $u \in V(G)$, we define the *vertex neighborhood of* u to be the set

$$N(u) := \{v \sim u : v \in V(G)\} \cup \{u\} \subset V(G).$$

The $q \times q$ identity matrix will be called Id_q. We define the *adjacency matrix* of G to be the matrix $\mathbf{A}_G$ indexed by the vertex set of G defined by

$$\mathbf{A}_G(u, v) = \begin{cases} 1 & u \sim v \\ 0 & \text{otherwise} \end{cases}. \tag{1}$$

Definition 1.2.1. Suppose J, K are two finite graphs without loops or multiple edges. We define the *Kronecker product graph* $J \times K$ by the vertex set $V(J) \times V(K)$ and the edge set

$$E(J \times K) := \{\{(u, v), (u', v)\} : (u, u') \in E(J)\} \cup \{\{(u, v), (u, v')\} : (v, v') \in E(K)\}.$$

Note here that this formulation of a product graph is by no means unique; this particular definition is merely conventional. It is used in both [4] and [6], which provide frameworks for our spectral theoretic and probabilistic solution techniques, respectively. It originated in [13], and bears the name Kronecker since the adjacency matrix of the product graph is the Kronecker product of the original adjacency matrices. Since this is the only notion of graph product we consider, this qualifier is dropped.

We shall term a proper subset of vertices $H \subsetneq V(G)$ a *subconnected* subset of $V(G)$ or G if it induces a connected subgraph. Let us impose on H a few anatomical structures. First, we define the *vertex boundary* of H to be given by

$$\partial H := \left\{x \in V(G) : \{x, y\} \in E(G),\ y \in H, x \notin H\right\} \tag{2}$$

which for our purposes is a good way to define the boundary of H. The *closure* of H is the set $H \cup \partial H \subset V(G)$. We supply an illustration of these definitions below.

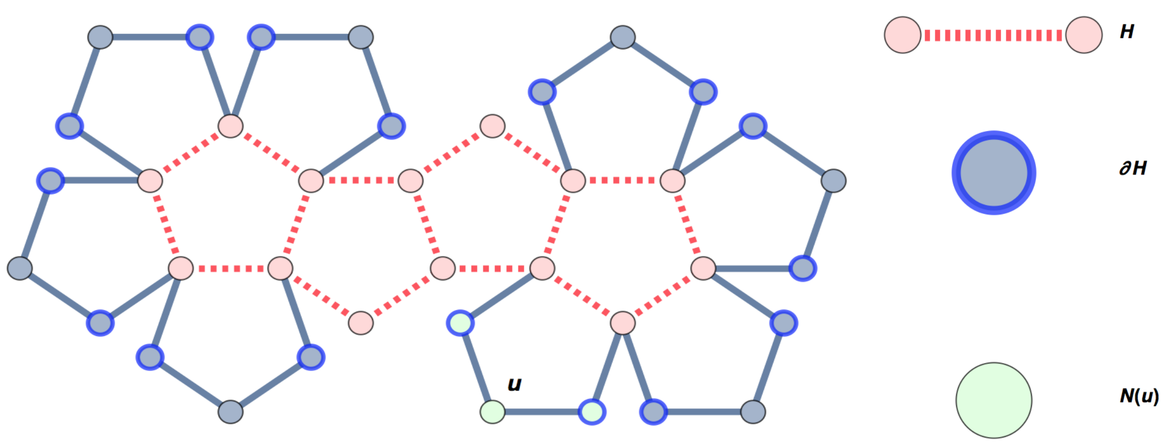

FIGURE 1. A dodecaheral platonic graph with subconnected subset H, its vertex boundary ∂H, and a particular vertex neighborhood $N(u)$.

We will work in function spaces of the form

$$\ell_2(V(G)) := \{f : V(G) \to \mathbb{C}\}$$

with obvious generalizations to any subset of $V(G)$. We equip this space with standard Hermitian inner product $\langle\cdot,\cdot\rangle_G$ given by

$$\langle f,g\rangle_G=\sum_{u\in V(G)}f(u)\overline{g(u)}$$

where the subscript on the bottom right of the bracket indicates the graph over which the inner product is being taken, if not obvious from the context. We observe that this space is naturally isomorphic to the finite-dimensional Hilbert space $\mathbb{C}^n$. In general, we identify functions in ℓ_2 with column vectors in $\mathbb{C}^n$.

Define the *oriented edge set* of G by

$$E^{\mathrm{or}}(G):=\big\{(u,v),\ (v,u):\{u,v\}\in E(G)\big\}.$$

By a *signature* on a graph G, we mean a map

$$\sigma:E^{\mathrm{or}}(G)\to\{z\in\mathbb{C}:|z|=1\}:(u,v)\mapsto\sigma_{uv}$$

satisfying the algebraic condition $\sigma_{vu}=\overline{\sigma_{uv}}=\sigma_{uv}^{-1}$. The *trivial signature* is given by $\sigma\equiv1$, and the *negative signature* is given by $\sigma\equiv-1$. By a *magnetic graph*, we mean a pair (G,σ) consisting of a graph G and a particular signature σ. We define the *signed adjacency matrix* of G to be the matrix $\mathbf{A}_G^\sigma$ indexed by the vertex set of G defined by

$$\mathbf{A}_G^\sigma(u,v)=\begin{cases}\sigma_{uv} & u\sim v\\ 0 & \text{otherwise}\end{cases}. \tag{3}$$

The following definition is a property achieved by some signatures which will be useful later.

Definition 1.2.2. Let (G,σ) be a magnetic graph. We say that σ is *balanced* if for every cycle $C:=\{u_0,u_1,...,u_{n-1}\}\subset V(G)$ where $u_i\sim u_{i+1}$ for $0\leqslant i\leqslant n$ and $u_{n+1}\equiv u_0$, we have

$$\prod_{i=0}^{n}\sigma_{u_iu_{i+1}}=1.$$

In other words, the product of the signature along every cycle comes to 1. Otherwise, we say σ is *unbalanced.*

We now specify a magnetic structure on a product graph.

Definition 1.2.3. Let $(J,\rho^J),(K,\rho^K)$ be two graphs as in (1.2.1), now equipped with signatures. For two adjacent vertices in $V(J\times K)$ of the form $((u,v),(u',v))$, and $((u,v),(u,v'))$, define a new signature ρ on the product graph by

$$\rho((u,v),(u',v))=\rho^J_{uu'}$$
$$\rho((u,v),(u,v'))=\rho^K_{vv'}$$

i.e. 'push' the signature back onto the graph from which the particular oriented edge originated.

The reader is invited to verify that this indeed forms a signature on the product graph.

