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Jacobians of Finite and Infinite Voltage Covers of Graphs

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JACOBIANS OF FINITE AND INFINITE VOLTAGE COVERS OF GRAPHS

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Abstract

The Jacobian group (also known as the critical group or sandpile group) is an important invariant of a graph X; it is a finite abelian group whose cardinality is equal to the number of spanning trees of X (Kirchhoff's Matrix Tree Theorem). This dissertation proves results about the Jacobians of certain families of covering graphs, Y, of a base graph X, that are constructed from an assignment of elements from a group G to the edges of X (G is called the *voltage group* and Y is called the *derived graph*). The principal aim is to relate the Jacobian of Y to that of X.

We develop the basic theory of derived graphs, including computational methods for determining their Jacobians in terms of X. Of particular interest is when the voltage assignment is given by mapping a generator of the cyclic group of order d to a single edge of X (all other edges are assigned the identity), called a *single voltage assignment*.

We show that, in general, the voltage group G acts as graph automorphisms of the derived graph Y, that the group of divisors of Y becomes a module over the group ring $\mathbb{Z}[G]$, and that the Laplacian endomorphism on the group of divisors of Y—which is used to compute the Jacobian of Y—can be described by a matrix with entries from $\mathbb{Z}[G]$, called the *voltage Laplacian*. Using this and matrix computations, we determine both the order and abelian group structure of the Jacobian of single voltage assignment derived graphs when the base graph X is the complete graph on n vertices, for every n and d.

When G is abelian, the determinant of the voltage Laplacian matrix is called the *reduced Stickelberger element*; and it is shown to be a power of two times the graph Stickelberger element defined in the literature in terms of Ihara zeta-functions. Also using zeta-functions, we develop some general product formulas that relate the order of the Jacobian of Y to that of X; these formulas, that involve the reduced Stickelberger element, become very simple and explicit in the special case of single voltage covers of X.

We adapt aspects of classical Iwasawa Theory (from number theory) to the study of towers of derived graphs. We obtain formulas for the orders of the Sylow p-subgroups of Jacobians in an infinite voltage p-tower, for any prime p, in terms of classical μ and λ invariants by using the decomposition of a finitely generated module over the Iwasawa Algebra.

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Chapter 1

Introduction and Basic Results from Graph Theory

We begin Section 1.1 by introducing the Jacobian. We discuss the many contexts in which it arises, such as physics, algebraic geometry, combinatorics, and cryptography. We then present motivating factors that support the results of this dissertation.

In Section 1.2, we present elementary definitions in graph theory, as well as Kirchhoff's foundational *Matrix Tree Theorem*. In Section 1.3, we introduce the *dollar game*, which is a sort of chip-firing game, where vertices of a finite graph trade dollars across edges in an effort to eliminate debt. This leads to the definition of the Divisor, Jacobian and Picard group. We show how the dollar game can be interpreted in terms of the Laplacian operator that is defined on $\mathbb{Z}$-valued functions on the vertices of the base graph X. We then describe a practical method of computing the Picard and Jacobian groups using the Smith Normal Form of the Laplacian.

Finally, in Section 1.5 we state the main results of the dissertation. For excellent references on topics presented in this dissertation, see [CP18], [Kli18], and [Ter11].

1.1 The Jacobian: Introduction and Motivation

The Jacobian is an algebraic invariant of a graph[1] X in the form of a finite abelian group whose size is equal to the number of spanning trees of X (this is well-known as the Matrix Tree Theorem). The Jacobian arises in contexts such as arithmetic geometry, statistical physics, combinatorics, and discrete dynamics (see [Lor08]). Because of the many contexts in which it arises, the Jacobian is often also referred to as the critical group, the sandpile group, the Picard group, or the group of components. In 1990, Dhar, who studied it in the realm of physics, referred to it as the sandpile group [Dha90]. In the context of algebraic curves, Bacher, de la Harpe and Nagnibeda referred to it as the Picard group or Jacobian group in 1997 [BHN97]. Biggs, who studied this group in the context of chip-firing as in [Big99], called it the critical group. Then more recently in 2007, Biggs discusses the uses of the critical group in cryptography [Big06]. In [RT13], Reiner and Tseng study the interaction between critical groups and graph coverings. More recently in 2017, Backman, Baker and Yuen study the Jacobian in the context of matroid theory in [BBY17], in which they give a bijection between the Jacobian of a regular matroid and the set of bases

[1] In this dissertation the term graph will mean simple graph (see Section 1.2) unless otherwise explicitly noted.

of the matroid.

The exact structure of the Jacobian is known for only a few classes of graphs. Some well-known families of graphs that the Jacobian is known for include the complete graph on n vertices, K_n [Hop14], the complete bipartite graph on $m+n$ vertices, $K_{m,n}$ [Mac11], and the generalized Petersen graph, $GP(n,k)$ ([KMM17]). We list some other papers containing calculations of Jacobians for families of graphs: Bai verifies the structure of the Jacobian of an n-cube in [Bai03], which was initially conjectured by Reiner. Jacobson computes the Jacobian of threshold graphs in [Jac03], which completed work that was started by Christianson and Reiner in [CR02]. Jacobson, Niedermaier and Reiner describe the Jacobian group structure for complete multipartite graphs, as well as the cartesian product of complete graphs (which generalizes results of Bai) in [JNR03]. Ducey and Jalil explain how, for various matrices associated to the Cayley graph, the spectrum can be used to give information on the Smith Normal Form in [DJ13]. In doing so, they re-prove results from [Bai03] and [JNR03]. Chandler, Sin and Xiang compute the Smith Normal Forms of the adjacency matrix and Laplacian matrix of Payley graphs in [CSX14]. Sin then goes on to add another class of computed examples, by applying some of the ideas used for Paley graphs to the Peisert graphs in [Sin16]. He obtains a complete description of the structure of the Jacobian of the Peisert graphs. Ducey, Hill and Sin determine the structure of the Kneser graph $KG(n,2)$ in [DHS18]. In [DDE$^+$19], Ducey et. al. give conditions that force the Sylow p-subgroup of the Jacobian of a strongly regular graph to take a specific form.

Overall, however, there are very few families of graphs for which the Jacobian has been found. Thus it is of some interest to compute the Jacobian of some well-known graph families. In this thesis we focus primarily on *voltage graphs*, especially ones in which the voltage assignment is the *single voltage assignment* and the *voltage group* is the cyclic group of order d. Even more specifically, we look at *single voltage covers* of the particular well-known graph families mentioned above. In doing so, we establish the structure and the order of the Jacobian for some of these families of graphs. Many of the results in Chapter 3 were obtained by first computing extensive tables via Sage and Mathematica, from which conjectures were formulated. Some of the conjectures, such as Conjecture 1, were then proven via lengthy matrix manipulations. A general formula for the order of the Jacobian of a single voltage cover is proven in Chapter 4, thereby other conjectures that were made in Chapter 3. In Chapter 5, we obtain growth formulas for the finite p-Jacobians of a cyclic voltage p-tower of graphs that is analogous to the results that Iwasawa established for the order formulas of class groups in a $\mathbb{Z}_p$-extension. The ideas in [HMSV19] and [Val20], as well as Dr. Sands' advice to pursue Iwasawa Theory for graphs, which is the foundation for the proofs, impelled the research that culminates in Chapter 5. More precise statements of the main results of this dissertation are given in Section 1.5. The contents of [Val20], by Daniel Vallières, are compared in greater detail with our results in Section 5.2.

For related results of interest, but tangential to this work, see: [LZ21], [DKM12], [GM13], [Woo17], [NW18].

1.2 Basic Definitions and Foundational Results in Graph Theory

In this section we present some basic definitions and foundational results from Graph Theory, such as *Kirchhoff's Matrix Tree Theorem* that pertain to this dissertation. Basic graph theory definitions, terminology, and results can be found in standard references such as [Wes00]—we include some of these here for completeness and convenience.

Definition 1 (Undirected Graph)**.** *A graph (not necessarily finite) is a pair* $X = (V, E)$*, where* $V = V(X)$ *is the set whose elements are called vertices, and* $E = E(X)$ *is a set of unordered pairs of vertices, whose elements are called edges. We depict these by*

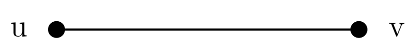

Figure 1.1: An edge in a graph X *with endpoints* u *and* v

where $u, v \in V(X)$ *and* $\{u, v\} \in E(X)$*. We call* u *and* v *the endpoints of edge* $\{u, v\}$*.*

We may also think of a graph as a topological space contained in Euclidean n-space which consists of a collection of points, called vertices (and where n is the number of vertices), and a collection of edges. Each edge is either homeomorphic to $[0, 1]$ and joins two distinct vertices, or it is homeomorphic to a circle and a joins a given vertex to itself (i.e. is a *loop*).

In an undirected graph, the edges indicate a two-way relationship in that each edge can be traversed in both directions.

Definition 2 (Multigraph)**.** *A multigraph is a graph which is permitted to have multiple edges (i.e. edges that are incident to the same two vertices).*

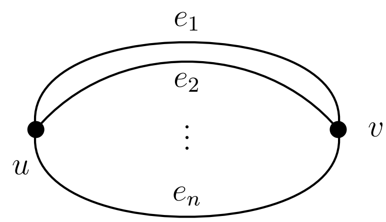

Figure 1.2: A multigraph

Definition 3. *Let $X = (V, E)$ be a graph (not necessarily finite).*

- Loop *A loop is an edge whose endpoints are equal.*

- Adjacent Vertices *Adjacent vertices are vertices that are endpoints of an edge.*

- Simple or Strict Graph *A graph is said to be simple if it contains no loops or multiple edges.*

For this dissertation the term "graph" will always mean "simple graph."

- Walk *A walk is a sequence of vertices $v_0, v_1, \cdots, v_n$ and edges $e_0, \cdots, e_{n-1}$ such that for $i = 0, \cdots, n-1$, the edge e_i has endpoints v_i and v_{i+1}.*

- Path *A path is a walk in which all vertices are distinct.*

- Cycle *A cycle is a path which begins and ends at the same vertex (so the only repeated vertices are the first and last).*

- Connected *A graph is (undirected) connected if and only if $\forall u, v \in V(X)$, there exists a path from u to v.*

- Degree of a Vertex *In a loopless graph, the degree of a vertex is the number of edges adjacent to the vertex.*

- Subgraph *H is a subgraph of X if $V(H) \subset V(X)$ and $E(H) \subseteq E(X)$.*

- Graph Isomorphism *A graph isomorphism from a graph G to a graph H is a bijection $f : V(G) \to V(H)$ such that $\{u, v\} \in E(G)$ if and only if $\{f(u), f(v)\} \in E(H)$.*

- Forest *A forest is a graph with no cycles.*

- Tree *A tree is a connected forest.*

- Spanning Tree *A spanning tree T of a graph X is a subgraph of X such that $V(T) = V(G)$.*

Definition 4. *Let $V(X) = \{v_1, \cdots, v_n\}$ denote the vertex set of a finite graph X.*

- Adjacency Matrix *The adjacency matrix of X, denoted by A_X, is an $n \times n$ matrix with i, j-entry*

$$a_{i,j} = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise} \end{cases}$$

- Degree Matrix *The degree matrix D_X of X is the $n \times n$ diagonal matrix with i, i-entry equal to the degree of v_i, $1 \leq i \leq n$.*

- Laplacian Matrix *The Laplacian matrix is the $n \times n$ matrix*

$$L_X = D_X - A_X$$

where D_X is the degree matrix of X and A_X is the adjacency matrix of X.

- Reduced Laplacian Matrix *A reduced Laplacian matrix, $\widetilde{L}_X$, is obtained by deleting row i and column i corresponding to vertex v_i of the Laplacian matrix, L_X, where L_X is as above.*

Theorem 1 (Kirchhoff's Matrix Tree Theorem)**.** *The determinant of any reduced Laplacian matrix is equal to the number of spanning trees of X, where X is a finite graph. In particular, the determinant is the same for all reduced Laplacians, and is nonzero if and only if X is connected.*

In the next section, we introduce the *dollar game*, which ultimately leads to the definition of the Jacobian group. We then supply further details on the Laplacian and reduced Laplacian. Finally, we discuss a computational method of computing the Picard and Jacobian groups using the Smith Normal Form of the Laplacian.

1.3 The Dollar Game

Chip-firing processes, defined by a commodity being exchanged between sites of a network according to simple local rules, have been introduced into the literature a number of times from various communities. One of many chip-firing games that has been explored by various authors in the literature is called the *dollar game*. It was introduced by Baker and Norine in [BN07], as a variant of an earlier version due to Biggs in [Big99]. Baker and Norine study the analogy between finite graphs and Riemann surfaces in the context of linear equivalence of divisors (i.e. the *Picard group*). As an application of their results, they characterize a winning (or non-winning) strategy for the dollar game. A great overview of the dollar game can be found in [Bak10].

Essentially, vertices of a finite graph trade dollars across edges in an effort to eliminate debt. We explain this in greater detail next.

We may think of vertices $V(X)$ as individuals, edges $E(X)$ as relationships between individuals, and the entire graph X as a community. As with most communities, the individuals represented in X are not equally wealthy. To record their varying degrees of wealth, we place an integer value next to each vertex, interpreting negative values as debt. Then we represent such a distribution as a formal sum of integer multiplies of vertices. To redistribute wealth, each individual $v \in V(X)$ may choose to lend or borrow a dollar along each edge incident to v (which allows for the possibility that either v or someone incident to v goes into debt). Note that v never lends along some edges, while borrowing along others (i.e., v either lends to all vertices it is adjacent to or borrows from all vertices it is adjacent to).

The goal of this game is to find a sequence of lending or borrowing moves so that everyone in the community X is debt-free. This is called the *dollar game* on X. If such a sequence exists, we say that the game is winnable.

1.3.1 The Divisor, Picard, and Jacobian Group

We now present ideas of the dollar game more formally. Further details can be found in both [CP18] and [Jen16].

Definition 5. *A divisor on a graph X (possibly infinite) is an element of the free* *abelian group on the vertices $V = V(X)$:*

$$Div(X) = \{ \sum_{v \in V(X)} a_v v \mid a_v \in \mathbb{Z}\}$$

where each $\sum_{v \in V(X)} a_v v$ is a formal linear combination of the vertices of X with integer coefficients, where only finitely many a_v are nonzero (in the case when X is an infinite graph). When $V(X) = \{v_1, \ldots, v_n\}$, we may write the elements of $Div(X)$ as $a_1v_1 + a_2v_2 + \cdots + a_nv_n$, where each $a_i \in \mathbb{Z}$.

Divisors represent distributions of wealth on X, where each person (i.e., vertex) v has a_v dollars (or debt, when $a_v < 0$).

The degree homomorphism is defined to be

$$\deg : \mathrm{Div}(X) \longrightarrow \mathbb{Z} \qquad \text{by} \qquad \deg(\sum_{v \in V(X)} a_v v) = \sum_{v \in V(X)} a_v$$

or, in the finite case, $\deg(a_1v_1 + \cdots + a_nv_n) = a_1 + \cdots + a_n$; in either case it is a finite sum.

Note that there are two uses of the word "degree"—it will always be clear from the context which usage is meant.

We are interested in equivalence classes of divisors on graphs, where the equivalence is given by *firing moves* (lending moves). Starting with a divisor D, we may *fire* a vertex which results in that vertex giving a dollar to each of its neighbors. We state this more formally in the following definition:

Definition 6. *A firing move based at a vertex v takes a divisor D to D', denoted $D \xrightarrow{v} D'$ where for all $w \in V(X)$, the coefficient (dollar amount) corresponding to vertex w is equal to*

$$a_w = \begin{cases} a_v - \deg(v) & \textit{if } w = v \\ a_v + 1 & \textit{if } w \textit{ is adjacent to } v \\ 0 & \textit{if } w \textit{ is not adjacent to } v \end{cases}$$

Definition 7. *Let $D, D' \in Div(X)$. Then we say D is linearly equivalent to D' if D' may be obtained from D by a sequence of firing moves for various vertices $v \in V(X)$. In this case, we write $D \sim D'$.*

The next Proposition can be found in [Jen16] as Lemma 1.4.

Proposition 1. *Linear Equivalence of divisors is an equivalence relation. Moreover, if $D_1 \sim D_2$ and $F_1 \sim F_2$, then $D_1 + F_1 \sim D_2 + F_2$.*

Definition 8. *The divisor class determined by $D \in Div(X)$ is*

$$[D] = \{D' \in Div(X) \mid D' \sim D\}.$$

We may think of a divisor class as a closed economy, where we begin with an initial distribution of wealth D, and through a sequence of firing moves, we arrive at the distribution D'. The collection of all possible distributions is the divisor class of D.

The next proposition can be found in [CP18], Section 1.2 (or it is an easy exercise from the definition of chip-firing).

Proposition 2. *Let $D, D' \in Div(X)$. If $D \sim D'$ then* $\deg(D) = \deg(D')$.

Thus, the degree map, which was initially defined on divisors, is also a well-defined map on the set of equivalence classes of divisors.

Now let $D, F \in \text{Div}(X)$. Then, by definition of a free abelian group, the sum of divisors D and F is defined vertex-wise. Moreover, this sum respects linear equivalence.

The degree map $\deg : \text{Div}(X) \to \mathbb{Z}$, is a surjective group homomorphism with kernel equal to the subgroup of $\text{Div}(X)$ of *divisors of degree 0*, denoted as $\text{Div}^0(X)$:

$$\text{Div}^0(X) = \{D \in \text{Div}(X) \mid \deg D = 0\}.$$

Let O denote the divisor whose vertex coefficients are all zero (i.e., the zero of the additive group $\text{Div}(X)$). The divisors that are equivalent to O are called the *principal divisors* on X and we denote this set by $\text{Pr}(X)$. It is an exercise that $\text{Pr}(X)$ is closed under addition and subtraction. So by Proposition 2, $\text{Pr}(X)$ is a subgroup of $\text{Div}^0(X)$.

From this, we get the following groups:

Definition 9. *The Picard group of X is the set of linear equivalence classes of divisors on X, i.e.,*

$$Pic(X) = Div(X)/Pr(X).$$

The Jacobian group of X is the subgroup of $Pic(X)$ consisting of divisor classes of degree 0 on X, i.e.,

$$\mathcal{J}(X) = Div^0(X)/Pr(X).$$

The next proposition can be found in [CP18] as Proposition 1.20.

Proposition 3. *Fix $v \in V(X)$. There is an isomorphism of groups*

$$Pic(X) \to \mathbb{Z} \oplus \mathcal{J}(X) \qquad \textit{given by} \qquad [D] \mapsto (\deg(D), [D - \deg(D)v]).$$

Thus, we see that

$$\mathrm{Pic}(X) = \mathrm{Div}(X)/\mathrm{Pr}(X) \cong \mathbb{Z} \oplus \mathcal{J}(X).$$

We now state the following important theorem.

Theorem 2. *If X is connected, then $\mathcal{J}(X)$ is a finite abelian group.*

1.3.2 The Laplacian and Reduced Laplacian

Firing moves may be expressed compactly via the graph Laplacian.

Definition 10. *Let $G = (V, E)$ be a graph with vertices $\{v_1, \cdots, v_n\}$. The graph Laplacian $L = L_X$ is the $n \times n$ matrix given by*

$$L_{i,j} = \begin{cases} \deg(v_i) & \textit{if } i = j \\ -1 & \textit{if } \; v_i \textit{ is adjacent to } v_j \\ 0 & \textit{if } \; i \neq j \textit{ and } v_i \textit{ is not adjacent to } v_j \end{cases}$$

By Definition 6, we see that the Laplacian matrix encodes all of the firing moves for X since a firing move by vertex v_j corresponds to subtracting the j^{th} column of L from a divisor.

The Laplacian is also the matrix representation of the following group homomorphism $\mathcal{L}$ defined below. For each fixed v_i define the principal divisor p_i based at v_i by

$$p_i = \deg(v_i)v_i - \sum_{\substack{j=1 \\ j\neq i}}^{n} \delta_{i,j} v_j$$

where $\delta_{i,j} = 1$ if v_j is adjacent to v_i and 0 otherwise. Then

$$\mathcal{L} : \mathrm{Div}(X) \to \mathrm{Div}(X) \qquad \text{where} \qquad \mathcal{L}(v_i) = p_i.$$

When extended by $\mathbb{Z}$-linearity to all of $\mathrm{Div}(X)$, this is a $\mathbb{Z}$-linear homomorphism from $\mathrm{Div}(X)$ to itself, whose image is $\mathrm{Pr}(X)$, the group of principal divisors. From this, we get the following important fact:

$$\mathrm{Pic}(X) = \mathrm{Div}(X)/\mathrm{im}(\mathcal{L}) = \mathrm{coker}(\mathcal{L}),$$

where, by definition, the cokernel of a homomorphism $f : A \to B$ is $B/f(A)$.

So it follows that the Picard group may be computed as the cokernel of the Laplacian. (For further details on this, see [CP18], Section 2.1.)

We now relate this to the Jacobian and the reduced Laplacian. Evidently, the matrix representation for $\mathcal{L}$ with respect to the basis of vertices is L, the Laplacian matrix.

A reduced Laplacian $\tilde{L}$ is the $(n-1) \times (n-1)$ integer matrix obtained by removing the row and column corresponding to any vertex v from the Laplacian matrix L. So the Jacobian group can be computed as the cokernel of the reduced Laplacian matrix

$$\mathcal{J}(X) \cong \mathbb{Z}^{n-1}/\mathrm{im}(\widetilde{L}) = \mathrm{coker}(\widetilde{L}),$$

where $\mathbb{Z}^{n-1}$ denotes the free $\mathbb{Z}$-module on the set $V(X) - \{v\}$ of rank $n-1$.

1.3.3 Computing the Jacobian

In this section, we give a computationally effective method for computing the Jacobian of a graph. We first begin by stating the Structure Theorem for Finitely Generated Abelian Groups, which can be found in [CP18] as Proposition 2.23.

Theorem 3. *[Structure Theorem for Finitely Generated Abelian Groups] A group is a finitely generated abelian group if and only if it is isomorphic to*

$$\mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_k\mathbb{Z} \oplus \mathbb{Z}^r \tag{1.1}$$

for some unique integers $d_1, \cdots, d_k$ with $d_i > 1$ for all i and some integer $r \geq 0$ that satisfy the following condition: $d_i \mid d_{i+1}$ $\forall i$. The d_i are the invariant factors of the group and r is called the free rank of the group.

Let A be a finitely generated abelian group as in (1.1). Then $\mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_k\mathbb{Z}$ is called the torsion subgroup of A, denoted A_{tor} (it consists of all the torsion elements—elements of finite order—in A) and $A/A_{\text{tor}} \cong \mathbb{Z}^r$. Hence, by the Structure Theorem, we have

$$A \cong A_{\text{tor}} \oplus A/A_{\text{tor}}.$$

Now we describe a computational method for computing the free rank and invariant factors of a finitely generated abelian group. Let $\{a_1, \cdots, a_m\}$ be any set of generators of A. Define the group homomorphism

$$\Phi : \mathbb{Z}^m \to A \qquad \text{by} \qquad e_i \mapsto a_i,$$

where e_i is the i^{th} standard basis vector of the Cartesian product of $\mathbb{Z}$ with itself m times, and extend by $\mathbb{Z}$-linearity. Since the a_i generate A, it follows that Φ is surjective and so

$$\mathbb{Z}^m / \ker \Phi \cong A.$$

Now since every subgroup of $\mathbb{Z}^m$ is finitely generated, there exists a finite set of generators for the kernel of Φ, say $\{b_1, \cdots, b_s\}$. Define

$$M : \mathbb{Z}^s \to \mathbb{Z}^m,$$

where M is the $m \times s$ matrix whose columns are $b_1, \cdots, b_s$. Putting these maps together, we get

$$\operatorname{coker}(M) = \mathbb{Z}^m / \operatorname{im}(M) \cong A.$$

So A is determined by the single matrix M, called a "relations matrix" for A. So every finitely generated abelian group is the cokernel of an integer matrix. Conversely, each integer matrix determines a finitely generated abelian group, namely, its cokernel. However, the construction of M depends on arbitrary choices for generators of A and $\ker \Phi$. Changing the choice of generators corresponds to integer changes of coordinates for the codomain and domain of M, or equivalently, to performing invertible integer row and column operations on M.

Definition 11. *The elementary integer row (and column, respectively) operations on an integer matrix consist of the following:*

(i) interchanging two rows (or columns),

(ii) negating a row (or column), and

(iii) adding one row (or column) to a different row (or column).

If a matrix M may be obtained from a matrix N through a sequence of elementary integer row and column operations, we write $M \sim N$. Note that $\sim$ is clearly an equivalence relation.

Suppose M is an $m \times n$ integer matrix with $M \sim N$. Start with the identity matrix $P = I_m$ and $Q = I_n$. Whenever a row operation is performed as part of a sequence to transform M to N, apply this same row operation to the P in that sequence. Likewise, whenever a column operation is performed to transform M to N, apply this same operation to the Q in that sequence. It follows that P and Q are invertible over $\mathbb{Z}$ and $PMQ = N$.

Conversely, given any integer matrices P and Q that are invertible over $\mathbb{Z}$ such that $PMQ = N$, it follows that $M \sim N$. (The proof of this converse requires the existence of the Smith Normal Form).

The next proposition can be found in [CP18] as Proposition 2.28.

Proposition 4. *Let M and N be $m \times n$ integer matrices. If $M \sim N$, then coker$(M) \cong$ coker(N).*

Definition 12. *An* $m \times n$ *integer matrix* M *is in Smith Normal Form (SNF) if* $M = diag(d_1, \cdots d_k, 0, \cdots, 0)$*, a diagonal matrix, where* $d_1, \cdots, d_k$ *are positive integers such that* $d_i \mid d_{i+1}\ \forall i$*. The* d_i *with* $d_i \geq 2$ *are called the invariant factors of* M*.*

We now illustrate the relevance of the Structure Theorem and Smith Normal Form in computing the Jacobian of graph. The next proposition can be found in [Kli18] as Proposition 4.5.2.

Proposition 5. *If* M *is a non-singular* $n \times n$ *integer matrix and the Smith normal form of* M *is* $diag(d_1, \cdots, d_k)$ *then*

$$coker(M) \cong \bigoplus_{i=1}^{k} \mathbb{Z}/d_i\mathbb{Z}.$$

So for any finite *connected* graph X with reduced Laplacian $\widetilde{L}$ (with respect to any vertex), since $\mathcal{J}(X) = \text{coker}(\widetilde{L})$, it follows that the invariant factors of $\mathcal{J}(X)$ are determined by the invariant factors of $\widetilde{L}$. Since $\widetilde{L}$ is invertible, none of its invariant factors are 0. So the free rank of $\mathcal{J}(X)$ is 0. The SNF factors of $\widetilde{L}$ equal to 1 have no effect on the isomorphism class of coker$(\widetilde{L})$. Suppose $d_1 \cdots, d_k$ are the invariant factors of $\widetilde{L}$ (the SNF factors that are greater than 1). The d_i are the same as the invariant factors of the finite abelian group $A = \mathcal{J}(X)$ as in Theorem 3, and hence,

$$\mathcal{J}(X) \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_k\mathbb{Z}.$$

The structure of $\text{Pic}(X)$ is then determined since $\text{Pic}(X) \cong \mathbb{Z} \oplus \mathcal{J}(X)$: the free rank of $\text{Pic}(X)$ is 1 and

$$\text{Pic}(X) \cong \mathbb{Z} \oplus \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_k\mathbb{Z}.$$

The next proposition can be found in [CP18] as Proposition 2.37.

Proposition 6. *For X a connected graph, the order of the Jacobian of X is*

$$|\mathcal{J}(X)| = \det(\widetilde{L}).$$

for any (hence every) reduced Laplacian $\widetilde{L}$ of X.

1.4 Voltage Graphs and Derived Graphs in a Nutshell

Fundamental concepts in this dissertation are the notions of a *voltage graph* and its associated *derived graph*, so we describe these, with slightly simplified restrictions for expository purposes. Precise general definitions are given in Chapter 2; and excellent books containing some information about them are [Ter11] and [GT87]. The terminology "voltage graph" comes from the origins of these concepts in electrical circuit theory (see [GT87]).

We start with a fixed, finite, connected graph X, with vertex set $v_1, v_2, \ldots, v_n$, and any group G (which could be infinite). The graph X is called the *base graph* and G is called the *voltage group*. For every edge joining v_i to v_j with $i < j$ we assign a group element $\alpha_{i,j}$ from G to that edge. (One can imagine that the "circuit voltage 'increases' by the amount $\alpha_{i,j}$" if the edge is traversed going from v_i to v_j.) We adopt the rule that we assign the inverse group element, $\alpha_{i,j}^{-1}$, to the edge from v_i to v_j if the edge is traversed backwards, namely from v_j to v_i, i.e., $\alpha_{j,i} = \alpha_{i,j}^{-1}$ (so the "reverse voltage *decreases* by the same amount.") The graph X together with a choice of group element voltages along its (oriented) edges is called a *voltage graph*.

A useful way of codifying (and defining more precisely) a voltage graph is by taking the ordinary adjacency matrix, A_X, of X and, wherever there is a 1 in entry i, j with $i < j$ (i.e., there is an edge between v_i and v_j) we assign some group element $\alpha_{i,j}$ from G; we then implement the rule that $\alpha_{i,j}^{-1}$ must be placed in the j, i-entry. This results in the *voltage adjacency matrix*, A_α, where α denotes a specific choice of voltage assignments to every edge. (Now one sees that changing the orientation on one or more of the edges, simply amounts to flipping the i, j and j, i entries of A_α, which codifies a different voltage assignment.)

Given a voltage graph as above, we now construct its *derived graph*, Y, as follows. Take $|\,G\,|$ copies of X, where these are indexed by the elements of G as "sheets" or "layers" that can be thought of as lying above each other. For each fixed g in G, the sheet for g has vertices labeled as $v_{1,g}, v_{2,g}, \dots, v_{n,g}$. These are the vertices of Y, and so there are $n|\,G\,|$ vertices in the derived graph.

Now create the edges of Y by the following rule. First erase all the edges in all the sheets that were initially taken as copies of X. Then, whenever there is an edge from v_i to v_j in the base graph X with assigned voltage $\alpha_{i,j}$, create edges that go from $v_{i,g}$ to $v_{j,g\alpha_{i,j}}$ in Y, for every $g \in G$, where $g\alpha_{i,j}$ is the group-product of these two group elements in G. So for fixed i, j this creates exactly one edge from each vertex $v_{i,g}$ to the vertex in the sheet that is "$\alpha_{i,j}$ volts above it", i.e., in the sheet indexed by $g\alpha_{i,j}$ (or in the sheet $g + \alpha_{i,j}$ if the group operation is addition). The rule for constructing edges describes each (*same, undirected*) edge of Y twice: once going "upward" from $v_{i,g}$ to $v_{j,h}$, where (by the edge-voltage rule) $h = g\alpha_{i,j}$, and again going "downward" from this $v_{j,h}$ to the original $v_{i,g} = v_{i,h\alpha_{j,i}}$; this is because of the "inverse rule:" namely, $\alpha_{j,i} = \alpha_{i,j}^{-1}$.

As mentioned, voltage graphs and their associated derived graphs are the main subject of this dissertation. The basic aim is to relate the invariants of X—more specifically the Jacobian of X, its order and its invariant factors—to those of Y. One can imagine that if the Jacobian of X is already both computationally and theoretically difficult to determine, then Jacobians of derived graphs—which are ostensibly much larger, depending on the size of G—are even more challenging. Also, there are many parameters that tend to make a completely general study intractable: a choice of any base graph X, a choice of any group G (possibly infinite, non-abelian, etc.), and any voltage assignment from G on edges of X, resulting in potentially arbitrarily complicated derived graphs Y. Nonetheless we make substantial inroads for important and quite general configurations.

In Chapter 2 we lay the foundations of the theory of voltage graphs, voltage adjacency matrices, voltage Laplacians, and, when G is abelian, the determinant of the voltage Laplacian, that we call the *reduced Stickelberger element* (for reasons explained later); the commutativity of G is only needed in order for the determinant to be well-defined, since the entries of the voltage Laplacian are from the integral group ring $\mathbb{Z}[G]$, which is a commutative ring if and only if G is an abelian group.

In particular, we determine various conditions under which the derived graph is connected. (For example, if all edge-voltages are assigned to be the identity element of G, the derived graph Y is just $|\,G\,|$ disjoint copies of X itself, which is certainly not connected when $|\,G\,| > 1$ even if X itself is connected.) Also in Chapter 2 we develop a method for writing out the ordinary adjacency matrix, A_Y, for the derived graph Y for any voltage assignment. (This was done independently, but later discovered to be in the literature in a more complicated form.) A powerful tool here is that G acts as *graph automorphisms* of the derived graph Y, and so $\mathrm{Div}(Y)$ becomes a module over the group ring $\mathbb{Z}[G]$. We exploit this $\mathbb{Z}[G]$-action to great advantage throughout the dissertation.

In Chapter 3 we carry out many matrix computations, both by computer and by hand, to determine the Jacobians and reduced Stickelberger elements for various base graphs X (families of X already mentioned, where we explicitly know $\mathcal{J}(X)$). We restrict the types of voltage assignments to ones that seem to result in derived graphs that are most closely related to the base graph. In particular, one such voltage assignment is what we call a *single voltage* assignment: where X has an edge between v_1 and v_2, G is the cyclic group of order d generated by τ, and $v_1 \to v_2$ is assigned voltage τ (so $v_2 \to v_1$ has voltage τ^{-1}); all other edges are assigned the identity element voltage. (Figure 2.7 is a picture of a single voltage derived graph.) For single voltage covers of certain X, such as complete graphs, we are able (by a lengthy hand computation) to work out all invariant factors of the Jacobians of all resulting derived graphs for every d. For other base graphs, however, we arrive at only *conjectured* order formulas and invariant factors.

In Chapter 4 we develop some general methods for relating the order of $\mathcal{J}(Y)$ to $\mathcal{J}(X)$, using *Ihara zeta-functions*. This is the "Fourier analysis" chapter of the dissertation. These formulas result in explicit (closed form) formulas for general (connected) base graphs X and arbitrary (connected) single voltage derived graphs Y in terms of the reduced Stickelberger element. This *verifies* many of the order formulas that were conjectured in Chapter 3.

Finally, in Chapter 5 we consider "Iwasawa Theory" for towers of voltage graphs,

$$X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots\cdots \leftarrow X_m \leftarrow \cdots\cdots$$

where each X_m is the derived graph of a voltage assignment on X from the cyclic group of order p^m, for some prime p (here p is fixed but the voltage groups and resulting derived graphs vary with m). The indicated maps are the "covering maps" between successive derived graphs, given explicitly by "reducing" the group elements indexing the sheets "modulo appropriate powers of p". For such towers we obtain *order formulas* for the order of the Jacobians $\mathcal{J}(X_m)$ in terms of the order of $\mathcal{J}(X)$ and certain "μ and λ invariants" that are completely analogous to the classical order formulas that Iwasawa established for class groups of Galois extensions of number fields in a $\mathbb{Z}_p$ tower. In hindsight we see that there is an "Iwasawa Decomposition of the p-adic completion of the Laplacian" of the given tower that is closely analogous to the Smith Normal Form decomposition for ordinary Laplacians of graphs, and so gives new insight into the theory of graph towers.

In the next section we single out the major results of each chapter in greater precision.

1.5 Statements of Main Results of the Dissertation

Following up on the preceding discussion, we collect the main results of the dissertation here. The numbering scheme uses the numbers in the body of this work.

Chapter 2: Covering Graphs and Voltage Graphs

In Theorem 9, we show that given an intermediate covering graph $\widetilde{X}$ corresponding to $H \trianglelefteq G$, it follows that $\widetilde{X}/X$ is always a voltage cover. This theorem plays a vital role in the theory of towers of voltage graphs in Section 5.1. Then in Theorem 10, for Y/X a single voltage cover, we show that there exists a choice of coset representatives of H in G such that $Y/\widetilde{X}$ is single a voltage cover.

Theorems 9 and 10:

Let (X, G, α) be a voltage graph with derived graph Y such that Y is connected. If $\widetilde{X}$ is an intermediate cover of Y/X corresponding to the normal subgroup H of G, then $\widetilde{X}/X$ is a voltage graph, whose voltage adjacency matrix is the voltage adjacency matrix of Y/X, but with nonzero entries reduced modulo H (thus has entries in $\mathbb{Z}[G/H]$). Furthermore, if Y/X is a single voltage cover, then there exists a choice of coset representatives of H in G such that $Y/\widetilde{X}$ is a voltage graph with single voltage assignment.

Theorem 11 gives that the reduced Stickelberger element, which is defined as the determinant of the matrix corresponding to the voltage Laplacian map on the divisors of Y, annihilates the Picard group. This theorem is pivotal in proving the main theorem in Chapter 5, Theorem 27, where we must construct a finitely generated torsion module over a complete p-adic group ring Λ.

Theorem 11:

Let G be any group. The voltage Laplacian $\mathcal{L}_\alpha : \mathrm{Div}(Y) \longrightarrow \mathrm{Div}(Y)$ is a $\mathbb{Z}[G]$-module homomorphism whose image is equal to the group of principal divisors and whose cokernel is equal to the Picard group, and its $n \times n$ matrix with respect to the $\mathbb{Z}[G]$-basis $v_{1,\tau_0}, \ldots, v_{n,\tau_0}$ is equal to $D_X - A_\alpha$, where τ_0 is the identity of G, D_X is the degree matrix for the base graph X and A_α is the voltage adjacency matrix of X. Furthermore, if G is abelian, then $\Theta_{Y/X} = \det(D_X - A_\alpha)$, the *reduced Stickelberger element*, annihilates the Picard group of any derived graph, hence it also annihilates the Jacobian (both as $\mathbb{Z}$ and $\mathbb{Z}[G]$-modules).

Chapter 2 also contains various results about connectedness of derived graphs that are not listed here.

Chapter 3: Covers of Complete Graphs and Other Graphs

Chapter 3 contains numerous conjectures regarding families of derived graphs that are computationally verified for small values.

Theorem 14 gives the exact structure of the Jacobian of single voltage cyclic covers of K_n. This gives a computationally effective method for computing the Smith

Normal Form of the Laplacian of Y. We first do elementary row and column operations on the *voltage* Laplacian, and then we tensor the entries with the regular representation of the group G to obtain the ordinary Laplacian for the derived graph Y. We further reduce by row and column operations over $\mathbb{Z}$ to obtain the Smith Normal Form for the Laplacian of Y.

Theorem 14:

Let Y be a single voltage cover of the complete graph K_n by the cyclic group of order d, where $n \geq 4$ and $d \geq 3$. Then

$$J(Y) \cong (\mathbb{Z}/n\mathbb{Z})^{(n-4)d+2} \oplus (\mathbb{Z}/n(n-2)\mathbb{Z})^{d-2} \oplus \mathbb{Z}/dn(n-2)\mathbb{Z},$$

where the exponents indicate the multiplicities of the (distinct) invariant factors. (The small cases when $n < 4$ or $d = 2$ are also classified.)

Theorem 18 gives partial results on the Jacobian of single voltage cyclic covers of $K_{n,n}$. To obtain Theorem 18, we first do elementary row and column operations on the voltage Laplacian matrix (and then on the ordinary Laplacian matrix) over $\mathbb{Z}_{(p)}$, the integers localized at p. This gives the primes p that divide the order of the Jacobian of the derived graph Y.

Theorem 18:

Let Y be a single voltage cover of the complete bipartite graph $K_{n,n}$ by the cyclic group of order d, where $n, d \geq 3$. For any prime p with $p \nmid n$, the Sylow p-subgroup, $\mathcal{J}_p(Y)$, of $\mathcal{J}(Y)$ has the following (elementary divisor) decomposition:

$$\mathcal{J}_p(Y) \cong (\mathbb{Z}/p^{2a}\mathbb{Z})^{d-2} \oplus (\mathbb{Z}/p^{2a+b}\mathbb{Z})^1$$

where p^a is the largest power of p dividing $n-1$ and p^b is the largest power of p dividing d. In particular, $|\mathcal{J}_p(Y)| = p^{2a(d-1)+b}$, and the p-rank of $\mathcal{J}(Y)$ is $d-1$.

Chapter 4: Zeta Functions of Voltage Graphs

Theorem 22 gives the order of the Jacobian of a derived graph Y in terms of the order of the Jacobian of the base graph X. From this, we are able to verify Conjectures 1-4(iii) from Chapter 3.

Theorem 22:

Let (X, G, α) be a voltage graph such that X is connected with $|\,E(X)\,| \neq |\,V(X)\,|$, G is abelian and the derived graph Y is connected. Then the order of the Jacobian of the derived graph Y is

$$|\mathcal{J}(Y)| = \frac{1}{d} \cdot |\mathcal{J}(X)| \prod_{\chi_t \neq \chi_0} \chi_t(\Theta_{Y/X}),$$

where the product is over all irreducible characters χ_t of G except the principal character χ_0, and $\Theta_{Y/X}$ is the reduced Stickelberger element (the determinant of the voltage Laplacian, which is an element of $\mathbb{Z}[G]$, so we may apply each χ_t to it).

In Theorem 23, we show that the reduced Stickelberger element is always of a specific form when the voltage assignment is given by the *single voltage* assignment. This result also proves Theorem 25 of Chapter 5, which we state below.

Theorem 23 and Corollary 13:

Assume X is connected with $|\,E(X)\,| \neq |\,V(X)\,|$, G is abelian and Y is a single voltage cover of X by the cyclic group $Z_d = \langle \tau \rangle$ such that Y is connected. Then the reduced Stickelberger element may be written in the following form:

$$\Theta_{Y/X} = K(\tau - 1)^2 \tau^{-1}, \qquad \text{for some nonzero integer } K \text{ independent of } d.$$

In addition, we have

$$|\mathcal{J}(Y)| = |\mathcal{J}(X)||K|^{d-1}d.$$

Chapter 5: Towers of Voltage Graphs and Iwasawa Theory

Theorem 25, which is a direct consequence of the preceding result, says that given a single voltage cyclic p-tower of derived graphs, we get an order formula for the Jacobian of each derived graph in the tower. Furthermore, this order formula is independent of the single voltage generator chosen.

Theorem 25:

Let $X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_m \leftarrow \cdots$ be a cyclic single voltage p-tower of derived graphs over base graph X, where all X_m are assumed to be connected. Then we have that

$$|\mathcal{J}(X_m)| = |\mathcal{J}(X)||K|^{p^m-1}p^m$$

where K is as in Theorem 23.

The main result of Chapter 5—which is the culmination of this dissertation—is Theorem 27, which establishes the order of the Sylow p-subgroups of the finite Jacobians of a cyclic voltage p-tower of graphs. It is analogous to the classical result in Iwasawa Theory which establishes the order of the Sylow p-subgroups of the class groups in towers of certain infinite extensions of a number field.

Theorem 27:

Let $X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_m \leftarrow \cdots$ be a cyclic voltage p-tower, where all X_m are connected. Let $\mathcal{J}_p(X_m)$ be the Sylow p-subgroup of $\mathcal{J}(X_m)$. Then there are nonnegative integers μ and λ and an integer ν such that

$$|\mathcal{J}_p(X_m)| = p^{e_m} \qquad \text{where} \qquad e_m = \mu p^m + \lambda m + \nu$$

for all $m \geq m_0$ for some $m_0 \geq 0$.

All Theorems are original to this dissertation unless explicitly stated (and cited) otherwise.

Chapter 2

Covering Graphs and Voltage Graphs

We begin Section 2.1 by defining a covering graph in general. In Section 2.2 we then go on to describe a specific type of covering graph, called a *derived graph*, that arises from what is called a *voltage graph*—where elements from a group (which may be finite or infinite) are assigned to the edges of a fixed base graph X. We then give a computationally effective way of constructing the adjacency matrix, hence Laplacian matrix, of a derived graph. We briefly describe a generalization of voltage graphs called *permutation voltage graphs*. We then remark on how graphs will be oriented for the remainder of this dissertation.

In Section 2.3, we determine conditions for when a derived graph is connected. We put this in terms of the *voltage adjacency matrix*. In Section 2.2.3, we define two important voltage assignments—*the constant voltage assignment* and *the single voltage assignment*. In Section 2.4, we give the definition of an *intermediate covering graph* $\widetilde{X}$ and then state the Fundamental Theorem of Galois Theory for graphs. We then show that given a voltage graph with derived graph Y such that Y is connected,

Y/X is a normal (i.e., Galois) extension, and conversely, if Y/X is a normal extension with Galois group G, then there exists a voltage assignment such that (X, G, α) is a voltage graph with derived graph Y. Lastly, we show that when $\widetilde{X}$ is an intermediate covering graph corresponding to $H \trianglelefteq G$, then $\widetilde{X}/X$ is always a voltage graph. Furthermore, we show that when Y/X is a single voltage cover, there exists a choice of coset representatives of H in G such that $Y/\widetilde{X}$ is a single voltage cover.

Finally, in Section 2.5, for Y a derived graph (which may be infinite), we consider *the group of divisors of* Y, *the group of principal divisors of* Y, *the Picard group of* Y, and *the Laplacian of* Y, all as $\mathbb{Z}$ and $\mathbb{Z}[G]$ modules, where G is any finite or infinite group. This leads to the definition of *the reduced Stickelberger element*, which we then relate to *the Stickelberger element* in [HMSV19].

2.1 Basic Theory of Covering Graphs and Voltage Graphs

We begin this section with some basic definitions.

Definition 13 (Neighborhood of a Vertex)**.** *Viewing X as a topological space, a neighborhood of a vertex v in a directed graph X is obtained by taking $\varepsilon = \frac{1}{3}$ of each edge at v, where each edge is assigned a length of 1. (Note here that ε was arbitrarily chosen to be $\frac{1}{3}$, but any $0 < \varepsilon < \frac{1}{2}$ will suffice.)*

Looking at a neighborhood of a vertex, v, is essentially "zooming" in on v—looking at what edges are coming into v and what edges are going out of v.

Definition 14 (Covering Graph)**.** *An undirected graph Y is a covering of an undirected graph X if, after arbitrarily directing the edges of X, there is an assignment of directions to the edges of Y and an onto graph homomorphism $\pi : Y \to X$ sending neighborhoods of Y one-to-one onto neighborhoods of X which preserve directions. We call such π a covering map.*

So a covering of X takes vertices and edges of Y to vertices and edges of X, while also preserving directions. We call $\pi^{-1}(x)$ the *fiber* of π above x. It is all of the vertices in Y that "lie above" $x \in X$, i.e.

$$\pi^{-1}(x) = \{y \in Y \mid \pi(y) = x\}.$$

Definition 15 (d-Sheeted Covering)**.** *A d-sheeted covering means every fiber contains exactly d elements, i.e.,*

$$|\pi^{-1}(x)| = d \ \forall x \in V(X).$$

One way to construct a covering map is to first start with an undirected, connected graph X. Next, fix an orientation of X (i.e. direct the edges of X). Now pick a spanning tree T of X. For a d-sheeted covering, make d copies of T. This gives the vertices of Y. So we can view Y as the set of points (x, i) where $x \in V(X)$ and $i = 0, \cdots, d-1$. Note, we could also take $d = \infty$ (any infinite cardinal). Lastly, we lift the edges of X left out of T to get the edges of Y (but in a way that preserves adjacency and direction).

Example 1. Let X be the triangle in Figure 2.1 below. We will construct a 2-sheeted covering of X. First orient the edges as in Figure 2.2 below. Next pick a spanning tree T of X (indicated in red), as shown in Figure 2.3. Now we make two copies of

T as shown in Figure 2.4. So we have that $\pi^{-1}(u) = \{u', u''\}, \pi^{-1}(w) = \{w', w''\}$ and $\pi^{-1}(v) = \{v', v''\}$. Lastly, we lift the edges of X left out of T to get the remaining edges in Y, but in a way that preserves direction and adjacency. So we have two choices: either we draw an edge from w' to u' (and so also from w'' to u''), which would result in two copies of X. Or we draw an edge from w' to u'' and hence an edge from w'' to u' as shown in Figure 2.5.

