Universidad de Medellín
Ciencia y Libertad

THESIS

In order to obtain the

DOCTORATE IN MODELING AND SCIENTIFIC COMPUTING

Delivered by:

Universidad de Medellín

Presented and defended by

Juan Pablo Fernández Gutiérrez

Defence date: 9 May, 2019

Title:

A Maximal Covering Location Problem Based Optimization of Complex Processes: A Novel Computational Approach

---

Faculty

Basic Sciences

Directors of thesis:

Dr. Vadim Azhmyakov (Universidad EAFIT, Medellín, Colombia)

Prof. Dr. Stefan Pickl (Universität der Bundeswehr, München, Germany)

Members of the Jury:

Prof. Wolfgang BEIN, University of Nevada, USA. President
Prof. Raymundo JUAREZ, Universidad Autonoma de Coahuila, Mexico.
Prof. Lai Soon LEE, Universiti Putra Malaysia, Malaysia.
Prof. Julien MONCEL, University of Toulouse, France.
Prof. Dr. Stefan Pickl, Representation of thesis supervisors.

To my family
A mi familia

"Gold there is, and rubies in abundance,
but lips that speak knowledge are a rare jewel."
Proverbs 20:15

"Hay oro y multitud de piedras preciosas;
Mas los labios prudentes son joya preciosa."
Proverbios 20:15

Resumen.

Ésta tesis basada en artículos analiza un nuevo enfoque computacional para el problema de localización de cobertura máxima (MCLP, sigla en inglés). Consideramos una formulación de tipo difuso del MCLP genérico y desarrollamos los aspectos teóricos y numéricos necesarios del Método de Separación (SM) propuesto. Una estructura específica del MCLP originalmente dado hace posible reducirlo a dos problemas auxiliares de tipo mochila (Knapsack). La separación equivalente que proponemos reduce esencialmente la complejidad de los algoritmos resultantes. Este algoritmo también incorpora una técnica de relajación convencional y el método de escalarización aplicado a un problema auxiliar de optimización multiobjetivo. La metodología de solución propuesta se aplica a continuación a la optimización de la cadena de suministro en presencia de información incompleta. Estudiamos dos ejemplos ilustrativos y realizamos un análisis riguroso de los resultados obtenidos.

El resultado anterior se extiende a un enfoque de optimización computacional recientemente desarrollado para una clase específica de problemas de localización de cobertura máxima (MCLP) con una estructura dinámica conmutada. La mayoría de los resultados obtenidos para el MCLP convencional abordan el caso "estático" donde una decisión óptima se determina en un período de tiempo fijo. En nuestra contribución consideramos una toma de decisiones óptima basada en MCLP dinámica y proponemos un método computacional efectivo para el tratamiento numérico del problema de localización de cobertura máxima dinámica de tipo conmutado (DMCLP). Una estructura geométrica genérica de las restricciones en cuestión hace posible separar el problema de optimización dinámica dado originalmente y reducirlo a una familia específica de problemas auxiliares relativamente simples. El método de separación (SM) generalizado para el DMCLP con una estructura conmutada finalmente conduce a un esquema de solución computacional. El algoritmo numérico resultante también incluye la relajación clásica de Lagrange. Presentamos un análisis formal riguroso de la metodología de optimización del DMCLP y también discutimos aspectos computacionales. El algoritmo basado en SM propuesto se aplica finalmente a un ejemplo orientado a la práctica. Ejemplo, a saber, de un diseño óptimo de una configuración de red móvil (dinámica).

Abstract.

This Ph.D. article-based thesis discusses a novel computational approach to the extended Maximal Covering Location Problem (MCLP). We consider a fuzzy-type formulation of the generic MCLP and develop the necessary theoretical and numerical aspects of the proposed Separation Method (SM). A specific structure of the originally given MCLP makes it possible to reduce it to two auxiliary Knapsack-type problems. The equivalent separation we propose reduces essentially the complexity of the resulting computational algorithms. This algorithm also incorporates a conventional relaxation technique and the scalarizing method applied to an auxiliary multiobjective optimization problem. The proposed solution methodology is next applied to Supply Chain optimization in the presence of incomplete information. We study two illustrative examples and give a rigorous analysis of the obtained results.

The previous result is extended to a newly developed computational optimization approach to a specific class of Maximal Covering Location Problems (MCLPs) with a switched dynamic structure. Most of the results obtained for the conventional MCLP address the "static" case where an optimal decision is determined on a fixed time-period. In our contribution we consider a dynamic MCLP based optimal decision making and propose an effective computational method for the numerical treatment of the switched-type Dynamic Maximal Covering Location Problem (DMCLP). A generic geome-

trical structure of the constraints under consideration makes it possible to separate the originally given dynamic optimization problem and reduce it to a specific family of relative simple auxiliary problems. The generalized Separation Method (SM) for the DMCLP with a switched structure finally leads to a computational solution scheme. The resulting numerical algorithm also includes the classic Lagrange relaxation. We present a rigorous formal analysis of the DMCLP optimization methodology and also discuss computational aspects. The proposed SM based algorithm is finally applied to a practically oriented example, namely, to an optimal design of a (dynamic) mobile network configuration.

Índice general

Introduction

The location theory is central to the design of logistic systems, especially in the supply chain. Classification of location problems allows understanding the different models there exist in this area. Even location and allocation decisions are generally intertwined, in this Ph.D. paper-based thesis, we split those decisions and concentrate on the pure location decision. This kind of decisions are strategic or tactical in all systems, that depends if location decision can be reversible in the medium term. Besides the decision affect demand volume because it may lead to the acquisition of customers who previously could not be served at a satisfactory level of service or lost some customers due to the closure of a facility. So, the study of these decisions is core in the design of the logistic network.

Currently, by the expansion of Operational Research, there are raising new applications, and those applications share the same mathematical models sometimes, although their parameters have a different meaning. In such a way, we work in the known Maximal Covering Location Problem (MCLP), and this model can be led to the telecommunication network. We can keep the same model structure, but their parameters shift their meaning because the facility here is the base stations that decision-maker has to select, the customer is a cell phone, and the covering concept is the distance traveled by telecommunication signal, instead of the distance traveled by the customer. In this case, the model allows selecting the optimal open base stations for a telecommunication network.