1.3. **Operators on ℓ_2.** In this section, we will formulate our Laplace operator of interest, the *magnetic Laplacian*, and introduce some other helpful operators which will be important later. Let G,H be as in the previous subsection. To be thorough, let us recall the *combinatorial Laplacian* associated to G (see [4, p. 2] for more detail):

Definition 1.3.1. The *combinatorial Laplacian* associated to G is the $n\times n$ matrix $\mathcal{L}_G$, with rows and columns indexed by $V(G)$, defined by

$$\mathcal{L}_G(u,v)=\begin{cases}d_v^G & u=v\\ -1 & u\sim v\\ 0 & \text{otherwise}\end{cases}.$$

If $f\in\ell_2(V(G))$, we may speak of its *combinatorial Laplacian* as the matrix product $\mathcal{L}_Gf$. We have the formula

$$(\mathcal{L}_Gf)(u)=\sum_{v\sim u}f(u)-f(v). \tag{4}$$

Now let us take (G, σ) to be a magnetic graph. Using the same framework as (1.3.1) and taking into account the signature, we have the following definition.

Definition 1.3.2. The *magnetic Laplacian* of G is the $n \times n$ matrix with rows and columns indexed by the vertex set of G with entries given by

$$\mathcal{L}_G^\sigma(u,v) = \begin{cases} d_v^G & u = v \\ -\sigma_{uv} & u \sim v \\ 0 & \text{otherwise} \end{cases}.$$

Similarly, if $f \in \ell_2(V(G))$ we have the magnetic Laplacian of f given by the product $\mathcal{L}_G^\sigma f$, and we have the equation

$$(5) \qquad (\mathcal{L}_G^\sigma f)(u) = \sum_{v \sim u} f(u) - \sigma_{uv} f(v).$$

As a matter of notation, since the combinatorial Laplacian will not be used beyond this introduction, *we often omit the superscript of σ when using the magnetic Laplacian* when the signature is clear from context. Also, if $H \subsetneq V(G)$ is a proper subset of vertices in G, then the symbol $\mathcal{L}_H$ will be used to refer to the magnetic Laplacian associated to the subgraph in G induced by H.

We may consider $\mathcal{L}_G^\sigma$ as a second order difference operator on $\ell_2(G)$. Immediately we see that as an operator, $\mathcal{L}_\sigma^G$ is formally self-adjoint, since its matrix representation is easily verified from the definition of σ to be Hermitian. If the Laplacian of a function vanishes at a vertex v, then we say the function is *harmonic* at v. If a function is harmonic at every vertex of a (sub)graph, then we say the function is harmonic on the (sub)graph.

Let us define two more operators, this time optimized for analysis on a subset of $V(G)$.

Definition 1.3.3. Suppose $H \subsetneq G$ as above. We define the *magnetic Dirichlet Laplacian* of H to be the principal submatrix L_H of $\mathcal{L}_G$ indexed by the vertex set of H.

L_H is self-adjoint on $\ell_2(H)$, inheriting this from $\mathcal{L}_G$. Finally,

Definition 1.3.4. Let (G, σ), $H \subsetneq V(G)$ be as before. We define *normal derivative* to be the operator $\frac{\partial}{\partial \eta}$ on $\ell_2(\overline{H})$ given by

$$\frac{\partial f}{\partial \eta}(u) = \sum_{\substack{v \sim u \\ v \in V(H)}} f(u) - \sigma_{uv} f(v)$$

for each $f \in \ell_2(\overline{H}), u \in \overline{H}$.

This derivative operator, in some sense, outputs a signed quantity measuring how much the function f flows inward towards a vertex u.

1.4. **MVP and Green's identity for the magnetic Laplacian.** In this section, we take a closer look at the magnetic Laplacian $\mathcal{L}_G$ to develop a useful mean value characterization of harmonic functions and a Green's identity, similar to the combinatorial ones developed in [1].

Theorem 1.1 (Magnetic Mean Value Property)**.** A function $f \in \ell_2(V(G))$ is harmonic on a magnetic graph (G, σ) if and only if at each vertex $u \in V(G)$, the following holds:

$$f(u) = \frac{1}{d_u^G}\left(\sum_{\substack{v \sim u \\ v \in G}} \sigma_{uv} f(v) \right).$$

Proof. We verify the claim directly:

$$\begin{aligned}
f \text{ is harmonic on } G &\iff \\
(\mathcal{L}_G f)(u) = 0 \text{ for each } u \in V(G) &\iff \\
\sum_{\substack{v\sim u\\ v\in G}} \big(f(u) - \sigma_{uv} f(v)\big) = 0 &\iff \\
f(u) = \frac{1}{d_u^G}\Bigg(\sum_{\substack{v\sim u\\ v\in G}} \sigma_{uv} f(v)\Bigg).&
\end{aligned}$$

□

We now state and verify a useful Green's identity adapted to the setting of magnetic graphs.

Theorem 1.2 (Magnetic Green's Identity)**.** Let (G,σ), H be as before and $f, g \in \ell_2(V(\overline{H}))$. Then the following holds:

$$\sum_{u\in H} (\mathcal{L}_{\overline{H}} f)(u)\overline{g(u)} - f(u)\overline{(\mathcal{L}_{\overline{H}} g)(u)} = \sum_{u\in\partial H} f(u)\overline{\frac{\partial g}{\partial\eta}(u)} - \frac{\partial f}{\partial \eta}(u)\overline{g(u)}.$$

Proof. We prove the identity by computation. Notice that

$$\begin{aligned}
&\sum_{u\in H} \mathcal{L}_{\overline{H}} f(u)\overline{g(u)} - \overline{\mathcal{L}_{\overline{H}} g(u)} f(u)\\
&= \sum_{u\in H} \overline{g(u)} \sum_{\substack{v\sim u\\ v\in\overline{H}}} \big(f(u) - \sigma_{uv} f(v)\big) - \sum_{u\in H} f(u) \sum_{\substack{v\sim u\\ v\in\overline{H}}} \big(\overline{g(u) - \sigma_{uv} g(v)}\big)\\
&= \sum_{u\in H} \sum_{\substack{v\sim u\\ v\in\overline{H}}} \big(\overline{g(u)} f(u) - \sigma_{uv}\overline{g(u)} f(v)\big) - \big(f(u)\overline{g(u)} - \sigma_{vu} f(u)\overline{g(v)}\big)
\end{aligned}$$

which yields the following:

$$\begin{aligned}
&\sum_{u\in H} \mathcal{L}_{\overline{H}} f(u)\overline{g(u)} - \overline{\mathcal{L}_{\overline{H}} g(u)} f(u)\\
&\qquad = \sum_{u\in H} \sum_{\substack{v\sim u\\ v\in\overline{H}}} \sigma_{vu} f(u)\overline{g(v)} - \sigma_{uv}\overline{g(u)} f(v).
\end{aligned} \tag{6}$$

One verifies that the summand on the R.H.S. is anti-symmetric. For any pair $u \sim v$ with $u, v \in H$, the two terms in the sum evaluated at these vertices in different order will cancel each other. Hence, all terms of the sum taken over edges strictly inside of H will vanish. The only terms which will not cancel are those evaluated on vertices inside of H with adjacent vertices in the vertex boundary ∂H. Hence, (6) reduces to

$$\sum_{u\in\partial H} \sum_{\substack{v\sim u\\ v\in H}} \sigma_{uv}\overline{g(u)} f(v) - \sigma_{vu} f(u)\overline{g(v)}.$$