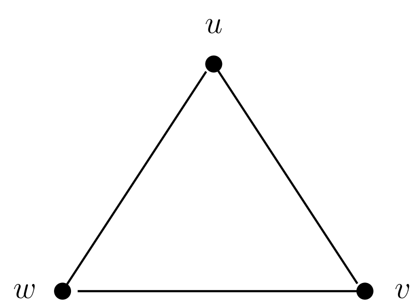

Figure 2.1: Base graph X

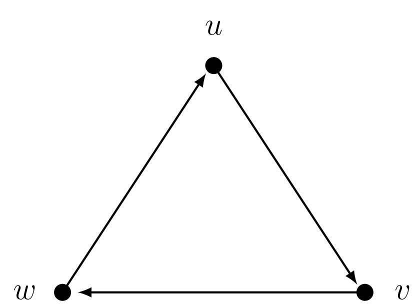

Figure 2.2: Base graph X with oriented edges

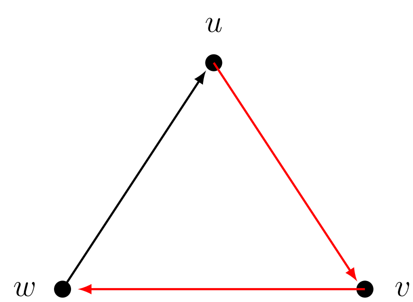

Figure 2.3: Spanning tree T of base graph X

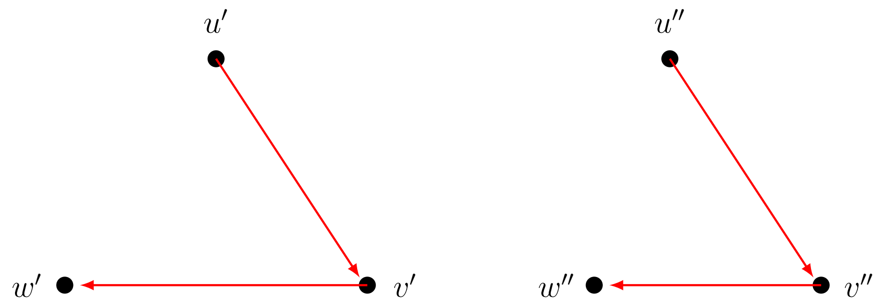

Figure 2.4: Two copies of tree T

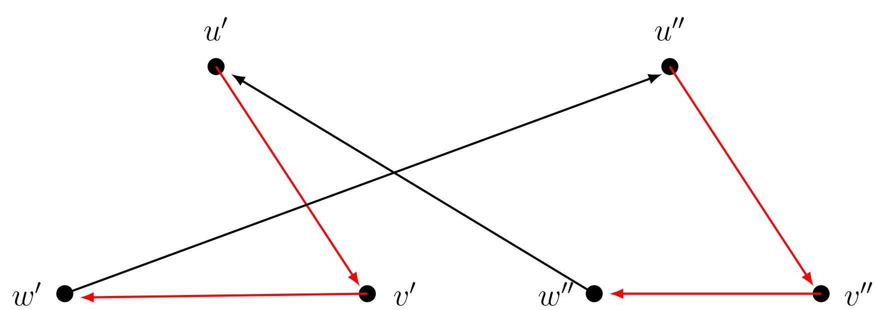

Figure 2.5: A 2-Sheeted Covering Y of X

We now give the definition of a *deck transformation*, which relates directly to Section 2.4, where we discuss Galois coverings.

Definition 16 (Deck Transformation)**.** *A deck transformation of a covering* $\pi : Y \to X$ *is a graph isomorphism* $\sigma : Y \to Y$ *such that* $\pi \circ \sigma = \pi$*. The set of all deck transformations of* π *forms a group under composition, called the deck transformation group* $Deck(Y/X)$*, also known as* $Aut(Y/X)$*.*

We will now look at a specific type of covering graph, called a *derived graph.*

2.2 Construction of Voltage Graphs and Examples

We first start with an undirected, finite and connected graph X and a group G (which may be finite or infinite). We assume, for convenience that X has no loops and no multiple edges; but the discussion easily generalizes to multigraphs. Next, we arbitrarily orient the edges of X. Then we label the forward-directed edges of X with elements from the group G. (Note: the edges of X do not have to labeled with different group elements. For instance, each edge could be assigned the group identity element.) These labels are referred to as the *voltages* and the assignment itself is called the *voltage assignment.* Furthermore, if we have a directed edge which goes from u to v labeled with the group element τ, then label the edge which goes from v to u by the group element's inverse, namely τ^{-1} :

$$u \bullet \xrightarrow{\quad\quad\tau\quad\quad} \bullet\ v$$

$$u \xleftarrow[\tau^{-1}]{} v$$

Definition 17. *Let X be a graph whose edges have been oriented, and let G be a group (finite or infinite). For a fixed orientation of the edges of X, let $E(X)^+$ denote the set of forward-directed edges of X; and let $E(X)^-$ denote the same edges but each with the reverse orientation (so each undirected edge of X becomes two edges in the disjoint union of $E(X)^+$ and $E(X)^-$). An (ordinary) voltage assignment is a map*

$$\alpha : E(X)^+ \cup E(X)^- \to G$$

such that if $e_{i,j} \in E(X)^+$ and $\alpha(e_{i,j}) = \alpha_{i,j} \in G$, then $e_{j,i} \in E(X)^-$ and $\alpha(e_{j,i}) = \alpha_{i,j}^{-1}$ (the inverse group element), where $e_{i,j}$ denotes the directed edge from v_i to v_j.

The triple (X, G, α) is called an ordinary voltage graph. The values of α are called the voltages and G is called the voltage group.

Note that a voltage assignment α is uniquely determined by its values on $E(X)^+$, so we will henceforth only specify α on the forward-directed edges of X.

Any such voltage assignment can be codified by its $n \times n$ *voltage adjacency matrix*:

$$A_\alpha = \left(\alpha_{i,j}\right)$$

where the i, j entry is zero if there is no edge between v_i and v_j; and the diagonal entries (for our graphs) are zero. (Note that the *voltage adjacency matrix* is also defined in [DZZ19] as Definition 2.14.)

The entries of A_α are from the integral group ring $\mathbb{Z}[G]$, and A_α is "inverse symmetric" in the sense that its transpose, A_α^t, is the same as A_α, but with all group elements in nonzero entries inverted. (As usual, we identify the integer k with the group ring element $k\tau_0$, where, for formality, τ_0 is the identity of G.) Changing the orientation of X simply results in a voltage assignment where some $\alpha_{i,j}$ are inverted.

(See Section 2.2.2 for more on orientation).

The purpose of assigning voltages to the graph X, called *the base graph*, is to obtain an object called *the derived graph*, which we will call Y here. To get the vertices of Y, we simply make $d = |G|$ copies of each vertex $x \in V(X)$ labeling them as $x_{\tau_0}, x_{\tau_1}, x_{\tau_2}, \cdots, x_{\tau_{d-1}}$where $\tau_0, \tau_1, \tau_2, \cdots, \tau_{d-1} \in G$ (again, we may take $d = \infty$, where the same formal construction works even if $|G|$ is uncountable). So this gives us $|G| \cdot |V(X)|$ vertices in Y. Now to construct the edges, the voltage assignment comes into play. Suppose $x \to u$ is a directed edge with voltage $\alpha(e_{x,u}) = \tau \in G$. So in X, we have the following:

$$x \bullet \xrightarrow{\quad\tau\quad} \bullet\ u$$

By definition, for each $\tau_i \in G$ we have the following edge in Y:

$$x_{\tau_i} \bullet \xrightarrow{\qquad\quad} \bullet\ u_{\tau_i \cdot \tau}$$

Thus we define the edges in Y to be (illustrated for $|\,G\,|$ finite):

$$\begin{array}{ccc}
x_{\tau_{d-1}} \bullet & \longrightarrow & \bullet\; u_{\tau_{d-1}\cdot\tau} \\
 & \vdots & \\
x_{\tau_1} \bullet & \longrightarrow & \bullet\; u_{\tau_1\cdot\tau} \\
x_{\tau_0} \bullet & \longrightarrow & \bullet\; u_{\tau_0\cdot\tau}
\end{array}$$

We do this for every fiber above $x \in V(X)$ to get the derived graph Y. If $|G| = d$, then $\pi : Y \to X$ is a d-sheeted covering map (where again, we allow d to be any infinite cardinal too). The sheets are indexed by the $\tau \in G$.

For every vertex $u \in V(X)$ the set of vertices $u \times G$ is called the fiber over u, where we denote the vertex (u, τ) in $u \times G$ by u_τ, for each $\tau \in G$. Similarly for every edge e between x and u in X, the set of edges, as constructed above, between any x_τ in the fiber over x and adjacent u_σ in the fiber over u in Y is called the fiber over e.

Note that the degree (valence) of each vertex $v_\tau = (v, \tau)$ of Y is the same as the degree of $v = \pi(v_\tau)$ in X. Also, no two vertices in the same fiber of π are adjacent in Y.

Reiterating the informal description of voltage and derived graphs, we may think of a voltage graph as taking $|G|$ copies of X, stacked vertically above each other. Then for each directed edge $v_i \to v_j$ with voltage τ, we do the following: for each sheet, indexed by $\sigma \in G$, we join the vertex $v_{i,\sigma}$ to the vertex $v_{j,\sigma\cdot\tau}$. So if we "look at the layer of sheets from above" we see vertex v_i joined to vertex v_j exactly $|G|$ times; and if v_i is not adjacent to v_j in X, then there are no edges "from above" between these two fibers. "Horizontally speaking," the edges in the derived graph Y between vertices in the fiber over v_i to vertices in the fiber over v_j are all "parallel" — they all "shift upward by the same τ." (However, some may "loop around" to "lower sheets" when τ may have finite order). This is illustrated in the following example.

Example 2. Let (K_4, Z_3, α) be such that $Z_3 = \langle\tau\rangle$ and $\alpha : E(X)^+ \to Z_3$ assigns the edge $e_{1,2}$ with voltage τ and with all other edges $e_{i,p}$ with voltage 1 $(i < p)$, as shown in Figure 2.6.

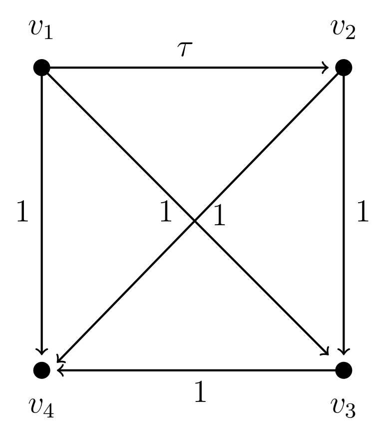

Figure 2.6: The Complete Graph on 4 vertices with voltage assignment $\alpha(e_{1,2}) = \tau$ and all other edges labeled with the identity

To get the derived graph Y, we make 3 copies of K_4. Then we remove edges $v_{1,\tau_i} \to v_{2,\tau_i}$ and add the edges in red, as shown in Figure 2.7.

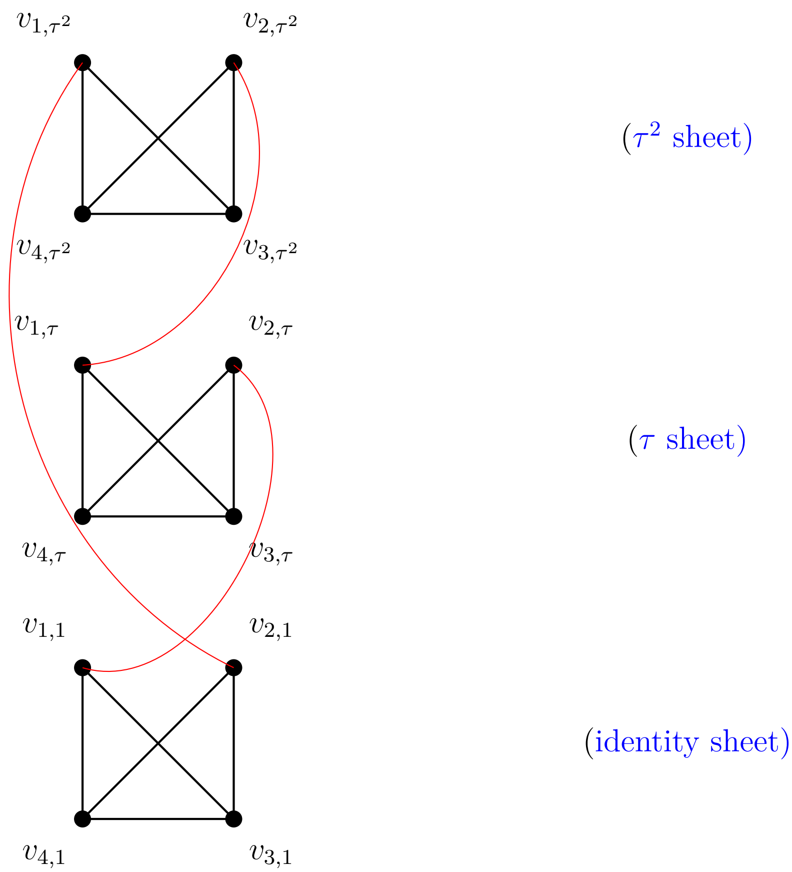

Figure 2.7: Derived graph Y corresponding to the voltage graph (K_4, Z_4, α) where α is defined above

2.2.1 Constructing the Laplacian for a Derived Graph Y

This subsection describes a computationally effective way of constructing the adjacency matrix, and hence, Laplacian for the derived graph Y from a given voltage graph (X, G, α). This construction is described in the language of $\mathbb{Z}[G]$-modules in Section 2.5. Note that G may be any finite group, not necessarily abelian.

Since group-voltages "multiply on the right" on vertices of Y, we view the Laplacian operator as acting on the right on divisors, $\vec{a}L$, and so we compute the Laplacian matrix from this perspective. Fix a listing of the vertices of X as $v_1, \cdots, v_n$ and let $G = \{\tau_0, \tau_1, \ldots, \tau_{d-1}\}$. The right regular representation of G is the homomorphism $\rho : G \to GL_d(\mathbb{Q})$, where for each $g \in G$ the $d \times d$ matrix $\rho(g)$ has a 1 in position i, j if and only if $\tau_i g = \tau_j$; all other entries are 0. Note that if G is abelian then the left and right regular representations are the same, and so ρ is just called the regular representation.

Fix any ordering of the group elements as $\tau_0, \cdots, \tau_{d-1}$. Next, for each i list the vertices of Y in the fiber over v_i as

$$v_{i,\tau_0}, v_{i,\tau_1}, \cdots, v_{i,\tau_{d-1}}.$$

Finally, list all the vertices of Y "lexicographically" with v_{i,τ_j} before v_{p,τ_q} if $i < p$ or if $i = p$ with $j < q$.

With respect to this ordering, the adjacency matrix of Y is the "tensor product" of A_α and ρ as follows: Create the $nd \times nd$ (block) matrix by replacing each nonzero entry $\alpha_{i,j}$ in A_α by the $d \times d$ matrix $\rho(\alpha_{i,j})$; replace each zero entry in A_α by the $d \times d$ zero matrix. Denote this matrix by A_Y. The fact that A_Y is an adjacency matrix for Y is the observation that, by definition of the derived graph, the adjacency matrix for Y can be written as an $n \times n$ matrix whose entries are $|\,G\,| \times |\,G\,|$ block matrices. When v_i is not adjacent to v_j, the i, j block is identically zero. When v_i is adjacent to v_j, i.e., there is a 1 in the i, j entry of A_X, then $v_{i,\tau}$ is adjacent to $v_{j,\tau\alpha_{i,j}}$, for every

$\tau \in G$; thus, with respect to the above labeling of the elements of G, (which we chose to be the same for all blocks) the i, j block of an adjacency matrix for Y is a matrix for $\rho(\alpha_{i,j})$ (again, keeping in mind that the matrix acts on the right on row vectors). This method is discussed briefly in [MS93], as well as in [FKL04].

Since each vertex v_σ in Y has the same degree as v in X, to create the $nd \times nd$ degree matrix of Y, likewise replace each entry $n_{i,j}$ of the degree matrix of X by the scalar matrix $n_{i,j} I_d$, where I_d is the $d \times d$ identity matrix (noting that all off diagonal entries are thus replaced by the zero matrix). Denote this (diagonal) matrix by D_Y.

The Laplacian of Y is then $D_Y - A_Y$; and the reduced Laplacian, $\widetilde{L}_Y$, is obtained from it by deleting the i^{th} row and column for any i (but *not* the i^{th} *block* row and column!).

Example 3. Let (X, G, α) where X is K_3, $G = Z_3 = \{1, \tau, \tau^2\}$ and $\alpha : E(X)^+ \to G$ as shown below in Figure 2.8. We compute the Laplacian matrix of the derived graph Y via the process described above. Then computing the Smith Normal Form of the Laplacian yields the Jacobian of Y.

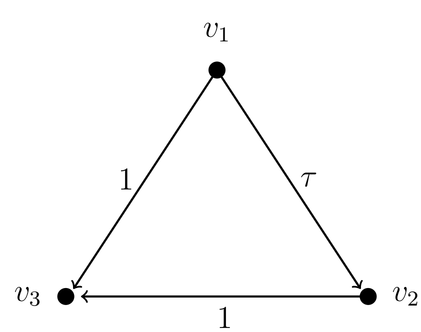

Figure 2.8: Voltage graph (K_3, Z_3, α)

Then we have that the voltage adjacency matrix is

$$A_\alpha = \begin{pmatrix} 0 & \tau & 1 \\ \tau^{-1} & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Now fix a listing of the elements of G as $\tau^2, \tau, 1$. Then we get that following matrices that correspond to the regular representation of G, $\rho : G \to GL_3(\mathbb{Q})$:

$$\rho(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\rho(\tau) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\rho(\tau^2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Next, list the vertices of Y as

$$v_{1,\tau^2}, \quad v_{1,\tau}, \quad v_{1,1}, \qquad v_{2,\tau^2}, \quad v_{2,\tau}, \quad v_{2,1}, \qquad v_{3,\tau^2}, \quad v_{3,\tau}, \quad v_{3,1}$$

Then the adjacency matrix of the derived graph, Y, is

$$\begin{aligned} A_Y &= A_\alpha \otimes \rho \\ &= \begin{pmatrix} 0_3 & \rho(\tau) & \rho(1) \\ \rho(\tau^{-1}) & 0_3 & \rho(1) \\ \rho(1) & \rho(1) & 0_3 \end{pmatrix} \end{aligned}$$

(where 0_3 denotes the 3×3 zero matrix). The degree matrix of the derived graph, Y, is

$$D_Y = \begin{pmatrix} 2I_3 & 0_3 & 0_3 \\ 0_3 & 2I_3 & 0_3 \\ 0_3 & 0_3 & 2I_3 \end{pmatrix}$$

From this, we get that the Laplacian of Y to be

$$
\begin{aligned}
L_Y &= D_Y - A_Y \\
&= \begin{pmatrix} 2I_3 & -\rho(\tau) & -\rho(1) \\ -\rho(\tau^{-1}) & 2I_3 & -\rho(1) \\ -\rho(1) & -\rho(1) & 2I_3 \end{pmatrix}
\end{aligned}
$$

$$
= \begin{pmatrix}
2 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\
0 & 2 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 2 & 0 & -1 & -0 & 0 & 0 & -1 \\
0 & -1 & 0 & 2 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 & 2 & 0 & 0 & -1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & -1 \\
-1 & 0 & 0 & -1 & 0 & 0 & 2 & 0 & 0 \\
0 & -1 & 0 & 0 & -1 & 0 & 0 & 2 & 0 \\
0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 2
\end{pmatrix}
$$

Computing the Smith Normal form via SAGE of $\widetilde{L}_Y$, the reduced Laplacian, (where, recall, this means deleting a row and its corresponding column), we get

$$\mathcal{J}(Y) = \mathbb{Z}/9\mathbb{Z}.$$

We describe a generalization of voltage graphs; however this generalization is not used in our dissertation.

A permutation voltage assignment for X is a function ϕ from the forward-directed edges of X into a symmetric group S_d, for any d (where $d = \infty$ also works). To each such permutation voltage graph there is associated a permutation derived graph Y, whose vertex set is $V(X) \times \{1, \cdots, d\}$ and whose edge set is $E(X) \times \{1, \cdots, d\}$. For each u in X, label the vertices above as u_i in Y for $i = 1, \cdots, d$. If the edge e of the base graph X goes from u to v and if the voltage on e is the permutation σ, then for $i = 1, \cdots, d$ the edge e_i of the derived graph Y is defined to go from the vertex u_i to the vertex $v_{\sigma(i)}$. Note that the cardinality of each fiber in the permutation derived graph is d, the number of permuted objects.

Although we observed this independently, we subsequently found this generalization in [GT87]. In this dissertation we will primarily deal with derived graphs where the group generated by all voltages acts transitively on the sheets and is abelian. Since any transitive permutation representation of an abelian group is equivalent to the regular representation, this generalization reduces to ordinary derived graphs in this case.

2.2.2 Orienting Graphs

At the outset we chose some labeling of the vertices of the base graph X as $v_1, \ldots, v_n$. This already imposes a natural lexicographic orientation on X, namely whenever there is an edge between v_i and v_j, orient the edge $v_i \to v_j$ if $i < j$. (Not all orientations are obtained this way, because X may have cycles.) With respect to the chosen labeling of vertices, the adjacency matrix, A, of the undirected graph X is uniquely determined. Any voltage assignment with respect to the lexicographic orientation α is now obtained by choosing group elements $\alpha_{i,j}$ to insert in the nonzero strictly upper triangular entries of A; the lower triangular entries are then forced to be the respective inverse group elements. For this fixed labeling of the vertices, but with any other orientation, a voltage is chosen for the oriented edge $v_p \to v_q$ and inserted in the p, q entry of A_α when $p < q$ or the q, p entry when $q < p$. So different orientations simply constitute whether one chooses to fill entry p, q or entry q, p (and the transpose entry is then forced). Thus, given a fixed labeling of the vertices of X, it is more systematic to describe how to fill the upper triangular portion of A_α. In other words, the notion of orientation can be dispensed with entirety, once one chooses an integer labeling of the vertices of the base graph. (Any other labeling of the vertices of X corresponds to a permutation of the rows and columns of A, namely PAP^{-1}, for some permutation matrix P.).

Any derived graph Y over X inherits a natural orientation from X. Namely, because no two vertices of Y in the same fiber of the covering map are adjacent in Y, any edge $u_\sigma \sim v_\tau$ in Y may be oriented $u_\sigma \to v_\tau$ if $u \to v$ is an edge in X (or $u_\sigma \leftarrow v_\tau$ if $u \leftarrow v$ is an edge in X). This is also compatible with π in the sense that π maps oriented edges of Y to edges of X with the same orientation.

The labeling of vertices in each fiber may be chosen, as we did above, in any way that is compatible with constructing $d \times d$ matrices for the regular representation of G.

For chains of coverings, adopting these consistencies are important for constructing examples, as we will see in Section 5.1.

We now go on to define two important voltage assignments in the next two sections.

2.2.3 The Constant and Single Voltage Assignments

Definition 18. *Let X be a connected base graph with vertices $v_1, \cdots, v_n$ and put the natural lexicographic resulting orientation on it, namely, $v_i \to v_j$ when $i < j$ as described in Section 2.2.2. Take G to be a cyclic group of order d generated by τ. Define the constant voltage assignment to be the one whose adjacency matrix puts τ in entry i, j for all $i < j$ whenever there is an edge between v_i and v_j (i.e. put τ in the upper triangular portion of the matrix whenever such an edge exists). Then put τ^{-1} in all entries of the lower triangular portion of the matrix, again whenever v_i is adjacent to v_j.*

Intuitively speaking, for constant voltage assignments, we are taking $|G|$ copies of the base graph X, but "rerouting" each edge by erasing it and connecting it to the "next sheet up". So for instance, if $v_i \to v_j$ in X, then $v_{i,\tau}$ goes to v_{j,τ^2} in Y.

We now define the *single voltage assignment.*

Definition 19. *Take a connected base graph X with vertices $v_1, \cdots, v_n$ and put the natural lexicographic resulting orientation on it, namely, $v_i \to v_j$ when $i < j$ as in Section 2.2.2. Take G to be a cyclic group of order d generated by τ. Define the single voltage assignment to be the one whose voltage adjacency matrix has τ in entry $1, 2$,*

τ^{-1} *in entry* $2, 1$, *and all other off-diagonal entries the identity.*

Intuitively speaking, for single voltage assignments, we are taking $|G|$ copies of the base graph X, but only "rerouting" the one edge $v_1 \to v_2$ by erasing it and connecting v_{1,τ^i} in sheet τ^i to $v_{2,\tau^{i+1}}$ in the "next layer up indexed by τ." The $d-1$st vertex $v_{1,\tau^{d-1}}$ "loops back" to be adjacent to v_{1,τ^0} in the identity sheet.

The choice of putting the single non-identity voltage τ on edge $v_1 \to v_2$ is for convenience, and allows arbitrary n and d. The Jacobians will, however, depend on which (directed) edge has the non-identity voltage. Note that Figure 2.7 is an example of a single voltage assignment derived graph. We will see more examples of this voltage assignment in Section 2.5.1.

2.3 Connectivity of Derived Graphs

Given a connected, undirected base graph X and a group G, we wish to determine when Y, the derived graph, will be connected. We will describe this in the case where $|G| < \infty$ only. Fix an orientation on X (as discussed in 2.2.2). We first define the following:

Definition 20. *Given a walk* W *on a voltage graph* (X, G, α)

$$W : w_1 \xrightarrow{\gamma_1} w_2 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_m} w_{m+1},$$

where w_i *are arbitrary (not necessarily distinct) vertices in* X *and* $\gamma_i \in G$ *are the* *voltages, the net voltage on W is defined to be the product of the voltages on the edges of W in the order and direction of the walk, denoted (by extending the definition of α) by*

$$\alpha(W) = \gamma_1\gamma_2 \cdots \gamma_m.$$

Definition 21. *A lift of W to Y is any walk*

$$\widetilde{W} : w_{1,\sigma_1} \to w_{2,\sigma_2} \cdots \to w_{m,\sigma_m}$$

such that $\pi(w_{i,\sigma_i}) = w_i \in V(X)$ (where $\pi : Y \to X$ is the covering map).

For each walk W in X (as above) and each $w_{i,\sigma} \in \pi^{-1}(w_i)$ there is a *unique* lift $\widetilde{W}$ of W to Y as follows: $w_{1,\sigma}$ is adjacent to exactly one vertex in Y above w_2, namely $w_{2,\sigma\gamma_1}$. Inductively, each $w_{i,\sigma\gamma_1\cdots\gamma_{i-1}}$ is adjacent to a unique vertex over w_{i+1} in X, namely $w_{i+1,\sigma\gamma_1\cdots\gamma_i}$. Continue this way to obtain $\widetilde{W}$.

This proves the special case of the following result for arbitrary coverings $\pi : Y \to X$ (not necessarily voltage coverings):

Proposition 7. *Suppose Y is a covering of X. Let W be a walk in X. Then W has a unique lift to $\widetilde{W}$, a walk in Y once the initial vertex of $\widetilde{W}$ is fixed.*

Proof. [Ter11] Proposition 13.1. □

Now let $g \in G$ and $\sigma \in G$ be a sheet index. If w_i is a vertex in X, then

$$g : w_{i,\sigma} \mapsto w_{i,g\sigma}$$

defines a *left group action* on the set of vertices in each fiber of π.

Remarks:

(i) This action by each $g \in G$ simultaneously on all fibers sends edges of Y to edges of Y.

(ii) This gives a (faithful) homomorphism

$$G \to \mathrm{Deck}(Y/X),$$

where we consider G acting trivially on X.

(iii) G acts as the left regular representation on each fiber (i.e., acts transitively and the stabilizer of any vertex is the identity subgroup of G).

The expression "regular representation" will mean "permutation isomorphic to the regular representation.

(iv) G permutes (transitively) the set of all lifts of W to walks $\widetilde{W}$ in Y.

Note that the preceding remarks are valid for $|G| = \infty$ too.

We now give the following definition, which leads to the conditions for when a derived graph Y is connected. For all $w \in V(X)$, define

$$G_w = \langle \alpha(C_w) \mid C_w \text{ is a cycle in } X \text{ starting and ending at } w \rangle \leq G.$$

In [GT87], this is called the *local voltage group.*

Observe that if W is a walk from $w = w_1$ to w_{m+1}, as above, and $t = \alpha(W)$ then

$$G_{w_{m+1}} = tG_w t^{-1}.$$

Although the next Theorem was proven independently, it can be found in [GT87] as Corollary 2 on page 88.

Theorem 4. *Let (X, G, α) be a voltage graph where X is connected and let Y be the derived graph. Then Y is connected if and only if $G_w = G$. (This does not depend on the choice of w or the orientation of X.)*

Proof. Since X is connected there is a path between any two vertices. Such paths can be lifted to Y, so there is a path in Y between any two fibers. If follows that Y is connected if and only if for any (hence every) fixed $w \in X$ and any $\sigma, \mu \in G$ there is a path in Y from w_σ to w_μ (i.e., the vertices in the same fiber $\pi^{-1}(w)$ are connected in Y). Such paths are lifts to Y by closed walks in X. Let C_w be a closed walk in X starting at w.

$$C_w : w = w_1 \xrightarrow{\gamma_1} w_2 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_m} w_{m+1} = w$$

where $w_i \in V(X)$ and $\gamma_i \in G$ represent the voltages. Then in Y we have

$$\widetilde{C}_w : w_\sigma = w_{1,\sigma} \to w_{2,\sigma\gamma_1} \to w_{3,\sigma\gamma_1\gamma_2} \to \cdots \to w_{m+1,\sigma\gamma_1\cdots\gamma_m} = w_{\sigma\gamma_1\cdots\gamma_m}$$

where $\widetilde{C}_w$ is the unique lift of C_w starting on sheet σ, $w_{i+1,\sigma\gamma_1\cdots\gamma_i}$ is in the fiber over w_{i+1} and $\sigma\gamma_1 \cdots \gamma_i$ indicates the sheet.

Now let $t = \alpha(C_w) = \gamma_1 \cdots \gamma_m$ be the product of all of the voltages taken over the closed walk, C_w, i.e. the net voltage. Letting $\mu = \sigma t$, we see that there is a path from w_σ to $w_\mu \in Y$.

We do this for every closed walk C_w starting at w, where w is fixed but arbitrary. Then we see that there exists a path from w_σ to w_μ $\forall \sigma, \mu \in G$ if and only if the subgroup generated by all of the net voltages for every closed path in X is G itself, i.e.

$$G_w = \langle \alpha(C_w) \mid C_w \text{ is a cycle in } X \text{ starting and ending at } w \rangle = G.$$

Note that this does not depend on the choice of w by the above observation. □

Corollary 1. *Fix any $w \in X$. Then Y is connected if and only if for every $\mu \in G$ there is a closed walk starting and ending at w whose net voltage is equals μ.*

Proof. By Theorem 4, Y is connected if and only if for every $\mu \in G$, we can write

$$\mu = \alpha(C_1) \cdots \alpha(C_k)$$

for some walks C_i starting and ending at w. Since the latter product is seen to equal $\alpha(C_1 \cdots, C_k)$ (the walks taken in succession) we get $\mu = \alpha(C_w)$ for some closed walk C_w. This gives the corollary (the converse being trivial). □

Exercises (these can be found in [GT87]): For X connected,

(i) The number of connected components of Y is $|G : G_w|$ for any $w \in V(X)$.

(ii) G acting on the left on Y as above permutes transitively the connected components of Y.

2.3.1 Connectedness and the Voltage Adjacency Matrix

Now we relate connectivity to the voltage adjacency matrix, A_α. First note the following

Proposition 8. *The number of walks of length k in X from v_i to v_j is the entry in position (i, j) of the matrix A^k.*

Proof. Refer to Lemma 2.5 in [Big93] □

We now relate this to the voltage adjacency matrix, where the elements of this matrix lie in the integral group ring $\mathbb{Z}[G]$. Raising the voltage adjacency matrix to the kth power, $(A_\alpha)^k$, gives the number of walks of length k as well as the net voltage associated to each of these walks, starting at v_i and ending a v_j.

Theorem 5. *Let A_α be a voltage adjacency matrix and let k be a positive integer. Suppose the i, j entry of A_α^k is the group ring element*

$$\kappa_{i,j}(k) = a_0\tau_0 + a_1\tau_1 + \cdots + a_{d-1}\tau_{d-1} \tag{2.1}$$

where the a_i are nonnegative integers (that depend on k) and $G = \{\tau_0, ..., \tau_{d-1}\}$. Then for each t with $0 \leq t \leq d-1$ there are exactly a_t distinct walks of length k from v_i to v_j that have net voltage τ_t. Thus the total number of walks of length k (of any net voltage) is $a_0 + a_1 + \cdots + a_{d-1}$.

Proof. When $k = 1$, the result follows since $A_\alpha^1 = A_\alpha$. Now assume it holds for $k = L$. There is a bijection between the number of walks of length $L + 1$ from v_i to v_j and the number of walks of length L from v_i to v_h, where v_h is adjacent (or equal) to v_j. We have that

$$\sum_{v_h \sim v_j} \kappa_{i,h}(L) = \kappa_{i,1}(L) + \kappa_{i,2}(L) + \cdots + \kappa_{i,n}(L)$$

is the sum of all net voltages over walks from v_i to v_h for all h. This gives the total number of walks from v_i to v_h, as well as the net voltage associated to each walk. Multiplying this sum on the right by $\alpha_{h,j}$, where this denotes the voltage on X going from vertex v_h to vertex v_j, we get

$$\sum_{v_h \sim v_j} \kappa_{i,h}(L)\alpha_{h,j} = \kappa_{i,j}(L + 1).$$

This completes the induction proof. □

Corollary 2. *Let A_Y be the adjacency matrix for Y as in Section 2.2.1 ($A_Y = A_X \otimes \rho$). In the notation of that construction, the number of walks of length k from v_{i,τ_p} to v_{j,τ_q} is equal to the $(v_{i,\tau_p}, v_{j,\tau_q})$-entry in A_Y^k.*

Proof. Since substitution with block matrices commutes with raising A_α to the kth power, the result follows immediately. □

Next, for any group ring element $\kappa_{i,j}(k)$ as in (2.1) define the group $\langle \kappa_{i,j}(k) \rangle$ to be the subgroup of G generated by all τ_t that have $a_t \neq 0$ in expression (2.1). (Adopt the convention that this subgroup is the identity if all $a_t = 0$.) So $\langle \kappa_{i,j}(k) \rangle$ is the subgroup generated by all net voltages over all walks of length k from v_i to v_j. In particular, $\langle \kappa_{i,i}(k) \rangle$ is the subgroup generated by the net voltages of all cycles of length k from v_i to itself.

Corollary 3. *The voltage graph Y is connected if and only if for any (hence every) i, G is generated by the collection of all subgroups $\langle \kappa_{i,i}(k) \rangle$, as k runs over all positive integers. (Note, the latter is just the subgroup we denoted as G_v, which again, [GT87] calls the Local Voltage Group).*

2.3.2 Covers of K_n

In this section, we now fix the base graph to be $X = K_n$, the complete graph on n vertices.

We have that every closed walk can be written as a walk over successive triangles since for any closed walk C in K_n,

$$w_1 \xrightarrow{e_{1,2}} w_2 \xrightarrow{e_{2,3}} \cdots w_m \xrightarrow{e_{m,1}} w_1$$

we can rewrite it as

$$w_1 \xrightarrow{e_{1,2}} w_2 \xrightarrow{e_{2,3}} w_3 \xrightarrow{e_{1,3}^{-1}} w_1 \xrightarrow{e_{1,3}} w_3 \xrightarrow{e_{3,4}} w_4 \xrightarrow{e_{1,4}^{-1}} w_1 \xrightarrow{e_{1,4}} \cdots \xrightarrow{e_{m-1,m}} w_m \xrightarrow{e_{m,1}} w_1$$

This yields the following Corollary to Theorem 4:

Corollary 4. *For $X = K_n$ and G any (finite) group, Y is connected if and only if the subgroup generated by all net voltages over all triangles in K_n is all of G.*

Now we fix the voltage group to be $G = Z_d$, the cyclic group of order d.

Corollary 5. *Let (K_n, Z_d, α) be a voltage graph where $\alpha : E(K_n)^+ \to Z_d$ and d is prime. We have that Y is disconnected if and only if α assigns every triangle net* *voltage 1.*

More generally,

Corollary 6. *Let $d = p_1^{\beta_1} \cdots p_r^{\beta_r}$ be the unique factorization of d into distinct prime powers. Then Y is connected if and only if for each i there is a triangle C_i in X containing w, such that $p_i^{\beta_i}$ divides the order of $\alpha(C_i)$.*

Proof. Y is connected if and only if the subgroup generated by the net voltage elements over all triangles in X starting and ending at a fixed w equals G. But the cyclic group Z_d is generated by $\tau_1, \cdots, \tau_r$ if and only if for each prime p_i dividing d, the Sylow p_i-subgroup of Z_d is generated by the set of Sylow p_i subgroups of $\langle\tau_1\rangle, \cdots, \langle\tau_r\rangle$. Since Z_d has a unique subgroup of order k for every $k|d$, the latter collection of Sylow p_i-subgroups generates the Sylow p_i-subgroup of Z_d if and only if one of them contains the full Sylow p_i-subgroup of Z_d, but this is if and only if one τ_j has order divisible by $p_i^{\beta_i}$, as claimed. □

Corollary 7. *In the notation of Section 2.3.1, for X a complete graph, a derived graph Y is connected if and only if $\langle\kappa_{1,1}(3)\rangle = G$. In particular, if d is a prime, Y is connected if and only if $\kappa_{1,1}(3)$ is nonzero.*

Example 4. (a) Let (K_5, Z_d, α) be a voltage graph where $\alpha : E(K_5)^+ \to Z_d$ is the constant voltage assignment. Then its voltage adjacency matrix is

$$A_\alpha = \begin{pmatrix} 0 & \tau & \tau & \tau & \tau \\ \tau^{-1} & 0 & \tau & \tau & \tau \\ \tau^{-1} & \tau^{-1} & 0 & \tau & \tau \\ \tau^{-1} & \tau^{-1} & \tau^{-1} & 0 & \tau \\ \tau^{-1} & \tau^{-1} & \tau^{-1} & \tau^{-1} & 0 \end{pmatrix}$$

and so

$$A_\alpha^3 = \begin{pmatrix} 6\tau + 6\tau^{-1} & 10\tau + 3\tau^{-1} & 12\tau + \tau^{-1} & \tau^3 + 12\tau & 3\tau^3 + 10\tau \\ 3\tau + 10\tau^{-1} & 6\tau + 6\tau^{-1} & 10\tau + 3\tau^{-1} & 12\tau + \tau^{-1} & \tau^3 + 12\tau \\ \tau + 12\tau^{-1} & 3\tau + 10\tau^{-1} & 6\tau + 6\tau^{-1} & 10\tau + 3\tau^{-1} & 12\tau + \tau^{-1} \\ \tau^{-3} + 12\tau^{-1} & \tau + 12\tau^{-1} & 3\tau + 10\tau^{-1} & 6\tau + 6\tau^{-1} & 10\tau + 3\tau^{-1} \\ 3\tau^{-3} + 10\tau^{-1} & \tau^{-3} + 12\tau^{-1} & \tau + 12\tau^{-1} & 3\tau + 10\tau^{-1} & 6\tau + 6\tau^{-1} \end{pmatrix}.$$

So in K_5 there are 12 walks of length 3 from v_1 to v_1: 6 with net voltage τ and 6 with net voltage τ^{-1}. There are 3 walks of length 3 from v_1 to v_2 with net voltage τ^{-1} and 10 walks with net voltage τ. In particular, we see that Y is connected by Corollary 4.

(b) Let (K_4, Z_4, α) be a voltage graph with $Z_4 = \{1, \tau, \tau^2, \tau^3\}$ and $\alpha : E(X)^+ \to Z_4$ is shown as in Figure 2.9.

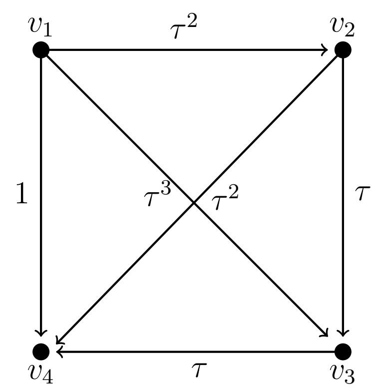

Figure 2.9: Voltage graph (K_4, Z_4, α)

Now the voltage adjacency matrix is equal to

$$A_\alpha = \begin{pmatrix} 0 & \tau^2 & \tau^3 & 1 \\ \tau^2 & 0 & \tau & \tau^2 \\ \tau & \tau^3 & 0 & \tau \\ 1 & \tau^2 & \tau^3 & 0 \end{pmatrix}$$

and so

$$A_\alpha^3 = \begin{pmatrix} 6 & 7\tau^2 & 7\tau^3 & 7 \\ 7\tau^2 & 6 & 7\tau & 7\tau^2 \\ 7\tau & 7\tau^3 & 6 & 7\tau \\ 7 & 7\tau^2 & 7\tau^3 & 6 \end{pmatrix}$$

Now because there are a total of 6 walks from v_i to v_i, for $i = 1, \cdots, 4$ all with net voltage 1_G (the identity element in G), it follows that Y is disconnected by Corollary 3.

(c) We now construct a voltage graph (K_n, Z_d, α) where the derived graph Y is *disconnected*. We then count the number of such voltage graphs. Fix the usual lexicographic orientation on K_n. Label the edges $v_1 \to v_2 \to v_3 \to \cdots \to v_{n-1}$ with elements from Z_d. This uniquely determines the remaining edge labels in such a way so that all triangles have net voltage equal to 1 as follows. Since the edge labels for $v_1 \to v_2 \to v_3 \to \cdots \to v_{n-1}$ are chosen, this uniquely determines the edge label for $v_1 \to v_n$, namely it is $\alpha(C)$, where C denotes the walk $v_1 \to v_2 \to v_3 \to \cdots \to v_{n-1}$. Similarly, the edge $v_1 \to v_3$ is labeled with the net voltage for the walk $v_1 \to v_2 \to v_3$. From this, the edge label for $v_1 \to v_4$ is determined by looking at the net voltage of the walk $v_1 \to v_3 \to v_4$. We continue this process for $v_1 \to v_5, \cdots, v_1 \to v_{n-1}$ in the same manner. Lastly, we get the remaining edge labels for

$$\begin{aligned}
&v_{n-2} \to v_n \\
&v_{n-3} \to v_n \\
&\vdots \\
&v_2 \to v_n
\end{aligned}$$

by looking at the net voltage of the walks

$$\begin{aligned}
&v_1 \to v_{n-2} \to v_n \\
&v_1 \to v_{n-3} \to v_n \\
&\vdots \\
&v_1 \to v_2 \to v_n
\end{aligned}$$

respectively.

There are $d \cdot d \cdot \cdots \cdot d \cdot d = d^{n-1}$ of these such graphs, including the case where all voltages are the identity.

2.4 Galois Theory of Covering Graphs and Voltage Graphs

We begin this section by first giving the definition of an intermediate covering graph and then state the Fundamental Theorem of Galois Theory for graphs. The following Theorems can be found in Chapter 13 and 14 of [Ter11].

Throughout this section, assume that X and all coverings of X are connected and finite.

Definition 22. *Suppose Y is a covering of X with projection map π. A graph $\widetilde{X}$ is an intermediate covering to Y/X if $Y/\widetilde{X}$ is a covering, $\widetilde{X}/X$ is a covering and the projection maps $\pi_1 : \widetilde{X} \to X$ and $\pi_2 : Y \to \widetilde{X}$ have the property that $\pi = \pi_1 \circ \pi_2$.*

If Y/X is a d-sheeted covering with projection map $\pi : Y \to X$, then it is normal or Galois if there are exactly d graph automorphisms $\sigma : Y \to Y$ such that $\pi \circ \sigma = \pi$. The Galois group is $G = Gal(Y/X) = \{\sigma : Y \to Y \mid \pi \circ \sigma = \pi\}$.

Observe that Y/X is Galois if and only if $|\mathrm{Aut}(Y/X)| = d = |G|$. By graph automorphism, we mean a one-to-one onto map of vertices and edges of Y that preserves adjacency.

Recall from Section 2.1 after Definition 15, the design of a general covering graph. Then for Y/X a normal covering graph with Galois group G, we choose one of the sheets of Y and call it sheet 1. The image of sheet 1 under an element g in G will be called sheet g. Any vertex $\tilde{x}$ in Y can then be uniquely denoted $\tilde{x} = (x, g)$, where $x = \pi(\tilde{x})$ and g is the sheet containing $\tilde{x}$. By [Ter11], the vertices of Y can also be simultaneously labeled in every fiber of π in this way so that if $u \to v$ is any directed edge in X and $(u, g_1) \to (v, g_2)$ is an edge lying over it in Y, then $(u, gg_1) \to (v, gg_2)$ is an edge in Y, for every $g \in G$ (hence these are all the edges in Y lying over $u \to v$).

By the remarks after Proposition 7, we have that for Y/X a normal covering, the Galois group $G = Gal(Y/X)$ acts transitively on the sheets of the covering and the stabilizer of any vertex is the identity of G. There are correspondences between the subgroups H of G and the intermediate graphs $\widetilde{X}$ to Y/X defined as follows.

Let $H \leq G$. The vertices of Y are of the form (x, g) where $x \in X$ and $g \in G$ indexes the sheets. The vertices of $\widetilde{X}$ are then defined to be (x, Hg). Create an edge from (a, Hr) to (b, Hs) for $a, b \in X$ and $r, s \in G$ if and only if there are $h, h' \in H$ such that there is an edge from (a, hr) to $(b, h's)$ in Y. The edge between (a, Hr) and (b, Hs) in $\widetilde{X}$ is given the direction of the projected edge between a and b in X. Check that $\widetilde{X}$ is well-defined, intermediate to Y/X and connected.

Conversely, let $\widetilde{X}$ be intermediate to Y/X, with projections $\pi : Y \to X$, $\pi_2 : Y \to \widetilde{X}$ and $\pi_1 : \widetilde{X} \to X$, as in Definition 22. Let $(x, 1) \in \pi^{-1}(x)$, where 1 indicates the 1st sheet. Then define $H = \{h \in G \mid \pi_2(x, h) = \pi_2(x, 1)\}$. Check that H is a subgroup of G.

The next result describes the Galois theory of (finite) Galois covers (it can be found in [Ter11] as Theorem 14.3).

Theorem 6 (Fundamental Theorem of Galois Theory for Graphs)**.** *Suppose Y/X is a normal covering with Galois group G and with covering map $\pi : Y \to X$.*

(i) Given a subgroup H of G, there exists a graph $\widetilde{X}$ intermediate to Y/X such that $H = Gal(Y/\widetilde{X})$.

(ii) Suppose that $\widetilde{X}$ is intermediate to Y/X. Then there is a subgroup H of G that equals $Gal(Y/\widetilde{X})$.

(iii) Two intermediate graphs $\widetilde{X}$ and $\widetilde{X'}$ are equal if and only if $Gal(Y/\widetilde{X}) = Gal(Y/\widetilde{X'})$.