Both paragraphs define the key task of mathematical modeling as well as the link between Operational Research and Math Modeling. Latter affirmation is the reason for doing a Ph.D. in Modeling and Scientific Computing. For doing this, Chapter one explains the research proposal in the first section including the objective and the methodology. The second section is an introduction of the traditional Maximal Covering Location Problem and the use of this model in both latter cases. Section three descriptive the Knapsack model, and their computational results related to this research. The latest section of the first chapter explains different resilient concepts in the supply chain, and why our proposal of the resilient model is different from that proposed in the known literature.

The key chapters, especially, chapter two and three contain the papers which are the result of the research. In the second chapter shows the resulting built model, the new computational procedure, and the application example in the supply chain. Chapter three shows the latest submitted paper where we move from static model to dynamic model based on the MCLP, the extended computational procedure similar to the static case, and the application to the dynamic telecommunication network.

The rest of the chapters contains future work and the conclusions of the Ph.D. based-papers thesis. The future work allows thinking in new research projects, including how can be applied this to a realistic case. The conclusions reinforce the findings of our research work. To inform we agree with the publishing policy, here it is the web address of the publishing policy of the journal where we published or submitted our work:

1. `https://www.elsevier.com/journals/ifac-papersonline/2405-8963/guide-for-authors#9001`
2. `https://aimsciences.org/journal/1547-5816/JIMO_Guide`

CHAPTER 1

Fundamental

1.1. Motivation and Research Proposal

The presented article-based work proposes a new effective numerical treatment of the conventional and extended Maximal Covering Location Problem (MCLP). We start our consideration by studying the classic (static) variant of the MCLP. The obtained conceptual results are next extended to the newly determined class of dynamic MCLP's. A specific linear-integer structure of a basic and advanced mathematical problems makes it possible to reduce the originally static or dynamic MCLP to two auxiliary optimization Knapsack-type problems. In either case, one can also include the lack of a complete information into the main mathematical model. The equivalent transformation (so-called separation) we propose provides a useful tool for an effective numerical treatment of the original MCLP and constitutes a conceptual basis for the new numerical approach we propose. This new computational approach reduces the total complexity of the resulting algorithms. The methodology we follow involves an additional relaxation procedure (for example, the celebrated Lagrange relaxation) in combination with some multiobjective optimization techniques. In the presented work we give a rigorous formal analysis of the proposed theoretical concepts and numerical algorithms. Finally, we apply the developed theory and numerical approach to practically oriented examples that includes several disciplines and areas among others optimization of logistics operations, general decision science, optimal manufacturing management system, and mobile networking system.

Constructive optimization of complex technological processes and the corresponding computer oriented methods and software are nowadays a usual and efficient methodology for the practical development of several real-world Management Systems (see e.g., [1, 2, 5, 6, 7, 9, 10, 14, 16, 17, 18, 22, 23, 24, 26, 25, 27]). The proposed Ph.D. paper-based Project will study mathematical aspects of two MCLPs in the presence of incomplete information. The requested optimal design of a real-world system under consideration can be formalized as a specific "disturbed" static or dynamic MCLP. The celebrated Maximal Covering Location Problem and the possible generalizations constitute a challenging mathematical problem with numerous applications in practice. We mostly deal with new constructive numerical approaches to this class of problems. Note that MCLP has a decisive role in the success of supply chains, with applications including the location of industrial plants, landfills, hubs, cross-docks, etc (see e.g., [1, 3, 8, 9, 10, 11, 12, 13, 14, 17, 21, 23]). It can also be applied to the general decision/management science (problems of economical, econometrical, social, financial nature). A well-known MCLP and the related supply chain activity involve the delivery of a manufactured product to the end customer or/and to a warehouse. In a classical MCLP, one seeks the location of a number of facilities on a network in such a way that the covered "population" is maximized [13, 15]. Let us also mention a possible application of the MCLP methodology in the optimization of mobile networks/communication science.

MCLP was first introduced by Church and ReVelle [13] on a network, and since then, several extensions to the original problem have been made. A variety of numerical approaches have been proposed to the practical treatment of distinct MCLP's. Let us mention here exact, heuristic and metaheuristic families of methods and also refer to [8, 9, 10, 11, 12, 13, 14, 17, 21] for some necessary details, concrete solution algorithms and further references.

Note that heuristics and metaheuristics have usually been employed in order to solve large size MCLP's (see e.g., [3, 12, 17, 19]). A recent interest to MCLP's has arisen out the uncertainty of model parameters, such as demands or/and locations of demand nodes [9, 10, 23].

The optimization approach we propose includes an equivalent transformation of the original MCLP that finally involves common Knapsack problems (see e.g., [15] and references therein). The developed approach reduces the complexity of an initially given MCPL and makes it possible to apply various exact methods to the original MCLP. We also use a generic relaxation (for example, the celebrated Lagrange scheme) for this purpose [11, 27]. Moreover, we also incorporate the standard multiobjective optimization techniques and some heuristic approaches into the resulting computational scheme. And, it should be noted already at this point that the MCLP based modeling approach we propose can be effectively implemented (at the optimization stage) in a concrete optimal design of some engineering, financial and social systems.

1.1.1. Motivation

The renewal of exact methods of Numerical Optimization ([34, 35, 36, 37, 38]) is currently a hard task. The numerical optimization methods permit consistency/convergence results, also looking be easy to program, and make it possible to reduce the principal problem to ones more studied problems. Because there exist difficult problems, the combination possibilities with other exact or heuristic techniques are important, which requires consistent algorithms, and are a source for engineering/economical application.

The possibility of a specific useful generalization for dynamic optimization problems (engineering and optimal control problem [4, 20]) is a so interesting area, which let us study how to transit from the static case into the dynamic case. MCLP has been applied traditionally to static problems in location theory, but the new applications to the location theory need to introduce the time dependence to make the decision. Both this motivation is caused by some limitations of existing methods.

1.1.2. Objective

The main aim of our Ph.D. project is with a strong theoretic foundation of a new effective numerical approach to MCLP's.