This will yield the identity as follows:

$$\begin{aligned}
&\sum_{u\in\partial H} \sum_{\substack{v\sim u\\ v\in H}} \sigma_{uv}\overline{g(u)} f(v) - \sigma_{vu} f(u)\overline{g(v)}\\
&= \sum_{u\in\partial H} \sum_{\substack{v\sim u\\ v\in H}} \big(f(u)\overline{g(u)} - \sigma_{vu} f(u)\overline{g(v)}\big) - \big(f(u)\overline{g(u)} - \sigma_{uv}\overline{g(u)} f(v)\big)\\
&= \sum_{u\in\partial H} f(u)\overline{\frac{\partial g}{\partial\eta}(u)} - \frac{\partial f}{\partial\eta}(u)\overline{g(u)}.
\end{aligned}$$

□

1.5. **Random walks on a graph.** Let (G,σ), H be as before, and let us further assume that G is a connected graph. In this paper, we will use the language of Markov chains to describe various random walk processes on G and H. We will generally follow the conventional notation from probability, see [7] for precise definitions of expectation and probability measure. For Markov chain theory outside the scope of this paper, see [10].

We consider a random walk on (G,σ) to be a Markov chain $\{S_t\}_{t\geqslant 0}$ on the state space $V(G)$, with initial state determined by initial distribution μ_0, and transitioning between adjacent vertices with uniform probability, generating a sequence of distributions $\{\mu_t\}_{t\geqslant 0}$. We think of distributions in this context as nonnegative real-valued functions on $V(G)$ whose values sum to 1, which at time t describe the probability of S_t being at any particular vertex. In the case where the initial distribution is δ_u, the unit impulse function at vertex u, we say the random walk starts at u. In order to have proper and effective formulations, we shall deal identify distributions with row vectors indexed by $V(G)$ following the typical conventions in this area. The process of transitioning from one step in the random walk to the next is described by the transition matrix associated to G, denoted $\mathbf{P}_G$.

Definition 1.5.1. The *transition matrix* associated to G is the $n \times n$ matrix $\mathbf{P}_G$ with rows and columns indexed by $V(G)$ defined by

$$\mathbf{P}_G(u,v) = \mathbb{P}[S_{n+1} = v \big| S_n = u] = \begin{cases} \frac{1}{d_u^G} & u \sim v \\ 0 & \text{otherwise} \end{cases}$$

where $\mathbb{P}[\cdot|\cdot]$ denotes conditional probability as usual.

The possible positions of a random walker at time step t are described by the t-th distribution μ_t. Recalling the classical result which asserts that any discrete Markov chain may be determined uniquely by its initial distribution and transition matrix, we have the following expression for μ_t:

$$\mu_t = \mu_{t-1}\mathbf{P}_G = \mu_0 \mathbf{P}_G^t \qquad t \geqslant 1 \tag{7}$$

2. Magnetic Poisson Problem via Spectral Theory

In this section, we will state and solve a Poisson-type boundary value problem on a magnetic graph. We will follow classical techniques due to Chung which have been adapted to this magnetic setting. These may be explored in the setting of the combinatorial Laplace operator in [4]. We will round out this section by exploring Green's functions briefly, and giving a formula for the Green's function for the product of two magnetic graphs, once again utilizing tools from [4].

2.1. Magnetic Poisson problem.

Let (G, σ) be a finite connected graph on n vertices without loops or multiple edges. Let $H \subsetneq V(G)$ be a proper subset of k vertices, which is assumed to induce a connected subgraph.

Problem 2.1.1 (Poisson problem)**.** Let $f \in \ell_2(H)$ and $g \in \ell_2(\partial H)$ be given functions. We wish to find a function $\Psi \in \ell_2(\overline{H})$ for which

$$\begin{cases} (\mathcal{L}_G \Psi)(u) = f(u) & u \in H \\ \Psi(u) = g(u) & u \in \partial H \end{cases} \tag{8}$$

Unsurprisingly, we will solve (8) by looking at two associated problems individually.

Problem 2.1.2 (Problem 1)**.** We consider a Dirichlet-type problem: let $g \in \ell_2(\partial H)$ be given as in (8). We wish to find a function $\psi \in \ell_2(\overline{H})$ for which

$$\begin{cases} (\mathcal{L}_G \psi)(u) = 0 & u \in H \\ \psi(u) = g(u) & u \in \partial H \end{cases} \tag{9}$$

Problem 2.1.3 (Problem 2)**.** We consider an nonhomogeneous-type problem: let $f \in \ell_2(H)$ be given as in (8). We wish to find a function $\phi \in \ell_2(\overline{H})$ for which

$$\begin{cases} (\mathcal{L}_G \phi)(u) = f(u) & u \in H \\ \phi(u) = 0 & u \in \partial H \end{cases} \tag{10}$$

If we may find unique solutions ψ, ϕ to (9), (10), respectively, then we may obtain a unique solution Ψ by setting $\Psi = \psi + \phi$.

Theorem 2.1 (Solution 1)**.** Let $\{e_i\}_{1\leqslant i\leqslant k}$ be an orthonormal basis of $\ell_2(H)$ of eigenvectors for L_H, associated to real eigenvalues $\{\lambda_i\}_{1\leqslant i\leqslant k}$, counted with multiplicity. Extend this system on H to a family $\{\widetilde{e_i}\}$ on $\overline{H}$ by setting

$\widetilde{e_i} \equiv 0$ on ∂H. Then the unique solution to (9) may be given by

$$\psi(w) = \begin{cases} -\sum_{i=1}^m \frac{e_i(w)}{\lambda_i}\Big[\sum_{u\in\partial H} \overline{\frac{\partial \widetilde{e_i}}{\partial\eta}(u)}g(u)\Big] & w \in H \\ g(w) & w\in\partial H \end{cases}. \tag{11}$$

Proof. Here, we adapt a technique due to Chung, seen in [4]. We know the values of ψ in ∂H, so we solve for the values of ψ in H. To this extent, for $w \in H$ we have a unique expression of ψ as a linear combination of e_i:

$$\psi(w) = \sum_{i=1}^m c_i e_i(w), \qquad w \in H.$$

Let us solve for c_i. First,

$$\begin{aligned} c_i &= \langle \psi, e_i\rangle_H \\ \lambda_i c_i &= \lambda_i \langle \psi, e_i\rangle_H \\ &= \langle \psi, L_H e_i\rangle_H \end{aligned} \tag{12}$$