(iv) We write $\widetilde{X} \leftrightarrow H$ for the above correspondences (which are inverses of each other) between graphs $\widetilde{X}$ intermediate to Y/X and subgroups H of the Galois group $G = Gal(Y/X)$.

(v) If $\widetilde{X_1} \leftrightarrow H_1$ and $\widetilde{X_2} \leftrightarrow H_2$ then $\widetilde{X_1}$ is intermediate to $Y/\widetilde{X_2}$ if and only if $H_1 \subset H_2$.

Definition 23. *Suppose we have*

$$\widetilde{X} \leftrightarrow H \subset G$$

$$\widetilde{X'} \leftrightarrow gHg^{-1} \qquad \textit{for some } g \in G$$

then we say $\widetilde{X}$ and $\widetilde{X'}$ are conjugate.

Theorem 7. *Suppose Y/X is a normal covering with Galois group G and $\widetilde{X}$ an intermediate covering corresponding to the subgroup H of G. Then $\widetilde{X}$ itself is a normal covering of X if and only if H is a normal subgroup of G, in which case $Gal(\widetilde{X}/X) \cong G/H$.*

$$\begin{tikzcd}
Y \arrow[d, dash, "\pi_2"] & 1 \arrow[d, dash] \\
\widetilde{X} \arrow[d, dash, "\pi_1"] & H \arrow[d, dash, "\trianglelefteq"] \\
X & G
\end{tikzcd}$$

Figure 2.10: Normal covering Y/X with Galois group G and intermediate covering $\widetilde{X}$ corresponding to $H \trianglelefteq G$

Note that if G is abelian, then all intermediate covers are normal.

Now we put this in terms of voltage graphs.

Theorem 8. *Let (X, G, α) be a voltage graph with Y the derived graph. If Y is connected, then Y/X is a normal cover with $Gal(Y/X) \cong G$. Conversely, given a normal (Galois) cover Y/X, with $G = Gal(Y/X)$, then Y/X is a voltage cover with the voltage group equal to $Gal(Y/X)$.*

Proof. Let (X, G, α) be a voltage graph and let Y be its derived graph. Assume Y is connected. We show that Y/X is Galois. Because G acts as automorphisms of Y that fix the fibers of π, $|\mathrm{Aut}(Y/X)| \geq |G|$. Thus, we need only to show that there are not more automorphisms of Y/X. Suppose by way of contradiction that $\lambda \in \mathrm{Aut}(Y/X)$. Then there exists $\sigma \in G$ such that $\lambda \circ \sigma$ fixes some v_μ in the fiber over $v \in X$. By Proposition 7, $\lambda \circ \sigma$ fixes all paths that start at v_μ, hence fixes the vertex at the end of each such path. Since Y is connected $\lambda \circ \sigma$ fixes all vertices of Y, so $\lambda \circ \sigma = id$ and so $\lambda = \sigma^{-1} \in G$.

Conversely, let Y/X ne a normal cover with $G = Gal(Y/X)$ of order d. As above, we may write the vertices of Y as (x, g) where $x \in X$ and $g \in G$. Suppose $u \to v$ is a directed edge in X. Then there are exactly d edges $(u, g_1) \to (v, g_2)$ in Y that map to the above edge in X under π. Since G acts transitively on the sheets of the covering, each vertex $(u, g_1) \in \pi^{-1}(u)$ is adjacent to a unique vertex $(v, g_2) \in \pi^{-1}(v)$. So for the base graph X, let G be the voltage group and define the voltage assignment by $\alpha(e_{u,v}) = g_1^{-1} g_2$. Note that $\alpha(e_{u,v})$ does not depend on the choice of edge above $u \to v$ because every edge is of the form $(u, gg_1) \to (v, gg_2)$ for some $g \in G$ and $(gg_1)^{-1}(gg_2) = g_1^{-1} g_2$. Do this for every edge in X. This yields the derived graph Y. (The element $g_1^{-1} g_2$ is called the *Frobenius element.*) □

For any Galois cover $\pi : Y \to X$ of a connected base graph X, the graph Y is necessarily connected except in the case where $G = \mathrm{Gal}(Y/X)$ is the cyclic group of order 2 and Y is two disjoint isomorphic copies of X interchanged by G. To see this, suppose $Y_1, \ldots, Y_k$ are the connected components of Y, and let H_i be the subgroup of G stabilizing Y_i for every i. Since G acts transitively on the vertices in each fiber of

π, it likewise transitively permutes the Y_i; and so all Y_i are isomorphic graphs and all H_i are isomorphic groups. Furthermore, by the transitive action, $k = |G : H_1|$. But then we may let the subgroups H_i act independently (with commuting actions) on Y, hence $H_1 \times \cdots \times H_k$ acts as automorphisms of Y; and we may let the symmetric group of degree k permute the Y_i, and hence correspondingly permute the component entries in the direct product group. All of these actions may be chosen to preserve the fibers of π as well. In other words the *wreath product* group, $H_1 \wr S_k$, is a subgroup of $\mathrm{Aut}(Y/X) \cong G$. This means

$$k|\, H_1\,| = |\, G\,| = |\, \mathrm{Aut}(Y/X)\,| \geq |\, H_1 \wr S_k\,| = |\, H_1\,|^k \cdot k!$$

If $k > 1$, we must have $k = 2$ and $|\, H_1\,| = 1$, which is the degenerate case described at the outset. Note that this degenerate case is also a *derived cover* for any X, by taking the voltage group to be the cyclic group of order 2 and all edge-voltages to be the identity element.

We choose to use "voltage cover" rather than Galois cover (when Y is connected) to emphasize the "voltage-assignment nature" of the cover.

We now show that for an intermediate covering $\widetilde{X}$ of Y/X, that corresponds to $H \trianglelefteq G$, it follows that $\widetilde{X}/X$ is always a voltage cover. We then show that for Y/X a single voltage cover, there exists a choice of coset representatives of H in G such that $Y/\widetilde{X}$ is single a voltage cover.

Theorem 9. *Let (X, G, α) be a voltage graph with derived graph Y such that Y*

is connected. If $\widetilde{X}$ is an intermediate cover of Y/X corresponding to the normal subgroup H of G, then $\widetilde{X}/X$ is a voltage graph, whose voltage adjacency matrix is the voltage adjacency matrix of Y/X, but with nonzero entries reduced modulo H (thus has entries in $\mathbb{Z}[G/H]$).

Proof. Let $\widetilde{X}$ be an intermediate cover of Y/X that corresponds to the normal subgroup H of G. So we have that $Y/\widetilde{X}$ and $\widetilde{X}/X$ are normal coverings with projection maps $\pi_1 : \widetilde{X} \to X$ and $\pi_2 : Y \to \widetilde{X}$ such that $\pi_2 \circ \pi_1 = \pi$.

Let $A_{\alpha,Y}$ be the voltage adjacency matrix for X corresponding to the derived graph Y. So $A_{\alpha,Y}$ has entries in $\mathbb{Z}[G]$. Now reduce the entries of $A_{\alpha,Y}$ modulo H, and denote this matrix A_{α_1}, which has entries in $\mathbb{Z}[G/H]$. This encodes the voltage graph $(X, G/H, \alpha_1)$, where $\alpha_1 : E(X)^+ \to G/H$. Since $Gal(\widetilde{X}/X) \cong G/H$ by Theorem 7, it follows that the derived graph of $(X, G/H, \tilde{\alpha})$ is $\widetilde{X}$ by the Fundamental Theorem of Galois Theory for Graphs. □

Corollary 8. *Let (X, G, α) be a voltage graph with $\alpha : E(X)^+ \to G$ the single voltage assignment. If $\widetilde{X}$ is an intermediate cover of Y/X corresponding to the normal subgroup H of G, then $\widetilde{X}/X$ is a voltage graph with single voltage assignment.*

Proof. This follows immediately from Theorem 9. □

Theorem 10. *Let (X, G, α) be a voltage graph with X connected, $\alpha : E(X)^+ \to G$ the single voltage assignment, $G \cong \langle \tau \rangle$ the cyclic group of order d. If $\widetilde{X}$ is an intermediate cover of Y/X corresponding to the normal subgroup H of G, then there exists a choice of coset representatives of H in G such that $Y/\widetilde{X}$ is a voltage graph* *with single voltage assignment.*

Proof. Let $H = \langle \tau^e \rangle$ where $1 < e < d$, $e|d$ and $\frac{d}{e} = f$. Fix a choice of coset representatives of $H \trianglelefteq G$ to be

$$1, \tau, \cdots, \tau^{e-1}.$$

Label the elements of H as

$$1, \tau^e, \tau^{2e}, \cdots, \tau^{(f-1)e}$$

So $G = \{\tau^i \tau^{je} = \tau^{i+je} \mid i = 0, 1 \cdots, e-1 \text{ and } j = 0, 1, \cdots, f-1\}$.

Now suppose X has vertices $v_1, \cdots, v_n$. We then describe the vertices of Y (in the ordinary way) as

$$v_{k,\tau^{i+je}} \qquad k = 1, \cdots, n \qquad i = 0, \cdots, e-1 \qquad \text{and} \qquad j = 0, 1, \cdots, f-1.$$

Let $\widetilde{X}$ be an intermediate cover of Y/X that corresponds to H. Then the vertices of $\widetilde{X}$ can be described as

$$\overline{v_{k,\tau^i}} \qquad k = 1, \cdots, n \qquad \text{and } i = 0, 1 \cdots, e-1.$$

We next define a voltage assignment $\widetilde{\alpha} : E(\tilde{X})^+ \to H$ in order to construct a derived graph $\widetilde{Y}$, that we show is isomorphic to Y.

Fixing the vertex $\overline{v_{k,\tau^i}} \in V(\widetilde{X})$, we have that the vertices in the fiber over this vertex in $\widetilde{Y}$ are of the form

$$\overline{v_{(k,\tau^i),\tau^{je}}} \qquad k = 1, \cdots, n \qquad i = 0, \cdots, e-1 \qquad \text{and} \qquad j = 0, 1, \cdots, f-1.$$

Now suppose that in X, we have the directed edge

$$v_1 \bullet \xrightarrow{\ \tau\ } \bullet\, v_2$$

(with all other edges labeled with the identity element in G).

Therefore in $\widetilde{X}$, we have the directed edge

$$\overline{v_{1,\tau^i}} \bullet \longrightarrow \bullet\, \overline{v_{2,\tau^{i+1}}}$$

for $i = 0, 1, \cdots, e-2$, and the directed edge

$$\overline{v_{1,\tau^{e-1}}} \bullet \longrightarrow \bullet\, \overline{v_{2,1}}.$$

Now we assign voltages from H to these edges in such a way that $Y \cong \widetilde{Y}$. First assign the identity voltage to the first $e-1$ directed edges as listed above; then assign τ^e to the last directed edge in $\widetilde{X}$

$$\overline{v_{1,\tau^{e-1}}} \bullet \xrightarrow{\ \tau^e\ } \bullet\, \overline{v_{2,1}}.$$

So for the first $e-1$ edges in $\widetilde{X}$, we get the edges in the derived graph $\widetilde{Y}$,

$$\overline{v_{(1,\tau^i),\tau^{je}}} \bullet \longrightarrow \bullet \ \overline{v_{(2,\tau^{i+1}),\tau^{je}}}.$$

Then for the "last" directed edge in $\widetilde{X}$, we get the following in $\widetilde{Y}$ for $j = 0, 1, \cdots, f-2$

$$\overline{v_{(1,\tau^{e-1}),\tau^{je}}} \bullet \longrightarrow \bullet \ \overline{v_{(2,1),\tau^{(j+1)e}}}$$

and for the "last" edge in $\widetilde{Y}$, we get

$$\overline{v_{(1,\tau^{e-1}),\tau^{(f-1)e}}} \bullet \longrightarrow \bullet \ \overline{v_{(2,1),1}}.$$

Now for all other edges in X we have

$$v_k \ \bullet \xrightarrow{\quad 1 \quad} \bullet \ v_l.$$

So in $\widetilde{X}$ we have

$$\overline{v_k, \tau^i} \bullet \longrightarrow \bullet \ \overline{v_l, \tau^i}.$$

Assigning these edges in $\widetilde{X}$ with the voltage $1 \in G/H$, we get the following edges in the derived graph $\widetilde{Y}$

$$\overline{v_{(k,\tau^i),\tau^{je}}} \bullet \longrightarrow \bullet \overline{v_{(l,\tau^i),\tau^{je}}}.$$

Using the correspondence

$$v_{k,\tau^{i+je}} \in V(Y) \qquad \longleftrightarrow \qquad \overline{v_{(k,\tau^i),\tau^{je}}} \in V(\widetilde{Y}),$$

we see that $Y \cong \widetilde{Y}$, as desired. $\square$

2.5 The Voltage Laplacian and Reduced Stickelberger Element

We next clarify the relationship between the $\mathbb{Z}$-module of all divisors on Y (where G, hence also Y may be infinite) and its structure as a $\mathbb{Z}[G]$-module. In each of the subsections below we describe the "$\mathbb{Z}$ structure" as item (1) and the "$\mathbb{Z}[G]$ structure" of the same object(s) as item (2). From this, we define *the reduced Stickelberger element*, and describe how it relates to the Stickelberger element defined in [HMSV19].

The group of divisors of Y:

1. By definition, the divisor group of Y, denoted as $\mathrm{Div}(Y)$, is the free $\mathbb{Z}$-module on the vertices of Y. Since the vertices are $\{v_{i,\sigma} \mid 1 \leq i \leq n, \sigma \in G\}$, these form a $\mathbb{Z}$-basis of $\mathbb{Z}$-rank $n|G|$.

2. Now let $g \in G$, $v_i \in V(X)$ and let $\sigma \in G$ be a sheet index. Then

$$g : v_{i,\sigma} \mapsto v_{i,g\sigma}$$

   defines a left group action on the set of vertices in each fiber of π. These disjoint orbits are $\pi^{-1}(v_i) = \{v_{i,\tau} \mid \tau \in G\}$ for $1 \leq i \leq n$. Since G acts as the left regular representation on each fiber, the $\mathbb{Z}$-span of each one is a free $\mathbb{Z}[G]$-module of rank 1; and $\mathrm{Div}(Y)$ is then a direct sum of these: a free $\mathbb{Z}[G]$-module of rank n, i.e.

$$\mathrm{Div}(Y) = \mathbb{Z}[G]v_{1,\tau_0} \oplus \mathbb{Z}[G]v_{2,\tau_0} \oplus \cdots \oplus \mathbb{Z}[G]v_{n,\tau_0} \tag{2.2}$$

where we, for simplicity, choose the identity representative from each orbit.

The group of principal divisors of Y:

1. To consider principal divisors in Y we need to find what vertices are adjacent to a given $v_{i,\tau}$ in Y. As in Section 2.3, $v_{i,\tau}$ is adjacent to $v_{j,\delta}$ in Y if and only if v_i is adjacent to v_j in X and one of the following holds: $v_i \xrightarrow{\alpha_{i,j}} v_j$ with $\tau\alpha_{i,j} = \delta$, or $v_j \xrightarrow{\alpha_{j,i}} v_i$ with $\delta\alpha_{j,i} = \tau$. Since $\alpha_{j,i}^{-1} = \alpha_{i,j}$ in G, in both cases $\delta = \tau\alpha_{i,j}$.

   This says (independent of the orientation on edges):

   - the unoriented degree of each vertex $v_{i,\tau}$ in Y is the same as the unoriented degree of v_i in X (call it n_i), and
   - the principal divisor "based at $v_{i,\tau}$" is, by definition,

     $$p_{i,\tau} = n_i v_{i,\tau} - \sum_{\substack{j=1 \\ v_i \sim v_j}}^{n} v_{j,\tau\alpha_{i,j}}.$$

     Then, by definition, the $\mathbb{Z}$-module of principal divisors, denoted by $\mathrm{Pr}(Y)$ is

     $$\mathrm{Pr}(Y) = \mathrm{Span}_{\mathbb{Z}}\{p_{i,\tau} \mid 1 \leq i \leq n, \tau \in G\}.$$

   Note that these principal generators are not necessarily a $\mathbb{Z}$-*basis* of $\mathrm{Pr}(Y)$.

2. Any graph automorphism permutes principal divisors, so $\mathrm{Pr}(Y)$ is a $\mathbb{Z}[G]$-module (not generally a free module). The left action of G partitions the above

set of principal divisors into orbits $\mathcal{O}_i = \{p_{i,\tau} \mid \tau \in G\}$, for $i = 1, 2, \ldots, n$, and again G acts as the regular permutation representation on each $\mathcal{O}_i$ (it acts by left multiplication on the subscript τ for $\tau \in G$). Thus if we pick a representative of each orbit, say for convenience p_{i,τ_0}, where τ_0 is the identity of G, then we get that

$$\mathrm{Pr}(Y) = \mathrm{Span}_{\mathbb{Z}[G]}\{p_{i,\tau_0} \mid 1 \leq i \leq n\}.$$

The Picard group of Y:

The Picard group is $\mathrm{Pic}(Y) = \mathrm{Div}(Y)/\mathrm{Pr}(Y)$. All these terms are both $\mathbb{Z}$ and $\mathbb{Z}[G]$-modules, as described above.

The Laplacian of Y:

1. By definition, if A_Y is the (ordinary) adjacency matrix for Y and D_Y is the degree matrix, then the Laplacian is $L_Y = D_Y - A_Y$, which, when $|G| = d$ is finite, is an $nd \times nd$ matrix with entries from $\mathbb{Z}$. By definition, L_Y can also be written as a $\mathbb{Z}$-module endomorphism of $\mathrm{Div}_{\mathbb{Z}}(Y)$ (even when $|G| = \infty$):

$$L_Y(v_{i,\tau}) = p_{i,\tau}, \qquad \text{for all } 1 \leq i \leq n \text{ and } \tau \in G$$

and this is extended by $\mathbb{Z}$-linearity to all of $\mathrm{Div}(Y)$. In particular, the $\mathbb{Z}$-module image of $\mathrm{Div}(Y)$ under the $\mathbb{Z}$-module homomorphism L_Y is $\mathrm{Pr}(Y)$, and its cokernel is $\mathrm{Pic}(Y) = \mathrm{Div}(Y)/\mathrm{Pr}(Y)$.

2. Consider the $\mathbb{Z}[G]$-module homomorphism which is defined on the above $\mathbb{Z}[G]$-

basis in (2.2) of $\mathrm{Div}(Y)$ by

$$\mathcal{L}_\alpha : \mathrm{Div}(Y) \longrightarrow \mathrm{Div}(Y) \qquad \text{by} \qquad \mathcal{L}_\alpha(v_{i,\tau_0}) = p_{i,\tau_0}, \quad 1 \leq i \leq n.$$

This map is now extended by $\mathbb{Z}[G]$-linearity to all of $\mathrm{Div}(Y)$, namely for all $\tau \in G$ by

$$\mathcal{L}_\alpha(\tau \cdot v_{i,\tau_0}) = \tau \cdot \mathcal{L}_\alpha(v_{i,\tau_0}) = \tau \cdot p_{i,\tau_0} = p_{i,\tau\cdot\tau_0} = p_{i,\tau}$$

and likewise for sums and differences of these.

Definition 24. *The $\mathbb{Z}[G]$-module homomorphism $\mathcal{L}_\alpha$ is called the voltage Laplacian of Y.*

Thus by the action of G we have

$$\mathcal{L}_\alpha(v_{i,\tau}) = p_{i,\tau} \qquad \text{for all } \tau \in G.$$

Since G acts transitively on the i^{th} G-orbit of both the vertices and the principal divisors in $\mathrm{Div}(Y)$ we see that

the image of the $\mathbb{Z}[G]$-module homomorphism $\mathcal{L}_\alpha$ is the $\mathbb{Z}[G]$-submodule $Pr(Y)$.

Finally, we compute the $n \times n$ matrix of $\mathcal{L}_\alpha$ with respect to the above $\mathbb{Z}[G]$-basis of $\mathrm{Div}(Y)$. In the j^{th} column of a matrix and i^{th} row we put the coefficient of v_{i,τ_0} in the expansion of $\mathcal{L}_\alpha(v_{j,\tau_0})$. By the above, this i,j-entry equals n_i if $i = j$ and $-\alpha_{j,i}$ if $i \neq j$ (and zero if v_i is not adjacent to v_j in X). Now this results in the i,j-entry of the *transpose* of $D_X - A_\alpha$. However, since the voltage "multiplies" on the right on divisors, we should be computing a matrix M representing $\mathcal{L}_\alpha$ where $\vec{a}M = \vec{b}$ (i.e., a row vector $\times M =$ a row vector). Such M is the transpose of the matrix we just computed (i.e., the matrix we computed acts on the left on column vectors).

This discussion shows that the voltage Laplacian endomorphism of $\mathrm{Div}(Y)$ is exactly the same as the ordinary Laplacian when we consider $\mathrm{Div}(Y)$ as a $\mathbb{Z}$-module rather than a $\mathbb{Z}[G]$-module. But note that the voltage Laplacian is always represented by an $n \times n$ matrix (with entries from $\mathbb{Z}[G]$), even when G is an infinite group; whereas the ordinary Laplacian of Y has a matrix representation (of degree $n|\,G\,|$) only when G is finite, as described explicitly in Section 2.2. We summarize this discussion by the following theorem.

Theorem 11. *Let G be any group. The voltage Laplacian $\mathcal{L}_\alpha : Div(Y) \longrightarrow Div(Y)$ is a $\mathbb{Z}[G]$-module homomorphism whose image is $Pr(Y)$ and cokernel is $Pic(Y)$, and its $n \times n$ matrix with respect to the $\mathbb{Z}[G]$-basis $v_{1,\tau_0}, \ldots, v_{n,\tau_0}$ is equal to $D_X - A_\alpha$, where τ_0 is the identity of G, D_X is the degree matrix for the base graph X and A_α is the voltage adjacency matrix of X.*

Definition 25. *Assume G is abelian. We call $\Theta_{Y/X} = \det(D_X - A_\alpha)$ the reduced Stickelberger element.*

Note that $\Theta_{Y/X}$ is an element of the integral group ring $\mathbb{Z}[G]$. We need G to be commutative only for the determinant to be well-defined (over a commutative ring).

In [HMSV19], their Stickelberger element is denoted by $\theta^*_{Y/X}(1)e$. By their Theorem 4.5, since their map ϕ is seen to be the same as our map $\mathcal{L}_\alpha$, their Stickelberger element relates to ours by

$$\theta^*_{Y/X}(1)e = 2^{r_X - 1}\Theta_{Y/X},$$

so our version of the Stickelberger element eliminates a 2-power factor (with nonnegative exponent). This leads to a stronger annihilation statement than their Theorem 4.7 (and explains the terminology "reduced"). To be more precise, a careful reading of the *proof* of their annihilation result (Theorem 4.7), which relies only on *exactly the same* (adjoint matrix) result that we use, reveals that they too actually achieve *the same* as the following corollary; however it is *stated* in that paper with the power of 2 still in place. Again, we use the expression "reduced Stickelberger element" merely to distinguish it from the version in [HMSV19].

Corollary 9. *For G abelian, $\Theta_{Y/X}$ annihilates the Picard group $Pic(Y)$ of any derived graph (viewed as a $\mathbb{Z}$-module or a $\mathbb{Z}[G]$-module), hence it also annihilates $\mathcal{J}(Y)$.*

Proof. This is immediate from Exercise 3 in Section 11.4 of [DF04] applied to $\mathcal{L}_\alpha$, viewing $\mathrm{Pic}(Y)$ as a module over the commutative ring $\mathbb{Z}[G]$. $\square$

In Section 3.2.2, we will see examples where the Jacobian is annihilated by a larger ideal in $\mathbb{Z}[G]$ than the one generated by the reduced Stickelberger element.

2.5.1 Example: K_n with an edge removed

Using SAGE, we calculate the reduced Stickelberger element of the voltage graph (X, Z_d, α) where $X = K_n - e_{i,j}$, (i.e., the complete graph K_n with an edge removed) and α is the single voltage assignment (but we let the edge on which τ is placed vary).

We first let $X = K_4 - e_{2,4}$, as shown in Figure 2.11.

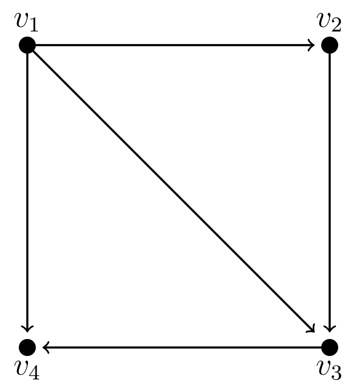

Figure 2.11: Base graph $K_4 - e_{2,4}$

We have the following data:

Table 2.1 The reduced Stickelberger element corresponding to base graph $K_4 - e_{2,4}$

τ label	$\Theta_{Y/X}$	$\deg(v_i, v_j),\ i < j$
$v_1 \to v_2$	$-3(\tau-1)^2\tau^{-1}$	$(3,2)$
$v_1 \to v_3$	$-4(\tau-1)^2\tau^{-1}$	$(3,3)$
$v_1 \to v_4$	$-3(\tau-1)^2\tau^{-1}$	$(3,2)$
$v_2 \to v_3$	$-3(\tau-1)^2\tau^{-1}$	$(2,3)$
$v_3 \to v_4$	$-3(\tau-1)^2\tau^{-1}$	$(3,2)$

Now we let $X = K_5 - e_{2,4}$. We have the following data:

Table 2.2 The reduced Stickelberger element corresponding to base graph $K_5 - e_{2,4}$

τ label	$\Theta_{Y/X}$	$\deg(v_i, v_j),\ i < j$
$v_1 \to v_2$	$-40(\tau-1)^2\tau^{-1}$	$(4,3)$
$v_1 \to v_3$	$-45(\tau-1)^2\tau^{-1}$	$(4,4)$
$v_1 \to v_4$	$-40(\tau-1)^2\tau^{-1}$	$(4,3)$
$v_1 \to v_5$	$-45(\tau-1)^2\tau^{-1}$	$(4,4)$
$v_2 \to v_3$	$-40(\tau-1)^2\tau^{-1}$	$(3,4)$
$v_2 \to v_5$	$-40(\tau-1)^2\tau^{-1}$	$(3,4)$
$v_3 \to v_4$	$-40(\tau-1)^2\tau^{-1}$	$(4,3)$
$v_3 \to v_5$	$-45(\tau-1)^2\tau^{-1}$	$(4,4)$
$v_4 \to v_5$	$-40(\tau-1)^2\tau^{-1}$	$(3,4)$

Now let $X = K_6 - e_{2,4}$. We have the following data:

Table 2.3 The reduced Stickelberger element corresponding to base graph $K_6 - e_{2,4}$

τ label	$\Theta_{Y/X}$	$\deg(v_i, v_j),\ i < j$
$v_1 \to v_2$	$-540(\tau-1)^2\tau^{-1}$	$(5,4)$
$v_1 \to v_3$	$-576(\tau-1)^2\tau^{-1}$	$(5,5)$
$v_1 \to v_4$	$-540(\tau-1)^2\tau^{-1}$	$(5,4)$
$v_1 \to v_5$	$-576(\tau-1)^2\tau^{-1}$	$(5,5)$
$v_1 \to v_6$	$-576(\tau-1)^2\tau^{-1}$	$(4,5)$
$v_2 \to v_3$	$-540(\tau-1)^2\tau^{-1}$	$(4,5)$
$v_2 \to v_5$	$-540(\tau-1)^2\tau^{-1}$	$(4,5)$
$v_2 \to v_6$	$-540(\tau-1)^2\tau^{-1}$	$(4,5)$
$v_3 \to v_4$	$-540(\tau-1)^2\tau^{-1}$	$(5,4)$
$v_3 \to v_5$	$-576(\tau-1)^2\tau^{-1}$	$(5,5)$
$v_3 \to v_6$	$-576(\tau-1)^2\tau^{-1}$	$(5,5)$
$v_4 \to v_5$	$-540(\tau-1)^2\tau^{-1}$	$(4,5)$
$v_4 \to v_6$	$-540(\tau-1)^2\tau^{-1}$	$(4,5)$
$v_5 \to v_6$	$-576(\tau-1)^2\tau^{-1}$	$(5,5)$

As noted in Section 2.2.3, we see that the reduced Stickelberger element, (and hence Jacobian) depends on the edge in which τ is placed.

Chapter 3

Covers of Complete Graphs and Other Graphs

In this chapter, we examine voltage graphs where the base graph is given the constant voltage assignment and the single voltage assignment. Furthermore, we consider when the base graph is K_n, the complete graph on n vertices, $K_{n,n}$, the complete bipartite graph on $2n$ vertices, $K_{n,2}$, the complete bipartite graph on $n+2$ vertices, and the Petersen graph. The reason for focusing on such base graphs is because it is known that the Jacobian of each of these base graphs has at most two distinct invariant factors (IFs): the Jacobian of K_n is $(\mathbb{Z}/n\mathbb{Z})^{n-2}$ ([Hop14] Theorem 1.0.6), which has one distinct IF, the Jacobian of $K_{n,n}$ and $K_{n,2}$ is $(\mathbb{Z}/n\mathbb{Z})^{2n-4} \oplus \mathbb{Z}/n^2\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}^{n-2} \oplus \mathbb{Z}/2n\mathbb{Z}$, respectively ([Mac11] Example 8), which each have two distinct IFs, and the Jacobian of the Petersen Graph is $\mathbb{Z}/2\mathbb{Z} \oplus (\mathbb{Z}/10\mathbb{Z})^3$ ([KMM17] Table 1), which also has two distinct IFs. Thus, we anticipate that derived graphs based on voltages assigned to such graphs with few invariant factors will likewise be simpler than general graphs. In addition to this, given the connectivity and symmetry of these graphs, they are relatively easy to work with, both theoretically and computationally. Much of the work in this chapter is computational in nature, either by hand or by computer calculations (and computations done by hand were also checked on a computer). They lead to both results of independent interest and conjectures that motivate work in Chapters 4 and 5. So in hindsight, the choice of these base graphs work as good "test cases."

We first consider the case when the base graph is K_n. In Sections 3.1 and 3.2, we gather data on the Jacobian of cyclic voltage covers of K_n with both the constant voltage assignment and the single voltage assignment. Then based on the data for single voltage cyclic covers, a conjecture is formulated for the rank, invariant factors, and order of the Jacobian. In Section 3.2.2, we determine what the reduced Stickelberger element is corresponding to this voltage graph. We then obtain a $\mathbb{Z}[G]$-module presentation of $\mathrm{Pic}(Y)$. Furthermore, we show that $\mathrm{Pic}(Y)$ is annihilated by a larger ideal in $\mathbb{Z}[G]$ than the one generated by the reduced Stickelberger element. We then prove the conjecture formulated in Section 3.2.1 via lengthy matrix manipulations.

In Section 3.3, we gather data on the Jacobian of cyclic voltage covers of $K_{n,n}$ with the single voltage assignment. Then a conjecture is again formulated for the rank, invariant factors, and order of the Jacobian. We obtain partial results for this conjecture in Section 3.4. In particular, we obtain the primes p (and their powers) that divide the order of the Jacobian of the derived graph Y, for p not dividing n.

Finally, in Sections 3.5 and 3.6, we gather data on the Jacobian of cyclic voltage covers of $K_{n,2}$ and the Petersen graph both with the single voltage assignment. Conjectures are again formulated for the rank, invariant factors, and order of the Jacobian. Lastly, we compute the reduced Stickelberger element for each of the derived graphs corresponding to these voltage graphs.

We briefly describe the general approach that is used to compute the Jacobian of a derived graph in this chapter. Recall, the Laplacian matrix is the *relations matrix* for the cokernel of the Laplacian map, viewed as a quotient of $\mathrm{Div}(Y)$. Viewing $\mathrm{Div}(Y)$ as a $\mathbb{Z}[G]$-module, elementary row and column operations on the *voltage* Laplacian changes the bases of $\mathrm{Div}(Y)$ in the domain and range as a $\mathbb{Z}[G]$-module (hence, a fortiori, they are also changes-of-bases as a $\mathbb{Z}$-module) — such computations are often easier to carry out because the voltage Laplacian is smaller in size, and independent of the size of G. Also, in the midst of manipulations of the voltage Laplacian, after reducing it to nearly diagonal (or nearly upper-triangular) form, it is often possible to compute the reduced Stickelberger element by hand. We sometimes divert to do so. (That determinant can be computed by row and column operations over the larger ring $\mathbb{Q}[G]$, unlike the need to work over $\mathbb{Z}[G]$ to obtain the invariant factors) Finally, once the voltage Laplacian has been sufficiently reduced, one may "tensor" its entries with the regular representation of G to obtain the ordinary Laplacian for the derived graph Y (as in Section 2.2.1). That matrix is then further reduced by row and column operations over $\mathbb{Z}$ to obtain the Smith Normal Form, which gives the invariant factors of $\mathcal{J}(Y)$.

3.1 The Constant Voltage Assignment on K_n

Let $X = K_n$ in Definition 18. Observe that the triangle $v_1 \xrightarrow{\tau} v_2 \xrightarrow{\tau} v_3 \xrightarrow{\tau^{-1}} v_1$ has net voltage τ. By Corollary 4, this immediately gives:

Proposition 9. *If (K_n, Z_d, α) is a voltage graph, where $\alpha : E(X)^+ \to Z_d$ is the constant voltage assignment, then the derived graph Y is connected.*

We now go on to gather data on the Jacobian and reduced Stickelberger element of the derived graph Y corresponding to such voltage graphs.

Using Sage, we compute the following table, which gives the rank, invariant factors, and order of the Jacobian of the derived graph corresponding to the voltage graph (K_4, Z_d, α), where α is the constant voltage assignment and $d = 1, \cdots, 20$. In this case, there are at most three distinct invariant factors (IFs), ignoring multiplicities, for every derived graph.

Table 3.1 Jacobian of K_4 with Constant Voltage Cover by Z_d (exponents represent multiplicities)

d		1st IF	2nd IF	3rd IF	order
1	1	4^2			2^4
2	1^4	2	8	24	$2^7 \cdot 3$
3	1^9	52	156		$2^4 \cdot 3 \cdot 13^2$
4	1^{12}	4	112	336	$2^{10} \cdot 3 \cdot 7^2$
5	1^{17}	724	3620		$2^4 \cdot 5 \cdot 181^2$
6	1^{20}	6	1560	4680	$2^7 \cdot 3^4 \cdot 5^2 \cdot 13^2$
7	1^{25}	10084	70588		$2^4 \cdot 7 \cdot 2521^2$
8	1^{28}	8	21728	65184	$2^{13} \cdot 3 \cdot 7^2 \cdot 97^2$
9	1^{33}	140452	1264068		$2^4 \cdot 3^2 \cdot 13^2 \cdot 37^2 \cdot 73^2$
10	1^{36}	2	302632	4539480	$2^7 \cdot 3 \cdot 5 \cdot 11^2 \cdot 19^2 \cdot 181^2$
11	1^{41}	1956244	21518684		$2^4 \cdot 11 \cdot 489061^2$
12	1^{44}	12	4215120	12645360	$2^{10} \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 13^2 \cdot 193^2$
13	1^{49}	27246964	354210532		$2^4 \cdot 13 \cdot 6811741^2$
14	1^{52}	2	58709048	1232890008	$2^7 \cdot 3 \cdot 7 \cdot 41^2 \cdot 71^2 \cdot 2521^2$
15	1^{57}	379501252	5692518780		$2^4 \cdot 3 \cdot 5 \cdot 13^2 \cdot 61^2 \cdot 181^2 \cdot 661^2$
16	1^{60}	16	817711552	2453134656	$2^{16} \cdot 3 \cdot 7^2 \cdot 31^2 \cdot 97^2 \cdot 607^2$
17	1^{65}	5285770564	89858099588		$2^4 \cdot 17 \cdot 1321442641^2$
18	1^{68}	18	11389252680	34167758040	$2^7 \cdot 3^7 \cdot 5^2 \cdot 13^2 \cdot 17^2 \cdot 37^2 \cdot 53^2 \cdot 73^2$
19	1^{73}	73621286644	1398804446236		$2^4 \cdot 19 \cdot 18405321661^2$
20	1^{76}	4	158631825968	2379477389520	$2^{10} \cdot 3 \cdot 5 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot 181^2 \cdot 37441^2$

Based on this data, no conjectures have yet been formulated for the rank, invariant factors, and order of the Jacobian.

Now using Mathematica we compute the reduced Stickelberger element for constant voltage cyclic covers of K_n for $3 \le n \le 7$.

Table 3.2 Reduced Stickelberger element of cyclic covers of K_n with constant voltage assignment

n	$\Theta_{Y/X}$
3	$-(\tau-1)^2\tau^{-1}$
4	$-(\tau-1)^2\left(\tau^2+14\tau+1\right)\tau^{-2}$
5	$-(\tau-1)^2\left(\tau^4+22\tau^3+204\tau^2+22\tau+1\right)\tau^{-3}$
6	$-(\tau-1)^2\left(\tau^6+32\tau^5+439\tau^4+3376\tau^3+439\tau^2+32\tau+1\right)\tau^{-4}$
7	$-(\tau-1)^2\left(\tau^8+44\tau^7+844\tau^6+9246\tau^5+63765\tau^4+9246\tau^3+844\tau^2+44\tau+1\right)\tau^{-5}$

From the above data, it appears that constant voltage covers give complicated derived graphs. Hence, we will now focus our attention on single voltage cyclic covers for the remainder of Chapter 3. Note that the reduced Stickelberger element will play an important role in Chapters 4 and 5.

We now go on to gather data on the Jacobian of cyclic voltage covers of K_n with the single voltage assignment.

3.2 Single Voltage Assignment on K_n

Now let $X = K_n$ in Definition 19.

By symmetry, the Jacobians will *not* depend on which (directed) edge has the non-identity voltage assignment. Putting the single non-identity voltage τ on edge $v_1 \to v_2$ is, again, for convenience. (Recall for general base graphs, the choice of the single voltage edge *does* depend on the edge chosen, as shown in Section 2.5.1.)

Observe that triangle $v_1 \xrightarrow{\tau} v_2 \xrightarrow{1} v_3 \xrightarrow{1} v_1$ has net voltage τ. By Corollary 4, this immediately gives:

Proposition 10. *If* (K_n, Z_d, α) *is a voltage graph, where* $\alpha : E(X)^+ \to Z_d$ *is the single voltage assignment, then the derived graph* Y *is connected.*

We now go on to gather data on the Jacobian of the derived graph Y corresponding to such voltage graphs.

3.2.1 Conjecture on the Jacobian of Single Voltage Cyclic Covers of K_n

Using Sage, we compute the following tables, which yields the rank, invariant factors, and order of the Jacobian of the derived graph corresponding to (K_n, Z_d, α), where α is the single voltage assignment and $n = 4, 5, 6$, respectively. Using this data, we make the following observations. We then go on to formulate Conjecture 1.

Table 3.3 Jacobian of K_4 with Single Voltage Cover by Z_d, (exponents represent multiplicities)

d		1st IF	2nd IF	3rd IF	order
1	1	4^2			2^4
2	1^4	4^2	16		2^8
3	1^7	4^2	8	24	$2^{10} \cdot 3$
4	1^{10}	4^2	8^2	32	2^{15}
5	1^{13}	4^2	8^3	40	$2^{16} \cdot 5$
6	1^{16}	4^2	8^4	48	$2^{20} \cdot 3$
7	1^{19}	4^2	8^5	56	$2^{22} \cdot 7$
8	1^{22}	4^2	8^6	64	2^{28}
9	1^{25}	4^2	8^7	72	$2^{28} \cdot 3^2$
10	1^{28}	4^2	8^8	80	$2^{32} \cdot 5$
11	1^{31}	4^2	8^9	88	$2^{34} \cdot 11$
12	1^{34}	4^2	8^{10}	96	$2^{39} \cdot 3$
13	1^{37}	4^2	8^{11}	104	$2^{40} \cdot 13$
14	1^{40}	4^2	8^{12}	112	$2^{44} \cdot 7$
15	1^{43}	4^2	8^{13}	120	$2^{46} \cdot 3 \cdot 5$
16	1^{46}	4^2	8^{14}	128	2^{53}
17	1^{49}	4^2	8^{15}	136	$2^{52} \cdot 17$
18	1^{52}	4^2	8^{16}	144	$2^{56} \cdot 3^2$
19	1^{55}	4^2	8^{17}	152	$2^{58} \cdot 19$
20	1^{58}	4^2	8^{18}	160	$2^{63} \cdot 5$

For $d \leq 20$, we observe the following from Table 3.3:

(i) For $d \geq 1$, the rank is $d + 1$.

(ii) For $d > 2$, there are 3 distinct invariant factors: the first invariant factor is 4 with multiplicity 2, the second invariant factor is 8 with multiplicity $d - 2$ and the third invariant factor is $d \cdot 8$ with multiplicity 1.

(iii) $|\mathcal{J}(Y)| = 4^{d+1} \cdot 2^{d-1} \cdot d$.

Table 3.4 Jacobian of K_5 with Single Voltage Cover by Z_d, (exponents represent multiplicities)

d		1st IF	2nd IF	3rd IF	order
1	1	5^3			5^3
2	1^4	5^4	30		$2 \cdot 3 \cdot 5^5$
3	1^7	5^5	15	45	$3^3 \cdot 5^7$
4	1^{10}	5^6	15^2	60	$2^2 \cdot 3^3 \cdot 5^9$
5	1^{13}	5^7	15^3	75	$3^4 \cdot 5^{12}$
6	1^{16}	5^8	15^4	90	$2 \cdot 3^6 \cdot 5^{13}$
7	1^{19}	5^9	15^5	105	$3^6 \cdot 5^{15} \cdot 7$
8	1^{22}	5^{10}	15^6	120	$2^3 \cdot 3^7 \cdot 5^{17}$
9	1^{25}	5^{11}	15^7	135	$3^{10} \cdot 5^{19}$
10	1^{28}	5^{12}	15^8	150	$2 \cdot 3^9 \cdot 5^{22}$

For $d \leq 10$ we observe the following from Table 3.4:

(i) For $d \geq 1$, the rank is $2d + 1$.

(ii) For $d > 2$, there are 3 distinct invariant factors: the first invariant factor is 5 with multiplicity $d + 2$, the second invariant factor is 15 with multiplicity $d - 2$ and the third invariant factor is $d \cdot 15$ with multiplicity 1.

(iii) $|\mathcal{J}(Y)| = 5^{2d+1} \cdot 3^{d-1} \cdot d$.

Table 3.5 Jacobian of K_6 with Single Voltage Cover by Z_d, (exponents represent multiplicities)

d		1st IF	2nd IF	3rd IF	order
1	1	6^4			6^4
2	1^4	6^6	48		$2^{10} \cdot 3^7$
3	1^7	6^8	24	72	$2^{14} \cdot 3^{11}$
4	1^{10}	6^{10}	24^2	96	$2^{21} \cdot 3^{13}$
5	1^{13}	6^{12}	24^3	120	$2^{24} \cdot 3^{16} \cdot 5$
6	1^{16}	6^{14}	24^4	144	$2^{30} \cdot 3^{20}$
7	1^{19}	6^{16}	24^5	168	$2^{34} \cdot 3^{22} \cdot 7$
8	1^{22}	6^{18}	24^6	192	$2^{42} \cdot 3^{25}$
9	1^{25}	6^{20}	24^7	216	$2^{44} \cdot 3^{30}$
10	1^{28}	6^{22}	24^8	240	$2^{50} \cdot 3^{31} \cdot 5$

For $d \leq 10$ we observe the following from Table 3.5:

(i) For $d \geq 1$, the rank is $3d + 1$.

(ii) For $d > 2$, there are 3 distinct invariant factors: the first invariant factor is 6 with multiplicity $2d+2$, the second invariant factor is 24 with multiplicity $d-2$ and the third invariant factor is $d \cdot 24$ with multiplicity 1.

(iii) $|\mathcal{J}(X)| = 6^{3d+1} \cdot 4^{d-1} \cdot d$.

Using the above observations, we formulate the following conjecture:

Conjecture 1. *For K_n the complete graph on n vertices with single voltage cover by Z_d, we have the following:*

1. *For $d \geq 1$, the rank is $(n-3)d + 1$.*

2. *For $d > 2$, there are 3 distinct invariant factors: the first invariant factor is n with multiplicity $(n-4)d + 2$, the second invariant factor is $n(n-2)$ with*

multiplicity $d-2$ and the third invariant factor is $d \cdot n(n-2)$ with multiplicity 1.

3. $|\mathcal{J}(Y)| = n^{(n-3)d+1} \cdot (n-2)^{d-1} \cdot d.$

We prove this conjecture in the next section.

3.2.2 The Jacobian of Single Voltage Cyclic Covers of K_n

In this section, we prove Conjecture 1, which gives the rank, invariant factors, and order of the Jacobian of single voltage cyclic covers of K_n. We first determine what the reduced Stickelberger element is of the derived graph Y that corresponds to this voltage graph. We then obtain a $\mathbb{Z}[G]$-module presentation of $\mathrm{Pic}(Y)$. Moreover, we show that $\mathrm{Pic}(Y)$ is annihilated by a larger ideal in $\mathbb{Z}[G]$ than the one generated by the reduced Stickelberger element.

We begin with two examples: letting $X = K_3$ and $X = K_4$. We then prove it in all generality.

Example 5. Let (X, G, α) be the voltage graph with X equal to K_3, $G = Z_d = \{1, \tau, \cdots, \tau^{d-1}\}$ and $\alpha : E(X)^+ \to G$ such that the directed edge from v_1 to v_2 is labeled with τ (hence the directed edge from v_2 to v_1 must be labeled with τ^{-1}). Label all other edges with the identity, as shown below in 3.1.

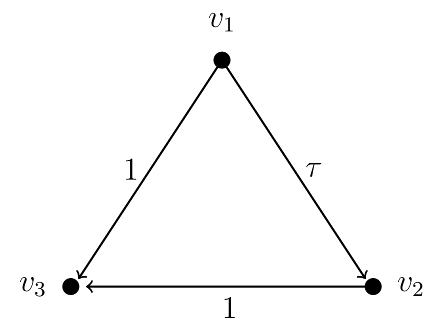

Figure 3.1: Voltage graph (K_3, Z_d, α) where α is the single voltage assignment

Then the reduced Stickelberger element of Y/X is

$$\begin{aligned}\Theta_{Y/X} &= \det\begin{pmatrix} 2 & -\tau & -1 \\ -\tau^{-1} & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \\ &= -\tau^{-1}(\tau-1)^2\end{aligned}$$

We now prove this in general, with $X = K_n$.

Theorem 12. *Let (K_n, Z_d, α) be as in Definition 19. Then the reduced Stickelberger element is*

$$\Theta_{Y/X} = -(n-2)n^{n-3}(\tau-1)^2\tau^{-1}.$$

Proof. For $X = K_n$ with single voltage assignment by Z_d, we have the voltage Laplacian matrix with $n-1$ down the diagonal, $-\tau$ in entry $(1,2)$, $-\tau^{-1}$ in entry $(2,1)$

and -1 elsewhere:

$$L_\alpha = \begin{pmatrix} n-1 & -\tau & -1 & -1 & \cdots & -1 \\ -\tau^{-1} & n-1 & -1 & -1 & \cdots & -1 \\ -1 & -1 & n-1 & -1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ -1 & -1 & -1 & -1 & \cdots & n-1 \end{pmatrix}$$

We will now use row and column operations in $\mathbb{Q}[\tau]$ to put the matrix in essentially upper triangular form. We denote row i and column j of the voltage Laplacian by C_i and R_j, respectively.