1.1.3. Methodology

Our methodology is based on deductive and mathematical modeling cycle, and in this article-based thesis project, furthermore, we introduce an aspect of resilience modeling in the context of supply chain and telecommunication network. The modeling cycle which is the principal step to simplify the real world problem into a mathematical model for the decision makers in different areas, and it is used to explain the parameters inside the main model. Finally, we propose a conceptual new algorithm for solving this problem, count the computational complexity, and show the final procedure for solving the optimization problem.

1.2. Maximal covering location problem

The Maximal Covering Location model ([13]) consider the following linear integer programming problem

$$\text{maximize} \quad J(z(y)) := \sum_{j=1}^{n} w_j z_j$$
$$\text{subject to} \quad \begin{cases} \sum_{i=1}^{l} y_i = k, \\ z_j \leq \sum_{i=1}^{l} a_{ij} y_i, \\ z \in \mathbb{B}^n, \; y \in \mathbb{B}^l \end{cases} \tag{1.1}$$

Here $w_j \in \mathbb{R}_+,\ j = 1, ..., n$ are given non-negative objective "weights" and decision variables $z_j,\ j = 1, ..., n$ determine the "facilities to be served". By y_i, where $i = 1, ..., l$, we define the generic decision variables of the problem under consideration and $k \in \mathbb{R}_+$ in (1.1) describes the total amount of the facilities to be located. Elements

$$a_{i,j} = \begin{cases} 1 & \textit{if the facility } i \textit{ covers the point } j \\ 0 & \textit{otherwise} \end{cases}$$

are components of the so called "eligibility matrix" $A := (a_{i,j})_{j=1,...,n}^{i=1,...,l}$ associated with the eligible sites that provide a covering of the demand points indexed by $j = 1, ..., n$. Note that the second index in (1.1), namely, $i = 1, ..., l$ is related to the given "facilities sites". Finally, the admissible sets $\mathbb{B}^n$ and $\mathbb{B}^l$ in the main problem (1.1) are defined as follows: $\mathbb{B}^n := \{0, 1\}^n,\ \ \mathbb{B}^l := \{0, 1\}^l$.

This is the core model we study in this Ph.D. project, understand its parameter, and the possible real application in two different environments. The traditional one, in the location-covering model, the aim is to locate a least-cost set of facilities in such a way that each user can be reached within a maximum travel time from the closest facility. From a logistic point of view, this model is used in the Public Sector, especially in services of firefighting, transport of the disabled, ambulance dispatching, etc.

On the other hand, this model can locate the best-fixed number of base stations under a limited budget allowing to maximize the proportion of demand nodes covered by the cells within the permitted range. Here in both field, we are assuming single-period, single-type facility, the single homogeneous customer (population, signal, to name a few) who are able to get a direct route to the single facility.

1.3. Knapsack problem

The Knapsack problem ([28]), one of the pioneering studies on this) which is really a family of combinatorial *NP*-hard problems known as this name, consider the following linear integer programming problem

$$\begin{aligned} \text{maximize} \quad & J(z) := \sum_{j=1}^{n} w_j z_j \\ \text{subject to} \quad & \sum_{i=j}^{n} u_j z_j \leq C, z \in \mathbb{B}^n \end{aligned} \tag{1.2}$$

Usually, this model is linked with a hitch-hiker who has to fill up his knapsack by selecting from among several possible objects those which will give him maximum comfort. Here, z_j is the binary decision variable which 1 is for a selected object j, 0 otherwise; w_j is a measure of the comfort given by object j, u_j its size and C the size of the knapsack. Readers, who preference another point view can think in resource allocation, where we want to invest a capital of C dollars, and you are considering n possible investments, each one with w_j of profit you expect from investment j, and u_j the number of dollars it requires.

The problem (1.2) is also known as $0-1$ knapsack problem. There are other problems related to Bounded Knapsack, Subset-sum, Change-making, $0-1$ multiple knapsack, generalized assignment, and Bin-packing. All of these problems are *NP*-hard, although there exist efficient algorithms as shown in [15]. These efficient algorithms can be included in a wide range of optimization problems, and the complexity theorem is useful for the operational count.

1.4. Resilience concept in supply chain

The main topic presented in this section is to discuss a resilience concepts approach to the extended Maximal Covering Location Problem (MCLP). We consider a fuzzy-type formulation of the generic MCLP and make a comparison among our resilient concept and other describe in ([29, 30, 31, 32, 33]).

Design resilient supply chain ([29, 32]) are planned in a range of options. Decision makers have the financial resources available, the type of network under consideration, their own risk preference, and other factors inside the context where the supply chain works. As disruptions are the principal source which gets in unreliable to the system in facility location, in such way, the objective of designing the supply chain networks that operate efficiently both normally and when a disruption occurs, it is considered resilient to disruptions. Facility's unavailability is another way of unreliability, which is generally caused by congestion and its maximum availability. In the latter, customers cannot be served, then customers should move for the next facility to be served. These facilities unavailable produces loss of goodwill for the company.

In design, we have assumed that no network is currently in place. Instead of, some supply chain already exists, and we want to fortify this network to make it more resilience, including fail, vulnerability or security of the facility. We are going to focus on the design resilient supply chain henceforth. Some

Facility Location Models are based on the classical Uncapacitated Fixed-charge Location Problem (UFLP, or known as simple plan location) as the Resilient Fixed-charge Location Problem (RFLP) which minimizes the sum of the fixed cost and the expected transportation and lost-sales costs, subject to r-resilient server to customers, which is modeled through "level-r" assignment among customer and r closer open facilities that allow introducing a "backup" facilities to understand the resilience.

An alternative model, based on the P-median problem for modeling the degree of coverage is a non-increasing step function of the distance to the nearest facility rather than the UFLP objective function. This model [30] generalized maximal covering location problem, called GMCLP, and is equivalent to UFLP where the fixed costs (normally non-zero) are set to zero. In particular, the "standard" MCLP problem is a special case of UFLP. In [31] face the same problem, in both cases they maximize the coverage of the demand points by determining one of the selected facility sites, which ensures maximum coverage level for each demand point, but [31] introduce a different concept in the level of coverage using a monotone decreasing function within $[0, 1]$ between total coverage distance S and a maximum partial coverage distance T, in this paper they name that problem as MCLP-P.