Let an extension $g_0 : \overline{H} \to \mathbb{C}$ of g be given by

$$g_0(u) = \begin{cases} 0 & u \in H \\ g(u) & u\in\partial H \end{cases}.$$

Then (12) becomes

$$\begin{aligned} \lambda_i c_i &= \langle \psi - g_0, L_H e_i\rangle_H \\ &= \langle L_H(\psi - g_0), e_i\rangle_H \\ &= \langle \mathcal{L}_{\overline{H}}(\psi - g_0), \widetilde{e_i}\rangle_{\overline{H}} \\ &= \langle -\mathcal{L}_{\overline{H}} g_0, \widetilde{e_i}\rangle_{\overline{H}} \\ &= \sum_{u\in\overline{H}} -\mathcal{L}_{\overline{H}} g_0(u)\overline{\widetilde{e_i}(u)} \\ &= \sum_{u\in H} g_0(u)\overline{\mathcal{L}_{\overline{H}}\widetilde{e_i}(u)} - \mathcal{L}_{\overline{H}} g_0(u)\overline{\widetilde{e_i}(u)} \end{aligned}$$

where $\mathcal{L}_{\overline{H}}$ is the magnetic Laplacian associated to the connected subgraph in G induced by $\overline{H}$. Moreover, the last equality follows by noting that $\widetilde{e_i} \equiv 0$ on ∂H, and $g_0 \equiv 0$ on H. By complicating the sum a bit, we have clear access to the Green's identity from Theorem (1.2), though the reader should note that a complex conjugation was applied. The computation is almost complete:

$$\begin{aligned} \lambda_i c_i &= \sum_{u\in\partial H} \overline{\widetilde{e_i}(u)}\frac{\partial g_0}{\partial\eta}(u) - \overline{\frac{\partial\widetilde{e_i}}{\partial\eta}(u)}g_0(u) \\ &= -\sum_{u\in\partial H}\overline{\frac{\partial\widetilde{e_i}}{\partial\eta}(u)}g(u) \end{aligned}$$

whence

$$\psi(w) = -\sum_{i=1}^m \frac{e_i(w)}{\lambda_i}\Big[\sum_{u\in\partial H}\overline{\frac{\partial\widetilde{e_i}}{\partial\eta}(u)}g(u)\Big], \qquad w\in H.$$

This completes the proof. ∎

Solving (10), it turns out, is merely some linear algebra relying on a key observation. In particular, we present the following lemma.

Lemma 2.1.1. Let (G, σ) and H be as in the beginning of this subsection. Then L_H is an invertible matrix.

Proof. Assume per contradiction that there exists a nonzero solution $h \in \ell_2(H)$ to the homogeneous linear system

$$L_H h \equiv 0.$$

Put $|h(u^*)| = \max_{u\in H}|h(u)| > 0$. Then from the definition of L_H, we have

$$\begin{aligned} |h(u^*)| &= \frac{1}{d^G_{u^*}}\left|\sum_{\substack{v\sim u^*\\ v\in H}} \sigma_{uv} h(v)\right| \\ &\leqslant \frac{1}{d^G_{u^*}} \sum_{\substack{v\sim u^*\\ v\in H}} |h(v)| \\ &\leqslant \frac{1}{d^H_{u^*}} \sum_{\substack{v\sim u^*\\ v\in H}} |h(u^*)| \\ &= |h(u^*)| \end{aligned}$$

where $d^H_{u^*}$ is the degree of the vertex u in the subgraph induced by H. From this we have forced two conclusions: (i) that the degree of u^* in G is the same as in the subgraph induced by H, and (ii) that $|h(v)| = |h(u^*)|$ for each v adjacent to u^* in H. Hence we may do the same computations on any vertex incident with u^* in H; in turn, the conclusions we made apply to all of these vertices as well, and their neighbors in H, and so on. Since H is connected as a subgraph of G, after finitely many iterations of this process we have the conclusion that the degree of each vertex $u \in H$ taken in H agrees with the degree of said vertex taken in G. This cannot be the case since we assumed H induces a proper subgraph of G, and that G is connected. The claim follows. □

We may now present a solution to the second problem.

Theorem 2.2 (Solution 2)**.** The unique solution ϕ to (10) may be written

$$\phi(w) = \begin{cases} (L_H^{-1} f)(w) & w \in H \\ 0 & w \in \partial H \end{cases} . \tag{13}$$

Proof. Notice that since ϕ is assumed to take the value 0 on ∂H, one verifies

$$\mathcal{L}_{\overline{H}}\phi \equiv L_H \phi$$

whence (10) is equivalently stated

$$\begin{cases} (L_H\phi)(u) = f(u) & u \in H \\ \phi(u) = 0 & u \in \partial H \end{cases}$$

Since L_H is invertible by (2.1.1), the claim follows immediately. □

In the next section, we will take a closer look at the matrix L_H^{-1}– notice that it can be loosely interpreted as a discrete Green's function. To wrap things up, we will explicitly write down the spectral solution to the Poisson problem as stated at the beginning of this section.

Theorem 2.3. The unique solution to (8) may be written $\Psi = \psi + \phi$, explicity given by

$$\Psi(w) = \begin{cases} (L_H^{-1} f)(w) - \sum_{i=1}^m \frac{e_i(w)}{\lambda_i}\left[\sum_{u\in\partial H} \overline{\frac{\partial \tilde{e_i}}{\partial\eta}(u)} g(u)\right] & w \in H \\ g(w) & w \in \partial H \end{cases}$$

where $e_i, \tilde{e_i}, \lambda_i$ are defined in (11).

2.2. **Greens' functions for magnetic graphs and products.** In this subsection, we explore an interesting interpretation of the matrix L_H^{-1} which showed up in (13), once again adapting techniques from [4] to this magnetic setting. In [4], Chung interprets this matrix as a discrete Green's function. Indeed, the matrix is a key part in constructing the solutions to both (10) and (8). We will now explicity construct this matrix from the eigensystem of the magnetic Laplacian, and show how we may be construct a Green's function for the product of two magnetic graphs, when their respective spectral systems are already identified.

Theorem 2.4. Let G be a finite connected graph without loops or multiple edges, and let $H \subsetneq V(G)$ be a proper subset of k vertices which is assumed to induce a connected subgraph. Let $\{e_i\}_{i=1}^m$, and $\{\lambda_i\}_{i=1}^m$ be the orthonormal system associated to L_H as in (11). Let L_H^{-1} be interpreted as a function on $H \times H$. We have the following

$$L_H^{-1}(i,j) = \sum_{k=1}^{m} \frac{1}{\lambda_k} e_k(i)\overline{e_k(j)}$$

where $(i,j) \in H \times H$ (we use slightly different notation for vertices in H here to be consistent with the $\{e_i\}$ notation).