First replace C_1 by $C_1 + C_3 + \cdots + C_n$ to get

$$\begin{pmatrix} 1 & -\tau & -1 & -1 & \cdots & -1 \\ -\tau^{-1} - n + 2 & n-1 & -1 & -1 & \cdots & -1 \\ 1 & -1 & n-1 & -1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & -1 & -1 & -1 & \cdots & n-1 \end{pmatrix}$$

Now replace $C_2, \cdots, C_n$ with $C_1 + C_2, C_1 + C_3, \cdots, C_1 + C_n$, respectively.

$$\begin{pmatrix} 1 & -\tau + 1 & 0 & 0 & \cdots & 0 \\ -\tau^{-1} - n + 2 & -\tau^{-1} + 1 & -\tau^{-1} - n + 1 & -\tau^{-1} - n + 1 & \cdots & -\tau^{-1} - n + 1 \\ 1 & 0 & n & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & n \end{pmatrix} \qquad (\dagger)$$

Now replace C_1 with $C_1 - \frac{1}{n}(C_3 + C_4 + \cdots + C_n)$ and simplify to obtain

$$\begin{pmatrix} 1 & -\tau + 1 & 0 & 0 & \cdots & 0 \\ \frac{-2\tau^{-1} - n + 2}{n} & -\tau^{-1} + 1 & -\tau^{-1} - n + 1 & -\tau^{-1} - n + 1 & \cdots & -\tau^{-1} - n + 1 \\ 0 & 0 & n & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n \end{pmatrix}$$

Let L' be the preceding matrix. Now use cofactor expansion along the first row of L' to get

$$\Theta_{Y/X} = \det L_\alpha = \det L' = 1 \cdot \det L'_{1,1} - (1 - \tau) \det L'_{1,2}$$

where $L'_{i,j}$ is the i, j minor of L'. To compute each of the minor determinants, use cofactor expansion along its first column (which has only one nonzero entry), to get:

$$\det L' = 1(1 - \tau^{-1})n^{n-2} - (1 - \tau)\frac{(-2\tau^{-1} - n + 2)}{n} n^{n-2}.$$

By elementary algebra this simplifies to the stated conclusion. $\square$

Corollary 10. *Under the hypothesis of the Theorem 12,* $(n-2)n^{n-3}(\tau-1)^2$ *annihilates the Picard group of* Y*,* $Pic(Y) = Div(Y)/Pr(Y)$*, hence annihilates its subgroup* $\mathcal{J}(Y) = Div^0(Y)/Pr(Y)$*.*

Proof. By Corollary 9 the reduced Stickelberger element $\Theta_{Y/X}$ annihilates the quotient as a $\mathbb{Z}[G]$-module. Since $-\tau^{-1}$ is a unit in the ring, the product of the remaining terms effects the annihilation. □

We will now do further operations on (†) from above to get a $\mathbb{Z}[G]$-module presentation of $\mathrm{Pic}(Y)$. We show that $\mathrm{Pic}(Y)$ is annihilated by a larger ideal in $\mathbb{Z}[G]$ than the one generated by the reduced Stickelberger element.

All of our row and column operations will now be in $\mathbb{Z}[\tau]$ (note that to compute the reduced Stickelberger element above, all of the row and column operations were in $\mathbb{Z}[\tau]$ until the final step). We begin with the matrix (†):

$$\begin{pmatrix} 1 & -\tau+1 & 0 & 0 & \cdots & 0 \\ -\tau^{-1}-n+2 & -\tau^{-1}+1 & -\tau^{-1}-n+1 & -\tau^{-1}-n+1 & \cdots & -\tau^{-1}-n+1 \\ 1 & 0 & n & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & n \end{pmatrix}$$

Replace R_2 with $(R_3 + \cdots + R_n) + R_2$ to get the matrix

$$\begin{pmatrix} 1 & -\tau+1 & 0 & 0 & \cdots & 0 \\ -\tau^{-1} & -\tau^{-1}+1 & -\tau^{-1}+1 & -\tau^{-1}+1 & \cdots & -\tau^{-1}+1 \\ 1 & 0 & n & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & n \end{pmatrix}$$

Next multiply R_2 by $-\tau$ and simplify to get

$$\begin{pmatrix} 1 & -\tau+1 & 0 & 0 & \cdots & 0 \\ 1 & -\tau+1 & -\tau+1 & -\tau+1 & \cdots & -\tau+1 \\ 1 & 0 & n & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & n \end{pmatrix}$$

Now replace R_2 with $R_2 - R_1$ to get

$$\begin{pmatrix} 1 & -\tau+1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -\tau+1 & -\tau+1 & \cdots & -\tau+1 \\ 1 & 0 & n & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & n \end{pmatrix}$$

Denote this matrix by M. Then we have

$$\mathbb{Z}^n / L_\alpha(\mathbb{Z}^n) = \mathbb{Z}^n / M(\mathbb{Z}^n).$$

For $G = Z_d$, let $e_1, e_2, ..., e_n$ be the standard $\mathbb{Z}[G]$-basis vectors. So e_i has a 1 in row i and zeros elsewhere. Then the relations that define the cokernel of M as a $\mathbb{Z}[G]$-module are

(i) $e_1 + e_3 + e_4 + ... + e_n = 0$,

(ii) $(1 - \tau)e_1 = 0$, and

(iii) $(1 - \tau)e_2 + ne_i = 0, \quad$ for $3 \le i \le n$.

Now (i) can be rewritten as

$$e_1 = -(e_3 + e_4 + ... + e_n)$$

and so e_1 is in the module spanned by $e_3, ..., e_n$, and therefore, the cokernel of M is generated by just $e_2, e_3, ..., e_n$. So the cokernel of M is the free $\mathbb{Z}[G]$-module generated by $e_2, e_3, ..., e_n$ subject to the relations (via reduction of the above relations):

(i) $(1 - \tau)(e_3 + e_4 + ... + e_n) = 0$, and

(ii) $(1 - \tau)e_2 + ne_i = 0, \quad$ for $3 \le i \le n$

Indeed these give a $\mathbb{Z}[G]$-module presentation of $\mathrm{Pic}(Y)$.

Adding (ii) together for all i, we get

$$(n - 2)(1 - \tau)e_2 = -n(e_3 + ... + e_n).$$

Multiplying both sides by $(1-\tau)$ and using relation (i), we get

$$(n-2)(1-\tau)^2 e_2 = n(1-\tau)(e_3 + ... + e_n) = 0,$$

hence $(n-2)(1-\tau)^2$ annihilates e_2. For $i \geq 3$ we have

$$ne_i = -(1-\tau)e_2.$$

Multiplying both sides by $(n-2)(1-\tau)$, we get

$$n(n-2)(1-\tau)e_i = -(n-2)(1-\tau)^2 e_2 = 0.$$

So $n(n-2)(1-\tau)$ annihilates e_i for $i = 3, \cdots, n$. This yields the following:

Theorem 13. *Under the hypothesis of the Theorem 12, $n(n-2)(\tau-1)^2$ annihilates $Pic(Y) = Div(Y)/Pr(Y)$. More specifically, $(n-2)(1-\tau)^2$ annihilates e_2 and $n(n-2)(1-\tau)$ annihilates e_i for $i = 3, \cdots, d$ where $e_2, \cdots, e_n$ generate the cokernel of L_α.*

Using these relations for $\mathrm{Pic}(Y)$, our goal is now to compute the Jacobian of Y.

We will now assume $n \geq 4$. We will rewrite our matrix relations by changing each e_i

to e_{i-1}. This yields the following $(n-1) \times (n-1)$ relations matrix

$$\begin{pmatrix} 0 & 1-\tau & 1-\tau & \cdots & 1-\tau \\ 1-\tau & n & 0 & \cdots & 0 \\ 1-\tau & 0 & n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1-\tau & 0 & 0 & \cdots & n \end{pmatrix}$$

We will put this matrix in block-diagonal form as follows. First replace $R_3, \cdots, R_{n-1}$ by $R_3 - R_2, \cdots, R_{n-1} - R_2$, respectively, to get

$$\begin{pmatrix} 0 & 1-\tau & 1-\tau & \cdots & 1-\tau \\ 1-\tau & n & 0 & \cdots & 0 \\ 0 & -n & n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -n & 0 & \cdots & n \end{pmatrix}$$

Now replace C_2 with $C_2 + C_3 + \cdots + C_{n-1}$ to get

$$\begin{pmatrix} 0 & (n-2)(1-\tau) & 1-\tau & \cdots & 1-\tau \\ 1-\tau & n & 0 & \cdots & 0 \\ 0 & 0 & n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n \end{pmatrix}$$

Now replace $C_4, \cdots, C_{n-1}$ with $C_4 - C_3, \cdots, C_{n-1} - C_3$, respectively, to get

$$\begin{pmatrix} 0 & (n-2)(1-\tau) & 1-\tau & 0 & \cdots & 0 \\ 1-\tau & n & 0 & 0 & \cdots & 0 \\ 0 & 0 & n & -n & \cdots & -n \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n \end{pmatrix}$$

Lastly, replace R_3 with $R_3 + \cdots + R_{n-1}$ to get

$$\begin{pmatrix} 0 & (n-2)(1-\tau) & 1-\tau & 0 & \cdots & 0 \\ 1-\tau & n & 0 & 0 & \cdots & 0 \\ 0 & 0 & n & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n \end{pmatrix} \qquad (*)$$

From this, we get a 3×3 block matrix in the upper left and a $(n-4) \times (n-4)$ scalar matrix with n on the diagonal on the lower right. Since the lower block is diagonal over $\mathbb{Z}[G]$, it remains to further reduce the upper submatrix. Do the following operations on the 3×3 matrix. Replace C_2 with $-(n-2)C_3 + C_2$ to get

$$\begin{pmatrix} 0 & 0 & 1-\tau \\ 1-\tau & n & 0 \\ 0 & n(2-n) & n \end{pmatrix}$$

Now interchange the columns to get

$$\begin{pmatrix} 1-\tau & 0 & 0 \\ 0 & 1-\tau & n \\ n & 0 & n(2-n) \end{pmatrix}$$

This appears to be the simplest $\mathbb{Z}[G]$-module relations matrix for $\mathrm{Pic}(Y)$.

Note that doing elementary row and column operations on the voltage Laplacian is the same as changing generators in the domain and range of the $\mathbb{Z}[G]$-module $\mathrm{coker}(L_\alpha)$. In particular, these also just change generators of its $\mathbb{Z}$-module structure. So in order to compute the Smith Normal Form of the Laplacian of Y, we may begin with the already reduced voltage Laplacian above. Thus, we will now tensor this matrix with ρ (as in Section 2.2.1) to get $d \times d$ block matrices in each entry, i.e.

$$\begin{pmatrix} \rho(1-\tau) & 0_d & 0_d \\ 0_d & \rho(1-\tau) & nI_d \\ nI_d & 0_d & n(2-n)I_d \end{pmatrix}$$

where I_d denotes the $d \times d$ identity matrix and 0_d denotes the $d \times d$ zero matrix. The resulting integer matrix is equivalent to the "upper left 3×3" portion of the

$\mathbb{Z}$-module Laplacian matrix. This yields the following $3d \times 3d$ matrix.

$$\begin{pmatrix}
1 & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
-1 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & -1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & -1 & n & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & -1 & 1 & \cdots & 0 & 0 & 0 & n & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & -1 & 1 & 0 & 0 & \cdots & n \\
n & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & n(2-n) & 0 & \cdots & 0 \\
0 & n & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & n(2-n) & \cdots & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & n & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & n(2-n)
\end{pmatrix} \tag{1}$$

We first do row and column operations that put the $1-\tau$ block in Smith Normal Form, where $\rho(1-\tau)$ is the matrix

$$\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & -1 \\
-1 & 1 & 0 & \cdots & 0 & 0 \\
0 & -1 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -1 & 1
\end{pmatrix} \tag{**}$$

in the upper left corner and middle. Do so by first replacing R_1 and R_{d+1} with $R_1 + R_2 + \cdots + R_d$ and $R_{d+1} + R_{d+2} + \cdots + R_{2d}$, respectively to get the first row to be all zeros. Then replace C_1 and C_{d+1} with $C_1+C_2+\cdots+C_d$ and $C_{d+1}+C_{d+2}+\cdots+C_{2d}$, respectively to get the $d+1$-st column to be all zeros.

Then to zero out all of the $-1'$s on the subdiagonal, replace $R_3, \cdots, R_d$ with $R_2 + R_3, \cdots, R_{d-1} + R_d$, respectively. Similarly, replace $R_{d+3}, \cdots, R_{2d}$ with $R_{d+2} + R_{d+3}, \cdots, R_{2d-1} + R_{2d}$, respectively. This yields the following matrix,

$$\begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & n & n & n & \cdots & n \\
0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & n & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & n & n & \cdots & n \\
n & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & n(2-n) & 0 & 0 & \cdots & 0 \\
n & n & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & n(2-n) & 0 & \cdots & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
n & 0 & \cdots & 0 & n & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & n(2-n)
\end{pmatrix} \tag{2}$$

a $(d-1)\times(d-1)$ identity block in the upper left corner and in the middle. We also get a $(d-1)\times(d-1)$ lower triangular block in the middle right with all entries equal to n.

Now interchange C_1 with C_{d+1} to get,

$$\begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & n & n & n & \cdots & n \\
0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & n & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & n & n & \cdots & n \\
0 & 0 & \cdots & 0 & 0 & n & 0 & \cdots & 0 & 0 & n(2-n) & 0 & 0 & \cdots & 0 \\
0 & n & \cdots & 0 & 0 & n & 0 & \cdots & 0 & 0 & 0 & n(2-n) & 0 & \cdots & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & 0 & n & n & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & n(2-n)
\end{pmatrix} \tag{3}$$

We will now remove the first row and column of this matrix, leaving us with a $(3d-1) \times (3d-1)$ matrix. By doing this, we are simply eliminating the invariant factor that is equal to zero.

Now we will zero out the nI_{d-1} matrix in the lower left corner by doing the following:

replace R_{2d+i} with $-nR_i + R_{2d+i}$ for all $i = 2, \cdots, d$ to get

$$\begin{pmatrix}
1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & n & n & n & \cdots & n \\
0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & n & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & n & n & \cdots & n \\
0 & \cdots & 0 & 0 & n & 0 & \cdots & 0 & 0 & n(2-n) & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & n & 0 & \cdots & 0 & 0 & 0 & n(2-n) & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & \cdots & 0 & 0 & n & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & n(2-n)
\end{pmatrix} \tag{4}$$

This gives a matrix a $(d-1) \times (d-1)$ and $2d \times 2d$ block.

From here on out, we will compute the Smith Normal Form of the $2d \times 2d$ block. Since the $(d-1) \times (d-1)$ block is an identity matrix, the invariant factors of the full matrix are the same as those of the lower right hand block.

$$\begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & n & n & n & \cdots & n \\
0 & 1 & 0 & \cdots & 0 & 0 & n & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 & n & n & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 & n & n & \cdots & n \\
n & 0 & 0 & \cdots & 0 & n(2-n) & 0 & 0 & \cdots & 0 \\
n & 0 & 0 & \cdots & 0 & 0 & n(2-n) & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
n & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & n(2-n)
\end{pmatrix} \tag{5}$$

To zero out the $(d-1) \times (d-1)$ lower triangular submatrix in rows 2 to d (with all entries equal to n), replace C_{d+j} with $-nC_i + C_{d+j}$ for all $i = 2, \cdots, d$ and for all $j = 2, \cdots, i$. This yields the following matrix

$$\begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & n & n & n & \cdots & n \\
0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\
n & 0 & 0 & \cdots & 0 & n(2-n) & 0 & 0 & \cdots & 0 \\
n & 0 & 0 & \cdots & 0 & 0 & n(2-n) & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
n & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & n(2-n)
\end{pmatrix} \tag{6}$$

Now move R_1 to be the last row and move C_1 to be the last column, i.e.,

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & n(2-n) & 0 & 0 & \cdots & 0 & n \\ 0 & 0 & \cdots & 0 & 0 & n(2-n) & 0 & \cdots & 0 & n \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & n(2-n) & n \\ 0 & 0 & \cdots & 0 & n & n & n & \cdots & n & 0 \end{pmatrix} \tag{7}$$

From this we have a $(d-1)\times(d-1)$ identity block in the upper left and a $(d+1)\times(d+1)$ block in the lower right (all other entries zero). From here, we will remove the $(d-1)\times(d-1)$ identity block for the same reasoning as above to obtain

$$\begin{pmatrix} n(2-n) & 0 & 0 & \cdots & 0 & n \\ 0 & n(2-n) & 0 & \cdots & 0 & n \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & n(2-n) & n \\ n & n & n & \cdots & n & 0 \end{pmatrix} \tag{8}$$

One way to reduce this to Smith Normal Form is as follows. Replace R_1 with $(n -$

$2)R_{d+1} + R_1$ to get

$$\begin{pmatrix} 0 & n(n-2) & n(n-2) & \cdots & n(n-2) & n \\ 0 & n(2-n) & 0 & \cdots & 0 & n \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & n(2-n) & n \\ n & n & n & \cdots & n & 0 \end{pmatrix} \tag{9}$$

Now replace $R_2, R_3, \cdots, R_d$ with $R_1 - R_2, R_1 - R_3, \cdots, R_1 - R_d$, respectively, to get

$$\begin{pmatrix} 0 & n(n-2) & n(n-2) & \cdots & n(n-2) & n \\ 0 & 2n(n-2) & n(n-2) & \cdots & n(n-2) & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & n(n-2) & 0 & \cdots & 2n(n-2) & 0 \\ n & n & n & \cdots & n & 0 \end{pmatrix} \tag{10}$$

where the $(d-1) \times (d-1)$ sub-matrix in the middle has all diagonal entries equal to $2n(n-2)$ and all other entries equal to $n(n-2)$.

Now replace R_3 with $R_2 + R_3$. Then replace R_4 with $R_3 + R_4$. Continue this process until we have replaced R_d with $R_{d-1} + R_d$.

$$\begin{pmatrix} 0 & n(n-2) & n(n-2) & \cdots & n(n-2) & n \\ 0 & 2n(n-2) & n(n-2) & \cdots & n(n-2) & 0 \\ 0 & 3n(n-2) & 3n(n-2) & \cdots & 2n(n-2) & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & nd(n-2) & nd(n-2) & \cdots & nd(n-2) & 0 \\ n & n & n & \cdots & n & 0 \end{pmatrix} \qquad (11)$$

where the $(d-1)\times(d-1)$ block in the middle has the following property: R_i has entry $i(n-2)$ in the first $(i-1)$-st columns and $(i-1)(n-2)$ in the remaining columns, for $i = 2, \cdots, d$.

Now replace C_i for $i = 3, \cdots, d$ with $C_2 - C_i$ to get

$$\begin{pmatrix} 0 & n(n-2) & 0 & \cdots & 0 & n \\ 0 & 2n(n-2) & n(n-2) & \cdots & n(n-2) & 0 \\ 0 & 3n(n-2) & 0 & \cdots & n(n-2) & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & n(d-1)(n-2) & 0 & \cdots & n(n-2) & 0 \\ 0 & nd(n-2) & 0 & \cdots & 0 & 0 \\ n & n & 0 & \cdots & 0 & 0 \end{pmatrix} \qquad (12)$$

This gives us a $(d-2)\times(d-2)$ upper triangular sub-matrix with all entries equal to $n(n-2)$.

Now use C_3 to zero out the remaining $n(n-2)$ entries in row 2, columns C_i for $i = 4, \cdots, d$ by replacing C_i with $C_i - C_3$.

$$\begin{pmatrix} 0 & n(n-2) & 0 & \cdots & 0 & n \\ 0 & 2n(n-2) & n(n-2) & \cdots & 0 & 0 \\ 0 & 3n(n-2) & 0 & \cdots & n(n-2) & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & n(d-1)(n-2) & 0 & \cdots & n(n-2) & 0 \\ 0 & nd(n-2) & 0 & \cdots & 0 & 0 \\ n & n & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{13}$$

Now use C_4 to zero out the remaining $n(n-2)$ in row 3, columns C_i for $i = 5, \cdots, d$ by replacing C_i with $C_i - C_4$.

$$\begin{pmatrix} 0 & n(n-2) & 0 & \cdots & 0 & n \\ 0 & 2n(n-2) & n(n-2) & \cdots & 0 & 0 \\ 0 & 3n(n-2) & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & n(d-1)(n-2) & 0 & \cdots & n(n-2) & 0 \\ 0 & nd(n-2) & 0 & \cdots & 0 & 0 \\ n & n & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{14}$$

Continue this process until only $n(n-2)$ is left on the diagonal of this $(d-2) \times (d-2)$ block and all other entries are equal to zero.

We will now zero out all the entries from R_2 to R_{d-1} in C_2. First replace C_2 with

$-2C_3 + C_2$. Then replace C_2 with $-3C_4 + C_2$. Continue this process until we have

$$\begin{pmatrix} 0 & n(n-2) & 0 & \cdots & 0 & n \\ 0 & 0 & n(n-2) & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & n(n-2) & 0 \\ 0 & nd(n-2) & 0 & \cdots & 0 & 0 \\ n & n & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{15}$$

Lastly, replace C_2 with $-C_1 + C_2$ and replace C_2 with $-(n-2)C_{d+1} + C_2$.

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & n \\ 0 & 0 & n(n-2) & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & n(n-2) & 0 \\ 0 & nd(n-2) & 0 & \cdots & 0 & 0 \\ n & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{16}$$

Permute the columns to obtain a diagonal matrix with the following entries on the diagonal: n with multiplicity 2, $n(n-2)$ with multiplicity $d-2$ and $nd(n-2)$ with multiplicity 1.

Finally, returning to the $(n-1) \times (n-1)$ reduced voltage Laplacian $(*)$, when we tensor (again, as in Section 2.2.1) the $(n-4) \times (n-4)$ matrix in the lower right with

ρ,

$$\begin{pmatrix} 0 & (n-2)(1-\tau) & 1-\tau & 0 & \cdots & 0 \\ 1-\tau & n & 0 & 0 & \cdots & 0 \\ 0 & 0 & n & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n \end{pmatrix}$$

we get the entry n on the diagonal with multiplicity $(n-4)d$. Putting these two matrices together yields the following result:

Theorem 14. *Let Y be a single voltage cover of the complete graph K_n by the cyclic group of order d, where $n \geq 4$ and $d \geq 3$. Then*

$$J(Y) \cong (\mathbb{Z}/n\mathbb{Z})^{(n-4)d+2} \oplus (\mathbb{Z}/n(n-2)\mathbb{Z})^{d-2} \oplus \mathbb{Z}/dn(n-2)\mathbb{Z},$$

where the exponents indicate the multiplicities of the (distinct) invariant factors. In particular, the order of the Jacobian of Y is $n^{(n-3)d+1} \cdot (n-2)^{d-1} \cdot d$ and its rank is $(n-3)d+1$.

Corollary 11. *For Y as above, the number of spanning trees of Y is $n^{(n-3)d+1} \cdot (n-2)^{d-1} \cdot d$*

Theorem 15. *For Y as above, but with $d = 2$, we have that*

$$J(Y) \cong (\mathbb{Z}/n\mathbb{Z})^{2(n-4)+2} \oplus (\mathbb{Z}/2n(n-2)\mathbb{Z})$$

where, again, the exponents indicate the multiplicities of the (distinct) invariant fac-

tors. When $n = 3$ and $d \geq 1$, we have that

$$J(Y) \cong \mathbb{Z}/3d\mathbb{Z}.$$

Proof. For $d = 2$, the same steps apply for matrices (1) through (10). For matrix (10), we have

$$\begin{pmatrix} 0 & n(n-2) & n \\ 0 & 2n(n-2) & 0 \\ n & n & 0 \end{pmatrix}$$

Now apply the same step as above to get matrix (16)

$$\begin{pmatrix} 0 & 0 & n \\ 0 & 2n(n-2) & 0 \\ n & 0 & 0 \end{pmatrix}$$

From $(*)$, we get n on the diagonal with multiplicity $2(n-4)$. Putting together $(*)$ and (16) yields the desired result.

Now when $n = 3$, X is a triangle and Y is a cycle of length $3d$. This case follows from the familiar Jacobian of a cycle. Alternatively, this can be done independently by corresponding (albeit easier) steps for the $n > 3$ reduction above, but starting with the 2×2 reduced voltage Laplacian

$$\begin{pmatrix} 1-\tau & 0 \\ 3 & 1-\tau \end{pmatrix}$$

Follow the same reductions that we did above to get (2) and (3). This yields the following $(2d-1)\times(2d-1)$ matrix

$$\begin{pmatrix}
1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
3 & 3 & \cdots & 3 & 3 & 3d & 0 & 0 & \cdots & 0 & 0 \\
3 & 0 & \cdots & 0 & 0 & 3 & 1 & 0 & \cdots & 0 & 0 \\
3 & 3 & \cdots & 0 & 0 & 6 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
3 & 3 & \cdots & 3 & 3 & 3(d-1) & 0 & 0 & \cdots & 0 & 1
\end{pmatrix}$$

We now, as usual, use the $(d-1)\times(d-1)$ identity block in the upper left to zero out all 3's in columns 1 to $d-1$. Then likewise use the $(d-1)\times(d-1)$ identity block in the lower right to zero out the entries in column d, rows $d+1$ to $2d-1$. This yields the following matrix, which is the asserted Smith Normal Form (after permuting rows and columns).

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 3d & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

□

Example 6. These are the corresponding outputs for $n = 4$ and $d = 7$. Note that each labeled matrix in this example is the special case of the correspondingly labeled matrix in the proof above, so we do not repeat how these are obtained.

$$\begin{pmatrix} 0 & 2(1-\tau) & 1-\tau & 0 & 0 & 0 \\ 1-\tau & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix} \tag{$*$}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix} \tag{$**$}$$

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8
\end{pmatrix} \tag{1}
$$

$$
\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&4&4&4&4&4&4&4\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&4&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&4&4&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&4&4&4&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&4&4&4&4&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&4&4&4&4&4&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&4&4&4&4&4&4\\
4&0&0&0&0&0&0&0&0&0&0&0&0&0&-8&0&0&0&0&0&0\\
4&4&0&0&0&0&0&0&0&0&0&0&0&0&0&-8&0&0&0&0&0\\
4&0&4&0&0&0&0&0&0&0&0&0&0&0&0&0&-8&0&0&0&0\\
4&0&0&4&0&0&0&0&0&0&0&0&0&0&0&0&0&-8&0&0&0\\
4&0&0&0&4&0&0&0&0&0&0&0&0&0&0&0&0&0&-8&0&0\\
4&0&0&0&0&4&0&0&0&0&0&0&0&0&0&0&0&0&0&-8&0\\
4&0&0&0&0&0&4&0&0&0&0&0&0&0&0&0&0&0&0&0&-8
\end{pmatrix}
\tag{2}
$$

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 4 & 4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 4 & 4 & 4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 4 & 4 & 4 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 4 & 4 & 4 & 4 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 4 & 4 & 4 & 4 & 4 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 4 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 4 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 4 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 4 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8
\end{pmatrix}
\tag{3}
$$

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 4 & 4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 4 & 4 & 4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 4 & 4 & 4 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 4 & 4 & 4 & 4 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 4 & 4 & 4 & 4 & 4 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8
\end{pmatrix} \tag{4}$$

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 4 & 4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 4 & 4 & 4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 4 & 4 & 4 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 4 & 4 & 4 & 4 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8
\end{pmatrix} \tag{5}
$$

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8
\end{pmatrix} \tag{6}
$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 0 \end{pmatrix} \tag{7}$$

$$\begin{pmatrix} -8 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & -8 & 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & -8 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & -8 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & -8 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & -8 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & -8 & 4 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 0 \end{pmatrix} \tag{8}$$

$$\begin{pmatrix} 0 & 8 & 8 & 8 & 8 & 8 & 8 & 4 \\ 0 & -8 & 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & -8 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & -8 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & -8 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & -8 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & -8 & 4 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 0 \end{pmatrix} \tag{9}$$

$$\begin{pmatrix} 0 & 8 & 8 & 8 & 8 & 8 & 8 & 4 \\ 0 & 16 & 8 & 8 & 8 & 8 & 8 & 0 \\ 0 & 8 & 16 & 8 & 8 & 8 & 8 & 0 \\ 0 & 8 & 8 & 16 & 8 & 8 & 8 & 0 \\ 0 & 8 & 8 & 8 & 16 & 8 & 8 & 0 \\ 0 & 8 & 8 & 8 & 8 & 16 & 8 & 0 \\ 0 & 8 & 8 & 8 & 8 & 8 & 16 & 0 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 0 \end{pmatrix} \tag{10}$$

$$\begin{pmatrix} 0 & 8 & 8 & 8 & 8 & 8 & 8 & 4 \\ 0 & 16 & 8 & 8 & 8 & 8 & 8 & 0 \\ 0 & 24 & 24 & 16 & 16 & 16 & 16 & 0 \\ 0 & 32 & 32 & 32 & 24 & 24 & 24 & 0 \\ 0 & 40 & 40 & 40 & 40 & 32 & 32 & 0 \\ 0 & 48 & 48 & 48 & 48 & 48 & 40 & 0 \\ 0 & 56 & 56 & 56 & 56 & 56 & 56 & 0 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 0 \end{pmatrix} \tag{11}$$

$$\begin{pmatrix} 0 & 8 & 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 16 & 8 & 8 & 8 & 8 & 8 & 0 \\ 0 & 24 & 0 & 8 & 8 & 8 & 8 & 0 \\ 0 & 32 & 0 & 0 & 8 & 8 & 8 & 0 \\ 0 & 40 & 0 & 0 & 0 & 8 & 8 & 0 \\ 0 & 48 & 0 & 0 & 0 & 0 & 8 & 0 \\ 0 & 56 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{12}$$

$$
\begin{pmatrix}
0 & 8 & 0 & 0 & 0 & 0 & 0 & 4 \\
0 & 16 & 8 & 0 & 0 & 0 & 0 & 0 \\
0 & 24 & 0 & 8 & 8 & 8 & 8 & 0 \\
0 & 32 & 0 & 0 & 8 & 8 & 8 & 0 \\
0 & 40 & 0 & 0 & 0 & 8 & 8 & 0 \\
0 & 48 & 0 & 0 & 0 & 0 & 8 & 0 \\
0 & 56 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 4 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \tag{13}
$$

$$
\begin{pmatrix}
0 & 8 & 0 & 0 & 0 & 0 & 0 & 4 \\
0 & 16 & 8 & 0 & 0 & 0 & 0 & 0 \\
0 & 24 & 0 & 8 & 0 & 0 & 0 & 0 \\
0 & 32 & 0 & 0 & 8 & 8 & 8 & 0 \\
0 & 40 & 0 & 0 & 0 & 8 & 8 & 0 \\
0 & 48 & 0 & 0 & 0 & 0 & 8 & 0 \\
0 & 56 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 4 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \tag{14}
$$

$$
\begin{pmatrix}
0 & 8 & 0 & 0 & 0 & 0 & 0 & 4 \\
0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 \\
0 & 56 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 4 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \tag{15}
$$

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\
0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 \\
0 & 56 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \tag{16}
$$

3.3 Conjecture on the Jacobian of Single Voltage Cyclic Covers of $K_{n,n}$

We now let $X = K_{n,n}$ in Definition 19. We partition the $2n$ vertices by: each v_i for i odd is adjacent to every v_j for j even, and vice versa. Note that the Jacobian does not depend on which edge is assigned the nontrivial voltage τ. However, as in Definition 19, we label the edge $v_1 \to v_2$ with voltage τ.

Observe that the closed walk $v_1 \xrightarrow{\tau} v_2 \xrightarrow{1} v_3 \xrightarrow{1} v_4 \xrightarrow{1} v_1$ has net voltage τ. By Corollary 4, this immediately gives:

Proposition 11. *If $(K_{n,n}, Z_d, \alpha)$ is a voltage graph, where $\alpha : E(X)^+ \to Z_d$ is the single voltage assignment, then the derived graph Y is connected.*

Using Sage, we compute the following tables, which yields the Jacobian of the derived graph corresponding to the voltage graph $(K_{n,n}, Z_d, \alpha)$, where α is the single voltage assignment and $n = 3, 4, 5$, respectively. From this, we formulate Conjecture 2 for the rank, invariant factors, and order of the Jacobian.

Table 3.6 Jacobian of $K_{3,3}$ with Single Voltage Cover by Z_d, (exponents represent multiplicities)

d		1st IF	2nd IF	3rd IF	order
1	1^2	3^2	9		3^4
2	1^6	3^4	72		$2^3 \cdot 3^6$
3	1^{10}	3^5	12	108	$2^4 \cdot 3^9$
4	1^{14}	3^6	12^2	144	$2^8 \cdot 3^{10}$
5	1^{18}	3^7	12^3	180	$2^8 \cdot 3^{12} \cdot 5$
6	1^{22}	3^8	12^4	216	$2^{11} \cdot 3^{15}$
7	1^{26}	3^9	12^5	252	$2^{12} \cdot 3^{16} \cdot 7$
8	1^{30}	3^{10}	12^6	288	$2^{17} \cdot 3^{18}$
9	1^{34}	3^{11}	12^7	324	$2^{16} \cdot 3^{22}$
10	1^{38}	3^{12}	12^8	360	$2^{19} \cdot 3^{22} \cdot 5$

For $d \leq 10$, we observe the following from Table 3.6:

(i) For $d \geq 1$, the rank is $2d + 1$.

(ii) For $d \geq 3$, there are 3 distinct invariant factors: the first IF is 3 with multiplicity $d + 2$, the second IF is 12 with multiplicity $d - 2$. The third IF is $12(3d)$ with multiplicity 1.

(iii) $|\mathcal{J}(Y)| = 3^{2d+2} \cdot 2^{2d-2} \cdot d$.

Table 3.7 Jacobian of $K_{4,4}$ with Single Voltage Cover by Z_d, (exponents represent multiplicities)

d		1st IF	2nd IF	3rd IF	order
1	1^2	4^4	16		4^6
2	1^6	4^8	288		$2^{21} \cdot 3^2$
3	1^{10}	4^{11}	36	432	$2^{28} \cdot 3^5$
4	1^{14}	4^{14}	36^2	576	$2^{38} \cdot 3^6$
5	1^{18}	4^{17}	36^3	720	$2^{44} \cdot 3^8 \cdot 5$
6	1^{22}	4^{20}	36^4	864	$2^{53} \cdot 3^{11}$
7	1^{26}	4^{23}	36^5	1008	$2^{60} \cdot 3^{12} \cdot 7$
8	1^{30}	4^{26}	36^6	1152	$2^{71} \cdot 3^{14}$
9	1^{34}	4^{29}	36^7	1296	$2^{76} \cdot 3^{18}$
10	1^{38}	4^{32}	36^8	1440	$2^{85} \cdot 3^{18} \cdot 5$

For $d \leq 10$, we observe the following from Table 3.7:

(i) For $d \geq 1$, the rank is $4d + 1$.

(ii) For $d \geq 3$, there are 3 distinct invariant factors: the first IF is 4 with multiplicity $3d + 2$, the second IF is 36 with multiplicity $d - 2$. The third IF is $36(4d)$ with multiplicity 1.

(iii) $|\mathcal{J}(Y)| = 4^{4d+2} \cdot 3^{2d-2} \cdot d$.

Table 3.8 Jacobian of $K_{5,5}$ with Single Voltage Cover by Z_d, (exponents represent multiplicities)

d		1st IF	2nd IF	3rd IF	order
1	1^2	5^6	25		5^8
2	1^6	5^{12}	800		$2^5 \cdot 5^{14}$
3	1^{10}	5^{17}	80	1200	$2^8 \cdot 3 \cdot 5^{20}$
4	1^{14}	5^{22}	80^2	1600	$2^{14} \cdot 5^{26}$
5	1^{18}	5^{27}	80^3	2000	$2^{16} \cdot 5^{33}$
6	1^{22}	5^{32}	80^4	2400	$2^{21} \cdot 3 \cdot 5^{38}$
7	1^{26}	5^{37}	80^5	2800	$2^{24} \cdot 5^{44} \cdot 7$
8	1^{30}	5^{42}	80^6	3200	$2^{31} \cdot 5^{50}$
9	1^{34}	5^{47}	80^7	3600	$2^{32} \cdot 3^2 \cdot 5^{66}$
10	1^{38}	5^{52}	80^8	4000	$2^{37} \cdot 5^{73}$

For $d \leq 10$, we observe the following from Table 3.8:

(i) For $d \geq 1$, the rank is $6d + 1$.

(ii) For $d \geq 3$, there are 3 distinct invariant factors: the first IF is 5 with multiplicity $5d + 2$, the second IF is 80 with multiplicity $d - 2$. The third IF is $80(5d)$ with multiplicity 1.

(iii) $|\mathcal{J}(Y)| = 5^{6d+2} \cdot 4^{2d-2} \cdot d$.

Using the above observations, we formulate the following conjecture:

Conjecture 2. *For $K_{n,n}$ the complete bipartite graph on $2n$ vertices with single voltage cover by Z_d, we have the following:*

(i) For $d \geq 1$, the rank is $2(n-2)d + 1$.

(ii) For $d \geq 3$, there are three distinct invariant factors: the first IF is n with multiplicity $(2n-5)d + 2$, the second IF is $n(n-1)^2$ with multiplicity $d - 2$, the third IF is $n(n-1)^2(nd)$ with multiplicity 1.

(iii) $|\mathcal{J}(Y)| = n^{(2n-4)d+2}(n-1)^{2d-2}d.$

In the next section, we obtain partial results for this conjecture.

3.4 Partial Results on the Jacobian of Single Voltage Cyclic Covers of $K_{n,n}$

Again, we let $X = K_{n,n}$ in Definition 19. We partition the $2n$ vertices by: each v_i for i odd is adjacent to every v_j for j even, and vice versa.

In this section, we compute the Smith Normal Form of the Laplacian matrix for the single voltage cyclic cover of $K_{n,n}$ over $\mathbb{Z}_{(p)}$, the integers localized at (p), for p not dividing n. This gives us the primes p (and their powers) that divide $|\mathcal{J}(Y)|$, for $p \nmid n$.

We first begin by proving what the reduced Stickelberger element is for the derived graph corresponding to single voltage cyclic covers of $K_{n,n}$.

Theorem 16. *Let $(K_{n,n}, Z_d, \alpha)$ be as in Definition 19. Then the reduced Stickelberger element is*

$$\Theta_{Y/X} = -(n-1)^2 n^{2n-4}(\tau-1)^2\tau^{-1}.$$

Proof. For $X = K_{n,n}$ with single voltage assignment by Z_d, we have the voltage Laplacian matrix with n down the diagonal, $-\tau$ in entry $(1,2)$, $-\tau^{-1}$ in entry $(2,1)$ and entry i,j equals -1 if $i+j$ is odd, and all other entries are zero. This gives us the following $2n \times 2n$ matrix:

$$L_\alpha = \begin{pmatrix} n & -\tau & 0 & -1 & 0 & \cdots & 0 & -1 \\ -\tau^{-1} & n & -1 & 0 & -1 & \cdots & -1 & 0 \\ 0 & -1 & n & -1 & 0 & \cdots & 0 & -1 \\ -1 & 0 & -1 & n & -1 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & -1 & 0 & -1 & 0 & \cdots & n & -1 \\ -1 & 0 & -1 & 0 & -1 & \cdots & -1 & n \end{pmatrix}$$

We will now use row and column operations in $\mathbb{Q}[\tau]$ to put the matrix in essentially upper triangular form. We denote row i and column j of the voltage Laplacian by C_i and R_j, respectively. First replace C_1 by $C_1 + C_3 + \cdots + C_{2n}$ to get

$$\begin{pmatrix} 1 & -\tau & 0 & -1 & 0 & \cdots & 0 & -1 \\ -\tau^{-1} - (n-1) & n & -1 & 0 & -1 & \cdots & -1 & 0 \\ 1 & -1 & n & -1 & 0 & \cdots & 0 & -1 \\ 0 & 0 & -1 & n & -1 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & -1 & 0 & -1 & 0 & \cdots & n & -1 \\ 0 & 0 & -1 & 0 & -1 & \cdots & -1 & n \end{pmatrix}$$

Now replace $C_2, C_4, C_6, \cdots, C_{2n}$ with $C_1 + C_2, C_1 + C_4, C_1 + C_6, \cdots, C_1 + C_{2n}$, respectively to get

$$\begin{pmatrix} 1 & 1-\tau & 0 & 0 & 0 & \cdots & 0 & 0 \\ -\tau^{-1}-(n-1) & -\tau^{-1}+1 & -1 & -\tau^{-1}-(n-1) & -1 & \cdots & -1 & -\tau^{-1}-(n-1) \\ 1 & 0 & n & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & n & -1 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & 0 & \cdots & n & 0 \\ 0 & 0 & -1 & 0 & -1 & \cdots & -1 & n \end{pmatrix}$$

This eliminates all of the -1's in all of the even-indexed columns.

We will now eliminate all of the -1's in all of the odd-indexed columns by doing the following:

Replace $C_3, C_5, C_7, \cdots, C_{2n-1}$ with $\frac{1}{n}C_{2n}+C_3, \frac{1}{n}C_{2n}+C_5, \frac{1}{n}C_{2n}+C_7, \cdots, \frac{1}{n}C_{2n}+C_{2n-1}$, respectively, to get

$$\begin{pmatrix} 1 & 1-\tau & 0 & 0 & \cdots & 0 & 0 \\ -\tau^{-1}-(n-1) & -\tau^{-1}+1 & \frac{-\tau^{-1}-2n+1}{n} & -\tau^{-1}-(n-1) & \cdots & \frac{-\tau^{-1}-2n+1}{n} & -\tau^{-1}-(n-1) \\ 1 & 0 & n & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & n & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & n & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & n \end{pmatrix}$$

This eliminates the -1's in row $2n$

Now replace $C_3, C_5, C_7, \cdots, C_{2n-1}$ with

$\frac{1}{n}C_{2n-2} + C_3, \frac{1}{n}C_{2n-2} + C_5, \frac{1}{n}C_{2n-2} + C_7, \cdots, \frac{1}{n}C_{2n-2} + C_{2n-1}$, respectively, to get

$$\begin{pmatrix} 1 & 1-\tau & 0 & 0 & \cdots & 0 & 0 \\ -\tau^{-1}-(n-1) & -\tau^{-1}+1 & \frac{-2\tau^{-1}-3n+2}{n} & -\tau^{-1}-(n-1) & \cdots & \frac{-2\tau^{-1}-3n+2}{n} & -\tau^{-1}-(n-1) \\ 1 & 0 & n & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & n & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & n & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & n \end{pmatrix}$$

This eliminates the -1's in row $2n-2$.

Now replace $C_3, C_5, C_7, \cdots, C_{2n-1}$ with

$\frac{1}{n}C_{2n-4} + C_3, \frac{1}{n}C_{2n-4} + C_5, \frac{1}{n}C_{2n-4} + C_7, \cdots, \frac{1}{n}C_{2n-4} + C_{2n-1}$, respectively, to get

$$\begin{pmatrix} 1 & 1-\tau & 0 & 0 & \cdots & 0 & 0 \\ -\tau^{-1}-(n-1) & -\tau^{-1}+1 & \frac{-3\tau^{-1}-4n+3}{n} & -\tau^{-1}-(n-1) & \cdots & \frac{-3\tau^{-1}-4n+3}{n} & -\tau^{-1}-(n-1) \\ 1 & 0 & n & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & n & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & n & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & n \end{pmatrix}$$

This eliminates the -1's in row $2n-4$.

Continue this process to zero out the remaining -1's in $R_{2n-6}, \cdots, R_4$ to get the following

$$\begin{pmatrix} 1 & 1-\tau & 0 & 0 & \cdots & 0 \\ -\tau^{-1}-(n-1) & -\tau^{-1}+1 & \frac{-(n-1)\tau^{-1}-n^2+n-1}{n} & -\tau^{-1}-(n-1) & \cdots & -\tau^{-1}-(n-1) \\ 1 & 0 & n & 0 & \cdots & 0 \\ 0 & 0 & 0 & n & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & n \end{pmatrix}$$

It only remains to zero out the 1's in C_1 from R_3 to R_{2n-1}. To do this, we first replace C_1 with $-\frac{1}{n}C_{2n-1}+C_1$. Then replace C_1 with $-\frac{1}{n}C_{2n-3}+C_1$. Continue this process until we replace C_1 with $-\frac{1}{n}C_3+C_1$. After simplifying, this yields the following matrix

$$\begin{pmatrix} 1 & 1-\tau & 0 & 0 & \cdots & 0 \\ \frac{(-2n+1)\tau^{-1}-(n-1)^2}{n^2} & -\tau^{-1}+1 & \frac{-(n-1)\tau^{-1}-n^2+n-1}{n} & -\tau^{-1}-(n-1) & \cdots & -\tau^{-1}-(n-1) \\ 0 & 0 & n & 0 & \cdots & 0 \\ 0 & 0 & 0 & n & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & n \end{pmatrix}$$

Let L' be the preceding matrix. Now use cofactor expansion along the first row of L'

to get

$$\Theta_{Y/X} = \det L_\alpha = \det L' = 1 \cdot \det L'_{1,1} - (1-\tau) \det L'_{1,2}$$

where $L'_{i,j}$ is the i, j minor of L'. To compute each of the minor determinants, use cofactor expansion along its first column (which has only one nonzero entry), to get:

$$\det L' = 1(1-\tau^{-1})n^{2n-2} - (1-\tau)\frac{(-2n+1)\tau^{-1} - (n-1)^2}{n^2} n^{2n-2}.$$

By elementary algebra this simplifies to the stated conclusion. $\square$

We now present the following Theorem from [Ger76], which we will use to obtain partial results for the Jacobian.

Theorem 17. *Let $A, B \in M_n(R)$, where $M_n(R)$ is the ring of $n \times n$ matrices with entries in the ring R. Then $A \sim B$ if and only if $A \sim_p B$ for all $p \in \mathcal{P}$ (where $\sim$ and $\sim_p$ denotes matrix equivalence over R and $R_{(p)}$, respectively); moreover,*

$$S(A) = \prod_{p \in \mathcal{P}} S_p(A).$$

where $S(A)$ and $S_p(A)$ denote the Smith Normal Form of A over R and $R_{(p)}$, respectively.

We will now compute the Smith Normal Form of the previous matrix over $\mathbb{Z}_{(p)}$, the integers localized at (p), for p not dividing n, so n becomes a unit.

Taking the last matrix from above, first replace R_2 by $n^2 R_2$. Then replace $R_3, \cdots, R_{2n}$

by $\frac{1}{n}R_3, \cdots, \frac{1}{n}R_{2n}$, respectively, to get

$$\begin{pmatrix} 1 & 1-\tau & \cdots & 0 \\ (-2n+1)\tau^{-1}-(n-1)^2 & -n^2\tau^{-1}+n^2 & \cdots & -n^2\tau^{-1}-n^3+n^2 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Now use the $(2n-2) \times (2n-2)$ block identity matrix to zero out the entries in row 2 from columns 3 through $2n$.