Finally, Lee in [33], provide a heuristic for the RFLP using a traditional objective function of UFLP subject to each customer should be served by either the primary facility or the secondary facility at any instant (additional constraint), the flow feasibility among each customer and the facility and facilities, each facility is used as either the primary or the secondary facility, but no both are enforced in a set of constraint of flow feasibility; and due to the unreliability of the facility, the demand of each customer cannot be satisfied by using only one facility. In all cases, mix assignment model with location has been the constant. We are proposing using the fuzzy parameter to include the customer preferences inside the constraint of location feasibility, and come back to the traditional MCLP, and follow the way location first, assignment after. In that manner, a customer is a coverage where several facilities have a sum of preference customer more than one, allowing to use the traditional coverage function of MCLP, and keeping the linearity of the model, but in any case, we can not avoid the *NP* problem in the models.

Now, we are able to propose a shifting in the use of disruption concept from the objective function into the covering concept, in that way, we are avoiding to mix to *NP* models as has been done until today.

CHAPTER 2

Papers on Static Model

We add in this chapter two papers on MCLPs applying to supply chain. The formal problem formulation for the classic MCLP under case, is given in papers [39, 40] anexed to this chapter in the sections 2.1 and 2.2.

2.1. IFAC Paper

The first paper is an advanced of our work. We combine MCLP with resilience concept, allowing to introduce disruption into MCLP in the constraint of covering. Such,

First We change the concept of covering radius for a fuzzy-type formulation in that constraint.

Second We keep the simplicity of the mathematical formulation, and allow to extended the model to another assumption.

Third We propose a conceptual new algorithm for solving it.

Fourth We present a toy example of this model applying to a resilient supply chain for a family of manufacturing plants-warehouses.

Available online at www.sciencedirect.com

ScienceDirect

IFAC PapersOnline CONFERENCE PAPER ARCHIVE

IFAC-PapersOnLine 49-31 (2016) 137–142

A Novel Numerical Approach to the MCLP Based Resilent Supply Chain Optimization

V. Azhmyakov[1], J. P. Fernández-Gutiérrez[1], S. K. Gadi[2], St. Pickl[3]

1 Departamento de Ciencias Basicas, Universidad de Medellin, Medellin, Colombia (e-mail: vazhmyakov@udem.edu.co; jpfernandez@udem.edu.co)
2 Facultad de Ingenieria Mecanica y Electrica, Universidad Autonoma de Coahuila, Torreon, Mexico (email: research@skgadi.com)
3 Department of Computer Science, Universität der Bundeswehr München, München, Germany (email: stefan.pickl@unibw.de)

Abstract: This paper deals with the Maximal Covering Location Problem (MCLP) for Supply Chain optimization in the presence of incomplete information. A specific linear-integer structure of a generic mathematical model for Resilient Supply Chain Management System (RSCMS) makes it possible to reduce the originally given MCLP to two auxiliary optimization Knapsack-type problems. The equivalent transformation (separation) we propose provides a useful tool for an effective numerical treatment of the original MCLP and reduces the complexity of algorithms. The computational methodology we follow involves a specific Lagrange relaxation procedure. We give a rigorous formal analysis of the resulting algorithm and apply it to a practically oriented example of an optimal RSCMS design.

1. INTRODUCTION

Constructive optimization of complex technological processes and the corresponding computer oriented methods and software are nowadays a usual and efficient methodology for the practical development of several real-world Management Systems (see e.g., [1,5-7,9,10,11,15,18,23,24]). Our paper studies mathematical aspects of a particular RSCMS model that involves incomplete information. The requested optimal design of a RSCMS can be formalized as a specific "disturbed" MCLP [10]. Recall that the celebrated Maximal Covering Location Problem is a challenging optimization problem with numerous applications in practice. It has a decisive role in the success of supply chains, with applications including location of industrial plants, landfills, hubs, cross-docks, etc (see e.g., [1,3,8-10,12-15,18,20,22,24]). A well-known MCLP and the related supply chain activity involve the delivery of a manufactured product to the end customer or/and to a warehouse. In a classical MCLP, one seeks the location of a number of facilities on a network in such a way that the covered "population" is maximized [14,24].

MCLP was first introduced by Church and ReVelle [14] on a network, and since then, several extensions to the original problem have been made. A variety of numerical approaches have been proposed to the practical treatment of distinct MCLPs. Let us mention here exact, heuristic and metaheuristic families of methods and also refer to [8-10,12-15,18,20,22] for some necessary details, concrete solution algorithms and further references. Note that heuristics and metaheuristics have usually been employed in order to solve large size MCLPs (see e.g., [3,13,18,20]). A recent interest to MCLPs has arisen out the uncertainty of model parameters, such as demands or/and locations of demand nodes [9,10,24].

The main aim of our contribution is with a strong theoretic foundation of the newly elaborated separation method. The optimization approach we propose includes an equivalent transformation of the original MCLP that finally involves a common Knapsack problem (see e.g., [16] and references therein). The developed approach reduces the complexity of an initially given MCPL and makes it possible to apply various exact methods to the original MCLP. We concretely use the well-known Lagrange relaxation scheme for this purpose [12,16]. And, it should be noted already at this point that the MCLP based optimization algorithm we propose can be effectively implemented (at the optimization stage) in a concrete RSCMS.

The remainder of our paper is organized as follows: Section 2 contains a formal problem statement and some necessary concepts. In Section 3 we prove our main separation result, namely, Theorem 1 and give a constructive characterization of the obtained auxiliary problems. Section 4 deals with the celebrated Lagrange relaxation scheme applied to the initially given MCLP as well as to the auxiliary Knapsack problem. We use our main theoretic results and propose a self-closed algorithm for an effective numerical treatment of the initially given MCLP. Section 5 contains a simplified computational example of an optimal RSCMS design. This practically oriented example illustretes the applicability of the proposed numerical scheme. Section 6 summarizes our paper.