Proof. Since L_H is Hermitian, it admits an elementary decomposition

$$L_H = \mathbf{U}D\mathbf{U}^*$$

where $D = \operatorname{diag}(\lambda_1, \lambda_2, ..., \lambda_m)$ is the diagonal matrix of eigenvalues and $\mathbf{U} = [e_1 \ e_2 \ ... \ e_m]$ is the matrix of eigenvectors, and $\mathbf{U}^*$ is its Hermitian conjugate. After carrying out a standard pointwise computation for the entries in the inverse matrix $L_H^{-1} = \mathbf{U}D^{-1}\mathbf{U}^*$, the claim follows. □

The preceding theorem is particularly useful in that an explicit construction of the Green's function for H depends only on the spectral system associated to H. Though this can be a difficult system to work with in practice, obtaining it using computation software for a particular graph is a tractable task. The identification of this system for general families of graphs remains an interesting, and quite open, research question.

The extension of the previous theorem to a product graph is not very difficult. Indeed, it requires only that we identify the spectral system associated to the product of two proper subsets. However, a technicality stands in our way of this identification. Recall that whenever G, H are not necessarily connected, we cannot always conclude that L_H is invertible. This becomes a problem because even if two graphs are connected, we have no reason to believe that their product is itself connected. As such, we must recall a result stated in [13] which clarifies the conditions required for a product to be connected.

Theorem 2.5 (Weichsel)**.** Let G_1, G_2 be connected graphs. Then the following are equivalent:

(a) The Kronecker product $G_1 \times G_2$ is connected.
(b) At most one of G_1 or G_2 is bipartite.
(c) There is at least one odd cycle in either G_1 or G_2.

Theorem 2.6. Let $(J, \rho^J), (K, \rho^K)$ be two signed, connected graphs and let $M \subsetneq V(J)$, $N \subsetneq V(K)$ be two proper subsets of m, n vertices, respectively, which induce connected subgraphs in their respective parent graphs. Assume both that $J \times K$ is connected, and that $M \times N \subset V(J \times K)$ induces a connected subgraph (i.e. each satisfies some condition in Theorem (2.5)). Finally, let $\{x_i\}_{i=1}^m$ and $\{y_j\}_{j=1}^n$ be orthonormal bases for $\ell_2(M), \ell_2(N)$ respectively, associated to eigenvalues $\{\mu_i\}, \{\nu_j\}$ counted with multiplicity. Then, the eigenvectors for the operator $L_{M\times N}$ on $\ell_2(M \times N)$ may be identified as $\{x_i y_j\}_{i,j}$, where $x_i y_j$ is defined pointwise. The eigenvalues are $\{\mu_i + \nu_j\}_{i,j}$.

Proof. Let us fix some $(p,q) \in M \times N$ and $1 \leqslant i, j \leqslant m, n$ (resp.) and compute

$$\begin{aligned}
\big(L_{M\times N} x_i y_j\big)(p,q) &= \sum_{\substack{(p',q')\sim(p,q)\\(p',q')\in M\times N}} x_i(p)y_j(q) - \rho((p,q),(p',q'))x_i(p')y_j(q') \\
&= \sum_{\substack{p'\sim p\\p'\in M}} x_i(p)y_j(q) - \rho^J_{pp'}x_i(p')y_j(q) \\
&\qquad\qquad + \sum_{\substack{q'\sim q\\q'\in N}} x_i(p)y_j(q) - \rho^K_{qq'}x_i(p)y_j(q') \\
&= y_j(q)\big(L_M x_i\big)(p) + x_i(p)\big(L_N y_j\big)(q) \\
&= (\mu_i + \nu_j)(x_i(p)y_j(q)).
\end{aligned}$$

This shows $\{x_i y_j\}_{i,j}$ are all eigenvectors with eigenvalues $\{\mu_i + \nu_j\}_{i,j}$. Moreover, since $\dim(\ell_2(M \times N)) = \dim \ell_2(M) \cdot \dim \ell_2(N)$, these account for all of the eigenvectors and, up to multiplicity, the eigenvalues associated to $L_{M\times N}$. □

We will now draw this section to a close by giving a formula for the Green's function on a product.

Theorem 2.7. Let $(J, \rho^J), (K, \rho^K)$, M, N, and $\{x_i y_j\}_{i,j=1}^{m,n}$, $\{\mu_i + \nu_j\}_{i,j}$ be as in (2.6). Recalling Theorem (2.4), we have the following expression for the Green's function $L^{-1}_{M\times N}$:

$$L^{-1}_{M\times N}\big((p,q),(p',q')\big) = \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{x_i(p)y_j(q)\overline{x_i(p')y_j(q')}}{\mu_i + \nu_j}.$$

3. Magnetic Dirichlet Problem via Random Walks

In this chapter, will will once again formulate a boundary value problem and give its solution. This time, our approach is that of random walks. The first subsection will cover probabilistic preliminaries, the second will introduce the notion of a magnetic 'lift,' the third will concern the formal statement of the problem and its solution, and the fourth will consider extensions of these results to the product case.

3.1. **Probabilistic preliminaries.** In this subsection, we will cover the probabilistic interpretation of the combinatorial Dirichlet problem and resolve questions of convergence and 'exiting.'

On a graph G, the random walk process is well studied. This might be because of the numerous recreational interpretations of the Dirichlet problem associated to the combinatorial Laplacian of G. If we have a subconnected subset $H \subsetneq V(G)$ of vertices which induces a connected subgraph, and we consider a random walk starting at some vertex $u \in H$, the expected value of the boundary condition at the vertex on which the random walker steps upon 'exiting' H becomes, in fact, the solution to the combinatorial Dirichlet problem. This idea is formalized in the classic 'Gambler's Ruin' problem [8, p. 14].

Let (G, σ) be a connected simple magnetic graph on n vertices. We consider a random walk $\{S_t\}_{t\geqslant 0}$. The first order of business is to give a meaning to the convergence of S_n. Notice from the formulation in 1.5 that we may characterize the set of all possible distributions on $V(G)$ as the probability simplex in $\mathbb{R}^n$, henceforth defined

$$\mathcal{S}_G := \big\{\nu : V(G) \to \mathbb{R} : \nu \geqslant 0, \quad \sum_{u\in V(G)} \nu(u) = 1\big\}$$

and similarly for any subconnected subset $H \subsetneq V(G)$. Note that $\mathcal{S}_G$ can be identified as a compact, convex subset of $\mathbb{R}^n$. We have many options for a norm or metric on this set. We will generally work with the commonly used L^1 distance, noting that this norm is trivial on $\mathcal{S}_G$. For each $\mu, \nu \in \mathcal{S}_G$ we set

$$||\mu - \nu||_1 = \sum_{u\in V(G)} |\mu(u) - \nu(u)|$$

In turn, a sequence of distributions $\{\nu_t\}_{t\geqslant 0} \subset \mathcal{S}_G$ converges to $\nu \in \mathcal{S}_G$ provided

$$\lim_{t\to\infty} ||\nu_t - \nu||_1 = 0. \tag{14}$$

More concretely put, we say a random walk $\{S_t\}_{t\geqslant 0}$ with initial distribution μ_0 will converge to a limit distribution μ provided

$$\lim_{t\to\infty} ||\mu_0 \mathbf{P}_G^t - \mu||_1 = 0.$$

By a *stationary distribution* $\eta \in \mathcal{S}_G$ on G we mean a left eigenvector for $\mathbf{P}_G$ with eigenvalue 1. That is, η is invariant under $\mathbf{P}_G$.