Note that doing elementary row and column operations on the voltage Laplacian is the same as changing generators in the domain and range of the $\mathbb{Z}_{(p)}[G]$-module $\operatorname{coker}(L_\alpha)$. In particular, these also just change generators of its $\mathbb{Z}_{(p)}$-module structure. So in order to compute the Smith Normal Form of the Laplacian of Y, we may begin with the already reduced voltage Laplacian above. Thus, we will now tensor this matrix with ρ (again, as in Section 2.2.1) to get the $d \times d$ block matrices in each entry.

$$\begin{pmatrix} I_d & \rho(1-\tau) \\ (-2n+1)\cdot\rho(\tau^{-1})-(n-1)^2 I_d & -n^2\cdot\rho(\tau^{-1})+n^2\cdot I_d \end{pmatrix}$$

where I_d denotes the $d \times d$ identity matrix,

$$\rho(1-\tau) = \begin{pmatrix} 1 & 0 & \cdots & 0 & -1 \\ -1 & 1 & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & -1 & 1 \end{pmatrix}$$

$$-n^2 \cdot \rho(\tau^{-1}) + n^2 \cdot I_d = \begin{pmatrix} n^2 & -n^2 & \cdots & 0 & 0 \\ 0 & n^2 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -n^2 \\ -n^2 & 0 & \cdots & 0 & n^2 \end{pmatrix}$$

and lastly,

$$(-2n+1)\cdot\rho(\tau^{-1}) - (n-1)^2 I_d = \begin{pmatrix} -(n-1)^2 & -2n+1 & \cdots & 0 & 0 \\ 0 & -(n-1)^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -(n-1)^2 & -2n+1 \\ -2n+1 & 0 & \cdots & 0 & -(n-1)^2 \end{pmatrix}$$

Putting these together, we get the following matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & -1 & 1 \\ -(n-1)^2 & -2n+1 & \cdots & 0 & n^2 & -n^2 & \cdots & 0 & 0 \\ 0 & -(n-1)^2 & \cdots & 0 & 0 & n^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -2n+1 & 0 & 0 & \cdots & n^2 & -n^2 \\ -2n+1 & 0 & \cdots & -(n-1)^2 & -n^2 & 0 & \cdots & 0 & n^2 \end{pmatrix}$$

We will now use row and column operations to put this matrix in diagonal form. To zero out the last column, replace C_{2d} with $C_{d+1} + C_{d+2} + \cdots C_{2d}$ to get

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & -1 & 0 \\ -(n-1)^2 & -2n+1 & \cdots & 0 & n^2 & -n^2 & \cdots & 0 & 0 \\ 0 & -(n-1)^2 & \cdots & 0 & 0 & n^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -2n+1 & 0 & 0 & \cdots & n^2 & 0 \\ -2n+1 & 0 & \cdots & -(n-1)^2 & -n^2 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Now replace R_{2d} with $R_{d+1} + R_{d+2} + \cdots + R_{2d}$ to get

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & -1 & 0 \\ -(n-1)^2 & -2n+1 & \cdots & 0 & n^2 & -n^2 & \cdots & 0 & 0 \\ 0 & -(n-1)^2 & \cdots & 0 & 0 & n^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -2n+1 & 0 & 0 & \cdots & n^2 & 0 \\ -n^2 & -n^2 & \cdots & -n^2 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

This gives us a matrix with entry $-n^2$ in $R_{2d},\ C_1$ through C_d and 0 from C_{d+1} through C_{2d}.

Now replace R_1 with $R_1 + R_2 + \cdots + R_d$ to get

$$\begin{pmatrix} 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & -1 & 0 \\ -(n-1)^2 & -2n+1 & \cdots & 0 & n^2 & -n^2 & \cdots & 0 & 0 \\ 0 & -(n-1)^2 & \cdots & 0 & 0 & n^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -2n+1 & 0 & 0 & \cdots & n^2 & 0 \\ -n^2 & -n^2 & \cdots & -n^2 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

This gives us a matrix with all entries equal to 1 in R_1, C_1 through C_d and 0 from C_d to C_{2d}.

To zero out the 1's in R_1, C_2 through C_d and to zero out the $-n^2$ in R_{2d} as well as columns C_2 through C_d, replace $C_2, C_3, \cdots, C_d$ with $C_2 - C_1, C_3 - C_1, \cdots, C_d - C_1$, respectively. This yields the following matrix.

$$\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & -1 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & -1 & 0 \\
-(n-1)^2 & n^2-4n+2 & (n-1)^2 & \cdots & (n-1)^2 & n^2 & -n^2 & \cdots & 0 & 0 \\
0 & -(n-1)^2 & -2n+1 & \cdots & 0 & 0 & n^2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -2n+1 & 0 & 0 & \cdots & n^2 & 0 \\
-n^2 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}$$

In R_d, we have $-(n-1)^2$ in the first column, $n^2 - 4n + 2$ in the second column, and $(n-1)^2$ in columns 3 through d.

Now use R_1 to zero out the remaining entries in C_1 to get the matrix

$$\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & -1 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & -1 & 0 \\
0 & n^2-4n+2 & (n-1)^2 & \cdots & (n-1)^2 & n^2 & -n^2 & \cdots & 0 & 0 \\
0 & -(n-1)^2 & -2n+1 & \cdots & 0 & 0 & n^2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -2n+1 & 0 & 0 & \cdots & n^2 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}$$

Now remove C_{2d} and R_{2d}, as they do not contribute any nonzero invariant factors. This leaves us with a $(2d-1) \times (2d-1)$ matrix.

$$\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 & -1 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & -1 \\
0 & n^2-4n+2 & (n-1)^2 & \cdots & (n-1)^2 & n^2 & -n^2 & \cdots & 0 \\
0 & -(n-1)^2 & -2n+1 & \cdots & 0 & 0 & n^2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & -2n+1 & 0 & 0 & \cdots & n^2
\end{pmatrix}$$

Now replace C_{2d-1} with $C_{d+1} + C_{d+2} + \cdots + C_{2d-1}$ to get

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & -1 \\ 0 & n^2-4n+2 & (n-1)^2 & \cdots & (n-1)^2 & n^2 & -n^2 & \cdots & 0 \\ 0 & -(n-1)^2 & -2n+1 & \cdots & 0 & 0 & n^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -2n+1 & 0 & 0 & \cdots & n^2 \end{pmatrix}$$

(this makes all entries in C_{2d-1} zero except for the entries in R_d and R_{2d-1}, which are -1 and n^2, respectively).

Now continue this process; replace $C_{2d-2}, \cdots, C_{d+2}$ with $C_{d+1}+C_{d+2}+\cdots+C_{2d-2}, \cdots, C_{d+1}+C_{d+2}$, respectively.

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & -1 \\ 0 & n^2-4n+2 & (n-1)^2 & \cdots & (n-1)^2 & n^2 & 0 & \cdots & 0 \\ 0 & -(n-1)^2 & -2n+1 & \cdots & 0 & 0 & n^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -2n+1 & 0 & 0 & \cdots & n^2 \end{pmatrix}$$

This gives us a $(d-1) \times (d-1)$ diagonal block in the upper right with -1's down the diagonal and a $(d-1) \times (d-1)$ diagonal block in the lower right with n^2's down the diagonal.

Now to zero out the $(d-1) \times (d-1)$ block in the upper right with -1's down the diagonal, replace $C_{d+1}, C_{d+2}, \cdots, C_{2d-1}$ with $C_2 + C_{d+1}, C_3 + C_{d+2}, \cdots, C_d + C_{2d-1}$

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & n^2-4n+2 & (n-1)^2 & \cdots & 0 & 2(n-1)^2 & (n-1)^2 & \cdots & (n-1)^2 \\ 0 & -(n-1)^2 & -2n+1 & \cdots & 0 & -(n-1)^2 & (n-1)^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -2n+1 & 0 & 0 & \cdots & (n-1)^2 \end{pmatrix}$$

In the lower right $(d-1) \times (d-1)$ block, we have entry $2(n-1)^2$ in column and row $d+1$, and then entry $(n-1)^2$ in row $d+1$, columns $d+2$ through $2d-1$. Then in columns $d+1$ through $2d-1$, row $d+2$ through row $2d-1$, we have $-(n-1)^2$ on the lower subdiagonal, and $(n-1)^2$ on the diagonal.

Use the identity $d \times d$ block in the upper left to zero out the $(d-1) \times (d-1)$

block in the lower left. This leaves us with the following matrix.

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 2(n-1)^2 & (n-1)^2 & \cdots & (n-1)^2 \\ 0 & 0 & 0 & \cdots & 0 & -(n-1)^2 & (n-1)^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & (n-1)^2 \end{pmatrix}$$

Since the $d \times d$ block in the upper left contributes no nontrivial factors, we will only consider the $(d-1) \times (d-1)$ block in the lower right

$$\begin{pmatrix} 2(n-1)^2 & (n-1)^2 & (n-1)^2 & \cdots & (n-1)^2 & (n-1)^2 \\ -(n-1)^2 & (n-1)^2 & 0 & \cdots & 0 & 0 \\ 0 & -(n-1)^2 & (n-1)^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & (n-1)^2 & 0 \\ 0 & 0 & 0 & \cdots & -(n-1)^2 & (n-1)^2 \end{pmatrix}$$

Now replace C_{d-1} with $C_1 + C_2 + \cdots + C_{d-1}$ to get

$$\begin{pmatrix} 2(n-1)^2 & (n-1)^2 & (n-1)^2 & \cdots & (n-1)^2 & d(n-1)^2 \\ -(n-1)^2 & (n-1)^2 & 0 & \cdots & 0 & 0 \\ 0 & -(n-1)^2 & (n-1)^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & (n-1)^2 & 0 \\ 0 & 0 & 0 & \cdots & -(n-1)^2 & 0 \end{pmatrix}$$

Then replace C_{d-2} with $C_1 + C_2 + \cdots + C_{d-2}$ to get

$$\begin{pmatrix} 2(n-1)^2 & (n-1)^2 & \cdots & (d-1)(n-1)^2 & d(n-1)^2 \\ -(n-1)^2 & n^2+2n-1 & \cdots & 0 & 0 \\ 0 & -(n-1)^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & -(n-1)^2 & 0 \end{pmatrix}$$

Continue this process to zero out the remaining $(n-1)^2$ terms on the diagonal. This yields the following matrix

$$\begin{pmatrix} 2(n-1)^2 & 3(n-1)^2 & \cdots & (d-1)(n-1)^2 & d(n-1)^2 \\ -(n-1)^2 & 0 & \cdots & 0 & 0 \\ 0 & -(n-1)^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & -(n-1)^2 & 0 \end{pmatrix}$$

Now use the $(d-2) \times (d-2)$ block diagonal matrix with entries $-(n-1)^2$ to zero out the entries in row 1, columns 1 through $d-2$ to get

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & d(n-1)^2 \\ -(n-1)^2 & 0 & \cdots & 0 & 0 \\ 0 & -(n-1)^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & -(n-1)^2 & 0 \end{pmatrix}$$

Now multiply each of $C_1, \cdots, C_{d-1}$ by the unit -1 and also move C_{d-1} to be the first column to get the diagonal matrix:

$$\begin{pmatrix} d(n-1)^2 & 0 & 0 & \cdots & 0 \\ 0 & (n-1)^2 & 0 & \cdots & 0 \\ 0 & 0 & (n-1)^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & (n-1)^2 \end{pmatrix}$$

This gives us the primes p (and their powers) that divide $|\mathcal{J}(Y)|$, for $p \nmid n$ by Theorem 17. We write this as the following theorem

Theorem 18. *Let Y be a single voltage cover of the complete bipartite graph $K_{n,n}$ by the cyclic group of order d, where $n, d \geq 3$. For any prime p with $p \nmid n$, the Sylow p-subgroup, $\mathcal{J}_p(Y)$, of $\mathcal{J}(Y)$ has the following (elementary divisor) decomposition:*

$$\mathcal{J}_p(Y) \cong (\mathbb{Z}/p^{2a}\mathbb{Z})^{d-2} \oplus (\mathbb{Z}/p^{2a+b}\mathbb{Z})^1$$

where p^a is the largest power of p dividing $n-1$ and p^b is the largest power of p dividing d. In particular, $|\mathcal{J}_p(Y)| = p^{2a(d-1)+b}$, and the p-rank of $\mathcal{J}(Y)$ is $d-1$.

3.5 Single Voltage Cyclic Covers of $K_{n,2}$: Conjectures and the Reduced Stickelberger Element

Now let $X = K_{n,2}$. We partition the $n+2$ vertices by: each v_i for $i = 1, 2, 3, \cdots, n$ is adjacent to v_{n+j} for $j = 1, 2$ and vice versa. For computational purposes, we break with the notation in Definition 19 and label the directed edge from v_1 to v_{n+1} with τ (and so the directed edge from v_{n+1} to v_1 must be labeled with τ^{-1}). Note that, however, the Jacobian does not depend on which edge is assigned the nontrivial voltage τ.

Observe that the closed walk $v_1 \xrightarrow{\tau} v_{n+1} \xrightarrow{1} v_n \xrightarrow{1} v_{n+2} \xrightarrow{1} v_1$ has net voltage τ. By Corollary 4, this immediately gives:

Proposition 12. *If $(K_{n,2}, Z_d, \alpha)$ is a voltage graph, where $\alpha : E(X)^+ \to Z_d$ is the single voltage assignment, then the derived graph Y is connected.*

Using Sage, we compute the following tables, which yields the Jacobian of the derived graph corresponding to $(K_{n,2}, Z_d, \alpha)$, where α is the single voltage assignment and $n = 3, 4, 5$, respectively. From this, we formulate Conjecture 3 for the rank, invariant factors, and order of the Jacobian. Following this, we compute the reduced Stickelberger element of such derived graphs.

Table 3.9 Jacobian of $K_{3,2}$ with Single Voltage Cover by Z_d, (exponents represent multiplicities)

d		1st IF	2nd IF	3rd IF	order
1	1^2	2	6		$2^2 \cdot 3$
2	1^6	2^2	24		$2^5 \cdot 3$
3	1^{10}	2^2	4	36	$2^6 \cdot 3^2$
4	1^{14}	2^2	4^2	48	$2^{10} \cdot 3$
5	1^{18}	2^2	4^3	60	$2^{10} \cdot 3 \cdot 5$
6	1^{22}	2^2	4^4	72	$2^{13} \cdot 3^2$
7	1^{26}	2^2	4^5	84	$2^{14} \cdot 3 \cdot 7$
8	1^{30}	2^2	4^6	96	$2^{19} \cdot 3$
9	1^{34}	2^2	4^7	108	$2^{18} \cdot 3^3$
10	1^{38}	2^2	4^8	120	$2^{21} \cdot 3 \cdot 5$

For $d \leq 10$, we observe the following from Table 3.9:

(i) The rank is $d + 1$.

(ii) For $d \geq 3$, the Jacobian of Y has three distinct invariant factors: the first invariant factor is 2 with multiplicity 2, the second invariant factor is 4 with multiplicity $d - 2$, and the third invariant factor is $d \cdot 12$ with multiplicity 1.

(iii) $|\mathcal{J}(Y)| = 2^{2d} \cdot 3 \cdot d$.

Table 3.10 Jacobian of $K_{4,2}$ with Single Voltage Cover by Z_d, (exponents represent multiplicities)

d		1st IF	2nd IF	3rd IF	order
1	1^2	2^2	8		2^5
2	1^6	2^4	48		$2^8 \cdot 3$
3	1^{10}	2^5	6	72	$2^9 \cdot 3^3$
4	1^{14}	2^6	6^2	96	$2^{13} \cdot 3^3$
5	1^{18}	2^7	6^3	120	$2^{13} \cdot 3^4 \cdot 5$
6	1^{22}	2^8	6^4	144	$2^{16} \cdot 3^6$
7	1^{26}	2^9	6^5	168	$2^{17} \cdot 3^6 \cdot 7$
8	1^{30}	2^{10}	6^6	192	$2^{22} \cdot 3^7$
9	1^{34}	2^{11}	6^7	216	$2^{21} \cdot 3^{10}$
10	1^{38}	2^{12}	6^8	240	$2^{24} \cdot 3^9 \cdot 5$

For $d \leq 10$, we observe the following from Table 3.10:

(i) The rank is $2d + 1$.

(ii) For $d \geq 3$, the Jacobian of Y has three distinct invariant factors: the first invariant factor is 2 with multiplicity $d+2$, the second invariant factor is 6 with multiplicity $d - 2$, and the third invariant factor is $d \cdot 24$ with multiplicity 1.

(iii) $|\mathcal{J}(Y)| = 2^{2d+3} \cdot 3^{d-1} \cdot d$.

Table 3.11 Jacobian of $K_{5,2}$ with Single Voltage Cover by Z_d, (exponents represent multiplicities)

d		1st IF	2nd IF	3rd IF	order
1	1^2	2^3	10		$2^4 \cdot 5$
2	1^6	2^6	80		$2^{10} \cdot 5$
3	1^{10}	2^8	8	120	$2^{14} \cdot 3 \cdot 5$
4	1^{14}	2^{10}	8^2	160	$2^{21} \cdot 5$
5	1^{18}	2^{12}	8^3	200	$2^{24} \cdot 5^2$
6	1^{22}	2^{14}	8^4	240	$2^{30} \cdot 3 \cdot 5$
7	1^{26}	2^{16}	8^5	280	$2^{34} \cdot 5 \cdot 7$
8	1^{30}	2^{18}	8^6	320	$2^{42} \cdot 5$
9	1^{34}	2^{20}	8^7	360	$2^{44} \cdot 3^2 \cdot 5$
10	1^{38}	2^{22}	8^8	400	$2^{50} \cdot 5^2$

For $d \leq 10$, we observe the following from Table 3.11

(i) The rank is $3d + 1$.

(ii) For $d \geq 3$, the Jacobian of Y has three distinct invariant factors: the first invariant factor is 2 with multiplicity $2d + 2$, the second invariant factor is 8 with multiplicity $d - 2$, and the third invariant factor is $d \cdot 40$ with multiplicity 1.

(iii) $|\mathcal{J}(Y)| = 2^{5d-1} \cdot 5d$.

The above data leads us to the following conjecture:

Conjecture 3. *For $K_{n,2}$ the complete bipartite graph on $n + 2$ vertices with single voltage cover by Z_d, we have the following:*

(i) For $d \geq 1$, the rank is $(n - 2)d + 1$.

(ii) For $d \geq 3$, there are three distinct invariant factors: the first IF is 2 with multiplicity $(n - 3)d + 2$, the second IF is $2(n - 1)$ with multiplicity $d - 2$, the

third IF is $2n(n-1)d$ *with multiplicity 1.*

(iii) $|\mathcal{J}(Y)| = 2^{d(n-2)+1}(n-1)^{d-1}nd$.

We now state and prove what the reduced Stickelberger element is of such derived graphs.

Theorem 19. *Let* $(K_{n,2}, Z_d, \alpha)$ *be as described at the outset of this section (i.e., we label the directed edge from* v_1 *to* v_{n+1} *with* τ *(and so the directed edge from* v_{n+1} *to* v_1 *must be labeled with* τ^{-1}*). Then the reduced Stickelberger element is*

$$\Theta_{Y/X} = -2^{n-2}(n-1)(\tau-1)^2\tau^{-1}.$$

Proof. For $X = K_{n,2}$ with single voltage assignment by Z_d, the voltage Laplacian matrix is the $(n+2)\times(n+2)$ matrix with an $n \times n$ diagonal block in the upper left with 2's down the diagonal, a 2×2 diagonal block in the lower right with n on the diagonal, $-\tau$ in entry $(1, n+1)$, $-\tau^{-1}$ in entry $(n+1, 1)$, and -1's elsewhere:

$$\begin{pmatrix} 2 & 0 & 0 & \cdots & 0 & -\tau & -1 \\ 0 & 2 & 0 & \cdots & 0 & -1 & -1 \\ 0 & 0 & 2 & \cdots & 0 & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & -1 \\ -\tau^{-1} & -1 & -1 & \cdots & -1 & n & 0 \\ -1 & -1 & -1 & \cdots & -1 & 0 & n \end{pmatrix}$$

We will now use row and column operations in $\mathbb{Q}[\tau]$ to put the matrix in essentially upper triangular form. We denote row i and column j of the voltage Laplacian by C_i

and R_j, respectively. First replace C_1 by $C_1 + C_2 + \cdots + C_n + C_{n+2}$ to get

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & -\tau & -1 \\ 1 & 2 & 0 & \cdots & 0 & -1 & -1 \\ 1 & 0 & 2 & \cdots & 0 & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 2 & -1 & -1 \\ -\tau^{-1} - (n-1) & -1 & -1 & \cdots & -1 & n & 0 \\ 0 & -1 & -1 & \cdots & -1 & 0 & n \end{pmatrix}$$

Now replace C_{n+1} and C_{n+2} with $C_1 + C_{n+1}$ and $C_1 + C_{n+2}$, respectively, to get

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 1-\tau & 0 \\ 1 & 2 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 2 & 0 & 0 \\ -\tau^{-1} - (n-1) & -1 & -1 & \cdots & -1 & 1-\tau^{-1} & -\tau^{-1} - (n-1) \\ 0 & -1 & -1 & \cdots & -1 & 0 & n \end{pmatrix}$$

Now permute R_2 through R_{n+1} to get

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 1-\tau & 0 \\ -\tau^{-1}-(n-1) & -1 & -1 & \cdots & -1 & 1-\tau^{-1} & -\tau^{-1}-(n-1) \\ 1 & 2 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 2 & 0 & 0 \\ 0 & -1 & -1 & \cdots & -1 & 0 & n \end{pmatrix}$$

For $i = 2, \cdots, n$, replace C_i with $\frac{1}{n}C_{n+2} + C_i$ to get

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 1-\tau & 0 \\ -\tau^{-1}-(n-1) & \frac{-\tau^{-1}-2n+1}{n} & \frac{-\tau^{-1}-2n+1}{n} & \cdots & \frac{-\tau^{-1}-2n+1}{n} & 1-\tau^{-1} & -\tau^{-1}-(n-1) \\ 1 & 2 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 2 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & n \end{pmatrix}$$

This zeros out the 1's in R_{n+2}.

To zero out the 1's in column 1, row 3 through $n+1$, do the following: replace

C_1 with $-\frac{1}{2}(C_2 + C_3 + \cdots + C_n)$ to get

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 1-\tau & 0 \\ \frac{(-n-1)\tau^{-1}-n+1}{2n} & \frac{-\tau^{-1}-2n+1}{n} & \frac{-\tau^{-1}-2n+1}{n} & \cdots & \frac{-\tau^{-1}-2n+1}{n} & 1-\tau^{-1} & -\tau^{-1}-(n-1) \\ 0 & 2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & n \end{pmatrix}$$

Lastly, permute columns 2 through $n+1$ to get

$$\begin{pmatrix} 1 & 1-\tau & 0 & \cdots & 0 & 0 & 0 \\ \frac{(-n-1)\tau^{-1}-n+1}{2n} & 1-\tau^{-1} & \frac{-\tau^{-1}-2n+1}{n} & \cdots & \frac{-\tau^{-1}-2n+1}{n} & \frac{-\tau^{-1}-2n+1}{n} & -\tau^{-1}-(n-1) \\ 0 & 0 & 2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 2 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & n \end{pmatrix}$$

Let L' be the preceding matrix. Now use cofactor expansion along the first row of L' to get

$$\Theta_{Y/X} = \det L_\alpha = \det L' = 1 \cdot \det L'_{1,1} - (1-\tau) \det L'_{1,2}$$

where $L'_{i,j}$ is the i, j minor of L'. To compute each of the minor determinants, use cofactor expansion along its first column (which has only one nonzero entry), to get:

$$\det L' = 1(1-\tau^{-1})2^{n-1}\cdot n - (1-\tau)\left(\frac{(-n-1)\tau^{-1}-n+1}{2n}\right)2^{n-1}\cdot n$$

By elementary algebra this simplifies to the stated conclusion.

□

3.6 Single Voltage Cyclic Covers of the Petersen Graph: Conjectures and the Reduced Stickelberger Element

We now let X be the Petersen graph in Definition 19. Then the Petersen graph with single voltage assignment is shown in Figure 3.2

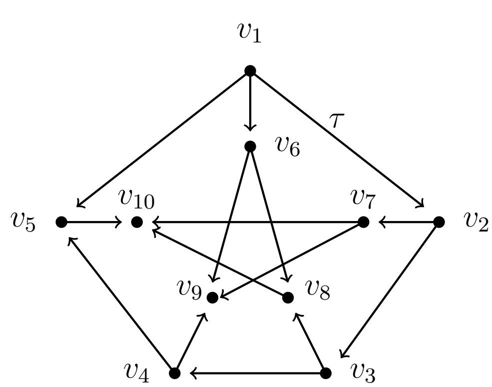

Figure 3.2: The Petersen graph with single voltage assignment

Using Sage, we compute the following table, which yields the Jacobian of the derived graph corresponding to the voltage graph (X, Z_d, α), where α is the single voltage assignment and X is the Petersen graph. From this, we formulate Conjecture 4 for the rank, invariant factors and order of the Jacobian. Following this, we compute the reduced Stickelberger element corresponding to such derived graphs.

Table 3.12 Jacobian of the Peterson graph with Single Voltage Cover by Z_d, (exponents represent multiplicities)

d		1st IF	2nd IF	3rd IF	4th IF	order
1	1^5	2	10^3			$2^4 \cdot 5^3$
2	1^{12}	2^2	10^4	80		$2^{10} \cdot 5^5$
3	1^{19}	2^3	10^5	40	120	$2^{14} \cdot 3 \cdot 5^7$
4	1^{26}	2^4	10^6	40^2	160	$2^{21} \cdot 5^9$
5	1^{33}	2^5	10^7	40^3	200	$2^{24} \cdot 5^{12}$
6	1^{40}	2^6	10^8	40^4	240	$2^{30} \cdot 3 \cdot 5^{13}$
7	1^{47}	2^7	10^9	40^5	280	$2^{34} \cdot 5^{15} \cdot 7$
8	1^{54}	2^8	10^{10}	40^6	320	$2^{42} \cdot 5^{17}$
9	1^{61}	2^9	10^{11}	40^7	360	$2^{44} \cdot 3^2 \cdot 5^{19}$
10	1^{68}	2^{10}	10^{12}	40^8	400	$2^{50} \cdot 5^{22}$

From Table 3.12 we formulate the following conjecture:

Conjecture 4. *For the Petersen graph with single voltage cover by Z_d, we have the following:*

(i) The rank is $3d + 1$.

(ii) For $d \geq 3$, the Jacobian of Y has four distinct invariant factors: the first invariant factor is 2 with multiplicity d, the second invariant factor is 10 with multiplicity $d + 2$, the third invariant factor is 40 with multiplicity $d - 2$, and the fourth invariant factor is $d \cdot 40$ with multiplicity 1.

(iii) $|\mathcal{J}(Y)| = 2^{5d-1} \cdot 5^{2d+1} \cdot d$.

We now compute the reduced Stickelberger element corresponding to this graph.

Theorem 20. *Let (X, G, α) be the voltage graph shown in Figure 3.2. Then the reduced Stickelberger element is*

$$\Theta_{Y/X} = -800(\tau - 1)^2\tau^{-1}.$$

Proof. Compute the determinant of the 10×10 voltage Laplacian matrix with 3's down the diagonal, $-\tau$ in entry $(1, 2)$, $-\tau^{-1}$ in entry $(2, 1)$, and 1's and 0's elsewhere:

$$\begin{pmatrix} 3 & -\tau & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ -\tau^{-1} & 3 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & -1 & 3 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 3 & 0 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 3 & 0 & -1 & -1 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & 3 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 3 \end{pmatrix}$$

□

Chapter 4

Zeta Functions of Voltage Graphs

In this chapter, all of our base graphs X will be finite and connected. We begin this chapter by stating definitions that relate to prime paths in a graph X and by stating the *three-term determinant formula*–all of which can be found in [Ter11].

In Section 4.1, we come up with a formula that relates the order of the Jacobian of a derived graph Y to the order of the Jacobian of the base graph X to which it corresponds. We do so by examining the relation between the zeta function of Y and the zeta function of X, as in Theorem 3.1 in [HMSV19]. From this, we define the *equivariant L-function* and the *reduced equivariant L-function*, which essentially just replaces the *Artinized adjacency matrix*, A_χ, in Theorem 3.1 of [HMSV19] with A_α, the voltage adjacency matrix. Factoring the equivariant L-function and expanding each term about $u = 1$, we find the lowest-degree nonzero coefficient. In doing so, we find that the lowest nonzero coefficient of the *reduced* equivariant L-function is in fact the reduced Stickelberger element. We use this to find the first non-vanishing Taylor coefficient at $u = 1$ of the reciprocal zeta function of Y. This, in turn, gives us a formula for the order of the Jacobian of the derived graph Y.

Then in 4.1.1, we specify the voltage group G to be the cyclic group of order d. This yields a more specific formula for the order of the Jacobian of the derived graph Y. We then go on to show that when the voltage assignment is the *single voltage assignment*, the reduced Stickelberger element is always of a specific form. This then leads to a formula for the order of the Jacobian of a derived graph Y given by the single voltage assignment. From this, Conjectures 1-4(iii) in Chapter 3 are verified.

We now introduce some terminology. These can also be found in [HMSV19].

Definition 26. *A closed path*

$$P : w = w_1 \xrightarrow{e_{1,2}} w_2 \xrightarrow{e_{2,3}} \cdots \xrightarrow{e_{m-1,m}} w_m = w$$

is called a prime path if it has no backtrack or tail and one may only go around the path once. For the closed path P, the equivalence class $[P]$ means the following

$$[P] = \{e_{1,2}e_{2,3}\cdots e_{m-1,m}, \quad e_{2,3}\cdots e_{m-1,m}e_{1,2}, \quad \cdots \quad , e_{m-1,m}e_{1,2}, \cdots e_{m-2,m-1}\}.$$

That is, two paths are equivalent if we can get one from the other by changing the starting vertex. A prime in a graph X is an equivalence class $[P]$ of prime paths. The length of P is $\nu(P) = m$, the number of edges in P.

Definition 27. *The Ihara zeta function of a finite connected graph X is defined to* *be*

$$\zeta_X(u) = \prod_{[P]} (1 - u^{\nu(P)})^{-1}$$

where the product is over all primes $[P]$ *in* X, $u \in \mathbb{C}$ *and* $|u|$ *is sufficiently small.*

The following theorem, which can be found in [Ter11], gives a formula for computing the zeta function.

Theorem 21 (Three-term determinant formula)**.** *Let* A *be the adjacency matrix of* X *and let* D *be the diagonal matrix of vertex degrees (which is a scalar matrix if* X *is a regular graph). Let*

$$r_X = r = |E(X)| - |V(X)| + 1$$

where $E(X)$ *and* $V(X)$ *denote the edges and vertices of* X*, respectively. Then we have the Ihara three-term determinant formula*

$$\zeta_X(u)^{-1} = (1 - u^2)^{r-1} \det(I - Au + (D - I)u^2).$$

4.1 Zeta Functions and the Order of the Jacobian

Throughout this section, for (X, G, α) a voltage graph, assume

(i) X is connected,

(ii) $r_X - 1 \neq 0$,

(iii) G is abelian, and

(iv) $\alpha : E(X)^+ \to G$ is such that Y is connected (as in Theorem 4).

By Theorem 2.11 in [HMSV19], we have that

$$\zeta_X^*(1) = (-1)^{r+1} 2^r (r-1)\kappa_X,$$

where $\zeta_X^*(1)$ denotes the first non-vanishing Taylor coefficient of $\zeta_X(u)^{-1}$ at $u = 1$ and κ_X is the number of spanning trees in X. Thus, rearranging

$$\zeta_X^*(1)/((-1)^{r+1} 2^r (r-1)) = \kappa_X = |\mathcal{J}(X)|. \tag{4.1}$$

We now seek a formula that relates $|\mathcal{J}(X)|$ to $|\mathcal{J}(Y)|$. The zeta function of a derived graph Y of a connected base graph X with voltage assignment by G, where G is abelian, is by Theorem 3.1 in [HMSV19],

$$\zeta_Y(u) = \zeta_X(u) \prod_{\chi_t \neq \chi_0} L(u, \chi_t) \tag{4.2}$$

where

$$L(u, \chi_t)^{-1} = (1-u^2)^{r-1} \det(I - A_{\chi_t} u + (D - I)u^2)$$

with the product of the χ_t over all the irreducible characters of G and where χ_0 is the trivial character, and A_χ (called the *Artinized adjacency matrix*) is defined to be

$$A_\chi = \sum_{\sigma \in G} \chi(\sigma) A(\sigma)$$

where (because X has no loops or multiple edges) the i, j-entry of $A(\sigma)$ simplifies to

$$a_{i,j}(\sigma) = \begin{cases} 1 & \text{if } v_{i,1} \to v_{j,\sigma} \text{ in } Y \\ 0 & \text{otherwise} \end{cases}$$

where $1 = \tau_0$ is the identity of G.

Since Y is the derived graph, $a_{i,j}(\sigma)$ is nonzero if and only if $v_i \xrightarrow{\tau} v_j$ in X and $\sigma = \tau$ (in which case $v_j \xrightarrow{\tau^{-1}} v_i$ and $a_{j,i}(\sigma^{-1}) \neq 0$ with $\sigma^{-1} = \tau^{-1}$). In other words, the i, j-entry of $A(\sigma)$ is nonzero for a unique σ in G, namely when σ equals the voltage of the edge $v_i \to v_j$. Thus

$$\sum_{\sigma \in G} \sigma A(\sigma) = A_\alpha \qquad \text{(the voltage adjacency matrix).}$$

Then for every irreducible character χ of G

$$A_\chi = \chi(A_\alpha)$$

where $\chi(A_\alpha)$ means evaluating every group element σ of the voltage adjacency matrix at χ (and $\chi(0) = 0$).

Now using A_α, the voltage adjacency matrix, in place of A_{χ_t} in $L(u, \chi_t)^{-1}$, we define

$$L(u, \alpha)^{-1} = (1 - u^2)^{r-1} \det(I - A_\alpha u + (D - I)u^2) \tag{4.3}$$

to be the *equivariant L-function.* From this, we now define the *reduced equivariant L function*

$$L^*(u,\alpha)^{-1} = \det(I - A_\alpha u + (D-I)u^2). \tag{4.4}$$

Then evaluate each group element entry in A_α at the degree-one character χ_t (the zero entries remain zero). When this matrix is substituted in place of A_α in (4.3)—since such evaluation is a ring homomorphism from $\mathbb{Z}[G]$ to $\mathbb{C}$—the resulting function is $L(u,\chi_t)^{-1}$; and likewise when we substitute the same matrix for A_α in (4.4) we get $L^*(u,\chi_t)^{-1}$.

Now observe that the equivariant L-function can be factored as

$$L(u,\alpha)^{-1} = (1-u)^{r_X-1}(1+u)^{r_X-1}\det(I - A_\alpha u + (D-I)u^2).$$

Expanding each of these terms of $(u-1)$, we get

$$(1-u)^{r_X-1} = (-1)^{r_X-1}(u-1)^{r_X-1} \qquad \text{and}$$

$$\begin{aligned}(1+u)^{r_X-1} &= (2+(u-1))^{r_X-1} \\ &= 2^{r_X-1} + 2^{r_X-2}(r_X-1)(u-1) + \cdots + (\text{higher powers of } (u-1)).\end{aligned}$$

The reduced equivariant L-function is of the form

$$c_0 + c_1(u-1) + c_2(u-1)^2 + \cdots \text{higher powers of } (u-1)$$

where the coefficients c_i are from the integral group ring $\mathbb{Z}[G]$.

So multiplying the above polynomials together, we get the lowest nonzero coefficient (which corresponds to the $(u-1)^{r_X-1}$ factor, which we know is nonzero) to be

$$(-1)^{r_X-1}2^{r_X-1}c_0.$$

For $G \neq 1$, letting $u=1$ in formula (4.4) for $L^*(u,\alpha)^{-1}$, we see that c_0 is exactly the reduced Stickelberger element, $\Theta_{Y/X}$. We then evaluate the degree-one irreducible characters χ_t at the group element $\Theta_{Y/X}$, which we denote by $\chi_t(\Theta_{Y/X})$, and take the product to get

$$\prod_{\chi_t \neq \chi_0} L(1,\chi_t) = \prod_{\chi_t \neq \chi_0} (-1)^{r_X-1}2^{r_X-1}\chi_t(\Theta_{Y/X}).$$

Lastly, in (4.2) we multiply by the coefficient corresponding to the L-function for X, i.e., ζ_X, which equals $(-1)^{(r_X+1)}2^{r_X}(r_X-1)|\mathcal{J}(X)|$ to get that the first non-vanishing Taylor coefficient of $\zeta_Y(y)^{-1}$ at $u=1$ is

$$\zeta_Y^*(1) = (-1)^{r_X+1}2^{r_X}(r_X-1)|\mathcal{J}(X)| \prod_{\chi_t \neq \chi_0} (-1)^{r_X-1}2^{r_X-1}\chi_t(\Theta_{Y/X}). \tag{4.5}$$

Now note that for any voltage cover Y of X of degree d, we have $r_Y - 1 = d(r_X-1)$. Also, the number of irreducible characters of G (abelian) is d. By formula (4.1) applied to Y, instead of X, we have

$$|\mathcal{J}(Y)| = \zeta_Y^*(1)/((-1)^{r_Y-1}2^{r_Y}(r_Y-1)).$$

Substituting the value of $\zeta_Y^*(1)$ from (4.5) into this and simplifying gives the following general formula.

Theorem 22. *Let (X, G, α) be a voltage graph satisfying the hypotheses at the outset of this subsection. Then the order of the Jacobian of the derived graph Y is*

$$|\mathcal{J}(Y)| = \frac{1}{d} \cdot |\mathcal{J}(X)| \prod_{\chi_t \neq \chi_0} \chi_t(\Theta_{Y/X}).$$

Note that the previous definitions of L and L^* can be made for arbitrary characters of G abelian, not necessarily irreducible.

In the next section, we let $G = Z_d$ be the cyclic group or order d.

4.1.1 Zeta Functions of Cyclic Graph Coverings

Assume throughout this subsection that (X, G, α) is a voltage graph such that

(i) X is connected,

(ii) $G = \langle \tau \rangle$ is cyclic of order d,

(iii) there is a closed walk in X with net voltage τ, and

(iv) $r_X - 1 \neq 0$.

By Corollary 1, these first three conditions ensure that all covering graphs by $Z_d = \langle \tau \rangle$ are connected.

Fix the primitive d^{th} root of unity $\lambda = e^{2\pi i/d}$. Then every irreducible character of Z_d maps τ to a dth root of unity, i.e.,

$$\chi_t : \tau \to \lambda^t, \qquad 0 \leq t \leq d-1,$$

where χ_0 is again the principal character of G.

Corollary 12. *Let X satisfy the hypotheses at the outset of this section, with voltage assignment by Z_d. Then the order of the Jacobian of the derived graph Y is*

$$|\mathcal{J}(Y)| = \frac{1}{d} \cdot |\mathcal{J}(X)| \prod_{t=1}^{d-1} \chi_t(\Theta_{Y/X}),$$

where $\chi_t : \tau \mapsto \lambda^t$.

Proof. This follows immediately from Theorem 22. □

We now go on to show that when (X, Z_d, α) is a voltage graph with α given by the *single* voltage assignment, as in Definition 19, then the reduced Stickelberger element, $\Theta_{Y/X}$, is of a specific form. We first prove the following lemma.

Lemma 1. *Let L be any symmetric $n \times n$ matrix with entries from $\mathbb{Z}$ with $n \geq 3$, and let x be an indeterminate over $\mathbb{Z}$. Assume L has 1 in position 1,2 (hence also in position 2,1) and* $\det L = 0$. *Let L_x be the same matrix L except with the 1,2 entry replaced by x and the 2,1 entry replaced by $1/x$, and let $\Theta(x) = \det L_x$ (so L_x is a matrix with entries from the localized polynomial ring $\mathbb{Z}[x, 1/x]$, and $\Theta(x)$ is an* *integer polynomial in the variables x and $1/x$). Then*

$$\Theta(x) = K(x-1)^2 x^{-1} \quad \textit{for some integer } K.$$

Since $\Theta(x) = K(x-1)^2 x^{-1}$, we also have $K = 2\Theta(2)$.

Note that the conclusions allow for the possibility that Θ is identically zero (i.e., $K = 0$).

Proof. By determinant formulas, it follows that $\det(L_x)$ is a *linear* function in the variables x and x^{-1} — this follows, for example, by looking at the individual terms in the symmetric group sum expansion for a determinant (see [DF04] Theorem 24, Section 11.4), where only a single factor of x or $1/x$ or $(x)(1/x)$ can appear in each term. Thus we may write

$$\Theta(x) = a + bx + cx^{-1} = (x^{-1})(ax + bx^2 + c) \qquad \text{for some integers } a, b, c.$$

Let $q(x)$ be the numerator of the right hand side above. If $q(x) = 0$ then $\Theta(x) = 0$ and so the lemma is true with K chosen to be zero. So assume $q(x) \neq 0$. Since $\det(L) = 0$, it follows that $x = 1$ is a root of q, and so all roots of q are rational numbers. Since q is nonzero but has a root, it is not a constant polynomial.

If s is any nonzero root of q, then substituting $x = 1/s$ into the (1,2 and 2,1) entries of L_x results in the matrix L_x^{tr} (the transpose of L_x) but with x evaluated at s in the latter. Both evaluated matrices have the same determinant, hence $1/s$ must be a root of q as well. Since q has at most two distinct roots, the only possibility for a different root would be $x = -1$. But if $\Theta(x) = K(x^2-1)/x$, then replacing x by $1/x$ yields

$K(1-x^2)x^{-1}$ and so Θ would not satisfy the symmetry condition $\Theta(x) = \Theta(1/x)$. If $q(x)$ had degree 1, then we would have $\Theta(x) = K(x-1)/x$, for some constant K. But again Θ would not satisfy the aforementioned symmetry (transpose) condition $\Theta(x) = \Theta(1/x)$, a contradiction. Thus, $\Theta(x) = K(x-1)^2x^{-1}$ for some integer K.

□

Theorem 23. *Assume X satisfies the hypotheses of this subsection and let Y be the single voltage cover of X by the cyclic group $Z_d = \langle\tau\rangle$. Then the reduced Stickelberger element may be written in the following form:*

$$\Theta_{Y/X} = K(\tau-1)^2\tau^{-1}, \qquad \textit{for some nonzero integer } K \textit{ independent of } d.$$

Proof. Let L_τ be the $n\times n$ voltage Laplacian matrix for the single voltage cover of X, so L_τ has entries in the integral group ring $\mathbb{Z}[\tau]$. The latter ring is the homomorphic image of the polynomial ring $\mathbb{Z}[x]$ where the indeterminate x is evaluated at τ. Since τ is a unit in the group ring, this algebra homomorphism induces a $\mathbb{Z}$-algebra homomorphism from the localization $\mathbb{Z}[x, 1/x]$ to $\mathbb{Z}[\tau]$ which sends $1/x$ to τ^{-1} (see [DF04], Theorem 36 and Examples 1 and 2 in Section 15.4). This further induces a $\mathbb{Z}$-algebra homomorphism from the $n\times n$ matrix ring over $\mathbb{Z}[x, 1/x]$ to the $n\times n$ matrix ring over $\mathbb{Z}[\tau]$. Since the determinant function is a polynomial in the entries of a matrix, determinants over the former ring map to determinants over the group ring by evaluating x at τ and $1/x$ at τ^{-1}. By invoking Lemma 1 and then evaluating x at τ we obtain that

$$\Theta_{Y/X} = K(\tau-1)^2\tau^{-1}, \quad \text{for some integer } K. \tag{4.6}$$

The entries of L_x depend only on the Laplacian for X. By definition of single voltage cyclic cover, the determinant of L_τ likewise depends only on the Laplacian for X and the choice of generator τ of the covering group. The above argument therefore shows that the reduced Stickelberger element is either zero or it may be written in a "canonical form", where both the form and the integer constant K are independent of the order, d, of the covering group (although d is inherent in the τ of this formula).

If $\Theta_{Y/X} = 0$, then it would follow that $|\mathcal{J}(Y)| = 0$ for all nontrivial (connected) single voltage covering graphs Y of X by Corollary 12, a contradiction. $\square$

Remark:

The reduced Stickelberger element does depend on the choice of generator. Suppose instead of using generator τ, we use $\mu = \tau^a$, where $(a, d) = 1$. Then writing the reduced Stickelberger element in terms of τ, we get a different reduced Stickelberger element–namely we get the reduced Stickelberger element

$$K(\mu - 1)^2\mu^{-1} = K(\tau^a - 1)^2\tau^{-a}.$$

Note, however, that K does not depend on the choice of generator since for any single voltage cyclic cover, we can write $\Theta_{Y/X}$ uniquely as a $\mathbb{Z}$-linear combination of the $\mathbb{Z}[G]-$basis elements $1, \tau, \cdots, \tau^{d-1}$. Then K is found to be the *gcd* of all coefficients of the linear combination (since if you replace τ by τ^a, the *gcd* of all the coefficients remains the same).

Corollary 13. *Let X satisfy the hypothesis at the outset of this subsection and assume*

α is the single voltage assignment by Z_d. Let K be as in Theorem 23. Then

$$|\mathcal{J}(Y)| = |\mathcal{J}(X)| \cdot |K|^{d-1} d.$$

Proof. By Corollary 12, we have that

$$\begin{aligned} |\mathcal{J}(Y)| &= \frac{|\mathcal{J}(X)| \prod_{t=1}^{d-1} \frac{|K|(e^{2\pi it/d}-1)^2}{e^{2\pi it/d}}}{d} \\ &= \frac{|\mathcal{J}(X)| \cdot |K|^{d-1} d^2}{d} \\ &= |\mathcal{J}(X)| \cdot |K|^{d-1} d, \end{aligned}$$

where by the factorization of $\frac{z^d-1}{z-1}$ over $\mathbb{C}$, the product of all $(e^{2\pi it/d}-1)$-terms in the numerator simplifies to d^2 and the product of $e^{2\pi it/d}$ in the denominator simplifies to ± 1. $\square$

The following corollaries verify Conjectures 1-4 part (iii).

Corollary 14. *Let $X = K_n$ with $n \geq 3$ and with single voltage assignment by Z_d. Then*

$$|\mathcal{J}(Y)| = n^{(n-3)d+1} \cdot (n-2)^{d-1} \cdot d.$$

Proof. As in Section 3.2, the hypotheses at the outset of this subsection are satisfied. By Theorem 12, we know that the reduced Stickelberger element for these covers is $\Theta_{Y/X} = (-(n-2)n^{(n-3)})(\tau-1)^2\tau^{-1}$. We also know that $|\mathcal{J}(X)| = n^{n-2}$. This yields the desired result (which agrees with Theorem 14). $\square$

Corollary 15. *Let $X = K_{n,n}$ with $n \geq 2$ and with the single voltage assignment by*

Z_d. *Then*

$$|\mathcal{J}(Y)| = n^{(2n-4)d+2}(n-1)^{2d-2}d.$$

Proof. By Theorem 16, we know that the reduced Stickelberger element for these covers is $\Theta_{Y/X} = -(n-1)^2 n^{2n-4}(\tau-1)^2\tau^{-1}$. Since $|\mathcal{J}(X)| = n^{2n-2}$, we get the desired result. $\square$

Corollary 16. *Let $X = K_{n,2}$ with $n \geq 2$ be as in the hypotheses of Theorem 19. Then*

$$|\mathcal{J}(Y)| = 2^{(n-2)d+1}(n-1)^{d-1}nd.$$

Proof. By Theorem 19, we have that the reduced Stickelberger element is $\Theta_{Y/X} = -2^{n-2}(n-1)(\tau-1)^2\tau^{-1}$. Since $|\mathcal{J}(X)| = 2^{n-1}n$, we get the desired result. $\square$

Corollary 17. *Let X be the Petersen graph with single voltage assignment by Z_d. Then*

$$|\mathcal{J}(Y)| = 2^{5d-1} \cdot 5^{2d+1} \cdot d.$$

Proof. By Theorem 20, we know that the reduced Stickelberger element is $\Theta_{Y/X} = -800(\tau-1)^2\tau^{-1} = -2^5 \cdot 5^2(\tau-1)^2\tau^{-1}$. Since $|\mathcal{J}(X)| = 2^4 \cdot 5^3$, we get the desired result. $\square$

Chapter 5

Towers of Voltage Graphs and Iwasawa Theory

We begin Section 5.1 by defining a cyclic p-tower of graphs. We then extend this definition to a cyclic *voltage* p-tower of graphs by using Theorems 8 and 9 from Section 2.4. From this, we get a "universal cover" of the tower by an infinite derived graph that we call X_{p^∞}; it is the derived graph obtained from the voltage graph $(X, \mathbb{Z}_p, \alpha)$, where the voltage group is the additive p-adic integers and the voltage assignment α is determined by the cyclic voltage p-tower. We call X_{p^∞} the *completion of the tower*.