2. PROBLEM FORMULATION AND SEPARATION

An optimal design of a complex logistics network can generally be implemented in two steps. Firstly one solves the location problem and next considers the corresponding demand allocation problem. Note that a conventional

2405-8963 © 2016, IFAC (International Federation of Automatic Control) Hosting by Elsevier Ltd. All rights reserved.
Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2016.12.175

MCLP does not constitute a "universal" solution approach under assumption of possible process disruptions (technical faults, maintenance and so on). This is specifically true with respect to the second sub-problem mentioned above. We next introduce a suitable analytic extension of the conventional MCLP that includes the possible updates of the demand allocation for the same location distribution. The extended modelling approach we propose can be expressed in the form of a (specific) linear integer program

$$\begin{aligned} &\text{maximize} \quad J(z(y)) := \sum_{i=1}^{n} w_j z_j \\ &\text{subject to} \quad \begin{cases} \sum_{i=1}^{l} y_i = k \in \mathbb{N}, \; l > k, \\ z_j \leq \sum_{i=1}^{l} a_{ij} y_i, \\ z \in \mathbb{B}^n, \; y \in \mathbb{B}^l \end{cases} \end{aligned} \tag{1}$$

Here $w_j \in \mathbb{R}_+, \; j = 1, ..., n$ are given nonnegative objective "weights" and variables $z_j, \; j = 1, ..., n$ determine the "facilities to be served". By y_i, where $i = 1, ..., l$, we define the generic decision variables of the problem under consideration and $k \in \mathbb{R}_+$ in (1) describes the total amount of the facilities to be located. Elements a_{ij}, where

$$1 \geq a_{ij} \geq 0, \; \sum_{i=1,...,l} a_{ij} \geq 1,$$

are components of the so called "eligibility matrix"

$$A := (a_{ij})_{j=1,...,n}^{i=1,...,l}$$

associated with the eligible sites that provide a resilient covering of the demand points indexed by $j = 1, ..., n$. Note that the second index in (1), namely, $i = 1, ..., l$ is related to the given "facilities sites". Finally, the admissible sets $\mathbb{B}^n$ and $\mathbb{B}^l$ in the main problem (1) are defined as follows:

$$\mathbb{B}^n := \{0,1\}^n, \;\; \mathbb{B}^l := \{0,1\}^l.$$

Note that the objective functional $J(\cdot)$ from (1) has a linear structure. We use the following natural notation $z := (z_1, ..., z_n)^T$ and $y := (y_1, ..., y_l)^T$. The implicit dependence

$$J(z(y)) = \langle w, z \rangle, \; w := (w_1, ..., w_n)^T$$

of the objective functional J on the vector y is given by the corresponding (componentwise) inequalities constraints $z \leq A^T y$ in (1). By $\langle \cdot, \cdot \rangle$ we denote here the scalar product in the corresponding Euclidean space. A vector pair (z, y) that satisfies all the constraints in (1) is next called an admissible pair for the main problem (1).

The abstract optimization framework (1) provides a constructive and modelling approach for various practically oriented problems (see e.g., [1,9,11,13,18,22,24]). Following [14] we next call the main optimization problem (1) a Maximal Covering Location Problem (MCLP). Let us also refer to [24] for a detailed discussion on the applied interpretation of the MCLP (1). Note that the main MCLP is formulated under the general (non-binary) assumption related to the elements a_{ij} of the eligibility matrix A. This corresponds to a suitable modelling approach under incomplete information (see e.g., [10] and references therein). Roughly speaking every selection of an admissible parameter a_{ij} in (1) has a "fuzzy" nature (similar to [8]). This fuzzy characterization of the MCLP under consideration provide an adequate modelling framework for the RSCMS (see Section 5).

The mathematical characterization of (1) can evidently be given in terms of the classic integer programming (see e., g. [11,16,19] for mathematical details). Let us note that (1) posesses an optimal solution (an optimal pair)

$$(z^{opt}, y^{opt}) \in \mathbb{B}^n \bigotimes \mathbb{B}^l,$$

where

$$z^{opt} := (z_1^{opt}, ..., z_n^{opt})^T, \;\; y^{opt} := (y_1^{opt}, ..., y_l^{opt})^T.$$

This fact is a direct consequence of the basic results from [11,16,19]. Our aim is to develop a simple and effective numerical approach to the sophisticated MCLP (1). We firstly "separate" the original optimization problem and introduce two auxiliary optimization problems. These formal constructions provide a necessary basis for the future numerical development we propose. The first auxiliary problem can be formulated as follows

$$\begin{aligned} &\text{maximize} \quad \sum_{j=1}^{n} \mu_j \sum_{i=1}^{l} a_{ij} y_i \\ &\text{subject to} \quad \begin{cases} \sum_{i=1}^{l} y_i = k, \;\; y \in \mathbb{B}^l, \\ \mu_j \in [0,1] \;\; \forall j = 1, ..., n \end{cases} \end{aligned} \tag{2}$$

The second auxiliary problem has the following specific form:

$$\begin{aligned} &\text{maximize} \quad J(z) := \sum_{j=1}^{n} w_j z_j \\ &\text{subject to} \quad \begin{cases} z_j \leq \sum_{i=1}^{l} a_{ij} \hat{y}_i \\ z \in \mathbb{B}^n \end{cases} \end{aligned} \tag{3}$$

where $\hat{y} \in \mathbb{B}^l$ is optimal solution of problem (2). The components of $\hat{y}$ are denoted as $\hat{y}_i, \; i = 1, ..., l$. The existency of an optimal solution for (2) is a direct consequence of the results from [11,19]. The same is also true with respect to the auxiliary problem (3). Let $\hat{z} \in \mathbb{B}^n, \; \hat{z} := (\hat{z}_1, ..., \hat{z}_n)^T$ be an optimal solution to (3). Evidently, problem (3) coincides with the originally given MCLP (1) in a specific case of a fixed variable $y = \hat{y}$. Let us note that in general $\hat{y} \neq y^{opt}$.