We wish to specify the conditions on G which are sufficient to guarantee that such a distribution exists. We recall a result set down by Chung in [5, p. 14].

Theorem 3.1. Let G be a connected simple graph. Then there is a stationary distribution $\eta \in \mathcal{S}_G$ for G if and only if the following two conditions hold:

(a) *irreducibility*, that is, for each $u, v \in V(G)$ there is $t \geqslant 0$ for which $\mathbf{P}_G^t(u,v) > 0$ $\iff$ G is connected.
(b) *aperiodicity*, that is, for each $u, v \in V(G)$, g.c.d.$\{t : \mathbf{P}_G^t(u,v) > 0\} = 1 \iff G$ is not bipartite.

Moreover, we may write down the stationary distribution as

$$\eta(v) = \frac{d_v}{2|E(G)|}$$

where $|E(G)|$ is the number of edges in G [5, p.15]. This can be checked, for instance using the Perron-Frobenius theorem applied to P^t for t large, to be the unique stationary distribution in $\mathcal{S}_G$. In fact, the conclusion is even stronger, in the sense that if G admits a stationary distribution, then every random walk process on G converges in the $||\cdot||_1$ metric to the stationary distribution (or any metric induced by a norm) [10, 1.22, p.34]. An important fact concerning the extension of these results to a product graph is the following theorem, borrowed from [6].

Theorem 3.2. Suppose G_1, G_2 admit stationary distributions η_1, η_2 as in Theorem (3.1). Moreover, assume the product graph $G_1 \times G_2$ is connected and not bipartite. Then the stationary distribution for the random walk process on $G_1 \times G_2$ may be given by $\eta_1\eta_2$, with multiplication defined pointwise.

We hedge a little bit on the detail of $G_1 \times G_2$ being bipartite since the graph theory literature appears to be a little unclear here. This is needed to guarantee the existence of a stationary distribution on the product, even if the stationary distributions on the coordinate graphs exist. Unlike Theorem (2.5), there does not appear to be a full characterization of a bipartite product in terms of the coordinate graphs. Some insight is given in [2] which asserts that a product *will* be bipartite if one of the coordinate graphs is bipartite.

The next order of business is to handle the issue of 'exiting' points. By this, we mean clarifying the notion of a random walker 'leaving' a particular region and the time at which the walker does so. Indeed, borrowing the language of [10], the random walk on G is irreducible and recurrent and as a consequence we may state a reformulation of the classic Markov Chain result from [10, 1.16, p.22] as follows.

Theorem 3.3. Suppose we have a random walk $\{S_t\}_{t \geqslant 0}$ on a connected simple graph G which starts in a subset of vertices $H \subset V(G)$, that is, $\mu_0 = \delta_u$ for some $u \in H$. Then there exists $T \geqslant 0$ for which

$$\mathbb{P}[S_T \in \partial H : S_0 \in H] = 1.$$

In other words, S_t will eventually leave the set H.

Supposing we have some random walk S_t originating at some vertex in H, in light of Theorem (3.3) we may define a modified random walk process by setting $T = \inf\{t : \mathbb{P}[S_t \notin H] = 1\}$ and $\widetilde{S}_t = S_{\min(t,T)}$, i.e. forcing the random walk to 'stop' at S_T. It is this modified random walk which we shall use to construct the solution to the magnetic Dirichlet problem.

3.2. **Magnetic lifts.** In this subsection, we will define and analyze a graph theoretic construction known as a magnetic lift graph. This construction seems to have originated as a discrete interpretation of a topological covering space, which was then reformulated and adapted to the setting of a magnetic graph. Some interesting exposition can be found in Biggs, [3, ch. 19]. At the present time, this construction is limited to the case where a magnetic graph is paired with a signature taking values in a subgroup of $S^1 \subset \mathbb{C}$. To this end, set

$$\mathbf{S}^1_p := \{z \in \mathbb{C} : z^p = 1\}.$$

Definition 3.2.1. Let (G, σ) be a magnetic graph. Further assume that σ takes values strictly in $\mathbf{S}^1_p$ for some integer $p \geqslant 2$. Write $\mathbf{S}^1_p = \{\omega_i\}_{i=0}^{p-1}$ We define the *lift* of G to be the non-magnetic graph $\widehat{G}$ consisting of vertex set $G \times \mathbf{S}^1_p$ and edges defined by

$$(u, \omega_i) \sim (v, \omega_j) \text{ in } \widehat{G} \iff u \sim v \text{ in } G \text{ and } \omega_j = \omega_i \sigma_{uv}.$$

The subsets $G \times \{\omega_i\} \subsetneq V(\widehat{G})$ for each fixed $\omega_i \in \mathbf{S}^1_p$ are called the *levels* of the magnetic graph.

As the reader will see in the next subsection, magnetic lifts are a useful setting in which to start a random walk. However, before attempting to do so, we must resolve some of the same issues that we encountered when extending Green's functions to products. Namely, knowing only that G is connected and non-bipartite, we do not necessarily know that its lift $\widehat{G}$ satisfies the same properties, and in turn we may not be able to apply the result in Theorem (3.1) to a random walk process on $\widehat{G}$ to obtain a stationary distribution. In fact, a complete answer in the spirit of Theorem (2.5) to characterize the conditions on G under which $\widehat{G}$ is bipartite has proven to be nontrivial and at the present time remains unclear. We present a strong implication of the connectedness of $\widehat{G}$, give a partial converse, and give some partial answers to the question of the bipartiteness of $\widehat{G}$. The reader may wish to revisit the definition of a balanced signature from the introduction which we have yet to use (1.2.2).

Theorem 3.4. Let (G, σ) be a magnetic graph, and assume σ takes values in $\mathbf{S}^1_p$ for some $p \geqslant 2$. Then if the magnetic lift $\widehat{G}$ is connected, then σ is unbalanced. Moreover, if $p = 2$, then the converse holds.