We show that given a cyclic *single* voltage p-tower of derived graphs, we obtain an order formula for the Jacobian of such a derived graph. We are then able to find the exact power of a prime p that divides the order of the Jacobian of a derived graph that lies in this tower. Moreover, we can explicitly write this power in terms of the μ and λ-invariants (the *Iwasawa Invariants*), where μ is the exact exponent for the p-power of K, the reduced Stickelberger coefficient, and $\lambda = 1$. We then illustrate this with some examples where we take the base graph to be $K_n, K_{n,n}, K_{n,2}$ (for some specific values of n), and the Petersen graph.

We begin Section 5.2 by stating the main result of this chapter, Theorem 27, which establishes the order of the finite p-Jacobians, $\mathcal{J}_p(X_m)$, of a cyclic voltage p-tower of graphs. Before proving it, we develop the theory of Iwasawa in a graph theoretic setting.

Sections 5.3 and 5.4 contain the culmination of the dissertation. In Section 5.3, we present important definitions and results pertaining to Λ-modules. Then in Subsection 5.3.1, we specify the Λ-modules be finitely generated. In Subsection 5.3.2 we construct a finitely generated torsion Λ-module, which we call Pic_Λ. This finitely generated Λ-module is the cokernel of the voltage Laplacian endomorphism on Div_Λ, that is annihilated by the *reduced p-Stickelberger element* Θ_{p^∞}.

Finally in Section 5.4 we prove the main theorem (Theorem 27) of this chapter; it gives an Iwasawa growth formula for the orders of the p-Jacobians, $\mathcal{J}_p(X_m)$ as $m \to \infty$. The way in which Theorem 27 is proven is by relating quotients of Λ-modules to quotients of R-modules, where R is the p-adic group ring of the voltage group $\mathbb{Z}_p$, and where the finite quotients are isomorphic to $\mathcal{J}_p(X_m)$. As a corollary to the main theorem, we show that the ranks of the Sylow p-subgroups, $\mathcal{J}_p(X_m)$ are bounded as m approaches infinity if and only if p does not divide the reduced p-Stickelberger element. In doing so, we see that the Iwasawa factors are related to the reduced p-Stickelberger element via the *characteristic polynomial* of Pic_Λ. We conclude this chapter with the example: single voltage p-towers over base graph $X = K_n$. We determine the exact Iwasawa factors of the Sylow p-subgroup of the Jacobian for every prime p.

Table 5.5 gives a list of definitions and terminology pertaining to Chapter 5, for convenience of reference.

5.1 p-Tower Covering Graphs

We begin by defining a cyclic p-tower of graphs (note that this definition can be found in section 4 of [Val20]). Throughout this section, we assume p is a fixed prime.

Definition 28. *A cyclic p-tower of graphs above a base graph X is a sequence of covering graphs*

$$X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_m \leftarrow \cdots$$

such that for $m \geq 0$, the cover X_m/X is normal with $Gal(X_m/X) \cong \mathbb{Z}/p^m\mathbb{Z}$.

Note that for $m \geq 0$, this implies that the cover X_{m+1}/X_m is normal with $Gal(X_{m+1}/X_m) \cong \mathbb{Z}/p\mathbb{Z}$ by the Fundamental Theorem of Galois Theory, along with the Third Isomorphism Theorem from [DF04].

We explicitly specify the given covering maps in the above tower (given by composing the successive covering maps):

$$\pi_{k,m} : X_k \longrightarrow X_m \qquad \text{for all } k \geq m$$

$$\text{where} \quad \pi_{k,m} = \pi_{t,m} \circ \pi_{k,t} \qquad \text{for all } k \geq t \geq m.$$

For each $m \geq 0$ let $G_m = \mathrm{Gal}(X_m/X_0)$. Then the covering maps above induce surjective group homomorphisms

$$\Pi_{k,m} : G_k \longrightarrow G_m \qquad \text{for all } k \geq m \tag{5.1}$$

that are compatible with the Galois permutation actions of each G_i on X_i. This is illustrated in Figure 5.1.

$$
\begin{array}{c} \vdots \\ \downarrow \end{array}
$$

$$
G_k\left\{\begin{array}{l} \left.\begin{array}{c} X_k \\ \downarrow \\ \vdots \\ \downarrow \end{array}\right\} Gal(X_k/X_m) \\ \quad X_m \\ \left.\begin{array}{c} \downarrow \\ \vdots \\ \downarrow \\ X = X_0 \end{array}\right\} G_m \end{array}\right.
$$

Figure 5.1: Voltage p-tower

Now we specialize Definition 28 to voltage graphs, where we will make all of these maps explicit.

Definition 29. *A cyclic voltage p-tower of graphs above a base graph X, where $r_X - 1 \neq 0$, is a sequence of derived graphs*

$$
X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_m \leftarrow \cdots
$$

such that for $m \geq 0$, X_m/X is a voltage graph with $Gal(X_m/X) \cong \mathbb{Z}/p^m\mathbb{Z}$.

Note that by Theorem 8, cyclic voltage p-towers are the same as cyclic p-towers in the case where $r_X - 1 \neq 0$ and the X_m are connected for all m.

Assume now we have a cyclic voltage p-tower as in Definition 29. We show how Theorems 8 and 9 in Section 2.4 extend to towers, so we may choose notation that describes the vertices, edges, voltage assignments and Galois actions on the graphs X_m in compatible ways that are determined by the covering maps. (It suffices to describe this inductively for just each X_{m+1}/X_m.) This choice of notation then leads to a "universal cover" of the tower, by an infinite derived graph that we call X_{p^∞}.

By Theorems 8 and 9 and Figure 2.10 in Section 2.4, for every $m \geq 0$ we have that G_m is a *quotient* group of G_{m+1}, namely $G_m = \prod_{m+1,m}(G_{m+1}) \cong G_{m+1}/H$, where H is the unique subgroup $\mathrm{Gal}(X_{m+1}/X_m)$ of order p in G_{m+1}, and moreover, the isomorphism is given by the natural projection map. Thus under the identification $G_{m+1} = \mathbb{Z}/p^{m+1}\mathbb{Z}$ we have $H = p^m G_{m+1}$ and $G_m = \mathbb{Z}/p^m\mathbb{Z}$ is obtained by reducing elements g in $\mathbb{Z}/p^{m+1}\mathbb{Z}$ modulo p^m to obtain elements in the group $\mathbb{Z}/p^m\mathbb{Z}$.

Labeling vertices of X_m in a compatible way:

Theorem 9 has an explicit rendition as follows. Fix any vertex v_i of X, and consider the vertices over v_i in the covers by the graphs X_m. For voltage covers of X by any group G, the vertices in the fiber over v_i are $\{v_{i,g} \mid g \in G\}$. For each $m \geq 0$, label the elements A_m in G_m with the integers from 0 to $p^m - 1$ representing the least nonnegative residue classes, written in their p-adic expansions:

$$A_m = a_0 + a_1 p + \cdots + a_{m-1} p^{m-1} \qquad \text{where all } a_i \in \{0, 1, \ldots, p-1\}. \tag{5.2}$$

With this labeling for every index m, the natural projection map from $\mathbb{Z}/p^{m+1}\mathbb{Z} \to$

$\mathbb{Z}/p^m\mathbb{Z}$ agrees with the covering map $\pi_{m+1,m} : V(X_{m+1}) \to V(X_m)$ map on vertices. Specifically, for a fixed vertex $v_{i,A_m} \in X_m$, the vertices $v_{i,B_{m+1}} \mapsto v_{i,A_m}$, where B_{m+1} is any of the p residue classes with integer representatives

$$B_{m+1} = A_m + b_m p^m \qquad \text{for } 0 \leq b_m \leq p-1.$$

This set of vertices in X_{m+1} is the fiber of $\pi_{m+1,m}$ over $v_{i,A_m} \in X_m$.

The Galois group acts on vertices of each X_m in a compatible way:
By Theorem 8, for every $m \geq 0$ the Galois action on the covers X_m of X is the same as the voltage group action on the vertices of X_m, namely the (additive) regular representation of $\mathbb{Z}/p^m\mathbb{Z}$: for arbitrary A_m as in (5.2)

$$b : v_{i,A_m} \longmapsto v_{i,b+A_m} \qquad \text{for each } b \in G_m = \mathbb{Z}/p^m\mathbb{Z},$$

where $b + A_m$ is reduced mod p^m. The above action does not depend on the choice of congruence class representatives. The action of G_{m+1} on X_{m+1} is thus clearly equal to the action of G_m on X_m after both the voltage group elements and voltage labels have been reduced mod p^m. (Alternatively, we may view G_{m+1} as acting on both graphs: faithfully on X_{m+1} but with kernel equal to $H = p^m G_{m+1}$ on X_m; and this action commutes with the covering map.)

Note that this explicit labeling can be chosen for any cyclic voltage p-tower. It is the *edges* between vertices in each X_m that distinguish the towers from one another.

Compatible voltage assignments to the edges of X determined by the tower cover-

ing maps:

Since each $\pi_m : X_m \to X_0$ is a voltage graph with Galois group $G_m = Gal(X_m/X_0) \cong \mathbb{Z}/p^m\mathbb{Z}$, this means there is a voltage assignment

$$\alpha_m : E^+(X) \longrightarrow G_m$$

that assigns voltages from G_m to each (oriented) edge of X. There must also be compatibility of these assignments as described in Section 2.4 (in the proof Theorem 9). So for all $k \geq m$ the following diagram on the left below commutes:

$$\begin{array}{ccccc} \alpha_k : E^+(X) & \longrightarrow & G_k & & X_k \\ \Vert & & \big\downarrow \Pi_{k,m} & & \big\downarrow \pi_{k,m} \\ \alpha_m : E^+(X) & \longrightarrow & G_m & & X_m \end{array}$$

This compatibility is most easily described by the voltage adjacency matrix: For each $m \geq 0$ the voltage adjacency matrix A_{α_m} for X_m/X is an $n \times n$ matrix with i, j entry $\alpha_m(i,j) \in G_m \cup \{0\}$. Then for $k \geq m$, the compatibility requires that when $v_i \to v_j$ in X

$$\Pi_{k,m}\alpha_k(i,j) = \alpha_m(i,j) \qquad \text{for all } 1 \leq i, j \leq n.$$

In other words, we can reduce the entries of A_{α_k} (mod p^m) to get the entries of A_{α_m}.

Thus for each fixed i, j with $1 \leq i, j \leq n$ we get a sequence

$$\alpha_1(i,j) \leftarrow \alpha_2(i,j) \leftarrow \alpha_3(i,j) \leftarrow \cdots$$

where each $\alpha_m(i,j) \in \mathbb{Z}/p^m\mathbb{Z}$ and the arrows are given by the natural projection homomorphism. Each of these sequences converges to an element of the additive group of p-adic integers, $\mathbb{Z}_p$. Therefore we may let

$$\alpha : E(X)^+ \longrightarrow \mathbb{Z}_p \qquad \text{by} \qquad \alpha(v_i \to v_j) = \lim_{m\to\infty} \alpha_m(i,j). \tag{5.3}$$

Definition 30. *Given a cyclic voltage p-tower as in Definition 29, with each X_m the derived graph for the voltage assignment $\alpha_m : E(X)^+ \to \mathbb{Z}/p^m\mathbb{Z}$, let X_{p^∞} be the derived graph obtained from the voltage graph $(X, \mathbb{Z}_p, \alpha)$, where the voltage group is the (additive) p-adic integers and voltage assignment α is determined by the tower as in (5.3). We call X_{p^∞} the completion of the tower.*

Finally, given the completion, X_{p^∞}, of the voltage p-tower, define the intermediate graphs $\overline{X}_m$ and covering maps $\overline{X}_m \xleftarrow{\Phi_m} X_{p^\infty}$ as follows. For each integer $m \geq 0$, for clarity let ϕ_m be the natural projection of $\mathbb{Z}_p$ onto $\mathbb{Z}_p/p^m\mathbb{Z}_p \cong \mathbb{Z}/p^m\mathbb{Z}$ (i.e., ϕ_m is reduction mod p^m for $\mathbb{Z}_p$). The vertices of $\overline{X}_m$ are defined to be

$$V(\overline{X}_m) = \{v_{i,\phi_m(g)} \mid 1 \leq i \leq n \text{ and } g \in \mathbb{Z}_p\},$$

and the edges of $\overline{X}_m$ are defined to be

$$E(\overline{X}_m) = \{v_{i,\phi_m(g)} \sim v_{j,\phi_m(g+\alpha(i,j))} \mid g \in \mathbb{Z}_p \text{ and } v_i \sim v_j \text{ in } X\}.$$

The (infinite degree) covering maps Φ_m are now just given by "reduction mod p^m"

on the vertices and edges of X_{p^∞} as follows. Define

$$\Phi_m : X_{p^\infty} \longrightarrow \overline{X}_m \qquad \text{by} \qquad \Phi_m(v_{i,g}) = v_{i,\phi_m(g)} \quad 1 \leq i \leq m \text{ and } g \in \mathbb{Z}_p.$$

When $m = 0$, the graph $\overline{X}_0$ is clearly isomorphic to X; and also there are clearly covering maps that are induced from the natural projections of $\mathbb{Z}_p/p^{m+1}\mathbb{Z}_p$ onto $\mathbb{Z}_p/p^m\mathbb{Z}_p$:

$$\overline{X}_{m+1} \longrightarrow \overline{X}_m \qquad \text{by} \qquad v_{i,\phi_{m+1}(g)} \longmapsto v_{i,\phi_m(g)} \quad \text{for } 1 \leq i \leq n \text{ and } g \in \mathbb{Z}_p.$$

By construction of α we have that $\phi_{m+1}(\alpha(i,j))$ maps to $\phi_m(\alpha(i,j))$ under the natural projection homomorphism from $\mathbb{Z}_p/p^{m+1}\mathbb{Z}_p$ onto $\mathbb{Z}_p/p^m\mathbb{Z}_p$; so the displayed maps on vertices are easily seen to take edges of $\overline{X}_{m+1}$ to edges of $\overline{X}_m$; and these maps are all p-fold covers.

Finally, note that $\mathbb{Z}_p$ acts (on the left) as graph automorphisms of both X_m and $\overline{X}_m$, for all $m \geq 0$, where the kernel of this action is $p^m\mathbb{Z}_p$. The voltage group $\mathbb{Z}_p/p^m\mathbb{Z}_p \cong G_m$ acts faithfully equivalent to the regular representation on each of the fibers of X_m/X and likewise on $\overline{X}_m/\overline{X}$. These actions may be viewed as actions by $\mathbb{Z}_p$, but with kernels equal to $p^m\mathbb{Z}_p$, and as such this $\mathbb{Z}_p$ action commutes with all covering maps.

The preceding discussion leads immediately to the following.

Theorem 24. *Let $X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_m \leftarrow \cdots$ be a cyclic voltage p-tower, let X_{p^∞} be the completion of the tower, and for each $m \geq 0$ let $\overline{X}_m$ be the associated intermediate graphs. Then for all $m \geq 0$ there are graph isomorphisms*

$\overline{X}_m \to X_m$, *depicted as the horizontal maps in Figure 5.2, such that all the maps in that figure commute and commute with the action of* $\mathbb{Z}_p$ *as automorphisms of each graph.*

$$
\begin{tikzcd}
X_{p^\infty} \arrow[d, phantom, "\vdots"] & \\
\vdots \arrow[d] & \vdots \arrow[d] \\
\overline{X}_k \arrow[r, "\cong"] \arrow[d] & X_k \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] \\
\overline{X}_m \arrow[r, "\cong"] \arrow[d] & X_m \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] \\
\overline{X}_0 \arrow[r, "\cong"] & X_0
\end{tikzcd}
$$

Figure 5.2: Cyclic voltage p-tower with completion X_{p^∞}

Proof. The horizontal maps in Figure 5.2 are the obvious ones. Namely, given the integer A_m in (5.2) representing any residue class in $G_m = \mathbb{Z}/p^m\mathbb{Z}$, view A_m as a p-adic integer; then define

$$V(X_m) \longrightarrow V(\overline{X}_m) \qquad \text{by} \qquad v_{i,A_m} \longmapsto v_{i,\phi_m(A_m)}, \quad 1 \leq i \leq n.$$

This does not depend on the choice of representative class for A_m in $\mathbb{Z}/p^m\mathbb{Z}$ or the lift of A_m to a p-adic integer because $\mathbb{Z}/p^m\mathbb{Z} \cong \mathbb{Z}_p/p^m\mathbb{Z}_p$. The remaining details to check in the proof, including compatibility of the action of $\mathbb{Z}_p$ on all graphs with respect to all these maps, are clear. □

There is a partial converse to Theorem 24. Namely, given any voltage assignment

$$\alpha : E(X)^+ \longrightarrow \mathbb{Z}_p$$

we obtain a "cyclic voltage p-tower" $X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots$ by simply taking X_m to equal $\overline{X}_m$ as above. However, depending on the choice of voltages, this tower may not result in *Galois* (normal) covers, i.e., the graphs $\overline{X}_m$ may not be connected (see Theorem 5). For example, if every edge has voltage zero, X_{p^∞} is evidently $|\,\mathbb{Z}_p\,|$ disjoint copies of X. The completion graph X_{p^∞} need not be connected even if all its "finite homomorphic images" $\overline{X}_m$ are connected.

Before closing this subsection with some special towers and examples, we specify notation that will be adopted henceforth for the completions. Because we will need to incorporate the p-adic group ring of $\mathbb{Z}_p$ in subsequent proofs, namely the ring $\mathbb{Z}_p[\mathbb{Z}_p]$, it would be confusing to have both p-adic coefficients and (additive) p-adic group elements as well. For this reason we will henceforth write the voltage group $\mathbb{Z}_p$ as the *multiplicative* profinite group Γ, where, as a profinite group, it is cyclic: it is the closure of an infinite (multiplicative) cyclic group $\langle\, \gamma\, \rangle$ under the p-adic metric topology, for some γ. For example, we may take $\gamma = 1 + p$ and Γ to be the multiplicative subgroup $1 + p\mathbb{Z}_p$ of the ring of p-adic integers. Another advantage of multiplicative notation voltages is to avoid confusion in voltage Laplacian matrices (whose entries lie in the group ring) between a p-adic identity element entry (which would be the p-adic zero in additive notation for the group $\mathbb{Z}_p$) versus a zero (integer) entry that would appear in position i, j when v_i is not adjacent to v_j.

Notation:

For fixed γ we may write each element of Γ in the form γ^A for some p-adic integer A. (This can be viewed as the usual way of relating additive and multiplicative notation; or as the p-adic exponential function to "base γ".) Now we shall write X_{p^∞} as the derived graph of the voltage graph (X, Γ, α), but where the vertex indices are written *additively* (in the second index) but the multiplicative voltage group Γ acts (on the left) as automorphisms of the graph by

$$\gamma^A : v_{i,g} \longmapsto v_{i,A+g} \qquad \text{for all } 1 \leq i \leq n \text{ and } g \in \mathbb{Z}_p.$$

Next, we specialize Definition 29 to single voltage graph towers.

Definition 31. *A cyclic single voltage p-tower of graphs above a base graph X, where $r_X - 1 \neq 0$, is a sequence of derived graphs*

$$X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_m \leftarrow \cdots$$

such that for $m \geq 0$, X_m/X is a single voltage graph with $Gal(X_m/X) \cong \mathbb{Z}/p^m\mathbb{Z}$.

By Definition 19 and Corollary 8, and by connectedness of each X_m we must have that each $\alpha_m(1, 2)$ reduces mod p^m to a generator of (the additive group) $\mathbb{Z}/p^m\mathbb{Z}$; and so the inverse limit of these elements is some p-adic integer that must generate $\mathbb{Z}_p$ as a topologically cyclic group. If we write that p-adic generator for Γ as τ, then we have the following edge in X :

$$v_1 \bullet \xrightarrow{\quad\tau\quad} \bullet\, v_2$$

where τ projects to a generator for the cyclic group G_m of order p^m, for all m. So we must also have the following edge in X:

$$v_2 \bullet \xrightarrow{\quad\tau^{-1}\quad} \bullet\, v_1$$

For each m, since X_m/X is a single voltage cyclic cover, Corollary 8 forces the intermediate covering graphs X_i/X for $i = 1, 2, \cdots, m$ to also be single voltage cyclic covers of X, and so the edge in X_i that has the nontrivial voltage is uniquely determined by the nontrivial voltage edge in X_m, regardless of the labeling of edges in X_i. (Since the X_i are all connected, the nontrivial voltage edge in X_m cannot project to an edge with trivial voltage in X_i, by Corollary 1.)

From this, we get the following theorem:

Theorem 25. *Let*

$$X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_m \leftarrow \cdots$$

be a cyclic single voltage p-tower of derived graphs over base graph X, where all X_m are assumed to be connected. Then we have that

$$|\mathcal{J}(X_m)| = |\mathcal{J}(X)||K|^{p^m-1}p^m$$

where $K = 2\Theta(2)$ is as in Lemma 1 and Theorem 23.

Proof. By Corollary 13, we have that

$$|\mathcal{J}(Y)| = |\mathcal{J}(X)| \cdot |K|^{d-1} d$$

for all covering graphs Y/X with single voltage assignment by $\mathbb{Z}/d\mathbb{Z}$. Then because X_m/X is a voltage graph for all m by Theorem, 9, we get the desired result. □

Corollary 18. *For the same hypotheses as in Theorem 25, we have that the exact power of p dividing $|\mathcal{J}(X_m)|$ is given by p^{e_m}, where*

$$e_m = \mu p^m + \lambda m + \nu$$

where μ is the exact exponent for the p-power of K, $\lambda = 1$ and $\nu = \beta - \mu$, where β is the exact exponent for the p-power of $|\mathcal{J}(X)|$.

Proof. Let p^β be the exact power of p dividing $|\mathcal{J}(X)|$ and let p^μ be the exact power of p dividing K. Then we have that the exact power of p dividing $|\mathcal{J}(X_m)|$ is given by

$$\begin{aligned} p^{e_m} &= p^\beta (p^\mu)^{p^m - 1} p^m \\ &= p^{\beta + \mu p^m - \mu + m} \\ &= p^{\mu p^m + m + (\beta - \mu)} \end{aligned}$$

□

Note that the definition of single voltage cover allows the possibility that the inverse limit generator τ, above, may be an irrational p-adic integer; however, the orders of the Jacobians are independent of the specific generator itself. Indeed neither the completion, X_{p^∞}, nor the p-adic single generator are used at all in the proof of Theorem 25.

Example 7.

(a) Let $X_0 = K_{10}$ and $p = 5$. Then we have that the exact p-power in $|\mathcal{J}(X_0)| = n^{n-2}$ is 5^8. The exact p-power in $|K| = (n-2)n^{n-3}$ is 5^7 (where K is given by Theorem 12). Thus

$$e_m = 7 \cdot 5^m + m + 1$$

for $m = 0, 1, 2, \ldots$.

(b) Let $X_0 = K_{6,6}$ and $p = 3$. Then we have that the exact p-power in $|\mathcal{J}(X_0)| = n^{2n-2}$ is 3^{10}. The exact p-power in $|K| = (n-1)^2 n^{2n-4}$ is 3^8 (where K is given by Theorem 16). Thus

$$e_m = 8 \cdot 3^m + m + 2$$

for $m = 0, 1, \ldots$.

(c) Let $X_0 = K_{16,2}$ and $p = 5$. Then we have that the exact p-power in $|\mathcal{J}(X_0)| = 2^{n-1}n$ is 5^0. The exact p-power in $|K| = 2^{n-2}(n-1)$ is 5^1 (where K is given by Theorem 19). Thus

$$e_m = 5^m + m - 1$$

for $m = 0, 1, \ldots$.

(d) Let X_0 be the Petersen graph and $p = 5$. Then we have that the exact p-power in $|\mathcal{J}(X_0)| = 2^4 \cdot 5^3$ is 5^3. The exact p-power in $|K| = 2^5 \cdot 5^2$ is 5^2 (where K is given by Theorem 20). Thus

$$e_m = 2 \cdot 5^m + m + 1$$

for $m = 0, 1, \ldots$. Note for all other odd primes, we have that the exact p-power in $|\mathcal{J}(X_m)|$ is m.

5.2 Iwasawa Theory: The Main Result and The Iwasawa "Program"

Iwasawa worked with $\mathbb{Z}_p$-extensions–infinite extensions K_∞ of a number field K with Galois group isomorphic to the additive p-adic integers, $\mathbb{Z}_p$, for some prime p. By using general theory of $\mathbb{Z}_p[[\Gamma]]$-modules, where $\Gamma = Gal(K_\infty/K)$, he was able to unravel the structure of the inverse limit of the p-Sylow subgroups of the class groups. This lead to him proving the following theorem, which can be found in [Was82].

Theorem 26. *Let K_∞/K be a $\mathbb{Z}_p$-extension. Let p^{e_m} be the exact power of p dividing the order of the class group of K_m, where K_m is the fixed field of the subgroup Γ^{p^m}. Then there exist nonnegative integers λ, μ and an integer ν such that*

$$e_m = \mu p^m + \lambda m + \nu$$

for all $m \geq m_0$ for some $m_0 \geq 0$.

We prove the analog of this in the graph theory setting:

Theorem 27. *Let*

$$X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_m \leftarrow \cdots$$

be a cyclic voltage p-tower, where all X_m are connected. Let $\mathcal{J}_p(X_m)$ be the Sylow p-subgroup of $\mathcal{J}(X_m)$. Then there are nonnegative integers μ and λ and an integer ν such that

$$|\mathcal{J}_p(X_m)| = p^{e_m} \qquad \text{where} \qquad e_m = \mu p^m + \lambda m + \nu$$

for all $m \geq m_0$ for some $m_0 \geq 0$.

This will be proven in Section 5.4.

In contrast with Theorem 27, [Val20] considers a Galois tower over a base graph X that is allowed to be a multigraph; however, his results also require X to be a *regular* multigraph (i.e., every vertex of X has the same valence, so the degree matrix, D_X, is a scalar matrix). Using zeta-function and character-theoretic methods similar to those in our Chapter 4, he obtains both *upper and lower asymptotic bounds* for the orders of the Sylow p-subgroups of the Jacobians in his towers, rather than an *exact* asymptotic (Iwasawa-type) formula. [Note: He calls his towers “abelian p-towers” although they are the same as our cyclic p-towers, since his definition also requires the Galois group of X_m/X to be *cyclic* of order p^m.] So this work is complementary to [Val20], and uses different strategies and methodologies.

We now describe the “Iwasawa Program” in a graph theoretic setting. From the cyclic voltage p-tower as in Figure 5.2

$$X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_m \leftarrow \cdots \leftarrow X_{p^\infty},$$

we get an inverse system of corresponding Galois groups

$$\Gamma_1 \leftarrow \Gamma_2 \leftarrow \Gamma_3 \leftarrow \cdots \leftarrow \Gamma_m \leftarrow \cdots \leftarrow \Gamma$$

where the inverse limit, Γ, is the additive p-adic integers, $\mathbb{Z}_p$ (which may also be viewed as the multiplicative profinite cyclic p-group, as described above). So we let $\Gamma = \langle \gamma \rangle$, where $\gamma \in \Gamma$ is a fixed topological generator. Then each $g \in \Gamma$ can be written uniquely as $g = \gamma^\alpha$, where $\alpha \in \mathbb{Z}_p$. Then under the above isomorphism, we may take $1 \in \mathbb{Z}_p$ to correspond to $\gamma \in \Gamma$ since 1 generates $\mathbb{Z}$ which is dense in $\mathbb{Z}_p$. Now because closed subgroups of $\mathbb{Z}_p$ are of the form $p^m\mathbb{Z}_p$, the closed subgroups of Γ are of the form Γ^{p^m}. Let $\Gamma_m = \Gamma/\Gamma^{p^m} \cong \mathbb{Z}_p/p^m\mathbb{Z}_p \cong Gal(X_m/X_0)$. So Γ_m is the cyclic group of order p^m generated by the image of γ. Now because Γ_m acts on the Jacobian of X_m, where the action is induced by the natural (left) action on the vertices, it follows that $\mathcal{J}(X_m)$ is a $\mathbb{Z}[\Gamma_m]$-module.

The covering map

$$X_{m+1} \to X_m$$

induces the surjective group homomorphism

$$\mathcal{J}(X_{m+1}) \to \mathcal{J}(X_m)$$

(surjectivity follows since X_{m+1}/X_m is a voltage graph and by Theorem 4.10 in [HMSV19] there is a surjection from $\mathcal{J}(X_{m+1})$ to $\mathcal{J}(X_m)$). From this, we get another inverse system

$$\mathcal{J}(X_0) \leftarrow \mathcal{J}(X_1) \leftarrow \mathcal{J}(X_2) \leftarrow \cdots \leftarrow \mathcal{J}(X_m) \leftarrow \cdots \leftarrow \mathcal{J}_\infty$$

where $\mathcal{J}_\infty = \varprojlim \mathcal{J}(X_m)$.

Since $\mathcal{J}(X_m)$ is a finite abelian group for all m, this induces an inverse system on their Sylow p-subgroups, $\mathcal{J}_p(X_m)$, i.e.

$$\mathcal{J}_p(X_0) \leftarrow \mathcal{J}_p(X_1) \leftarrow \mathcal{J}_p(X_2) \leftarrow \cdots \leftarrow \mathcal{J}_p(X_m) \leftarrow \cdots \leftarrow \mathcal{J}_p(X_{p^\infty})$$

where we denote $\varprojlim \mathcal{J}_p(X_m) = \mathcal{J}_p(X_{p^\infty})$, which by definition, is a profinite p-group.

Since each of the finite $\mathcal{J}_p(X_m)$ is a $\mathbb{Z}_p$-module it follows that the inverse limit $\mathcal{J}_p(X_{p^\infty})$ is also a $\mathbb{Z}_p$-module. Likewise, since there is a Γ-action on each finite $\mathcal{J}_p(X_m)$, there is a Γ action on the inverse limit, $\mathcal{J}_p(X_{p^\infty})$.

Now for $m \geq n \geq 0$ there is a natural map

$$\phi_{m,n} : \mathbb{Z}_p[\Gamma_m] \to \mathbb{Z}_p[\Gamma_n]$$

induced by the map

$$\Gamma_m \to \Gamma_n.$$

For the indeterminate T, we also have that

$$\mathbb{Z}_p[\Gamma_m] \cong \mathbb{Z}_p[T]/((1+T)^{p^m} - 1)$$

where this isomorphism is defined by

$$\gamma \bmod \Gamma^{p^m} \mapsto 1 + T \bmod ((1+T)^{p^m} - 1).$$

Since $(1+T)^{p^n} - 1$ divides $(1+T)^{p^m} - 1$, there is a natural map in the polynomial rings corresponding to $\phi_{m,n}$. So for the inverse system

$$\mathbb{Z}_p[\Gamma_1] \leftarrow \mathbb{Z}_p[\Gamma_2] \leftarrow \mathbb{Z}_p[\Gamma_3] \leftarrow \cdots \leftarrow \mathbb{Z}_p[\Gamma_m] \leftarrow \cdots$$

we have that the inverse limit $\varprojlim \mathbb{Z}_p[\Gamma_m]$ with respect to the maps $\phi_{m,n}$ is $\mathbb{Z}_p[[\Gamma]] = \Lambda$, the profinite group ring of Γ, as shown in [Was82].

Note that the ordinary group ring $\mathbb{Z}_p[\Gamma]$ is contained in Λ since an element $\alpha \in \mathbb{Z}_p[\Gamma]$ gives a sequence of elements $\alpha_n \in \mathbb{Z}_p[\Gamma_n]$ such that $\phi_{m,n}(\alpha_m) = \alpha_n$. However, Λ contains more elements. [Was82] shows that Λ is the compactification of $\mathbb{Z}_p[\Gamma]$, under the profinite topology defined by the open subgroups Γ_m.

The following two useful theorems about $\Lambda = \mathbb{Z}_p[[\Gamma]]$ can be found in [Was82]. These theorems will referenced in the sections that follow.

Theorem 28. $\mathbb{Z}_p[[\Gamma]] \cong \mathbb{Z}_p[[T]]$ *with the isomorphism being induced by* $\gamma \mapsto T + 1$.

Theorem 29. $\Lambda = \mathbb{Z}_p[[\Gamma]]$ *is a Noetherian local ring.*

Remark:

$\mathbb{Z}_p[\Gamma]$ is not Noetherian. Let $G = \Gamma \cong \mathbb{Z}_p$, the additive group of p-adic integers. Then G has an infinite strictly increasing chain of subgroups,

$$H_1 < H_2 < H_3 < \cdots ,$$

which can be seen explicitly by letting $H_i = \frac{1}{q^i}\mathbb{Z}$, where q is a fixed prime not equal to p (hence is unit in $\mathbb{Z}_p$). Let $S = \mathbb{Z}_p$. Then $S[G] = \mathbb{Z}_p[\Gamma]$ denotes the usual group ring over S. Then for all i, there are ring homomorphisms

$$S[G] \longrightarrow S[G/H_i]$$

obtained by just reducing the elements of G modulo H_i and extending this group homomorphism by S-linearity to a ring homomorphism on all of $S[G]$.

Let K_i be the kernel of the i^{th} homomorphism above. Then

$$K_1 \subset K_2 \subset K_3 \subset \cdots$$

is a strictly increasing chain of ideals, where the containments are proper since the subgroups H_i are all distinct. Thus, $S[G]$ does not satisfy the A.C.C. on ideals.

We now present the following lemma, which will be used in proving Theorem 27 in Section 5.4.

Lemma 2. *If $A \cong \mathbb{Z}_p$ as a $\mathbb{Z}_p$-module and B is a $\mathbb{Z}_p$-submodule of A, then either*

$B = 0$ or A/B is finite.

Proof. By the Fundamental Theorem of Finitely Generated Modules over a PID (see [DF04], Section 12.1, Theorem 4) applied to the cyclic module A, there is a generator a of A and element $\beta \in \mathbb{Z}_p$ such that βa generates B. Multiplying a and β by the same unit ($= 1/a$) we may assume $a = 1$, so (with the new β) we have $B = \beta\mathbb{Z}_p$. If $\beta = 0$ then $B = 0$. If $\beta \neq 0$ then $\beta = up^k$ for some unit u and $k \geq 0$; and then $A/B \cong \mathbb{Z}_p/p^k\mathbb{Z}_p$ is finite. $\square$

If we followed the "Iwasawa program" directly, we would try to show that the Sylow p-subgroup of $\mathcal{J}_\infty$ is a finitely generated torsion Λ-module. However, in the next few sections we take a somewhat different tack. Since our towers are voltage towers, we can use divisors to create modules over $\mathbb{Z}_p[\Gamma]$ and then over Λ to construct a finitely generated torsion Λ-module that maps onto each finite Jacobian, which we will do in Subsection 5.3.2. In this way, we can ultimately prove our Main Theorem, Theorem 27, on the growth of the orders of these finite Jacobians. So we achieve the same goal as the Iwasawa program via this route.

For the moment we leave aside the question as to whether our Jacobian Λ-module, constructed via divisors actually coincides with the inverse limit of the Sylow p-subgroups of the finite Jacobians, $\mathcal{J}_p(X_{p^\infty})$, since there may be unresolved issues of inverse limits and tensor products.

In the next section, we give important results and definitions pertaining to Λ-modules. Many of the modules over Λ that occur in Iwasawa theory are finitely generated torsion modules. Thus, in Subsections 5.3.1 and 5.3.1, the Λ-modules that we consider will be finitely generated.

5.3 Iwasawa Modules

The overall strategy for the proof of Theorem 27 is modeled on a "method of descent" that is reminiscent of the techniques used in the Inverse Galois Groups problem via the "rigidity method." The latter problem is to show that every finite group is a Galois group over the rational numbers, $\mathbb{Q}$. An excellent reference for the Inverse Galois Groups problem is [Vol96]. For a general overview of significant results pertaining to it, [RR15] is another great reference.

The general approach to the Inverse Galois Groups problem, undertaken by Thompson, Belyi, Fried and others, is to first realize an arbitrary finite group as a Galois group over the analytic field $\mathbb{C}(t)$, where t is transcendental over the complex numbers. It can be shown that every finite group G is a Galois group of some finite extension K of $\mathbb{C}(t)$. This result is "classical," and its proof relies on the theory of branched covers of Riemann surfaces: The fields K are function fields of Riemann surfaces; and there are topological covers of the Riemann sphere with any specified ramification type–hence with any Galois group type–by the Riemann Existence Theorem.

Any given Galois group that is realized over $\mathbb{C}(t)$ can then be "descended" to be a Galois group over a field that is the "algebraic analogue of $\mathbb{C}(t)$," namely as a Galois group over $\overline{\mathbb{Q}}(t)$, where $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ (again t is transcendental over $\overline{\mathbb{Q}}$). The field $\overline{\mathbb{Q}}$ does not have the same completeness/analytic properties as $\mathbb{C}$, which are essential to the first step; however it is "closer" to the ultimate goal of realizing G over just $\mathbb{Q}(t)$.

The second step, descent from $\overline{\mathbb{Q}}(t)$ to $\mathbb{Q}(t)$, involves the so-called "rigidity method." It applies, unfortunately, only to certain (nearly simple) groups; and so the full Inverse Galois Groups problem cannot entirely be solved by these two successive descents. The final descent from realizing a Galois group over $\mathbb{Q}(t)$ to realizing it over $\mathbb{Q}$ itself is achieved by the classical Hilbert Irreducibility Theorem, which says that there are infinitely many specializations (i.e., evaluations) of t to a rational number that maps $\mathbb{Q}(t)$ to $\mathbb{Q}$ and "preserves" a given Galois extension.

In our setting we begin with a given cyclic voltage p-tower, which has an associated tower of Jacobian p-subgroups, $\mathcal{J}_p(X_m)$. By the methods of Section 5.1, we first pass to the completion of this tower, X_{p^∞}. This is the analog of the "algebraic stage" where $\overline{\mathbb{Q}}(t)$ was the corresponding main actor in the Inverse Galois problem. The "profinite Jacobian" of X_{p^∞} carries the necessary information about the finite Jacobians: it is a subgroup of the Picard group, $\mathrm{Div}_R/\mathrm{Pr}_R$, of X_{p^∞} (namely M_R/Pr_R), where $R = \mathbb{Z}_p[\Gamma]$ is the usual group algebra of Γ with coefficients from $\mathbb{Z}_p$. It is, however, difficult to work directly with the "algebraic" divisor group, $\mathrm{Div}_R = \mathrm{Div}_{\mathbb{Z}_p}(X_{p^\infty})$, and its quotients. They are modules over the non-Noetherian (and incomplete) ring R (see the remark after Theorem 29). We circumvent this shortcoming by passing to the larger, complete ring, the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[\Gamma]]$, which plays the role analogous to $\mathbb{C}(t)$ in the Galois group problem. In contrast, Λ is a complete, Noetherian, Unique Factorization Domain.

Moreover, there is a "classical" general Structure Theorem for finitely generated modules over Λ, namely Theorem 30. To avail ourselves of this, we extend scalars of Div_R and Pic_R to obtain modules over Λ, denoted by Div_Λ and Pic_Λ respectively. This Structure Theorem (Theorem 30) then applies to these Λ-modules. Indeed, the Iwasawa Structure Theorem plays the same role for Pic_Λ as the Smith Normal Form Theorem does for $\mathrm{Pic}_{\mathbb{Z}}(X)$, for any graph X. The "guts" of the proof of Theorem 27 now involves a two-fold descent to carry the Iwasawa decomposition from the "analytic" level of modules over Λ, to the "algebraic" level of modules over $\mathbb{Z}_p[\Gamma]$, and finally to the "finite" layers level, where we ultimately achieve our (asymptotic) growth formulas for the finite p-Jacobians, $\mathcal{J}_p(X_m)$.

One technical obstacle is to pick out the correct "divisors of degree zero" submodule of the Λ-module Div_Λ (recall that we need to take the quotient of the submodule of divisors of degree zero by principal divisors in order to obtain the Jacobian). The naïve approach of taking a degree map on Div_Λ with respect to the chosen basis of divisors for this free module does not work (that quotient is too big). It turns out that the module we denote as M_Λ is the right choice—it is basically divisors of degree zero together with the augmentation ideal of Λ times all divisors.

The key result that undergirds the technical aspects of the first stage descent proof is that certain finite quotients of the "analytic" ring Λ are isomorphic to corresponding quotients of the "algebraic" ring R (see Proposition 13). This generalizes from rings to the modules that we need to "descend" (Proposition 14). The descent from the "algebraic" modules over R to the finite modules over $\mathbb{Z}_p$ is achieved by "reducing modulo p," once we precisely determine what the kernels of these reduction maps are, as well as their relationships to M_R and Pr_R. Ultimately, a variant of the classical Iwasawa Decomposition Theorem—the one in our Theorem 32—exactly matches the setup we have maneuvered to, and it provides the desired conclusion.

We begin by defining what it means for two Λ-modules to be *pseudo-isomorphic*. We then define when a nonconstant polynomial is *distinguished*. This leads the way to results (in particular, Proposition 14 and Corollary 19) that will be used in proving the claims in Section 5.4.

Definition 32. *Two Λ-modules M and M' are said to be pseudo-isomorphic, written*

$$M \sim M'$$

if there is a homomorphism $M \to M'$ with finite kernel and co-kernel.

Definition 33. *A nonconstant polynomial $P(T) \in \Lambda$*

$$P(T) = T^n + a_{n-1}T^{n-1} + \cdots + a_0$$

is called distinguished if $p \mid a_i$ for all $0 \leq i \leq n-1$.

The following proposition can be found in [Ouy] as Corollary 2.4.

Proposition 13. *Let $F(T)$ be a distinguished polynomial in $\mathbb{Z}_p[T]$. Then*

$$\mathbb{Z}_p[T]/F(T)\mathbb{Z}_p[T] \cong \mathbb{Z}_p[[T]]/F(T)\mathbb{Z}_p[[T]],$$

where the isomorphism is as $\mathbb{Z}_p[T]$-modules. The isomorphism is the natural one, namely for $r \in \mathbb{Z}_p[T]$, the coset $r + F(T)\mathbb{Z}_p[T]$ maps to $r + F(T)\mathbb{Z}_p[[T]]$.

Let $\Lambda = \mathbb{Z}_p[[\Gamma]]$ and fix a topological generator γ for Γ; so by Theorem 28 the map $\gamma \mapsto T + 1$ extends to an isomorphism from Λ to $\mathbb{Z}_p[[T]]$. Let $Z = \langle \gamma \rangle$ be the (abstract, not topological) cyclic group generated by γ; so $Z \cong \mathbb{Z}$, the infinite cyclic group. Under this isomorphism the polynomial ring $\mathbb{Z}_p[T]$ corresponds to the p-adic integral group ring $\mathbb{Z}_p[Z] = \mathbb{Z}_p[\gamma]$. For $m \geq 0$ let $\omega_m = \gamma^{p^m} - 1$ and let $\eta_m = 1 + \gamma + \cdots + \gamma^{p^m-1}$.

Lemma 3. *For $m \geq 1$ both ω_m and η_m map to distinguished polynomials in $\mathbb{Z}_p[T]$.*

Proof. By definition, ω_m maps to $(T+1)^{p^m} - 1$. Thus

$$(T+1)^{p^m} - 1 \equiv (T^{p^m} + 1^{p^m}) - 1 \equiv T^{p^m} \pmod{p\mathbb{Z}_p[T]},$$

which establishes the first claim. Since $\omega_m = (\gamma - 1)\eta_m$ in the Unique Factorization Domain $\mathbb{Z}_p[\gamma]$, and since $\gamma - 1 \mapsto T$, we have $\eta_m \mapsto T^{p^m-1} \pmod{p\mathbb{Z}_p[T]}$, which gives the second claim $\square$

Note: this works for $\omega_0 = \gamma - 1$ too.

Let $R = \mathbb{Z}_p[\Gamma]$. Fix $m \geq 0$.

Definition 34. *Let D be any Λ-module and let B be any subset of D. For every $m \geq 0$, define*

$$\Omega_m^D(B) = B \cap \omega_m D.$$

Define $R_m = R/\Omega_m^\Lambda(R) = R/R \cap \omega_m \Lambda$.

In the special case when B is an R-submodule of D (where D is considered as an R-module), we have that $\Omega_m^D(B)$ is an R-submodule of B containing $\omega_m B$.

The sets $\Omega_m^D(B)$ define relatively open subsets of B in the "ω-adic topology" on D. They obey the appropriate transitive property: If B and C are subset of D with $C \subseteq B$, then

$$\Omega_m^D(B) \cap C = \Omega_m^D(C).$$

It is not true in general that $\omega_m B \cap C = \omega_m C$ however.

Proposition 14. *Let D be a Λ-module, let A be any Λ-submodule of D and let B be any R-submodule of A, where A is considered as an R-module. Then the map*

$$\phi : B/\Omega_m^D(B) \longrightarrow A/\Omega_m^D(A) \qquad by \qquad \phi(x + \Omega_m^D(B)) = x + \Omega_m^D(A)$$

is a well-defined and injective R-module homomorphism. If B contains a set of Λ-module generators for A, then ϕ is an isomorphism and $A = B + \Omega_m^D(A)$; and if additionally $\omega_m D \subseteq A$ then $A = B + \omega_m D$.

Proof. We first simplify notation by denoting $\Omega_m^D(C)$ by just $\Omega_m(C)$ for every subset C of D throughout the proof. The map $B \to A/\Omega_m(A)$ by $x \mapsto x + \Omega_m(A)$ is a well-defined R-module homomorphisms, and since $\Omega_m(B) \subseteq \Omega_m(A)$, its kernel clearly contains $\Omega_m(B)$. This map therefore factors through $B/\Omega_m(B)$, giving the homomorphism ϕ. We also have

$$\begin{aligned}\ker\phi &= \{x+\Omega_m(B) \mid x \in B \text{ and } x+\Omega_m(A) = 0+\Omega_m(A)\}\\ &= \{x+\Omega_m(B) \mid x \in B \text{ and } x \in \Omega_m(A)\}\\ &= (B \cap \Omega_m(A))/\Omega_m(B)\\ &= (B \cap (A \cap \omega_m D))/\Omega_m(B)\\ &= (B \cap \omega_m D)/\Omega_m(B) = \Omega_m(B)/\Omega_m(B) = 1,\end{aligned}$$

so ϕ is injective. It remains to show if B contains a set of Λ-module generators for A, then ϕ is surjective. Assuming this hypothesis, every $y \in A$ can be written as

$$y = \alpha_1 b_1 + \cdots + \alpha_n b_n, \qquad \text{for some } \alpha_1, \ldots, \alpha_n \in \Lambda \text{ and } b_1, \ldots, b_n \in B.$$

By Proposition 13 and Lemma 3, for each α_i there is some $r_i \in \mathbb{Z}_p[\gamma] \subseteq R$ such that $\alpha_i - r_i \in \omega_m\Lambda$. Let $y' = r_1 b_1 + \cdots + r_n b_n \in B$. By construction,

$$y - y' = (\alpha_1 - r_1)b_1 + \cdots + (\alpha_n - r_n)b_n \in A \cap (\omega_m\Lambda)B \subseteq A \cap \omega_m D = \Omega_m(A).$$

Thus $\phi(y' + \Omega_m(B)) = y' + \Omega_m(A) = y + \Omega_m(A)$, and so ϕ is surjective, hence an isomorphism. Also, surjectivity of ϕ implies that $A = B + \Omega_m(A)$. If $\omega_m D \subseteq A$, then $\Omega_m(A) = \omega_m D$, so the last assertion holds too.