The first auxiliary problem, namely, problem (2) is a usual linear scalarization of the following multiobjective optimization problem (vector optimization):

$$\begin{aligned} &\text{maximize} \quad \{\sum_{i=1}^{l} a_{i1} y_i, ..., \sum_{i=1}^{l} a_{in} y_i\} \\ &\text{subject to} \quad \begin{cases} \sum_{i=1}^{l} y_i = k, \\ y \in \mathbb{B}^l \end{cases} \end{aligned} \tag{4}$$

Recall that a scalarizing of a multi-objective optimization problem is an adequate numerical approach, which means formulating a single-objective optimization problem such that optimal solutions to the single-objective optimization problem are Pareto optimal solutions to the multi-objective optimization problem. We next assume that the multipliers $\mu_j, \; j = 1, ..., n$ in (2) are chosen by such a way that problems (2) and (4) are equivalent (see e.g., [2,11,19] for necessary details). In this particular case we call (2) an adequate scalarizing of (4). Moreover, problems (2) and (3) have a structure of a so-called Knapsack problem (see [16] and references therein). Various efficient numerical algorithms are recently proposed for a generic Knapsack problem. We refer to [16] for a comprehensive overwiev about the modern implementable numerical approaches to this basic optimization problem.

3. THE SEPARATION BASED SOLUTION APPROACH

The relevance and main motivation of the auxiliary optimization problems (2) and (3) introduced in Section 2 can be stated by the following abstract result.

Theorem 1. Assume (z^{opt}, y^{opt}) is an optimal solution of (1) and (2) is an adequate scalarizing of (4). Let $\hat{y}$ be an optimal solutions of (2) and $\hat{z}$ be an optimal solution of the auxiliary problem (3). Then (1) and (3) possess the same optimal values, that is

$$J(z^{opt}(y^{opt})) = J(\hat{z}). \tag{5}$$

Moreover, in the case problems (1), (2), and (3) possess unique solutions we additionally have $(z^{opt}, y^{opt}) = (\hat{z}, \hat{y})$.

Proof: Since

$$\sum_{i=1}^{l} \hat{y}_i = k, \quad \hat{z}_j \le \sum_{i=1}^{l} a_{ij}\hat{y}_i,$$

we conclude that $(\hat{z}, \hat{y})$ is an admissible pair for the original MCLP (1). Taking into account the definition of an optimal pair for problem (1), we next deduce

$$J(\hat{z}(\hat{y})) \le J(z^{opt}(y^{opt})). \tag{6}$$

Let

$$\Gamma = \Gamma_z \bigotimes \Gamma_y \subset \mathbb{B}^n \bigotimes \mathbb{B}^l$$

be a solutions set (the set of all optimal solutions) for problem (1). We also define the solutions sets $\Gamma_{(2.2)} \subset \mathbb{B}^l$ and $\Gamma_{(2.3)} \subset \mathbb{B}^n$ of problems (2) and (3), respectively. From (6) it follows that

$$\Gamma_{(2.3)} \bigotimes \Gamma_{(2.2)} \subset \Gamma. \tag{7}$$

Taking into account the restrictions associated with the variable y in (1) and (2), we next obtain

$$\Gamma_y \equiv \Gamma_{(2.2)}. \tag{8}$$

Since (2) is an adequate scalarization of the multi-objective maximization problem (4), we deduce

$$z_j \le \max_{\begin{cases} \sum_{i=1}^{l} y_i = k, \\ y \in \mathbb{B}^l \end{cases}} \sum_{i=1}^{l} a_{ij} y_i.$$

This fact implies

$$\Gamma_z \subset \Gamma_{(2.3)}. \tag{9}$$

Inclusions (7), (9) and the basic equivalence (8) now imply the following crucial equivalence

$$\Gamma_{(2.3)} \bigotimes \Gamma_{(2.2)} \equiv \Gamma. \tag{10}$$

Taking into account the same form of the objective functionals in (1) and (2.3), we immediately obtain the basic relation (5). In a specific case of one point sets Γ, $\Gamma_{(2.3)}$ and $\Gamma_{(2.2)}$ the expected relation $(z^{opt}, y^{opt}) = (\hat{z}, \hat{y})$ is a direct consequence of (10). The proof is completed. □

Theorem 1 makes it possible to separate (equivalently) the originally given sophisticated problem (1) into two simple optimization problems. It provides a theoretical basis for effective numerical approaches to the abstract MCLPs and to corresponding applications.

We now observe that the first auxiliary optimization problem, namely, problem (2) has a trivial combinatorial structure and can be easily solved:

$$\hat{y}_i = 1 \text{ if } i \in \hat{I}; \quad \hat{y}_i = 0 \text{ if } i \in \{1, ..., l\} \setminus \hat{I}, \tag{11}$$

where

$$\begin{aligned} &\hat{I} := \{1 \le i \le l \mid S_{\mathcal{A}_i} \in \max_k \{S_{\mathcal{A}_1}, ..., S_{\mathcal{A}_l}\}\}, \\ &S_{\mathcal{A}_i} := \sum_{j=1}^{n} \mu_j a_{ij}, \ \mathcal{A}_i := (a_{i1}, ..., a_{in})^T. \end{aligned} \tag{12}$$

Here $\mathcal{A}_i$ is a vector of i-row of the eligibility matrix A and operator $\max_k$ determines an array of k-largest numbers from the given array. Evidently, the choice (11)-(12) determines an optimal solution of (2). Roughly speaking the combinatorial algorithm (11)-(12) assigns the maximal value $\hat{y}_i = 1$ for all vectors $\mathcal{A}_i$ which sum of components $S_{\mathcal{A}_i}$ belongs to the array of k-largest sums of compoents of all vectors $\mathcal{A}_i$, $i = 1, ..., l$. It is easy to see that for the given eligibility matrix A with the specific elements a_{ij} (determined in Section 2) the sum of components $S_{\mathcal{A}_i}$ constitutes a specific norm of the given vector $\mathcal{A}_i$. Let us also note that the total complexity of the combinatorial algorithm (11)-(12) is equal to

$$O(l \times \log k) + O(k)$$

(see e.g., [16] for details).

Let us denote

$$c := \sum_{j=1}^{n} \sum_{i=1}^{l} a_{ij}\hat{y}_i.$$

Then the inequality constraints in (3) imply the generic Knapsack-type constraint with uniform weights

$$\sum_{j=1}^{n} z_j \le c.$$

We now present a fundamental solvability result for the second auxiliary optimization problem, namely, the Knapsack problem (3).

Theorem 2. The Knapsack problem (3) can be solved in $O(nc)$ time and $O(n + c)$ space.