Proof. Assume $\widehat{G}$ is connected and $p \geqslant 2$. Fix any $u \in V(G)$ and look at the path starting at $(u,1) \in V(\widehat{G})$ and terminating at (u,ω) for some $\omega \in \mathbf{S}_p^1 \backslash \{1\}$. By projecting this path onto the original graph G, that is, viewing the first coordinates of the path in $\widehat{G}$ as a path in G, we find that it in fact is a cycle. Moreover, since this path began on one level in $\widehat{G}$ and ended on another level in the lift graph, it follows from the definition of the edge set of a magnetic lift that the product of the signatures along the associated cycle in G cannot be equal to 1; in other words, σ must be unbalanced. For the partial converse, let us now assume that σ is unbalanced and $p = 2$. Let $(u,s_1),(v,s_2) \in V(\widehat{G})$ be fixed. Since G is connected, we may find a fixed path $R := \{u,...,v\} \subset V(G)$ connecting u and v. Also, since σ is unbalanced, there exists some cycle $C' := \{u_0,u_1,...,u_n\}$ in G for which

$$\prod_{i=0}^{n} \sigma_{u_i,u_{i+1}} = -1$$

where $u_{n+1} \equiv u_0$. Now by viewing the path R in the lift graph as a path beginning at $(u,s_1) \in V(\widehat{G})$, one may construct a new path terminating at (v,s) for some $s = \pm 1$. If $s = s_2$ then we have found the desired path connecting the two vertices in the lift. If $s \neq s_2$, then proceed by looking at the cycle in G obtained by concatenating a fixed path connecting v and u_0 with the cycle C', and then the path connecting u_0 to v, the reverse of the fixed one connecting v to u_0. Because the cycle contains the one identified as having a signature product of -1, when viewed as a path in the lift, it originates at (v,s) and since $s \neq s_2$ and $p = 2$, it must terminate at (v,s_2). This completes the proof. $\square$

We shall give some interesting results concerning the question of what happens when $\widehat{G}$ is bipartite. Unfortunately these results do not (yet) provide a satisfying answer. In furtherance of this research question, the following theorem gives a formula for the adjacency matrix of the magnetic lift, an important tool in the study of bipartite graphs. We need a matrix first, however.

Definition 3.2.2. Let (G,σ^G) be a simple magnetic graph on n vertices, and assume σ takes values in some $\mathbf{S}_p^1$, $p \geqslant 2$. Define the *signed adjacency matrix* associated to the magnetic lift to be the matrix indexed by $V(\widehat{G})$ given by

$$\widehat{\mathbf{A}^\sigma}((u,s_1),(v,s_2)) = \begin{cases} \sigma_{uv}^G & (u,s_1) \sim (v,s_2) \\ 0 & \text{otherwise} \end{cases}. \tag{15}$$

This definition is a little tricky since $\widehat{G}$ doesn't 'technically' have a magnetic structure, but it is rather trivial since there would be no $\widehat{G}$ without one. It is useful because including the signature in the entries encodes $\widehat{\mathbf{A}^\sigma}$ with a little additional information.

Theorem 3.5. Let (G,σ) be a simple magnetic graph on n vertices, and assume σ takes values in some $\mathbf{S}_p^1 := \{\omega_j\}_{j=0}^{p-1}$, $p \geqslant 2$, where as usual $\omega_j = e^{\frac{2\pi i j}{p}}$. Let $\widehat{\mathbf{A}^\sigma}$ represent the signed adjacency matrix for the magnetic lift $\widehat{G}$, and let $\mathbf{A}_G^\sigma$ be the signed adjacency matrix for G. Decompose $\mathbf{A}_G^\sigma$ into the sum of matrices A_{ω_j} which take the same value as $\mathbf{A}_G^\sigma$ whenever $\sigma_{uv} = \omega_j$. That is,

$$\mathbf{A}_G^\sigma = \sum_{j=0}^{p-1} A_{\omega_j}. \tag{16}$$

Let $\{P_j\}_{j=0}^{p-1}$ represent the family of permutation matrices of size $p \times p$. Then the signed adjacency matrix of the lift $\widehat{\mathbf{A}^\sigma}$ satisfies

$$\widehat{\mathbf{A}^\sigma} = \sum_{j=0}^{p-1} P_j \otimes A_{\omega_j}.$$

where $\otimes$ represents Kronecker matrix product [13].

Proof. One verifies that in a manner similar to (16), we have

$$\widehat{\mathbf{A}^\sigma} = \sum_{j=0}^{p-1} \widehat{A_{\omega_j}}$$

where $\widehat{A_{\omega_j}}$ corresponds to the restriction of (15) the to the edges which have signature ω_j. Indeed, if we chose to enumerate the vertices of $\widehat{G}$ by fixing an enumeration of $V(G)$ and then ordering the 'copies' of $V(G)$ in $V(\widehat{G})$ by the indices of ω_j, each of the matrices $\widehat{A}_{\omega_j}$ is in fact revealed to have a block structure described by Id_p blown up by A_{ω_j} via Kronecker product, then subjected to a block permutation through right multiplication by the matrix $P_j \otimes \mathrm{Id}_n$. That is,

$$\widehat{A_{\omega_j}} = (\mathrm{Id}_p \otimes A_{\omega_j})(P_j \otimes \mathrm{Id}_p) = (\mathrm{Id}_p P_j) \otimes (A_{\omega_j} \mathrm{Id}_n) = P_j \otimes A_{\omega_j}$$

where we make use of a multiplicative property of the Kronecker product: for matrices A, B, C, D,

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$$

whenever the products AC, BD make sense. Summing over j yields the result. □

This decomposition of the signed adjacency matrix for the lift yields the next theorem which is more concrete, if a bit limited.

Theorem 3.6. Let (G, σ) be a simple magnetic graph on n vertices, and assume σ takes values values in a proper subgroup of $\mathbf{S}^1_p$, $p \geqslant 2$, of order $q := \frac{p}{2}$. Then $\widehat{G}$ is bipartite.

Proof. The matrix A_{ω_j} as in (16) is identically 0 for each $\omega_j \notin \mathbf{S}^1_q$. In turn, we apply Theorem (3.5) to obtain

$$\widehat{\mathbf{A}^{\sigma}} = \sum_{\omega_j \notin \mathbf{S}^1_q} P_j \otimes A_{\omega_j}$$

where P_j is a permutation matrix as in Theorem (3.5). This in fact completes the proof, since up to a block permutation (i.e. relabeling the vertices), this adjacency matrix is of the form corresponding to a bipartite graph (though this is not technically a classical adjacency matrix, its nonzero entries correspond to one) [3, p.11, 2c]. □

3.3. **Dirichlet problem via random walks.** In this subject we will formulate the main problem and give two ways to solve it. The first technique will be for general simple and connected magnetic graphs, and the second, slightly slicker technique will be for magnetic graphs with signature structures on a group $\mathbf{S}^1_p$ for some p, utilizing magnetic lifts.

Problem 3.3.1 (Dirichlet)**.** Let (G, σ) be a simple and connected magnetic graph and let H be a proper subconnected subset in G. Let $f \in \ell_2(\partial H)$ be a given boundary condition. We wish to find a function $\Psi \in \ell_2(\overline{H})$ for which

$$\left\{ \begin{array}{ll} (\mathcal{L}_G \Psi)(u) = 0 & u \in H \\ \Psi(u) = f(u) & u \in \partial H \end{array} \right. . \tag{17}$$

We have the following result.