□

Corollary 19. *For $R = \mathbb{Z}_p[\Gamma]$, we have that ϕ induces an R-module isomorphism*

$$R_m \cong \mathbb{Z}_p[\Gamma_m].$$

Proof. As in Definition 34, we have $R_m = R/\Omega_m(R) = R/(R \cap \omega_m \Lambda)$. Now $\mathbb{Z}_p[T]$ corresponds to $\mathbb{Z}_p[\gamma]$ in the isomorphism between $\mathbb{Z}_p[[T]]$ and Λ, where γ is a fixed topological generator for Γ. So we have

$$\begin{aligned} R_m &= R/(R \cap \omega_m \Lambda) && \text{by definition} \\ &\cong \Lambda/\omega_m \Lambda && \text{by Proposition 14} \\ &\cong \mathbb{Z}_p[\gamma]/(\omega_m) && \text{by Proposition 13} \\ &\cong \mathbb{Z}_p[\gamma]/(\gamma^{p^m} - 1) \\ &\cong \mathbb{Z}_p[\Gamma_m], \end{aligned}$$

where the last isomorphism follows since $\mathbb{Z}_p[\gamma]/(\gamma^{p^m} - 1)$ is isomorphic to the group ring of the cyclic group $\mathbb{Z}/p^m\mathbb{Z} \cong \Gamma_m$. Hence $R_m \cong \mathbb{Z}_p[\Gamma_m]$.

□

In the next subsection, we now specify our Λ-modules to be finitely generated. Theorem 32 will be used in Section 5.4 to ultimately prove Theorem 27.

5.3.1 Finitely Generated Λ-Modules

The following theorem can be found in [Was82] as Theorem 13.12.

Theorem 30. *[Structure Theorem for Iwasawa modules] For any finitely generated*

Λ-module M, we get the following pseudo-isomorphism:

$$M \sim \Lambda^r \left(\oplus_{i=1}^{s} \Lambda/(p^{k_i})\right) \oplus \left(\oplus_{j=1}^{t} \Lambda/(g_j(T)^{m_j})\right)$$

where $r = rank(M)$, s, t, k_i and $m_j \in \mathbb{Z}$ and g_j is distinguished and irreducible. This decomposition is uniquely determined by M. If M is a torsion module, then $r = 0$.

We can see where the order formula comes from by examining the relevant finite quotients of the individual factors in Theorem 30. These may be found in [Was82], Section 13.3.

Case 1) $M = \Lambda/(p^{\mu})$:

Here $M/\omega_m M$ is isomorphic to $\mathbb{Z}_p[\Gamma_m]$, which is an abelian group isomorphic to $(\mathbb{Z}/p^{\mu}\mathbb{Z})^{p^m}$. It has order $p^{\mu p^m}$ and rank p^m.

and

Case 2) $M = \Lambda/(g(T))$ for g a distinguished polynomial of degree λ :

Here M is isomorphic to $\mathbb{Z}_p[T]/g(T)\mathbb{Z}_p[T]$, which is isomorphic to the $\mathbb{Z}_p$-module of polynomials over $\mathbb{Z}_p$ of degree $\leq \lambda$; so, for sufficiently large m, $M/\omega_m M$ has a subgroup of constant index, c, that is isomorphic to $(\mathbb{Z}/p^m\mathbb{Z})^{\lambda}$. It has order $p^{m\lambda+c}$, and rank $\leq \lambda + c$.

Definition 35. *As in the notation of Theorem 30, we define the Iwasawa invariants of M by*

$$\mu = \sum_{i=1}^{s} k_i \qquad \textit{and} \qquad \lambda = \sum_j m_j \deg g_j.$$

Definition 36. *Let M be any finitely generated torsion Λ-module with p^{k_i} and $g_j^{m_j}$, as in Theorem 30. The characteristic polynomial of M, denoted by $Char(M)$, is the product:*

$$Char(M) = p^{k_1+\cdots+k_s} g_1^{m_1} \cdots g_t^{m_t},$$

where $Char(M) = 1$ if M is finite.

We record some basic facts about finitely generated torsion Λ-modules. These may be found in [For20], Section 1.1.

Proposition 15. *Let P be any finitely generated torsion Λ-module.*

1. *The relation "pseudo-isomorphism" is an equivalence relation on any set of finitely generated torsion Λ-modules.*

2. *For any Λ-module M, the characteristic polynomial is an invariant of the pseudo-isomorphism equivalence class of M.*

3. *If M is a submodule of P, then $Char(P) = Char(M)Char(P/M)$. In particular, $Char(M) \Big| Char(P)$.*

The following Theorem can be found in [Sha] as Theorem 2.4.7. In it, $\mathcal{O}$ is a valuation ring of a p-adic field. We simplify his statement by taking $\mathcal{O} = \mathbb{Z}_p$.

Theorem 31. *Let M be a finitely generated, torsion, Λ-module, and let $n_0 \geq 0$ be such that $Char(M)$ and $\omega_{m,n_0} = \omega_m/\omega_{n_0}$ are relatively prime for all $n \geq n_0$. Set $\lambda(M) = \lambda$ and $\mu(M) = \mu$. Then there exists an integer ν such that*

$$|M/\omega_{m,n_0}M| = q^{e_m} \qquad \textit{where} \qquad e_m = \mu p^m + \lambda m + \nu$$

for all sufficiently large $n \geq 0$.

This theorem is used to prove Theorem 32. But first we need an elementary lemma to circumvent an extraneous hypothesis in Theorem 31. We phrase this lemma in the language of elementary number theory rather than ideals; here "divides", "gcd" etc. means "up to associates".

Lemma 4. *Let U be any Unique Factorization Domain and let $d \in U$ with $d \neq 0$. Suppose $\{a_m\}_{m=0}^{\infty}$ is any sequence of nonzero elements of U with $a_m \,\big|\, a_{m+1}$ for all $m \geq 0$. Then there exists some $n_0 \geq 0$ such that*

$$gcd(a_{n_0}, d) = gcd(a_m, d) \qquad \text{for all } m \geq n_0, \quad \text{and}$$
$$gcd(a_m/a_{n_0}, d) = 1 \qquad \text{for all } m \geq n_0.$$

Proof. This is an easy exercise. The key point is that d has only finitely many divisors, so the chain of $gcd(a_m, d)$ must stabilize after finitely many steps. □

Theorem 32. *Let P be a nonzero finitely generated torsion Λ-module. Let $P_m = \omega_m P$, for all $m \geq 0$. Assume there is a Λ-submodule N of P such that $P_m \subseteq N$ and $|\,N/P_m\,| < \infty$, for all $m \geq 0$. Then there are nonnegative integers μ and λ and an integer ν such that*

$$|\,N/P_m\,| = p^{e_m} \qquad \text{where} \quad e_m = \mu p^m + \lambda m + \nu,$$

for all $m \geq m_0$, for some constant $m_0 \geq 0$.

Proof. First let $d = \mathrm{Char}(P)$ and apply Lemma 4 in $U = \Lambda$ to $a_m = \omega_m$, for all $m \geq 0$. Let n_0 be as provided by the conclusion of that lemma. For any $m \geq n_0$, define $\omega_{m,n_0} = \omega_m/\omega_{n_0} \in \Lambda$. Let $M = P_{n_0}$.

Note that for all $m \geq n_0$ we have

$$\omega_{m,n_0} M = (\omega_m/\omega_{n_0})(\omega_{n_0} P) = \omega_m P = P_m.$$

By hypotheses then, for all $m \geq n_0$,

$$\begin{aligned} |\, M/\omega_{m,n_0} M \,| &= |\, P_{n_0}/P_m \,| \\ &= \frac{|N/P_m|}{|N/P_{n_0}|} \leq |\, N/P_m \,| < \infty. \end{aligned}$$

By Lemma 4 we have that $\omega_{m,n_0} = \omega_m/\omega_{n_0}$ is relatively prime to $\mathrm{Char}(P) = d$, for all $m \geq n_0$. By Proposition 15(3) we have that ω_{m,n_0} is relatively prime to $\mathrm{Char}(M)$ as well.

We now have the hypotheses of Theorem 31 above. This theorem proves that there are μ, λ, and some ν' such that

$$|\, M/\omega_{m,n_0} M \,| = p^{e_m} \qquad \text{where} \quad e_m = \mu p^m + \lambda m + \nu',$$

for all m greater than or equal to some fixed $m_0 \geq n_0$.

Now, as noted above, $\omega_{m,n_0} M = P_m$, and so for all $m \geq m_0$, by Lagrange we have

$$\begin{aligned} |\, N/P_m \,| &= |\, N/P_{n_0} \,| \cdot |\, P_{n_0}/P_m \,| \\ &= |\, N/M \,| \cdot |\, M/\omega_{m,n_0} M \,| \\ &= p^k \cdot p^{e_m} \qquad \text{where} \quad p^k = |\, N/M \,| \quad \text{and} \quad e_m = \mu p^m + \lambda m + \nu'. \end{aligned}$$

Finally, let $\nu = k + \nu'$ to obtain the conclusion to the theorem. □

The goal of the next subsection is to construct a finitely generated torsion Λ-module, $P = \mathrm{Pic}_\Lambda$. We will then apply Theorem 32 to Pic_Λ in Section 5.4.

5.3.2 Constructing a Finitely Generated Λ-Module

Let $R = \mathbb{Z}_p[\Gamma]$ be the usual group ring of Γ with coefficients from $\mathbb{Z}_p$. The objects we are concerned with were described in Section 2.5, so the reader may wish to review this section first. For the given voltage p-tower let X_{p^∞} be its completion, so by Theorem 24 we may henceforth identify the intermediate graphs of X_{p^∞}/X with corresponding graphs in the tower.

Fix the following subset of X_{p^∞}:

$$\mathcal{B} = \{v_{i,0} \mid 1 \leq i \leq n\},$$

where 0 is the additive identity of $\mathbb{Z}_p$, so these vertices are taken from the "zeroth sheet". We fix the identification of X and X_0 with $\mathcal{B}$ by v_i is identified with $v_{i,0}$.

We first take the free $\mathbb{Z}$-module on basis $\mathcal{B}$, $\mathrm{Div}_{\mathbb{Z}}(X)$, and extend scalars (see [DF04], Section 10.4, Corollary 18) to the free $\mathbb{Z}_p$-module with the same basis, now viewed over $\mathbb{Z}_p$. Denote this module by $\mathrm{Div}_{\mathbb{Z}_p}(X_0)$. We can do likewise for each of the graphs X_m and for X_{p^∞} too. We obtain the free $\mathbb{Z}_p$-modules of divisors

$$\mathrm{Div}_{\mathbb{Z}_p}(X_m) = \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathrm{Div}_{\mathbb{Z}}(X_m), \quad m \geq 0$$
$$\mathrm{Div}_{\mathbb{Z}_p}(X_{p^\infty}) = \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathrm{Div}_{\mathbb{Z}}(X_{p^\infty}).$$

Now for every $m \geq 0$, each $\mathrm{Div}_{\mathbb{Z}}(X_m)$ is a free $\mathbb{Z}[\Gamma_m]$-module on the set $\mathcal{B}$ too, once we consider the group indices for vertices in X_m to be p-adic indices reduced to $\mathbb{Z}_p/p^m\mathbb{Z}_p \cong \mathbb{Z}/p^m\mathbb{Z}$ (as in Theorem 24); and so $\mathrm{Div}_{\mathbb{Z}_p}(X_m)$ is a free module of rank n over $\mathbb{Z}_p[\Gamma_m]$. We may do likewise for X_{p^∞} to obtain that $\mathrm{Div}_{\mathbb{Z}_p}(X_{p^\infty})$ is a free R-module, also of rank n (on basis $\mathcal{B}$). In order to emphasize the free, rank n nature of these respective modules, we adopt the following notation:

$$\mathrm{Div}_{R_m} = \mathrm{Div}_{\mathbb{Z}_p}(X_m) \qquad \text{and} \qquad \mathrm{Div}_R = \mathrm{Div}_{\mathbb{Z}_p}(X_{p^\infty}).$$

Since $\mathrm{Div}_{R_m} = \mathrm{Div}_{\mathbb{Z}_p}(X_m)$ is a free $\mathbb{Z}_p$-module on the basis of vertices of X_m, $\{v_{i,g} \mid 1 \leq i \leq n,\ g \in \Gamma_m\}$, we may define the usual degree zero divisors with respect to this $\mathbb{Z}_p$-basis, and denote this by

$$\mathrm{Div}^0_{\mathbb{Z}_p}(X_m) = \left\{ \sum_{i,g} a_{i,g} v_{i,g} \mid a_{i,g} \in \mathbb{Z}_p \text{ and } \sum_{i,g} a_{i,g} = 0 \right\}$$

where these sums are for $1 \leq i \leq n$ and $g \in \Gamma_m$.

Next we extend scalars from R to Λ. Since Div_R is a free R-module of rank n, its extension is a free Λ module of rank n, denoted by

$$\mathrm{Div}_\Lambda = \Lambda \otimes_R \mathrm{Div}_R.$$

Since R is a subring of Λ we may simply view the elements of Div_Λ as Λ-linear combinations of $\mathcal{B}$ and Div_R as the subset of these consisting of R-linear combinations of $\mathcal{B}$.

Next, as in Section 2.5, we define the Laplacian endomorphism:

$$\mathcal{L}_{p^\infty} : \mathrm{Div}_R \longrightarrow \mathrm{Div}_R \qquad \text{by} \qquad \mathcal{L}_{p^\infty}(v_{i,0}) = p_{i,0} \quad 1 \le i \le n,$$

where $p_{i,0}$ (as in Section 2.5) is the principal divisor "based at $v_{i,0}$." This is extended by R-linearity to all of Div_R. Because Γ acts transitively on vertices in each fiber of X_{p^∞}/X, as usual we have that the image of $\mathcal{L}_{p^\infty}$ is the $\mathbb{Z}_p$-span of the set of *all* principal divisors. We encapsulate this by the following notation (definition):

$$\mathrm{Pr}_R = \mathcal{L}_{p^\infty}(\mathrm{Div}_R).$$

By taking the "same map", but defined on the basis $\mathcal{B}$ of the free Λ-module Div_Λ we denote this by

$$\widehat{\mathcal{L}}_{p^\infty} : \mathrm{Div}_\Lambda \longrightarrow \mathrm{Div}_\Lambda \qquad \text{by} \qquad \widehat{\mathcal{L}}_{p^\infty}(v_{i,0}) = p_{i,0} \quad 1 \le i \le n,$$

extended now by Λ-linearity. (Formally, $\widehat{\mathcal{L}}_{p^\infty} = 1 \otimes \mathcal{L}_{p^\infty}$.) Now we just define

$$\mathrm{Pr}_\Lambda = \widehat{\mathcal{L}}_{p^\infty}(\mathrm{Div}_\Lambda).$$

Likewise, as in Section 2.5, because Γ_m acts transitively on the vertices of X_m, using the same $\mathcal{L}_{p^\infty}$, but instead reading the vertices $v_{i,0}$ as lying in Div_{R_m} (i.e., with the vertex indices reduced to $\mathbb{Z}_p/p^m\mathbb{Z}_p$), and extended by R_m-linearity—call this map $\mathcal{L}_m$—defines the usual Laplacian endomorphism of Div_{R_m}. Its image is the R_m-module of principal divisors of Div_{R_m}, denoted as

$$\mathrm{Pr}_{R_m} = \mathcal{L}_m(\mathrm{Div}_{R_m}).$$

We now define the appropriate Picard groups as follows:

$$\begin{aligned}
\mathrm{Pic}_{R_m} &= \mathrm{Div}_{R_m}/\mathrm{Pr}_{R_m} && \text{(an } R_m\text{-module)}\\
\mathrm{Pic}_{R} &= \mathrm{Div}_{R}/\mathrm{Pr}_{R} && \text{(an } R\text{-module)}\\
\mathrm{Pic}_{\Lambda} &= \mathrm{Div}_{\Lambda}/\mathrm{Pr}_{\Lambda} && \text{(a } \Lambda\text{-module).}
\end{aligned}$$

So these modules are *cokernels* of the respective module endomorphisms.

It is important to note that, with respect to the basis $\mathcal{B}$ of both Div_R (as an R-basis) and Div_Λ (as a Λ-basis), the two maps, $\mathcal{L}_{p^\infty}$ and $\widehat{\mathcal{L}}_{p^\infty}$, have *the same matrix representation.* Thus we define

$$\Theta_{p^\infty} = \det \mathcal{L}_{p^\infty} = \det \widehat{\mathcal{L}}_{p^\infty},$$

which is an element of $R = \mathbb{Z}_p[\Gamma]$ that plays the role of the reduced Stickelberger element. In particular, by the same argument as in Section 2.5 we have

$$\Theta_{p^\infty} \text{ annihilates both } \mathrm{Pic}_R \text{ and } \mathrm{Pic}_\Lambda.$$

Note that if $\Theta_{p^\infty} = 0$, its projection onto the finite reduced Stickelberger elements Θ_{p^m} would also be zero, but this contradicts Corollary 12. So $\Theta_{p^\infty} \neq 0$.

Next, we identify the Λ-submodule that plays the role of "degree zero divisors" in the proof of Theorem 27. (Using divisors of Λ-degree-zero with respect to basis $\mathcal{B}$ is "too big" a submodule, since it would have quotient isomorphic to the coefficient ring, Λ.)

Definition 37. *Let*

$$S_1 = \{v_{i,0} - v_{j,0} \mid 1 \le j < i \le n\} \qquad \textit{and}$$
$$S_2 = \{p_{i,0} \mid 1 \le i \le n\}$$

Let M_Λ be the Λ-submodule of Div_Λ generated by S_1, S_2 and $(\gamma - 1)Div_\Lambda$, and let $M_R = Div_R \cap M_\Lambda$ and $N_\Lambda = M_\Lambda / Pr_\Lambda$.

It turns out that M_Λ is actually generated by just S_1 and $(\gamma - 1)\mathrm{Div}_\Lambda$ (see Claim 2(1) in the next subsection); but we include the redundant generators for expository clarity.

Since Div_Λ is a finitely generated Λ-module and Λ is Noetherian, all of its submodules are finitely generated, and so it follows that the quotient modules $\mathrm{Div}_\Lambda/\mathrm{Pr}_\Lambda = \mathrm{Pic}_\Lambda$ and $M_\Lambda/\mathrm{Pr}_\Lambda = N_\Lambda$ are also finitely generated as Λ-modules.

Theorem 33. *Let Div_Λ, Pr_Λ, Pic_Λ and N_Λ be as above. Then Pic_Λ is a finitely generated torsion module over the Iwasawa Algebra $\Lambda = \mathbb{Z}_p[[\Gamma]]$ and therefore so is its submodule N_Λ.*

5.4 The Main Theorem

We now go on to prove Theorem 27. The way in which we do so is by relating quotients of Λ-modules to quotients of R-modules. In particular, we are able to show that $M_\Lambda/(\omega_m \mathrm{Div}_\Lambda + \mathrm{Pr}_\Lambda) \cong M_R/(\ker \pi_m + \mathrm{Pr}_R)$, where the latter is isomorphic to $\mathrm{Div}_{R_m}/\mathrm{Pr}_{R_m} = \mathcal{J}_p(X_m)$, and where $\pi_m : \mathrm{Div}_R \to \mathrm{Div}_{R_m}$. We first prove Claims 1-4 below, which lead to the lattice and map diagram in Figure 5.2. Then by comparing the first column (Λ-level) with the second column (R-level), we are able to show that $N_\Lambda/\omega_m \mathrm{Pic}_\Lambda \cong \mathcal{J}_p(X_m)$. We are then in a position to apply Theorem 32 which establishes a growth formula for the order of the Jacobians, $\mathcal{J}_p(X_m)$ $\forall m$.

Consider the reduction map

$$\pi_m : \mathrm{Div}_R \to \mathrm{Div}_{R_m} \quad \text{by} \quad v_{i,g} \mapsto v_{i,\overline{g}}$$

where $g \in \Gamma$ and $\overline{g} \in \Gamma_m$ is the reduction of g to $\Gamma/\Gamma^{p^m} \cong \mathbb{Z}_p/p^m\mathbb{Z}_p$, (and recall $\mathrm{Div}_{R_m} = \mathrm{Div}_{\mathbb{Z}_p}(X_m)$). Here we are really defining π_m on the free R-basis vectors on the zeroth sheet, and then extending by R-linearity to all of Div_R. It is helpful to keep in mind that for all $m \geq 0$, by the above map and by the previous subsection we have

$$\mathrm{Div}_R \text{ is an } R\text{-submodule of } \mathrm{Div}_\Lambda, \text{ and}$$

$$\mathrm{Div}_{R_m} \text{ is an } R\text{-quotient module of } \mathrm{Div}_R.$$

In the next claim, we let $D = \mathrm{Div}_\Lambda$ as in Proposition 14, but we simplify notation by writing $\Omega_m(\mathrm{Div}_R)$ to denote $\Omega_m^D(\mathrm{Div}_R)$.

Claim 1

The kernel of π_m is $\Omega_m(\mathrm{Div}_R) = \mathrm{Div}_R \cap \omega_m \mathrm{Div}_\Lambda$, where $\omega_m = \gamma^{p^m} - 1$.

Proof. By Proposition 14 and Corollary 19, we get the following isomorphisms, where the composition of these isomorphisms is the induced map on Div_R mod $\ker \pi_m$:

$$\begin{aligned}
\mathrm{Div}_R/\Omega_m(\mathrm{Div}_R) &\cong \mathrm{Div}_\Lambda/\omega_m \mathrm{Div}_\Lambda \\
&\cong (\Lambda \oplus \Lambda \oplus \cdots \oplus \Lambda)/(\omega_m(\Lambda \oplus \Lambda \oplus \cdots \oplus \Lambda)) \\
&\cong (\Lambda/(\omega_m)) \oplus \cdots \oplus (\Lambda/(\omega_m)) \\
&\cong R_m \oplus \cdots \oplus R_m \\
&\cong \mathbb{Z}_p[\Gamma_m] \oplus \cdots \oplus \mathbb{Z}_p[\Gamma_m] \\
&\cong \mathrm{Div}_{R_m}
\end{aligned}$$

the free $\mathbb{Z}_p[\Gamma_m]$-module of rank n. Thus, the kernel of π_m is $\Omega_m(\mathrm{Div}_R)$, as claimed. $\square$

Now let

$$K_m = \ker \pi_m.$$

Claim 2

As R-modules we have the following:

1. $\mathrm{Div}_\Lambda/M_\Lambda \cong \mathbb{Z}_p$ and M_Λ is the Λ-submodule of Div_Λ generated by $S_1 \cup (\gamma-1)\mathrm{Div}_\Lambda$ (where S_1 is as in Definition 37)

2. $\omega_m \mathrm{Div}_\Lambda + \mathrm{Div}_R = \mathrm{Div}_\Lambda$. In particular, $M_\Lambda + \mathrm{Div}_R = \mathrm{Div}_\Lambda$ and $\omega_m \mathrm{Div}_\Lambda + M_R = \ M_\Lambda$.

3. $\mathrm{Div}_R/M_R \cong \mathbb{Z}_p$

4. $\ker \pi_m \subseteq M_R$

5. $\mathrm{Div}^0_{\mathbb{Z}_p}(X_m) = \pi_m(M_R)$

Proof. Keep in mind throughout that all the maps following are valid as R-module maps, as well as abelian group maps.

To prove (1): We may obtain $\mathrm{Div}_\Lambda/M_\Lambda$ as follows: first factor Div_Λ by $(\gamma-1)\mathrm{Div}_\Lambda$. By Corollary 19 applied with $m=0$, and as in the proof of Claim 1, we have that

$$\mathrm{Div}_\Lambda/(\gamma-1)\mathrm{Div}_\Lambda \cong (\Lambda/\omega_0\Lambda) \oplus \cdots \oplus (\Lambda/\omega_0\Lambda) \cong \underbrace{\mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p}_{n \text{ of these}},$$

where the divisors $v_{1,0}, \ldots, v_{n,0}$ map to a basis of this free $\mathbb{Z}_p$-module of rank n. Now factor out the submodule generated by the images of all $v_{i,0} - v_{j,0}\ \forall i,j$ from the quotient $\mathrm{Div}_\Lambda/(\gamma-1)\mathrm{Div}_\Lambda$. By doing this, we are simply identifying all the basis vectors with each other, leaving the rank-1 $\mathbb{Z}_p$-module quotient. The latter process is the same as modding $\mathrm{Div}_{\mathbb{Z}_p}(X_0)$ by the degree zero divisors in $\mathrm{Div}_{\mathbb{Z}_p}(X_0)$. So this tells us that $p_{i,0}$ for $1 \leq i \leq n$ must already be contained in the Λ-submodule generated by just $S_1 \cup (\gamma - 1)\mathrm{Div}_\Lambda$. This proves (1).

To prove (2), apply Proposition 14 to $D = A = \mathrm{Div}_\Lambda$ and $B = \mathrm{Div}_R$. Its final assertion gives the first claim of (2). The second assertion is immediate because $\omega_m \mathrm{Div}_\Lambda \subseteq M_\Lambda$ by definition. Finally, by the first result of (2) and the latter observation, we have

$$\begin{aligned} M_\Lambda &= M_\Lambda \cap (\omega_m \mathrm{Div}_\Lambda + \mathrm{Div}_R) \\ &= \omega_m \mathrm{Div}_\Lambda + (M_\Lambda \cap \mathrm{Div}_R) \\ &= \omega_m \mathrm{Div}_\Lambda + M_R \end{aligned}$$

as needed for the third assertion.

To prove (3): First note that by definition $M_R = M_\Lambda \cap \mathrm{Div}_R$. By the previous two parts, we have that

$$\mathrm{Div}_\Lambda / M_\Lambda = (\mathrm{Div}_R + M_\Lambda)/M_\Lambda.$$

Then by Figure 5.3 we see that

$$\mathrm{Div}_R / M_R \cong \mathbb{Z}_p.$$

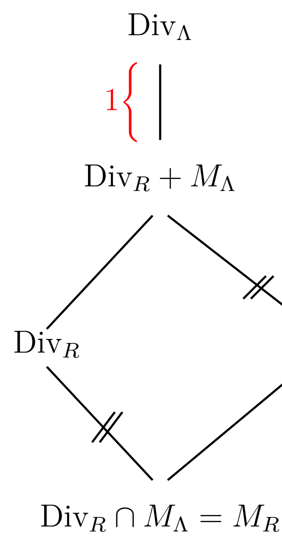

Figure 5.3: $(Div_R + M_\Lambda)/M_\Lambda \cong Div_R/M_R$ *by the Diamond Isomorphism Theorem*

To prove (4): By Claim 1 we have $\ker \pi_m = \mathrm{Div}_R \cap \omega_m \mathrm{Div}_\Lambda$. Since $\omega_m = (\gamma - 1)\eta_m$, we have

$$\omega_m \mathrm{Div}_\Lambda = (\gamma - 1)\eta_m \mathrm{Div}_\Lambda \subseteq (\gamma - 1)\mathrm{Div}_\Lambda \subseteq M_\Lambda.$$

Thus we have $\ker \pi_m \subseteq \mathrm{Div}_R \cap M_\Lambda = M_R$, as needed for (4).

To prove (5): By the above discussion, it is clear that

$$\mathcal{S}_1 \cup (\gamma - 1)\mathrm{Div}_R \subseteq \mathrm{Div}_R \cap M_\Lambda = M_R.$$

Since the image of $\mathcal{S}_1 \cup (\gamma - 1)\mathrm{Div}_R$ under π_m generates $\mathrm{Div}^0_{\mathbb{Z}_p}(X_m)$ as a $\mathbb{Z}_p$-module, it follows that

$$\mathrm{Div}^0_{\mathbb{Z}_p}(X_m) \subseteq \pi_m(M_R). \tag{5.4}$$

To show the reverse containment, let $D = \pi_m^{-1}(\mathrm{Div}^0_{\mathbb{Z}_p}(X_m))$, (the complete preimage). By basic properties of homomorphisms (part of the Lattice Isomorphism Theorem) and by (4) we have:

$$\pi_m^{-1}(\pi_m(M_R)) = M_R + \ker \pi_m = M_R.$$

By applying π^{-1} to (5.4) we get $D \subseteq M_R$. By the Lattice Isomorphism Theorem we have that π_m induces an isomorphism

$$\mathrm{Div}_R/D \cong \mathrm{Div}_{R_m}/\pi_m(D) = \mathrm{Div}_{R_m}/\mathrm{Div}^0_{\mathbb{Z}_p}(X_m) = \mathrm{Div}_{\mathbb{Z}_p}(X_m)/\mathrm{Div}^0_{\mathbb{Z}_p}(X_m) \cong \mathbb{Z}_p.$$

Since $D \subseteq M_R$ we get that Div_R/M_R is a quotient $\mathbb{Z}_p$-module of the $\mathbb{Z}_p$-module Div_R/D. By (3) we also have $\mathrm{Div}_R/M_R \cong \mathbb{Z}_p$. This is illustrated in Figure 5.4:

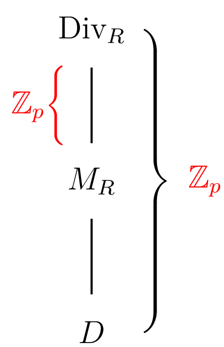

Figure 5.4: $\mathrm{Div}_R/M_R \cong \mathbb{Z}_p$ is a quotient $\mathbb{Z}_p$-module of the $\mathbb{Z}_p$-module Div_R/D

However, the only $\mathbb{Z}_p$-module quotient of $\mathbb{Z}_p$ that is also isomorphic to $\mathbb{Z}_p$ is the quotient by the zero submodule (this follows by Lemma 2 in Section 5.2) i.e., we must have $M_R = D$; and so $\pi_m(M_R) = \pi_m(D) = \mathrm{Div}^0_{\mathbb{Z}_p}(X_m)$, as needed for (5). $\square$

From Claim 2(4) it follows that the kernel of the map π_m restricted to M_R (which we simply denote by π_m too) is also equal to $K_m = \mathrm{Div}_R \cap \omega_m \mathrm{Div}_\Lambda$. From the definition in Section 5.3.2 we introduce the new notation:

$$M_{R_m} = \mathrm{Div}^0_{\mathbb{Z}_p}(X_m).$$

Now π_m induces the surjective map

$$\overline{\pi}_m : M_R/\mathrm{Pr}_R \to M_{R_m}/\mathrm{Pr}_{R_m}$$

where $M_R/\mathrm{Pr}_R = N_R$ and $M_{R_m}/\mathrm{Pr}_{R_m} = \mathcal{J}_p(X_m)$, and note that $\mathrm{Pr}_{R_m} \cong \mathrm{Pr}_{\mathbb{Z}_p}(X_m)$. This is defined by the following commutative diagram in Figure 5.5:

$$\begin{tikzcd}
M_R \arrow[rr, "\pi_m"] \arrow[dr, "\mathrm{proj}"'] & & M_{R_m} \arrow[r, "\mathrm{proj}"] & M_{R_m}/\mathrm{Pr}_{R_m} = \mathcal{J}_p(X_m) \\
& M_R/\mathrm{Pr}_R = N_R \arrow[urr, "\overline{\pi}_m"] & &
\end{tikzcd}$$

Figure 5.5: The map $\overline{\pi}_m : N_R \to \mathcal{J}_p(X_m)$ commutes with the natural projection map

Claim 3

For $\overline{\pi}_m : M_R/\mathrm{Pr}_R \to M_{R_m}/\mathrm{Pr}_{R_m}$, we have that $\ker \overline{\pi}_m = (K_m + \mathrm{Pr}_R)/\mathrm{Pr}_R$.

Proof. First note that for all $a \in M_R$, we have that $\overline{\pi}_m(\overline{a}) = \overline{\pi}_m(a + \mathrm{Pr}_R) = \pi_m(a) + \mathrm{Pr}_{R_m}$.

Let $a \in M_R$. We use bar notation to emphasize that elements of N_R are cosets of Pr_R in M_R. Then for $\overline{a} \in N_R$, with $\overline{a} \in \ker \overline{\pi}_m$, we have that

$$\begin{aligned}\overline{\pi}_m(\overline{a}) &= \overline{\pi}_m(a + \mathrm{Pr}_R) \\ &= \pi_m(a) + \mathrm{Pr}_{R_m} \\ &= 0 + \mathrm{Pr}_{R_m}.\end{aligned}$$

Thus $\overline{a} \in \ker \overline{\pi}_m$ if and only if $\pi_m(a) \in \mathrm{Pr}_{R_m}$. So if $a \in K_m$, then $a + \mathrm{Pr}_R \in \ker \overline{\pi}_m$. Hence, $(K_m + \mathrm{Pr}_R)/\mathrm{Pr}_R \subseteq \ker \overline{\pi}_m$.

For the reverse containment, let $\overline{a} \in \ker \overline{\pi}_m$. Then by the above, $\pi_m(a) \in \mathrm{Pr}_{R_m}$. Now since $\pi_m : \mathrm{Pr}_R \to \mathrm{Pr}_{R_m}$ is surjective (by the transitive action of Γ on the fibers of the covering map $X_{p^\infty} \to X$, Pr_R contains all principal divisors), there exists $b \in \mathrm{Pr}_R$ such that $\pi_m(a) = \pi_m(b)$ and so $\pi_m(a - b) = 0$. Thus, $a - b = k$, for some $k \in K_m = \mathrm{Div}_R \cap \omega_m \mathrm{Div}_\Lambda$. So we have that $k \in \mathrm{Div}_R$ and $k = \omega_m d$ for some $d \in \mathrm{Div}_\Lambda$. Therefore, $a = \omega_m d + b$ and so $a \in (K_m + \mathrm{Pr}_R)/\mathrm{Pr}_R$. So we have shown that

$$\ker \overline{\pi}_m = (K_m + \mathrm{Pr}_R)/\mathrm{Pr}_R.$$

□

Define $Q_R = \mathrm{Pr}_\Lambda \cap \mathrm{Div}_R$.

Claim 4

We have that $Q_R + K_m = \mathrm{Pr}_R + K_m = (\omega_m \mathrm{Div}_\Lambda + \mathrm{Pr}_\Lambda) \cap \mathrm{Div}_R$.

Proof. By Proposition 14 applied with $D = \mathrm{Div}_\Lambda$, $A = \mathrm{Pr}_\Lambda + \omega_m \mathrm{Div}_\Lambda$ and $B = \mathrm{Pr}_R$, since Pr_R and Pr_Λ are both generated (as R- and Λ-modules, respectively) by the same generators, they both have the same image in $\mathrm{Div}_\Lambda/\omega_m\mathrm{Div}_\Lambda$ as in Claim 1. So, by the last sentence of Proposition 14,

$$\mathrm{Pr}_R + \omega_m\mathrm{Div}_\Lambda = \mathrm{Pr}_\Lambda + \omega_m\mathrm{Div}_\Lambda. \tag{5.5}$$

Then since

$$\mathrm{Pr}_R \subseteq Q_R \subseteq \mathrm{Pr}_\Lambda,$$

by (5.5), we get

$$\mathrm{Pr}_R + \omega_m\mathrm{Div}_\Lambda = Q_R + \omega_m\mathrm{Div}_\Lambda = \mathrm{Pr}_\Lambda + \omega_m\mathrm{Div}_\Lambda. \tag{5.6}$$

Now because Pr_R and Q_R are contained in Div_R, intersecting the subgroups in (5.6) with Div_R gives

$$\begin{aligned}(\mathrm{Pr}_R + \omega_m\mathrm{Div}_\Lambda) \cap \mathrm{Div}_R &= \mathrm{Pr}_R + (\omega_m\mathrm{Div}_\Lambda \cap \mathrm{Div}_R) = \mathrm{Pr}_R + K_m \\ &= Q_R + (\omega_m\mathrm{Div}_\Lambda \cap \mathrm{Div}_R) = Q_R + K_m \\ &= (\mathrm{Pr}_\Lambda + \omega_m\mathrm{Div}_\Lambda) \cap \mathrm{Div}_R,\end{aligned}$$

which gives the desired result. □

The following checklist is for ease of verifying the lattice properties depicted in Figure 5.6.

Table 5.1: Checklist for Figure 5.6

Columns 1 and 2
Containments: $\mathrm{Pr}_\Lambda \subseteq \omega_m \mathrm{Div}_\Lambda + \mathrm{Pr}_\Lambda \subseteq M_\Lambda \subseteq \mathrm{Div}_\Lambda$ *Reason:* Clear. $\mathrm{Pr}_R \subseteq Q_R \subseteq K_m + Q_R \subseteq M_R \subseteq \mathrm{Div}_R$ *Reason:* By Claim 2(4) $K_m \subseteq M_R$, so all containments are clear.
Intersections: $\mathrm{Div}_R \subseteq \mathrm{Div}_\Lambda$ *Reason:* Section 5.3.2. $\mathrm{Div}_R \cap M_\Lambda = M_R$ *Reason:* By definition. $(\omega_m \mathrm{Div}_\Lambda + \mathrm{Pr}_\Lambda) \cap \mathrm{Div}_R = K_m + Q_R$ *Reason:* Claim 4. $\mathrm{Pr}_\Lambda \cap \mathrm{Div}_R = Q_R$ *Reason:* By definition.
Joins: $M_\Lambda + \mathrm{Div}_R = \mathrm{Div}_\Lambda$ *Reason:* Claim 2(2). $(\omega_m \mathrm{Div}_\Lambda + \mathrm{Pr}_\Lambda) + M_R = M_\Lambda$ *Reason:* By Claim 2(2). $\mathrm{Pr}_\Lambda + (K_m + Q_R) = \omega_m \mathrm{Div}_\Lambda + \mathrm{Pr}_\Lambda$ *Reason:* Not actually needed, so left as an exercise.

Columns 2 and 3
$\mathrm{Pr}_{R_m} \subseteq M_{R_m} \subseteq \mathrm{Div}_{R_m}$
Reason: This is clear since $M_{R_m} = \mathrm{Div}^0_{\mathbb{Z}_p}(X_m)$.
$\pi_m : \mathrm{Div}_R \to \mathrm{Div}_{R_m}$ is well-defined and surjective
Reason: By definition of π_m.
$\pi_m(M_R) = M_{R_m}$ and $M_R = \pi_m^{-1}(M_{R_m})$
Reason: Claim 2(5); and its proof gives the second assertion.
$\pi_m(K_m + Q_R) = \mathrm{Pr}_{R_m}$ and $\pi_m^{-1}(\mathrm{Pr}_{R_m}) = K_m + Q_R$
Reason: Follows from Claims 3 and 4 because $\pi_m(\mathrm{Pr}_R) = \mathrm{Pr}_{R_m}$ and $K_m = \ker \pi_m$.

The following lattice and map diagram summarizes Claims 1-4.

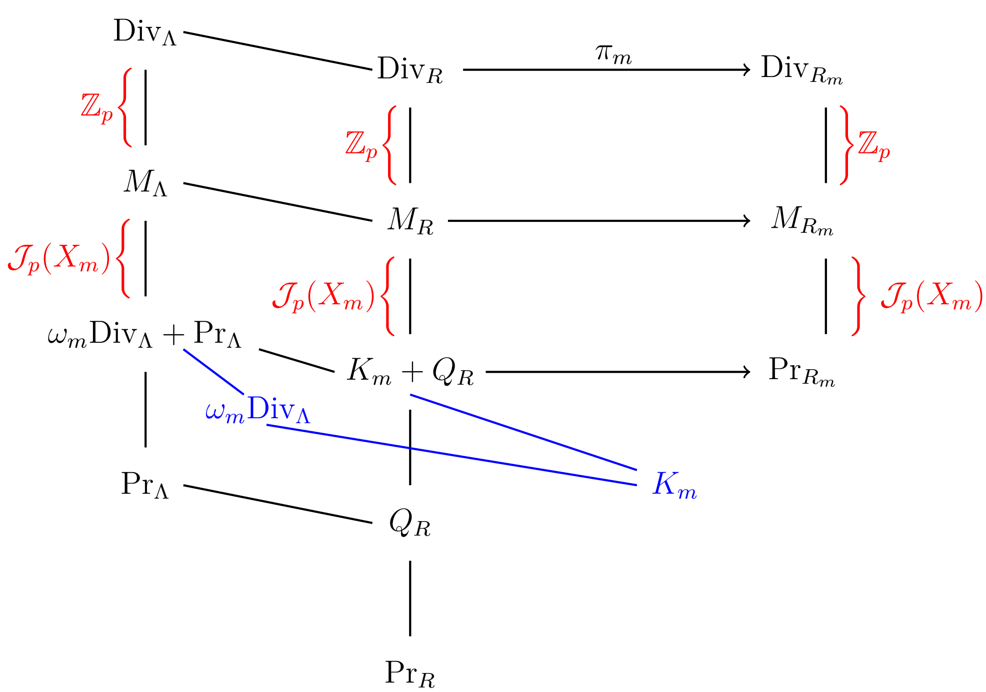

Figure 5.6: Lattice and map diagram showing Claims 1-4

For each subgroup A of Div_Λ let $\widetilde{A}$ denote the image of A under the natural projection map

$$\sim : \mathrm{Div}_\Lambda \longrightarrow \mathrm{Div}_\Lambda/\mathrm{Pr}_\Lambda$$

(which is both a Λ- and an R-module homomorphism). Since $\sim$ is a Λ-module homomorphism, the image of $\omega_m \mathrm{Div}_\Lambda + \mathrm{Pr}_\Lambda$ under it is $\omega_m \mathrm{Pic}_\Lambda$. Since Div_R is an R-submodule of Div_Λ, we may apply $\sim$ to it as well, and to its submodules.

By the Diamond Isomorphism Theorem, since we've checked all the appropriate intersections from column 1 to column 2 in Figure 5.6, this natural projection, by inspection, gives the first two columns in Figure 5.7 as well as all intersections (depicted, as usual, by horizontal lines) between their subgroups in column 2. To get the third column of Figure 5.7, factor the third column of Figure 5.6 by Pr_{R_m}. The horizontal lines—which are homomorphisms—relating column 2 to column 3 in Figure 5.7 are obtained by taking images of the subgroups in column 2 under $\overline{\pi}_m$. By Claim 4, $\overline{\pi}_m$ is a well-defined group homomorphism from the second column of Figure 5.7 to its third column. By simple inspection of the claims, all the horizontal group homomorphisms from column 2 to their images in column 3 of Figure 5.7 are valid too. Note that there are no direct "horizontal line" relationships from column 1 to column 3, and so nothing to check in that regard. By the Lattice Isomorphism Theorem, the (already established) quotient groups (in red) are consequently also preserved when passing between columns (thus also transitively from column 1 to column 3). We only need these to be abelian group isomorphisms; but they are, in fact, R- and Λ-module isomorphisms.

$$
\begin{tikzcd}
\mathrm{Pic}_\Lambda \arrow[dr, dash] \arrow[dd, dash, "\mathbb{Z}_p"'] & & \\
& \widetilde{\mathrm{Div}}_R \arrow[r, "\overline{\pi}_m"] \arrow[dd, dash, "\mathbb{Z}_p"'] & \mathrm{Pic}_{R_m} \arrow[dd, dash, "\mathbb{Z}_p"] \\
N_\Lambda \arrow[dr, dash] \arrow[dd, dash, "\mathcal{J}_p(X_m)"'] & & \\
& \widetilde{M}_R \arrow[r] \arrow[dd, dash, "\mathcal{J}_p(X_m)"'] & \mathcal{J}_p(X_m) \arrow[dd, dash, "\mathcal{J}_p(X_m)"] \\
\omega_m \mathrm{Pic}_\Lambda \arrow[dr, dash] \arrow[dd, dash] & & \\
& \widetilde{K}_m \arrow[r] \arrow[dd, dash] & \overline{0} = \overline{\mathrm{Pr}}_{R_m} \\
0 \arrow[dr, dash] & & \\
& \widetilde{0} &
\end{tikzcd}
$$

Figure 5.7: The natural projection homomorphism from Div_Λ to Pic_Λ indicated in the first two columns and the passage from π_m to $\overline{\pi}_m$ indicated in the third column

We are now in a position to directly apply Theorem 32, with $P = \mathrm{Pic}_\Lambda$ and $N = N_\Lambda = M_\Lambda/\mathrm{Pr}_\Lambda$ used as P and M in its hypotheses. Since X_m is connected and $\omega_m \mathrm{Pic}_\Lambda \subseteq N_\Lambda$ for all m by (5.7), we have

$$|N_\Lambda/\omega_m \mathrm{Pic}_\Lambda| = |\mathcal{J}_p(X_m)| < \infty.$$

It is here that we need the crucial hypothesis that all X_m are connected, so the Jacobians are finite (the rest of the arguments up to this point have not used this fact!). This leads immediately to the conclusion of Theorem 27.

Corollary 20. *Under the hypothesis and notation of Theorem 32, the ranks of $\mathcal{J}_p(X_m)$ are bounded as $m \to \infty$ if and only if p does not divide Θ_{p^∞} in Λ (or in $\mathbb{Z}_p[\Gamma]$).*

Proof. By definition, Pic_Λ is the cokernel of the voltage Laplacian, $\mathcal{L}_{p^\infty} : \mathrm{Div}_\Lambda \to \mathrm{Div}_\Lambda$, where $\Theta_{p^\infty} = \det \mathcal{L}_{p^\infty}$. In the notation of Theorem 30, let p^μ be the product of the p^{k_i}. Then the characteristic polynomial, as in Definition 36, is equal to

$$p^{\mu} \prod_{j=1}^{t} g_j^{m_j} = \mathrm{Char}(\mathrm{Pic}_{\Lambda}). \tag{5.7}$$

Let $M = \omega_{m_0}\mathrm{Pic}_{\Lambda}$ where $m_0 \geq 0$ is fixed. Then since Pic_{Λ}/M has finite p-rank (fixed, independent of $m \to \infty$), the ranks of Pic_{Λ} and M differ by a constant, and one is bounded as $m \to \infty$ if and only if the other is bounded.

We now compare μ invariants for Pic_{Λ} and M, as follows. By Proposition 15(3), we have

$$\mathrm{Char}(M) = \frac{\mathrm{Char}(\mathrm{Pic}_{\Lambda})}{\mathrm{Char}(\mathrm{Pic}_{\Lambda}/M)}. \tag{5.8}$$

The Λ-module Pic_{Λ}/M is a quotient of the module $\mathrm{Div}_{\Lambda}/(\omega_{m_0}\mathrm{Div}_{\Lambda})$; and as in Claim 1,

$$\mathrm{Div}_{\Lambda}/(\omega_{m_0}\mathrm{Div}_{\Lambda}) \cong \underbrace{(\Lambda/\omega_{m_0}\Lambda) \oplus \cdots \oplus (\Lambda/\omega_{m_0}\Lambda)}_{n \text{ of these}}.$$

But by Lemma 3 we know ω_{m_0} maps to a distinguished polynomial in $\mathbb{Z}_p[[T]] \cong \Lambda$, so $(\Lambda/(\omega_{m_0}))^n$ is already in Iwasawa decomposition form, and it clearly has characteristic polynomial $\omega_{m_0}^n$ (again, under the identification $\gamma \mapsto T+1$). Once more useage of Proposition 15(3) gives that

$$\mathrm{Char}(\mathrm{Pic}_{\Lambda}/M) \,\Big|\, \omega_{m_0}^n,$$

so $\mathrm{Char}(\mathrm{Pic}_{\Lambda}/M)$ is relatively prime to p (since the distinguished polynomial ω_{m_0} is).