The formal proof of Theorem 2 can be forund in [16].

4. LAGRANGE RELAXATION AND CONSTRUCTIVE NUMERICAL TREATMENT OF THE ORIGINAL MCLP

Our main analytic results, namely, Theorem 1, the combinatorial choice algorithm (11)-(12) and Theorem 2 provide a theoretic basis for a novel exact solution scheme for the originally given MCLP (1). Finally we need to define a suitable and implementable procedure for an effective numerical treatment of (3). This auxiliary optimization problem, which is $\mathcal{NP}$-hard, has been comprehensively studied in the last few decades and several exact algorithms for its solution can be found in the literature (see [16] and the references therein). Constructive algorithms for Knapsack problems are mainly based on two basic approaches: branch-and-bound and dynamic programming. Let us also mention here the celebrated "combined" approach.

In this paper we apply the well-known Lagrange relaxation scheme to the second auxiliary problem (problem (3)). "Relaxing a problem" has various meanings in applied mathematics, depending on the areas where it is defined, depending also on what one relaxes (a functional, the underlying space, etc.). We refer to [2,4-7,12, 21] for various implementable relaxation techniques. Introducing the Lagrange function

$$\mathcal{L}(z,\lambda) := \sum_{j=1}^{n} w_j z_j - \sum_{j=1}^{n} \lambda_j \big(z_j - \sum_{i=1}^{l} a_{ij}\hat{y}_i\big)$$

associated with the Knapsack problem (3), we next consider the following relaxed problem

$$\begin{array}{ll} \text{maximize} & \mathcal{L}(z,\lambda) \\ \text{subject to} & z \in \mathbb{B}^n \end{array} \qquad (13)$$

The relaxed problem (13) does not contain the unpleasant inequality constraints which are included in the objective function (3.17) as a penalty term

$$\sum_{j=1}^{n} \lambda_j \big(z_j - \sum_{i=1}^{l} a_{ij}\hat{y}_i\big).$$

Recall that all feasible solutions to (3) are also feasible solutions to (13). The objective value of feasible solutions to (3) is not larger than the objective value in (13) (see [16] for the necessary proofs). Thus, the optimal solution value to the relaxed problem (13) is an upper bound to the original problem (3) for any vector of nonnegative multipliers $\lambda := (\lambda_1, ..., \lambda_n)^T$, $\lambda_j \geq 0$. For a concrete numerical solution of the relaxed problem (13) we use here the classic branch-and-bound method (see e.g., [11,16]). In a branch-and-bound algorithm we are interested in achieving the tightest upper bound in (13). Hence, we would like to choose a vector of nonnegative multipliers

$$\hat{\lambda}^{\mathcal{L}} := (\hat{\lambda}_1^{\mathcal{L}}, ..., \hat{\lambda}_n^{\mathcal{L}})^T, \ \hat{\lambda}_j^{\mathcal{L}} \geq 0$$

such that (13) is minimized. This evidently leads to the generic Lagrangian dual problem

$$\begin{array}{ll} \text{minimize} & \mathcal{L}(z,\lambda) \\ \text{subject to} & \lambda \geq 0 \end{array} \qquad (14)$$

It is well-known that the Lagrangian dual problem (14) yields the least upper bound available from all possible Lagrangian relaxations. The problem of finding an optimal vector of multipliers $\hat{\lambda}^{\mathcal{L}} \geq 0$ in (14) is in fact a linear programming problem [11,19]. In a typic branch-and-bound algorithm one will often be satisfied with a sub-optimal choice of multipliers $\lambda \geq 0$ if only the bound can be derived quickly. In this case subgradient optimization techniques can be applied [19]. The following analytic result is an immediate consequence of our main Theorem 1 and of the basic properties of the primal-dual system (13)-(14).

Theorem 3. Let $(\hat{z}^{\mathcal{L}}, \hat{\lambda}^{\mathcal{L}})$ be an optimal solution of the primal-dual system (13)-(14) associated with the auxiliary problem (3). Assume that all conditions of Theorem 1 be satisfied. Then

$$J(z^{opt}(y^{opt})) \leq J(\hat{z}^{\mathcal{L}}). \qquad (15)$$

and the obtained inequality (15) constitutes a tightest upper bound.

We are now ready to formulate a complete algorithm for an effective numerical treatment of the basic MCLP (1).

Algorithm 1.

I. Given an initial MCLP (1) separate it into two auxiliary problems (2) and (3);
II. Apply the combinatorial algorithm (11)-(12) and compute $\hat{y}$;
III. Using $\hat{y}$, construct the Lagrange function $\mathcal{L}(z,\lambda)$ and solve the primal-dual system (13)-(14).

The numerical consistency of the proposed Algorithm 1 is established by our main theoretic results, namely, by Theorem 1 - Theorem 3.

Finally let us note that the Lagrange relaxation scheme is usually applied to the original problem (1) (see e.g., [12,16]). In that case the resulting (relaxed) problem and the corresponding Lagrangian dual problem possess a higher complexity in comparison with the proposed "partial" Lagrange relaxation (13)-(14) of the original MCLP (1). This is an immediate consequence of the proposed separation method (Section 3) that reduces the initial problem (1) to two auxiliary optimization problem (2)-(3).

5. APPLICATION TO THE OPTIMAL DESIGN OF A RESILENT SUPPLY CHAIN MANAGEMENT SYSTEM

This section is devoted to a practical application of the proposed novel numerical approach to the MCLP (1). We use the basic MCLP model and optimize a Resilient Supply Chain for a family of manufacturing plants - warehouses. Note that the "resilience" of a Supply Chain Management

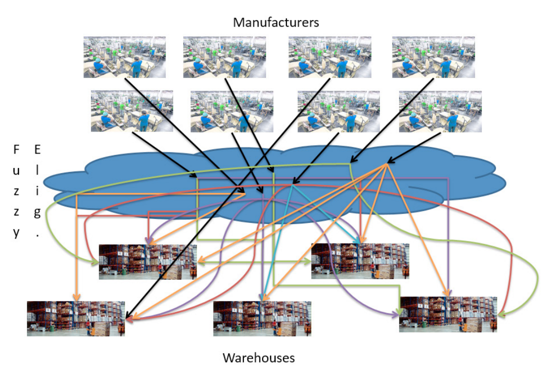


Fig. 1. Fuzzy eligibility model

System is modelled here by an eligibility matrix A with the fuzzy-type components a_{ij} (see Section 2). The conceptual Supply Chain scheme that include $l = 8$ manufacturing plants and $n = 5$ warehouses is indicated on Fig. 1.