Theorem 3.7 (Dirichlet Solution 1)**.** Let G, H, f, Ψ be as in (17). Let S_t be a random walk with associated initial distribution μ_0. The unique solution to (17) may be given by

$$\Psi(u) = \mathbb{E}[f(\widetilde{S_T}) \prod_{i=1}^{T} \sigma_{S_{i-1}S_i} : \mu_0 = \delta_u], \quad u \in \overline{H} \tag{18}$$

where $\widetilde{S}_t$ is the modified random walk process formulated at the end of subsection 3.1.

Proof. Uniqueness follows from applying the MVP in Theorem (1.1) to two the difference of two solutions $\Psi_1 - \Psi_2$. We now simply check that the solution as stated in (18) indeed solves (17). Let Ψ be given by (3.7). If $u \in \partial H$,

$$\begin{aligned} \Psi(u) &= \mathbb{E}[f(\widetilde{S_T}) \prod_{i=1}^{T} \sigma_{S_{i-1}S_i} : \mu_0 = \delta_u] \\ &= \mathbb{E}[f(u) : \mu_0 = \delta_u] = f(u). \end{aligned}$$

Now if $u \in H$, we have

$$\begin{aligned}\Psi(u) &= \mathbb{E}[f(\widetilde{S_T})\prod_{i=1}^{T}\sigma_{S_{i-1}S_i} : \mu_0 = \delta_u]\\ &= \sum_{v\sim u}\mathbb{P}[\mu_1 = \delta_v : \mu_0 = \delta_u]\mathbb{E}[f(\widetilde{S_T})\prod_{i=1}^{T}\sigma_{S_{i-1}S_i} : \mu_0 = \delta_u, \mu_1 = \delta_v]\\ &= \sum_{v\sim u}\frac{1}{d_u^G}\mathbb{E}[f(\widetilde{S_T})\sigma_{uv}\prod_{i=2}^{T}\sigma_{S_{i-1}S_i} : \mu_0 = \delta_u, \mu_1 = \delta_v]\\ &= \sum_{v\sim u}\frac{1}{d_u^G}\sigma_{uv}\mathbb{E}[f(\widetilde{S_T})\sigma_{uv}\prod_{i=1}^{T}\sigma_{S_{i-1}S_i} : \mu_0 = \delta_v]\\ &= \frac{1}{d_u^G}\left(\sum_{v\sim u}\sigma_{uv}\Psi(v)\right).\end{aligned}$$

In the second line we used a fact about the expectation operator for random variables on finite proability spaces; namely, if $\{A_i\}_i$ forms a disjoint partition of some probability space on which a random variable X is defined, we have $\mathbb{E}[X] = \sum_i \mathbb{P}[A_i]\mathbb{E}[X : A_i]$.

We once again appeal to the mean value characterization of harmonic functions in Theorem (1.1) to see that Ψ is harmonic on H. This concludes the proof. $\square$

The main drawback of this formulation for the solution to (17) is that the random variable in the expectation from (3.7) depends both on the position of the random walker and the product of the signatures along the path which the walker takes. Magnetic lifts seek to circumvent this problem, in that the position of a walker on a lift graph itself encodes information about the signature product in (3.7). To this end, we present the following alternative to (3.7).

Theorem 3.8. Let G, H, f, Ψ be as in (17). Assume further that σ takes values in some $\mathbf{S}_p^1 = \{\omega_i\}_{i=0}^{p-1}$ for $p \geqslant 2$ and that the lift $\widehat{G}$ is connected and not bipartite. Let S_t be a random walk on $\widehat{G}$ on vertices of the form $S_t = (u_t, \sigma_t)$, with associated initial distribution μ_0 on the lift $\widehat{G}$. The unique solution to (17) may be given by

$$\Psi(u) = \mathbb{E}[f(\widetilde{u_T})\sigma_T : \mu_0 = \delta_{(u,\omega_0)}], \quad u \in \overline{H} \tag{19}$$

where $\widetilde{S}_t$ is the modified random walk process formulated at the end of subsection 3.1.

Proof. This is just a special case of the previous solution derivation. All that need be checked is (i) that $\sigma(T)$ is equal to the signature product in (3.7), which is easy to verify from the definition of the edge set of $\widehat{G}$, and (ii) that $f(\widetilde{u_T})$ is a well-defined random variable when the walk is on $\widehat{G}$ and not G, which follows from Theorem (3.3). $\square$

4. Concluding Remarks

Combinatorial-type Laplace operators for graphs without magnetic structure have played a surprisingly important role in the worlds of geometry processing [12], image analysis, probability theory [11], and information theory. This author hopes that as research on the magnetic Laplace operator moves forward, its applications in some of the quantum cousins of the preceding applied topics will become more apparent. Also of interest is the resolution of the unanswered questions on magnetic lifts and graph products. In particular, if the second stochastic technique presented in the last section is to be useful, then a better understanding of the graph theoretic properties of a magnetic lift is essential. At the present time, it can be rather mysterious. Underpinning the partial results we covered is the conjecture of the author that a magnetic lift might have a more convenient representation as some sort of graph product as opposed to its usual interpretation as a 'covering' graph [3, ch. 19], though the precise formulation of this conjecture is undetermined at the present time.

References

1. E. Bendito, A. Carmona, A. Encinas, *Solving Boundary Value Problems on Networks Using Equilibrium Measures*, J. Fun. Anal., no. 171, 2000, pp. 155-176.
2. P. Bhaskarudu, *Some Results on Kronecker Product of Two Graphs*, Intl. Jrl. of Math. Trends and Technology, vol. 3, 2012.

3. N. Biggs, *Algebraic Graph Theory* 2ed., Cambridge Univ. Press, 1996.
4. F. Chung, *Discrete Green's Functions*, J. Combin. Theory Ser. A, no. 91, 2000, pp. 191-214.
5. F. Chung, *Spectral Graph Theory*, Regonial Conference Srs., no. 92, Amer. Math. Soc., Providence, RI, 1997.
6. B. González-Arévalo, J. Luis-Palacios, *Expected hitting times for random walks on weak products of graphs*, Stat. & Prob. Letters, no. 43, 1999.
7. J. Jacod, P. Protter, *Probability Essentials*, Springer-Verlag, Berlin, 2000.
8. G. Lawler, *Random Walks and the Heat Equation*, AMS Student mathematical library no. 55, Amer. Math. Soc., Providence, RI, 2010.
9. E. Lieb, M. Loss, *Fluxes, Laplacians, and Kasteleyn's theorem*, Duke Mathematical Journal vol. 71 no. 2, 1993.
10. B. Sericulo, *Markov Chains: Theory, Algorithms, and Applications*, Wiley, Hoboken, NJ, 2013.
11. J. Solomon, *Optimal Transport on Discrete Domains*, AMS Short Course Lecture Notes, preprint, 2018.
12. M. Wardetzky et. al., *Discrete Laplace operators: No free lunch*, Eurographics Symposium on Geometry Processing, 2007.
13. P. Weichsel, *The Kronecker product of graphs*, Proceedings of the American Mathematical Society, vol. 13, p.47-52, 1962.

University of Oklahoma
E-mail address: `sawyerjack@ou.edu`