By (5.8), this shows

$$p \,\Big|\, \mathrm{Char}(\mathrm{Pic}_\Lambda) \quad \Longleftrightarrow \quad p \,\Big|\, \mathrm{Char}(M).$$

If $\mu(\mathrm{Pic}_\Lambda) = 0$, then $\mathrm{Char}(\mathrm{Pic}_\Lambda) = \Theta_{p^\infty}$ by Proposition 10.23 in [KKS12]. In this case, p does not divide Θ_{p^∞} by definition of $\mathrm{Char}(\mathrm{Pic}_\Lambda)$. By Lemma 4.32 in [Bro], we have that the ranks of the finite Λ-module quotients of a finitely generated torsion Λ-module stay bounded if and only if the μ invariant of the Iwasawa decomposition is zero. So if the ranks of $\mathcal{J}_p(X_m)$ stay bounded as $m \to \infty$, then p does not divide Θ_{p^∞}.

Conversely, we show that if the ranks of $\mathcal{J}_p(X_m)$ don't stay bounded as $m \to \infty$, then p does divide Θ_{p^∞} in Λ. So if the rank of $\mathcal{J}_p(X_m) \to \infty$ as $m \to \infty$, then the μ-invariant of the submodule M, and hence also of Pic_Λ, must be nonzero. i.e., the Iwasawa decomposition of Pic_Λ in (5.8) must have at least one factor of the form $\Lambda/(p^a)$, for some $a \geq 1$. This forces p to divide Θ_{p^∞} as follows: If p did not divide Θ_{p^∞}, then Θ_{p^∞} would be relatively prime to p in the UFD Λ; but then Θ_{p^∞} would not annihilate the Iwasawa factor $\Lambda/(p^a)$ of Pic_Λ. This would contradict Theorem 9 that says that Θ_{p^∞} annihilates Pic_Λ (hence every submodule and quotient module of Pic_Λ too—hence Θ_{p^∞} annihilates anything pseudo-isomorphic to Pic_Λ too). Thus, p must divide Θ_{p^∞}. □

5.5 Example: Single Voltage p-Towers over K_n

The purpose of this section is to put Corollary 18 into the context of the preceding "Iwasawa Theory" for graph Jacobians. In the special case of $X_0 = K_n$, the complete graph on n vertices, we can furthermore exactly determine the "Iwasawa invariant factors" of the p-adic Jacobian, for every prime p (although for convenience, we only treat $p \geq 5$).

First fix $n \geq 4$ and $p \geq 5$ (so we can just quote Theorem 14; if $p = 2$ or 3, Theorem 15 is needed). Let $X_0 = K_n$, the complete graph on n vertices. By Theorem 30, we have that since N_Λ is finitely generated as a Λ-module

$$N_\Lambda \sim \Lambda^r \oplus (\bigoplus_{i=1}^{s} \Lambda/(p^{k_i})) \oplus (\bigoplus_{j=1}^{t} \Lambda/(g_j(T)^{m_j})),$$

where $\mu = \sum k_i$ and $\lambda = \sum m_j \deg g_j$. First note that $r = 0$ since N_Λ is torsion. By Corollary 18, we know that μ is equal to the exact exponent for the p-power of K, the reduced Stickelberger element coefficient, and that $\lambda = 1$. Since $\lambda = 1$ we must have $t = 1, m_1 = 1$ and $\deg(g_1) = 1$, i.e.

$$g_1(T) = g(T) = T + a_0,$$

where $p \mid a_0$ (since $g(T)$ is distinguished). Thus, we have

$$N_\Lambda \sim \bigoplus_{i=1}^{s} \Lambda/(p^{k_i}) \oplus \Lambda/(T + a_0).$$

According to Theorem 14, we have the exact abelian group structure of the Jacobian of all (finite) single voltage covers of K_n $\forall n$. Each $\mathcal{J}(X_m)$ has three distinct invariant factors:

(i) n with multiplicity $(n-4)d+2$,

(ii) $n(n-2)$ with multiplicity $d-2$ and

(iii) $d \cdot n(n-2)$ with multiplicity 1

We describe the abelian group structure of N_Λ in each of the following cases.
Case 0) $p \nmid n(n-2)$.

Since p does not divide n or $(n-2)$, it follows that $\mu = \beta = \nu = 0$, where μ, β and ν are as in Corollary 18, and the order of the Sylow p-subgroup of $\mathcal{J}(X_m)$, denoted by $\mathcal{J}_p(X_m)$, is

$$|\mathcal{J}_p(X_m)| = p^{e_m}, \qquad \text{where } e_m = m.$$

However, by Theorem 14, we have the exact Invariant Factor decomposition, which gives us the distinct elementary divisor, and so

$$\mathcal{J}_p(X_m) \cong Z_{p^m}$$

which has multiplicity 1. Now if we take inverse limits of these abelian groups as $m \to \infty$, we see by Theorem 30 that the inverse limit is

$$N_\Lambda \cong \Lambda/(g(T))$$

where

$$\Lambda/g(T)) \cong \mathbb{Z}_p.$$

Case 1) $p|n$, so $p \nmid (n-2)$ since $p > 3$.

Let p^a be the exact power of p dividing n. Since $p \nmid (n-2)$, we have that $\mu = a(n-3)$ and $\beta = a(n-2)$ where β and μ are as above. Thus, it follows that $\nu = \beta - \mu = a$. So then for all $m \geq 0$, we have that the order of the Sylow p-subgroup of $\mathcal{J}(X_m)$ is

$$|\mathcal{J}_p(X_m)| = p^{e_m} \qquad \text{where } e_m = a(n-3)p^m + m + a.$$

However, by Theorem 14, we have the exact Invariant Factor decomposition, which gives us the distinct elementary divisors, and so

$$\mathcal{J}_p(X_m) \cong (Z_{p^a})^{N_m} \times Z_{p^{m+a}}, \qquad \text{where } N_m = (n-4)p^m + 2 + p^m - 2 = (n-3)p^m$$

and N_m gives us the multiplicity. The second elementary divisor has multiplicity equal to 1. Note that these elementary divisors are distinct since $m + a > a$ for all $m \geq 1$.

Now if we take inverse limits of these abelian groups as $m \to \infty$, we see by Theorem 30 that the inverse limit is

$$N_\Lambda \cong (\Lambda/(p^a))^{(n-3)p^m} \times \Lambda/(g(T))$$

where $(n-3)p^m$ denotes the multiplicity of the factor, and by the same reasoning as above, we have that $\Lambda/(g(T)) \cong \mathbb{Z}_p$.

Case 2) $p|(n-2)$, so $p \nmid n$.

Let p^b be the exact power of p dividing $n-2$. Since $p \nmid n$, we have that $\mu = b$ and $\beta = 0$. Thus, it follows that $\nu = \alpha - \mu = -b$. So then for all $m \geq 0$, we have that the order of the Sylow p-subgroup of $\mathcal{J}(X_m)$ is

$$|\mathcal{J}_p(X_m)| = p^{e_m}$$

by Corollary 18 where $e_m = bp^m + m - b$. However, by Theorem 14, we have the exact Invariant Factor decomposition, which gives us the distinct elementary divisors, and so

$$\mathcal{J}_p(X_m) \cong (Z_{p^b})^{N_m} \times Z_{p^{m+b}}$$

where $N_m = p^m - 2$ gives the multiplicity. Note that these elementary divisors are distinct since $m + b > b$ for all $m \geq 1$.

Now if we take inverse limits of these abelian groups as $m \to \infty$, we see by Theorem 30 that the inverse limit is

$$N_\Lambda \cong (\Lambda/(p^b))^{p^m-2} \times \Lambda/(g(T))$$

where $p^m - 2$ denotes the multiplicity of the factor, and by the same reasoning as above, we have that $\Lambda/(g(T)) \cong \mathbb{Z}_p$.

Example 8. We let $p = 5$ and consider when $n = 4, 5$ and 7, respectively (for $m = 1, 2, 3, 4, 5$).

Table 5.2 Jacobian of K_4 with Single Voltage Cover by $\mathbb{Z}/5^m\mathbb{Z}$,

m	The Jacobian
1	$(Z_4)^2 \times (Z_8)^3 \times Z_8 \times Z_5$
2	$(Z_4)^2 \times (Z_8)^{23} \times Z_8 \times Z_{5^2}$
3	$(Z_4)^2 \times (Z_8)^{123} \times Z_8 \times Z_{5^3}$
4	$(Z_4)^2 \times (Z_8)^{623} \times Z_8 \times Z_{5^4}$
5	$(Z_4)^2 \times (Z_8)^{3123} \times Z_8 \times Z_{5^5}$

Table 5.3 Jacobian of K_5 with Single Voltage Cover by $\mathbb{Z}/5^m\mathbb{Z}$,

m	The Jacobian
1	$(Z_5)^7 \times (Z_{15})^3 \times Z_3 \times Z_{5^2}$
2	$(Z_5)^{27} \times (Z_{15})^{23} \times Z_3 \times Z_{5^3}$
3	$(Z_5)^{127} \times (Z_{15})^{123} \times Z_3 \times Z_{5^4}$
4	$(Z_5)^{627} \times (Z_{15})^{623} \times Z_3 \times Z_{5^5}$
5	$(Z_5)^{3127} \times (Z_{15})^{3123} \times Z_3 \times Z_{5^6}$

Table 5.4 Jacobian of K_7 with Single Voltage Cover by $\mathbb{Z}/5^m\mathbb{Z}$,

m	The Jacobian
1	$(Z_7)^{17} \times (Z_{35})^3 \times Z_7 \times Z_{5^2}$
2	$(Z_7)^{77} \times (Z_{35})^{23} \times Z_7 \times Z_{5^3}$
3	$(Z_7)^{377} \times (Z_{35})^{123} \times Z_7 \times Z_{5^4}$
4	$(Z_7)^{1877} \times (Z_{35})^{623} \times Z_7 \times Z_{5^5}$
5	$(Z_7)^{9377} \times (Z_{35})^{3123} \times Z_7 \times Z_{5^6}$

By Theorem 12, the reduced Stickelberger element for X_{p^∞} is $\Theta_{p^\infty} = -(n-2)n^{n-3}(\gamma-1)^2\gamma^{-1}$ (because we may treat τ as a polynomial indeterminate in this matrix calculation). In the notation of Corollary 18, we have (ignoring the minus sign)

$$K = (n-2)n^{n-3}.$$

Note that K and $\gamma - 1$ are relatively prime in the UFD, $\Lambda = \mathbb{Z}_p[[T]]$. Also, under the above isomorphism,

$$\gamma - 1 \leftrightarrow T.$$

Moreover, $K = p^\mu \cdot u$, where u is a unit in $\mathbb{Z}_p$. Thus, we see that the $g(T)$ Iwasawa λ-invariant factor must divide T, hence $a_0 = 0$ and $g(T) = T$.

Table 5.5: List of Notation for Chapter 5

1	p is a fixed prime and $\mathbb{Z}_p$ is the additive group of p-adic integers (also viewed as a ring according to context); a topological generator for $\mathbb{Z}_p$ may be taken to be 1
2	Γ is the topological group isomorphic to $\mathbb{Z}_p$ written multiplicatively and γ is a fixed topological generator for Γ (where we may let γ map to 1 under the isomorphism of Γ with $\mathbb{Z}_p$)
3	X is a connected graph with $r_X - 1 \neq 0; V(X) = \{v_1, v_2, ..., v_n\}$
4	$X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots$ is a cyclic voltage p-tower with X_m connected for all m, as in Definition 29
5	X_{p^∞} is the infinite derived graph obtained from the voltage graph $(X, \mathbb{Z}_p, \alpha)$, where the voltage group is the additive p-adic integers and the voltage assignment α is determined by the cyclic voltage p-tower in point (4); We call X_{p^∞} the *completion of the tower*
6	Γ acts as automorphisms of X_{p^∞} and acts on the fibers of the covering map $X_{p^\infty} \to X$ in a way that is permutation isomorphic to the regular representation (transitive with the stabilizer of a point equal to the identity)
7	$\pi_m : X_{p^\infty} \to X_m \; \forall m$ is the reduction map (so X_m is obtained by reducing the elements of X_{p^∞} modulo p^m by $v_{i,g} \mapsto v_{i,\overline{g}}$, where $\overline{g} \in \mathbb{Z}_p/p^m\mathbb{Z}_p$)
8	$\Gamma_m = \Gamma/\Gamma^{p^m}$, a cyclic group of order p^m; and $\Gamma_m \cong Gal(X_m/X_0)$

9	For any intermediate graph X_m defined in point (4), $\mathrm{Div}_{\mathbb{Z}}(X_m)$ is a free $\mathbb{Z}[\Gamma_m]$-module of rank n (and a free $\mathbb{Z}$-module of rank $n \cdot p^m$); by extension of scalars of free modules, $\mathrm{Div}_{\mathbb{Z}_p}(X_m) \cong \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathrm{Div}_{\mathbb{Z}}(X_m)$ is a free $\mathbb{Z}_p[\Gamma_m]$-module of rank n
10	$\mathcal{J}(X_m)$ is the usual Jacobian of X_m
11	$\mathcal{J}_p(X_m)$ is the Sylow p-subgroup of $\mathcal{J}(X_m)$, i.e., $\mathcal{J}_p(X_m) = \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{J}(X_m)$, which is a $\mathbb{Z}_p$-module
12	$R = \mathbb{Z}_p[\Gamma]$ is the usual group algebra of Γ over $\mathbb{Z}_p$
13	$\mathrm{Div}_R = \mathrm{Div}_{\mathbb{Z}_p}(X_{p^\infty})$ is a free R-module of rank n (and a free $\mathbb{Z}_p$-module of infinite rank)
14	$R_m \cong \mathbb{Z}_p[\Gamma_m]$
15	From point (9) and point (14), we have $\mathrm{Div}_{R_m} = \mathrm{Div}_{\mathbb{Z}_p}(X_m)$ is a free R_m-module of rank n (and a free $\mathbb{Z}_p$-module on the vertices of X_m)
16	With respect to the $\mathbb{Z}_p$-basis in point (15), $\mathrm{Div}^0_{\mathbb{Z}_p}(X_m)$ is the usual divisors of degree zero; we let $M_{R_m} = \mathrm{Div}^0_{\mathbb{Z}_p}(X_m)$
17	$\mathcal{L}_{p^\infty} : \mathrm{Div}_R \to \mathrm{Div}_R$ is the voltage p-Laplacian defined on the R-basis of Div_R, where $\mathcal{L}_{p^\infty}(v_{i,0}) = p_{i,0}$ $1 \leq i \leq n$
18	Pr_R is the image of $\mathcal{L}_{p^\infty}$
19	Pic_R is the cokernel of $\mathcal{L}_{p^\infty}$; it can also be defined as the quotient $\mathrm{Div}_R/\mathrm{Pr}_R$
20	$\Lambda = \mathbb{Z}_p[[\Gamma]]$ is the completion of $\mathbb{Z}_p[\Gamma]$, under the profinite topology defined by the open subgroups Γ^{p^m}
21	the free R-module Div_R becomes a Λ-module by extension of scalars; so Div_Λ is the corresponding module $\Lambda \otimes_R \mathrm{Div}_R$

22	$\widehat{\mathcal{L}}_{p^\infty} : \mathrm{Div}_\Lambda \to \mathrm{Div}_\Lambda$ is the voltage p-Laplacian defined on the R-basis of Div_R (but here viewed as a Λ-module basis) where $\widehat{\mathcal{L}}_{p^\infty}(v_{i,0}) = p_{i,0}$ $1 \leq i \leq n$
23	Pr_Λ is the image of $\widehat{\mathcal{L}}_{p^\infty}$
24	Pic_Λ is the cokernel of $\widehat{\mathcal{L}}_{p^\infty}$; it can also be defined by the quotient $\mathrm{Div}_\Lambda/\mathrm{Pr}_\Lambda = \mathrm{Pic}_\Lambda$
25	Pr_{R_m} is the image of the Laplacian endomorphism of Div_{R_m}, denoted by $\mathcal{L}_m$ (here we're using the same map $\mathcal{L}_{p^\infty}$ as above, but instead reading the vertices $v_{i,0}$ as lying in Div_{R_m} and extending by R_m-linearity)
26	$\mathrm{Pic}_{R_m} = \mathrm{Div}_{R_m}/\mathrm{Pr}_{R_m}$ is the cokernel of $\mathcal{L}_m$
27	M_Λ is the Λ-submodule of Div_Λ generated by S_1, S_2 and $(\gamma - 1)\mathrm{Div}_\Lambda$ where $S_1 = \{v_{i,0} - v_{j,0} \mid 1 \leq j < i \leq n\}$ and $S_2 = \{p_{i,0} \mid 1 \leq i \leq n\}$
28	$M_R = \mathrm{Div}_R \cap M_\Lambda$
29	$N_\Lambda = M_\Lambda/\mathrm{Pr}_\Lambda$ is a submodule of Pic_Λ
30	$\Theta_{p^\infty} = \det \widehat{\mathcal{L}}_{p^\infty} = \det \mathcal{L}_{p^\infty}$ is the reduced p-Stickelberger element for X_{p^∞}/X; it is as an element of $\mathbb{Z}_p[\Gamma]$
31	$\omega_m = \gamma^{p^m} - 1 = (\gamma - 1)\eta_m$, where $\eta_m = 1 + \gamma + \gamma^2 + \cdots + \gamma^{p^m - 1}$, where γ as above is a fixed topological generator of Γ
32	For D any Λ-module and B be any subset of D, we define $\Omega_m^D(B) = B \cap \omega_m D$
33	For R_m as in point (15), define $R_m = R/\Omega_m^\Lambda(R) = R/R \cap \omega_m\Lambda$

Chapter 6

Conclusion

In retrospect, we see that the development of this dissertation rests on a number of themes—the Picard group, the Jacobian group, the (voltage) Laplacian, the Smith Normal Form, and the reduced Stickelberger element. We survey these ideas in Section 6.1. Then in Section 6.2, we present various pathways for future work.

6.1 An Encapsulation

The Picard and Jacobian groups of a graph X are invariants that may be defined in terms of the Laplacian matrix of X. In particular, the Picard group is the cokernel of the Laplacian L, considered as a $\mathbb{Z}$-module endomorphism of the $\mathbb{Z}$-module of divisors of X, $\mathrm{Div}_{\mathbb{Z}}(X)$, which is free over $\mathbb{Z}$ of rank $n = |V(X)|$. Then for the voltage graph with voltage group G and derived graph Y, the $\mathbb{Z}$-module of divisors of Y becomes a free module of the same rank n over the larger ring $\mathbb{Z}[G]$. The voltage adjacency matrix captures the "voltage adjacencies" across the different sheets of the covering of X by Y; and these sheets are acted on by G as the regular representation. The voltage Laplacian $\mathcal{L}$ is then an endomorphism of the free $\mathbb{Z}[G]$-module $\mathrm{Div}_{\mathbb{Z}}(Y) = \mathrm{Div}_{\mathbb{Z}[G]}(X)$, which agrees with the ordinary Laplacian endomorphism for Y; it can be represented by an $n \times n$ matrix with entries in $\mathbb{Z}[G]$, or an $n|\,G\,| \times n|\,G\,|$ matrix of integers—the ordinary Laplacian matrix for Y—by forming $\mathcal{L} \otimes \rho$, where ρ is a matrix representation for the regular representation of G. This is all described in Chapter 2.

For a connected graph X, the Jacobian of X is the torsion subgroup of the $\mathbb{Z}$-module $\mathrm{Pic}_{\mathbb{Z}}(X)$. The Jacobian is a finite abelian group whose decomposition into invariant factor cyclic subgroups is its *Smith Normal Form*; and the product of these invariant factors, which is the determinant of the reduced Laplacian matrix, is the order of the Jacobian of X, the important *tree number* of the graph X. Likewise, for Y the derived graph as above, the Picard group $\mathrm{Pic}_{\mathbb{Z}}(Y)$ may also be viewed as the cokernel of the voltage Laplacian endomorphism of the $\mathbb{Z}[G]$-module $\mathrm{Div}_{\mathbb{Z}[G]}(X)$; and when G is abelian, the determinant of the voltage Laplacian—called (by us) the *reduced Stickelberger element*—also annihilates the Picard group of Y (as a $\mathbb{Z}$- or a $\mathbb{Z}[G]$-module). So the reduced Stickelberger element—which is an element of the group ring $\mathbb{Z}[G]$ rather than just an integer—plays a role closely analogous to the tree number for ordinary graphs X (again, assuming G is abelian). Moreover, we saw in Chapter 4 that this reduced Stickelberger element may also be interpreted as an "equivariant L-function" whose values at various complex roots of unity can be used to compute the order of the Jacobian of Y explicitly in certain cases. This is the "Fourier analysis" section of the dissertation.

In Chapter 5 we take this development a step further: For a fixed prime p, given an infinite cyclic p-tower of derived graphs over the base graph X, there is a natural p-adic completion of this tower to an infinite (uncountable) graph X_{p^∞} that can nonetheless be described by an $n \times n$ voltage adjacency (or Laplacian) matrix, but with voltages from the p-adic group Γ. In this setting, the voltage modules of divisors and principal divisors as well as the Picard group can all be extended to become modules over the group ring $\mathbb{Z}_p[\Gamma]$. We may then extend scalars to obtain corresponding modules over the completed group ring, the Iwasawa algebra Λ. The advantage that accrues is that the Picard group of X_{p^∞}, when extended to the Λ-module Pic_Λ, is still the cokernel of the (same matrix) voltage Laplacian endomorphism, but is now a finitely generated, torsion Λ-module. Classical Iwasawa theory then gives an *Iwasawa Decomposition* of this module into its "Iwasawa invariant factors" in a way completely analogous to how the Smith Normal Form gives a decomposition of $\mathrm{Pic}_{\mathbb{Z}}(X)$. When we "pick off" the subgroup of Pic_Λ that maps to the torsion subgroups of the finite Picard groups, this Iwasawa decomposition "descends" to give powerful information about the Jacobians of the graphs in the original tower: their asymptotic orders and ranks.

Again, the reduced Stickelberger element for X_{p^∞} annihilates the Picard groups for the modules over both $\mathbb{Z}_p[\Gamma]$ and Λ, and plays a critical role in the theory. In certain cases the reduced Stickelberger element equals the product of the Iwasawa invariant factors, in the same way as for the tree number above. So this dissertation, taken as a whole, indicates how some of the key invariants of a graph generalize naturally to voltage graphs, where the group action on the derived graph provides analogous invariants that may be exploited to considerable advantage. It also points to many new, possible extensions and applications of these ideas.

6.2 Topics to Consider

Constructing more examples:

In Chapter 2, we considered single and constant voltage assignments. However, conjectures were not made for constant voltage assignments even though some data was gathered (see Section 3.1). Re-visiting this assignment on various graph families may lead to provable conjectures.

Constructing voltage assignments that are "similar" to the single voltage assignment would be valuable, as the single voltage assignment lead to a plethora of results. We may also explore other families of graphs with the single voltage assignment by the cyclic group of order d, such as Paley graphs (where vertices come from a fixed finite field and two vertices are connected by an edge if their difference is a square in the field), transposition graphs, the complete bipartite graph on $m+n$ vertices (we only considered the case when $m=2$), and other various strongly regular graphs.

Then instead of having multiple variables, we may look at single voltage cyclic covers of all known small graphs (graphs with less than or equal to 13 vertices). To obtain further results on the Petersen graph, it may be of value to work over $\mathbb{Z}_{(p)}$, as we did for $K_{n,n}$, for $p \neq 2$ or 5, to put the voltage Laplacian matrix (and then ordinary Laplacian) in diagonal form.

Edge Artin L-Functions and edge zeta-functions:

In Chapter 4, we used results about zeta functions and L-functions from [HMSV19] and [Ter11] to obtain an order formula for the Jacobian of the derived graph Y. Could one use the edge adjacency matrix, edge zeta function, or edge Artin L-function to obtain similar results?

Extending results to multigraphs:

Throughout this dissertation we restricted the base graph X in the voltage graph (X, G, α) to be *simple*. However, the construction of voltage graphs easily generalizes to multigraphs (or graphs with loops). Allowing multiple edges or loops complicates things. However, both [HMSV19] and [Ter11] allow their graphs to have loops and multiple edges. Therefore, it may be possible to extend the results from Chapter 4 to such multigraph bases. This would require a more complicated definition of both the *Artinized adjacency matrix* and the *voltage adjacency matrix*.

Further results when G is abelian:

Many of the tools developed in this dissertation were proved for arbitrary (finite) abelian voltage groups, although we often specialized to cyclic groups, where explicit results were more tractable. For other families of non-cyclic abelian voltage groups—such as elementary abelian p-groups—which methods and theorems, especially in Chapters 4 and 5, lead to new results? In particular, are there families of voltage covers that are "natural" generalizations of single voltage covers? Also, theorems on the asymptotic growth of Jacobians in general abelian p-towers may be obtainable from theorems about finitely generated modules over the multivariable formal power series ring $\mathbb{Z}_p[[T_1, \ldots, T_k]]$. Some results in this vein have already been achieved by Daniel Vallieres in [Val20].

Voltage graphs where G is non-abelian:

We might also consider voltage graphs where the voltage group is non-abelian. In this case, *single voltage assignments* would not result in a connected derived graph. It might be interesting to consider the relationship between different voltage assignments on a particular base graph and how it determines the resulting derived graph and Jacobian.

Regarding Chapter 4, when G is an abelian group, all of the irreducible representations are degree one. Thus, the Artin Ihara L-functions are easier to work with. So if we consider G non-abelian, this would again result in a more complicated definition of the Artinized adjacency matrix. Recall, the reduced Stickelberger element played a significant role in obtaining the order formula. So when the voltage group is non-abelian, how does one define the reduced Stickelberger element?

Resolving the relationship between $\mathcal{J}_p(X_{p^\infty})$ and N_Λ:

Recall from Chapter 5: $\mathcal{J}_\infty = \varprojlim \mathcal{J}(X_m)$ and $\mathcal{J}_p(X_{p^\infty}) = \varprojlim \mathcal{J}_p(X_m)$. Instead of working with $\mathcal{J}_p(X_{p^\infty})$, we used divisors over Λ to construct what we call N_Λ. Thus, we wish to determine whether or not $\mathcal{J}_p(X_{p^\infty})$ actually coincides with the Jacobian that we constructed, N_Λ.

Theorem 27 and Classical Iwasawa Theory (over general number fields and elliptic curves):

Examining whether the ideas in the proof of Theorem 27 provide new insight into classical Iwasawa Theory for extensions of number fields is an intriguing line of investigation. If there is a voltage tower "lurking" in the background in this classical setting, the results and/or methods of this dissertation may be applied to obtain new results in number theory, perhaps pertaining to elliptic curves or algebraic curves. In this way our dissertation, which was inspired by the goal of seeking a graph-theoretic analog to classical Iwasawa Theory could, in turn, show how graph theory may be used to "return the favor" by giving new insights into Number Theory!

Bibliography

[AM69] M. F. Atiyah and I. G. McDonald *Introduction to Commutative Algebra*, Addison-Wesley Publishing Company, 1969.

[Bai03] Hua Bai *On the critical group of the n-cube*, Linear Algebra Appl. 369 (2003), 251?261.

[Bak10] Matthew Baker *The Dollar Game on a Graph (With 9 Surprises!)*, Linear Algebra Appl. 369 (2003), 251?261.

[BBY17] Spencer Backman, Matthew Backer, Chi Ho Yuen *Geometric Bijections for Regular Matroids, Zonotopes, and Ehrhart Theory*, https://arxiv.org/abs/1701.01051, 2017.

[BHN97] Roland Bacher, Pierre de la Harpe, Tatiana Nagnibeda *The lattice of integral flows and the lattice of integral cuts on a finite graph*, http://people.math.gatech.edu/ mbaker/pdf/g4g9.pdf, 2010.

[Big93] Norman Biggs *Algebraic Graph Theory, 2nd Ed.*, Cambridge University Press, 1993.

[Big99] Norman Biggs *Chip-firing and the critical group of a graph,* Algebraic Combin. 9 (1999), no. 1, 25-45.

[Big06] Norman Biggs *The Critical group from a cryptographic perspective,* http://www.cdam.lse.ac.uk/Reports/Files/cdam-2006-07.pdf, 2006.

[BN07] Matthew Baker and Serguei Norine *Riemann-Roch and Abel-Jacobi theory on a finite graph*, Adv. Math. 215 (2007), no. 2, 766?788.

[Bro] Jim Brown *An Introduction to Iwasawa Theory.*, http://www.math.caltech.edu/ jimlb/iwasawa.pdf.

[CP18] Scott Corry and David Perkinson *Divisors and Sandpiles*, American Mathematical Society, 2018.

[CR02] H. Christianson and V. Reinter *The Critical Group of a Threshold Graph*, Linear Algebra Appl. 394 (2002), 233-244.

[CSX14] David B. Chandler, Peter Sin, and Qing Xiang *The Smith and Critical Groups of Paley Graphs*, https://arxiv.org/pdf/1401.8260.pdf, 2014.

[DDE+19] Joshua Ducey, David Duncan, Wesley Engelbrecht, Jawahar Madan, Eric Piato, Christina Shatford, and Angela Vichitbandha *Critical Group Structure from the Parameters of a Strongly Regular Graph*, https://arxiv.org/pdf/1910.07686.pdf, 2019.

[DF04] D. Dummit and R. Foote *Abstract Algebra, 3rd ed.*, Wiley, 2004.

[Dha90] Deepak Dhar *Self-organized critical state of sandpile automaton models*, Phys. Rev. Lett. 64 (1990), no. 14, 1613-1616.

[DHS18] Joshua Ducey, Ian Hill, and Peter Sin *The Critical Group of the Kneser Graph on 2-subsets of an n-Element Set.*, https://arxiv.org/pdf/1707.09115.pdf, 2018.

[DJ13] Joshua E. Ducey and Deelan M. Jalil *Integer invariants of abelian Cayley graphs*, https://arxiv.org/abs/1308.2335, 2013.

[DKM12] Art M. Duval, Caroline J. Kilvans, and Jeremy L. Martin *Cuts and Flows of Cell Complexes*, http://www.dam.brown.edu/people/cklivans/cutflow.pdf, 2012.

[DZZ19] Christopher Dowd, Valerie Zhang, and Sylvester Zhang *Arborescences of Covering Graphs*, http://www-users.math.umn.edu/ reiner/REU/-DowdZhangZhang2019.pdf, 2019.

[FKL04] Rongquan Feng, Jin Ho Kwak and Jaeun Lee *Characteristic Polynomials of Graph Coverings*, Bill. Austral. Math. Soc. Vol. 69 (2004) [133-136].

[For20] Bence Forras *Iwasawa Theory*, https://bforras.eu/docs/Forras “underscore” Iwasawa “underscore” Theory ”underscore” blue.pdf, 2020.

[Ger76] Larry J Gerstein *A Local Approach to Matrix Equivalence*, Santa Barbara, California, 1976.

[GM13] Darren B. Class and Criel Merino *Critical Groups of Graphs with Dihedral Actions*, https://arxiv.org/pdf/1304.6011.pdf, 2013.

[GT87] J. Gross and T.Tucker *Topological Graph Theory, 1st Ed.*, Wiley, 1987.

[HMSV19] K. Hammer, T. Mattman, J. Sands and D. Valliers *Special Value $u = 1$ of Artin-Ihara l-functions*, https://arxiv.org/abs/1907.04910, 2019.

[Hop14] Tessa Hoppenfeld *The Structure of the Jacobian Group of a Graph*, http://people.reed.edu/ davidp/homepage/students/hoppenfeld.pdf, 2014.

[Jac03] Brian Jacobson *Critical Groups of Graphs*, http://www-users.math.umn.edu/ reiner/HonorsTheses/Jacobson "underscore" thesis.pdf, 2003.

[Jen16] David Jensen *Tropical Brill-Noether Theory*, https://web.ma.utexas.edu/users/sampayne/pdf/Math665Lecture1.pdf, 2016.

[JNR03] Brian Jacobson, Andrew Niedermaier, and Victor Reiner *Critical groups for complete multipartite graphs and Cartesian products of complete graphs*, https://onlinelibrary.wiley.com/doi/abs/10.1002/jgt.10139, 2003.

[Kli18] Caroline J. Klivans *The Mathematics of Chip-Firing*, CRC Press, 2018.

[KMM17] Y.S. Kwon, A.D. Mednykh, I.A. Mednykh, *On Jacobian group and complexity of the generalized Petersen graph $GP(n, k)$ through Chebyshev polynomials*, https://arxiv.org/abs/1612.03372, 2016.

[KKS12] Nobushige Kurokawa, Masato Kurihara and Takeshi Saito *Number Theory 3. Translations of Mathematical Monographs: Iwanami Series in Modern Mathematics 242*, Providence, Rhode Island: American Mathematical Society, 2012. isbn: 978-0-8218- 1355-3.

[Lor08] Dino Lorenzini *Smith Normal Form and Laplacians*, https://www.sciencedirect.com/science/article/pii/S0095895608000282, 2008.

[LZ21] Yoav Len and Dmitry Zakharov *Kirchhoff's Theorem for Prym Varieties*, https://arxiv.org/pdf/2012.15235.pdf, 2021.

[Mac11] John Machacek *The Critical Group of Line Graph: The Bipartite Case*, http://www-users.math.umn.edu/ reiner/HonorsTheses/Machacek "underscore" thesis.pdf, 2011.

[MS93] Hirobumo Mizuno and Iwao Sato *Characteristic polynomials of some graph coverings*, https://core.ac.uk/download/pdf/82110246.pdf, 1993.

[NW11] Serguei Norine and Peter Whalen *Jacobians of Nearly Complete and Threshold Graphs*, https://doi.org/10.1016/j.ejc.2011.04.003, 2011.

[NW18] Hoi H. Nguyen and Melanie Matchett Wood *Random Integral Matrices: Universality of Surjectivity and the Cokernel*, https://arxiv.org/pdf/1806.00596.pdf, 2018.

[Ouy] Yi Ouyang *Introduction to Iwasawa Theory*, https://www.math.unipd.it/ algant/iwasawa.pdf.

[RR15] Fariba Ranjbar and Saeed Ranjbar *Inverse Galois Problem and Significant Methods*, https://arxiv.org/abs/1512.08708, 2015.

[RT13] Victor Reiner and Dennis Tseng *Critical Groups of Coverings, Voltage and Signed Graphs*, https://arxiv.org/abs/1301.2977, 2013.

[Sha] Romyar Sharifi *Iwasawa Theory*, https://www.math.ucla.edu/ sharifi/iwasawa.pdf.

[Sin16] Peter Sin *The Critical Groups of the Peisert Graphs $P^*(G)$*, https://arxiv.org/pdf/1606.00870.pdf, 2016.

[Ter11] Audrey Terras *Zeta Functions of Graphs, 1st Ed.*, Cambridge University Press, 2011.

[Val20] Daniel Vallieres *On Abelian l-Towers of Multigraphs*, https://arxiv.org/abs/2006.14012, 2020.

[Vol96] Helmut Volklein *Groups as Galois Groups*, Cambridge University Press, 1996.

[Was82] Lawrence Washington *Introduction to Cyclotomic Fields, 1st Ed.*, Springer-Verlag, 1982.

[Wes00] Douglas West *Introduction to Graph Theory, 2nd Ed.*, Prentice Hall, 2000.

[Woo17] Melanie Wood *The Distribution of Sandpile Groups of Random Graphs*, Journal of the American Mathematical Society 30 (2017), pp. 915-958.

Appendix A

Sample SAGE Code

This code is used to compute the adjacency matrix for a derived graph Y. Once we (separately) compute the degree matrix for Y, we use the output of this code to compute the Laplacian. Then putting the Laplacian matrix into Smith Normal Form gives the invariant factors (and their multiplicities) of the Jacobian, as in Table 3.3, Table 3.4, Table 3.5, etc.

Listing A.1: Computing the Adjacency Matrix of a Derived Graph

```
def group_op(S,f,l,l_e):
    """Does the group operation of S on
    the elements l and l_e.

    Parameters
    ----------
    input variable : variable type
        Description of variable

    S : Group
        The group
    f : dict
        This maps the string version
        of group element to the indices
        in S.list(). More specifically
        the keys are the string version
        of group elements and the values
        are the indices.
    l : str
        the initial vertex in the derived graph
    l_e : str
        the edge going from l to l_u

    Returns
    -------
```

```
    output_group_element : same as elements of S
        The output of the group operation on l_e
        and l.

    Examples
    --------
    >>> S[f['(1,2)']]
    (1,2)

    >>> S[f['(1,2)']]*S[f['(2,3)']]
    (1,3,2)

    """
    return S[f[l_e]]*S[f[l]]

def get_out_edges(G,v):
    """Tells you which vertices are adjacent to v in the base graph

    Parameters
    ----------
    input variable : vertex


    G : Base Graph
    v : vertex

    Returns
    -------
    output : tuple
        initial vertex, terminal vertex, and edge label

    Examples
    --------
    >>> get_out_edges(G,'v')
    ('v', 'u', '(1,2)')

    """
    out_edges = []
    for e in G.edges():
        if v == e[0]:
            out_edges.append(e)
    return out_edges

def voltage(G,v,l,S,f):
    """Tells you which edges are adjancent to v_l in the derived graph

    Parameters
    ----------
```

```
    input variable : variable type
        Description of variable

    G : Graph
        Base Graph
    v : str
        intial vertex
    l : str
        the initial vertex in the derived graph
    l_e : str
        the edge going from l to l_u
    f : dict
        This maps the string version
        of group element to the indices
        in S.list(). More specifically
        the keys are the string version
        of group elements and the values
        are the indices.

    l_u: str
       which vertex is adjacent to v_l via the group operation of l on l_e

    Returns
    -------
    output: tuple
        the terminal vertex and the group element

    Examples
    --------
    >>> voltage(G,'v','(1,3)',S,f)
    ('u', (1,2,3))

    >>> voltage(G,'v','(1,2)',S,f)
    ('u', ())

    """
    # e tells us which vertices are adjacent to v in the base graph
    voltages = []
    for e in get_out_edges(G,v):
        u=e[1]
        l_e=e[2]
        l_u=group_op(S,f,l_e,l)
        voltages.append((u,str(l_u)))
    return voltages

def get_derived(G,S):
    """Gives the adjacency matrix for the derived graph

```

```
    Parameters
    ----------
    input variable : variable type
        Description of variable

    G : Graph
        Base Graph
    S : Group
        the group
    f : dict
        This maps the string version
        of group element to the indices
        in S.list(). More specifically
        the keys are the string version
        of group elements and the values
        are the indices.

    Returns
    -------
    output: Matrix
        the adjacency matrix for the derived graph


    """
    f={str(t):i for i,t in enumerate(S.list())}
    derived_vertices=[(x,str(i)) for x in G.vertices() for i in S.list()]
    Adjacency=Matrix.zero(len(G.vertices())*S.order())
    for v in G.vertices():
        for label in S.list():
            if get_out_edges(G,v) is None:
                continue
            row = derived_vertices.index((v,str(label)))
            vert_and_label = voltage(G,v,str(label),S,f)
            for x in vert_and_label:
                col = derived_vertices.index(x)
                Adjacency[row,col]=1
    return Adjacency, derived_vertices

# Example
S=CyclicPermutationGroup(4) #this is the cyclic group of order 4
G=DiGraph()
G.add_edges([('v_1','v_2','(1,2,3,4)')])
G.add_edges([('v_1','v_3','()')])
G.add_edges([('v_1','v_4','()')])
G.add_edges([('v_2','v_3','()')])
G.add_edges([('v_2','v_4','()')])
G.add_edges([('v_3','v_4','()')])
# Base graph is K_4 single voltage assignment by element '(1,2,3,4)'
```

```
A,F=get_derived(G,S)
print A,F # this prints the directed adjacency matrix for the derived graph
A1=A.transpose()
A2=A1+A
print A2 # this print the undirected adj matrix of the derived graph
```

Appendix B

Sample Mathematica Code

This code was used to check the row and column operations in the proof of Theorem 14. It was then used to construct the matrices in Example 6.

Here is the matrix for ρ(1-τ)

$b = \text{ConstantArray}[0, d];$

$b[[d]] = 1;$

$T = \text{Transpose}[\text{ToeplitzMatrix}[b, \text{RotateRight}[\text{Reverse}[b]]]];$

$B = \text{IdentityMatrix}[d] - T;$

$\text{MatrixForm}[B]$

The next command creates matrix (1).

$M = \text{ConstantArray}[0, \{3d, 3d\}];$

$\text{Part}[M, 1;;d, 1;;d] = B;$

$\text{Part}[M, d+1;;2d, d+1;;2d] = B;$

$\text{Part}[M, d+1;;2d, 2d+1;;3d] = n\text{IdentityMatrix}[d];$

$\text{Part}[M, 2d+1;;3d, 1;;d] = n\text{IdentityMatrix}[d];$

$\text{Part}[M, 2d+1;;3d, 2d+1;;3d] = n(2-n)\text{IdentityMatrix}[d];$

$\text{MatrixForm}[M]$

Next do all the row and column operations to produce matrix (2).

$\mathrm{M1} = M;$

$\mathrm{M1}[[1]] = \sum_{i=1}^{d} \mathrm{M1}[[i]];$

$\mathrm{M1}[[d+1]] = \sum_{i=1}^{d} \mathrm{M1}[[d+i]];$

$\mathrm{M2a} = \mathrm{Transpose}[\mathrm{M1}];$

$\mathrm{M2a}[[1]] = \sum_{i=1}^{d} \mathrm{M2a}[[i]];$

$\mathrm{M2a}[[d+1]] = \sum_{i=1}^{d} \mathrm{M2a}[[d+i]];$

$\mathrm{M2} = \mathrm{Transpose}[\mathrm{M2a}];$

$\mathrm{For}[i = 3, i < d+1, i{+}{+}, \mathrm{M2}[[i]] = \mathrm{M2}[[i-1]] + \mathrm{M2}[[i]]];$

$\mathrm{For}[i = 3, i < d+1, i{+}{+}, \mathrm{M2}[[d+i]] = \mathrm{M2}[[d+i-1]] + \mathrm{M2}[[d+i]]];$

$\mathrm{MatrixForm}[\mathrm{M2}]$

Next do the column switch to produce matrix (3).

$\mathrm{M3} = \mathrm{Transpose}[\mathrm{M2}];$

$\mathrm{M3}[[d+1]] = \mathrm{M3}[[1]];$

$\mathrm{For}[i = 1, i < 3d+1, i{+}{+}, \mathrm{M3}[[1, i]] = 0];$

$\mathrm{M3} = \mathrm{Transpose}[\mathrm{M3}];$

$\mathrm{MatrixForm}[\mathrm{M3}]$

The next command produces matrix (4).

$\mathrm{M4} = \mathrm{Part}[\mathrm{M3}, 2;;3d, 2;;3d];$

$\mathrm{For}[i = 1, i < d, i{+}{+}, \mathrm{M4}[[2d+i]] = \mathrm{M4}[[2d+i]] - n\mathrm{M4}[[i]]];$

$\mathrm{MatrixForm}[\mathrm{M4}]$

The next command produces matrix (5).

$\mathrm{M5} = \mathrm{Part}[\mathrm{M4}, d;;3d-1, d;;3d-1];$

$\mathrm{MatrixForm}[\mathrm{M5}]$

The next command produces matrix (6).

$\mathrm{M6} = \mathrm{M5};$

$\mathrm{For}[i = 2, i < d+1, i{+}{+},$

$\mathrm{For}[j = d+2, j < 2d+1, j{+}{+}, \mathrm{M6}[[i, j]] = 0]];$

$\mathrm{MatrixForm}[\mathrm{M6}]$

The next command produces matrix (7).

$\mathrm{M7a} = \mathrm{ConstantArray}[0, \{2d, 2d\}];$

$\mathrm{M7a}[[2d]] = \mathrm{M6}[[1]];$

$\mathrm{Part}[\mathrm{M7a}, 1;;2d-1, 1;;2d] = \mathrm{Part}[\mathrm{M6}, 2;;2d, 1;;2d];$

$\mathrm{M7b} = \mathrm{Transpose}[\mathrm{M7a}];$

$\mathrm{M7c} = \mathrm{ConstantArray}[0, \{2d, 2d\}];$

$\mathrm{M7c}[[2d]] = \mathrm{M7b}[[1]];$

$\mathrm{Part}[\mathrm{M7c}, 1;;2d-1, 1;;2d] = \mathrm{Part}[\mathrm{M7b}, 2;;2d, 1;;2d];$

$\mathrm{M7} = \mathrm{Transpose}[\mathrm{M7c}];$

$\mathrm{MatrixForm}[\mathrm{M7}]$

The next command produces matrix (8).

$\mathrm{M8} = \mathrm{Part}[\mathrm{M7}, d;;2d, d;;2d];$

$\mathrm{MatrixForm}[\mathrm{M8}]$

The next command produces matrix (9).

$\mathrm{M8}[[1]] = \mathrm{M8}[[1]] + (n-2)\mathrm{M8}[[d+1]];$

$\mathrm{M9} = \mathrm{M8};$

$\mathrm{MatrixForm}[\mathrm{M9}]$

The next command produces matrix (10).

$\mathrm{For}[i = 2, i < d + 1, i{+}{+}, \mathrm{M9}[[i]] = \mathrm{M9}[[1]] - \mathrm{M9}[[i]]];$

$\mathrm{M10} = \mathrm{M9};$

$\mathrm{MatrixForm}[\mathrm{M10}]$

The next command produces matrix (11).

$\mathrm{For}[i = 3, i < d + 1, i{+}{+}, \mathrm{M10}[[i]] = \mathrm{M10}[[i]] + \mathrm{M10}[[i - 1]]];$

$\mathrm{M11} = \mathrm{M10};$

$\mathrm{MatrixForm}[\mathrm{M11}]$

The next command produces matrix (12).

$\mathrm{M12a} = \mathrm{Transpose}[\mathrm{M11}];$

$\mathrm{For}[i = 3, i < d + 1, i{+}{+}, \mathrm{M12a}[[i]] = \mathrm{M12a}[[2]] - \mathrm{M12a}[[i]]];$

$\mathrm{M12} = \mathrm{Transpose}[\mathrm{M12a}];$

$\mathrm{MatrixForm}[\mathrm{M12}]$

The next command produces matrix (13).

$\mathrm{M13a} = \mathrm{Transpose}[\mathrm{M12}];$

$\mathrm{For}[i = 4, i < d + 1, i{+}{+}, \mathrm{M13a}[[i]] = \mathrm{M13a}[[i]] - \mathrm{M13a}[[3]]];$

$\mathrm{M13} = \mathrm{Transpose}[\mathrm{M13a}];$

$\mathrm{MatrixForm}[\mathrm{M13}]$

The next command produces matrix (14).

$\mathrm{M14a} = \mathrm{Transpose}[\mathrm{M13}];$

For$[i = 5, i < d + 1, i{+}{+}, \mathrm{M14a}[[i]] = \mathrm{M14a}[[i]] - \mathrm{M14a}[[4]]]$;

M14 = Transpose[M14a];

MatrixForm[M14]

The next command finishes the zeroing-out process after matrix (14).
(We'll just zero the entries out instead of doing column operations.
This matrix does not appear in the notes.)

For$[i = 2, i < d + 1, i{+}{+},$

For$[j = i + 2, j < d + 1, j{+}{+}, \mathrm{M14}[[i, j]] = 0]]$;

MatrixForm[M14]

The next command finishes the zeroing-out process to produce matrix (15).
(We'll just zero the entries out instead of doing operations on C_2.)

For$[i = 2, i < d, i{+}{+}, \mathrm{M14}[[i, 2]] = 0]$;

M15 = M14;

MatrixForm[M15]

The next command produces matrix (16).

M16a = Transpose[M15];

$\mathrm{M16a}[[2]] = \mathrm{M16a}[[2]] - \mathrm{M16a}[[1]] - (n - 2)\mathrm{M16a}[[d + 1]]$;

M16 = Transpose[M16a];

MatrixForm[M16]