Here i' is an index that corresponds to a "resilient" cover of demand point. We also assume that $a_{ij} + a_{i'j} \geq 1$ for $i = 1, ...5$ $j = 1, ...8$. The last condition means that at least two feasible facilities (warehouse) cover a given demand point (the manufacturing plants). The corresponding (transposed) eligibility matrix A is given as follows:

$$A^T = \begin{pmatrix} 0.81286 & 0.0 & 0.0 & 0.62968 & 0.0 \\ 0.25123 & 0.58108 & 0.32049 & 0.89444 & 0.79300 \\ 0.0 & 0.0 & 0.64850 & 0.91921 & 0.94740 \\ 0.54893 & 0.90309 & 0.74559 & 0.50869 & 0.99279 \\ 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.77105 & 0.27081 & 0.65883 & 0.60434 & 0.23595 \\ 0.0 & 0.51569 & 0.0 & 0.0 & 0.57810 \\ 0.64741 & 0.91733 & 0.60562 & 0.63874 & 0.71511 \end{pmatrix}$$

Recall that the objective weights $w_j \in \mathbb{R}_+,\ j = 1, ..., n$ indicates a priority and are assumed to be equal to

$$w^T = (32.0,\ 19.0,\ 41.0,\ 26.0\ 37.0\ 49.0\ 50.0\ 11.0)^T$$

Note that the fifth demand point in this example has no "resilient" character (only one facility covers this point). We assume that the Supply Chain decision maker is interested opens $k = 2$ facilities. Moreover, we also calculate from (12)

$$S_{\mathcal{A}_1} = 8.06295\ S_{\mathcal{A}_2} = 5.86033\ S_{\mathcal{A}_3} = 5.30955$$
$$S_{\mathcal{A}_4} = 7.47098\ S_{\mathcal{A}_5} = 6.99921$$

Application of the basic *Algorithm 1* leads to the following computational results:

$$\begin{aligned} z^{opt} &= (1,\ 1,\ 0,\ 1,\ 1,\ 1,\ 0,\ 1)^T, \\ y^{opt} &= (1,\ 0,\ 0,\ 1,\ 0)^T, \end{aligned} \tag{16}$$

The corresponding (maximal) value of the objective functional is equal to

$$J(z^{opt}(y^{opt})) = \max_{Problem(1)} J(z(y)) = 174.0$$

Let us also note that the computed scalarizing multiplier μ in the auxiliary problem (2) for the given problem data is equal to

$$\mu = (2.0,\ 2.0\ 1.0\ 2.0\ 2.0\ 2.0\ 1.0\ 2.0)^T.$$

The practical implementation of the computationall *Algorithm 1* was carried out by using the standard Python package and an author-written program.

For comparison, the given MCLP problem was also solved by a direct application of the standard CPLEX optimization package. We use the concrete problem parameters given above and obtain the same optimal pair as in (16). The CPLEX integer programming solver proceeds with 6 MIP simplex iterations and 0 branch-and-bound nodes for in total 13 binary variables and 9 linear constraints. Let us finally note that all the customers (except the fifth) are covered and moreover, could still be covered if one of the facilities is closed.

6. CONCLUDING REMARKS

In this contribution, we proposed a conceptually new numerical approach to a wide class of Maximal Covering Location Problems with the fuzzy-type eligibility matrices. This computational algorithm is next applied to the optimal design of a practically motivated Resilient Supply Chain Management System. The developed computational scheme is based on a novel separation approach to the initially given maximization problem. The separation scheme we propose makes it possible to reduce the original sophisticated problem to two Knapsack-type optimization problems. The first one constitutes a generic linear scalarization of a multiobjective optimization problem and the second auxiliary problem is a simple version of the classic Knapsack formulation. Application of the conventional Lagrange relaxation in combination with a specific combinatorial algorithm leads to an implementable algorithm for the given Maximal Covering Location Problem as well as for the optimal design of a Resilient Supply Chain.

Theoretical and computational methodologies we present in this contribution can be applied to various generalizations and extensions of the basic MCLP and also to several optimization problems associated with the RSCMS design. One can combine the elaborated separation scheme with the conventional branch-and-bound method, with the celebrated dynamic programming approach or/and with an alternative exact or heuristic numerical algorithm. Let us finally note that we discussed here only theoretic aspects of the newly elaborated approach and presented the corresponding conceptual solution procedure. The basic methodology we developed needs a comprehensively numerical examination that includes solutions of several MCLPs and simulations of the corresponding optimal RSCMSs.

REFERENCES

[1] G. Alexandris, I. Giannikos, A new model for maximal coverage exploiting GIS capabilities, European Journal of Operational Research, vol. 202(2), 2010, pp. 328 – 338.

[2] K. Atkinson, W. Han, *Theoretical Numerical Analysis*. Springer, New York 2005.

[3] H. Aytug, C. Saydam, Solving large-scale maximum expected covering location problems by genetic algorithms: a comparative study, *European Journal of Operational Research*, vol. 141(3), 2002, pp. 480 – 494.

[4] V. Azhmyakov and W. Schmidt, Approximations of relaxed optimal control problems, *Journal of Optimization Theory and Applications*, vol. 130, 2006, pp. 61 - 77,

[5] V. Azhmyakov, M. Basin, C. Reincke-Collon, Optimal LQ-type switched control design for a class of linear systems with piecewise constant inputs, in: *Proceedings of the 19th IFAC World Congress*, Cape Town, South Africa, 2014, pp. 6976 – 6981.

[6] V. Azhmyakov, J. Cabrera, A. Poznyak, Optimal fixed - levels control for non - linear systems with quadratic cost functionals, *Optimal Control Applications and Methods*, to appear in 2016.

[7] V. Azhmyakov, R. Juarez, On the projected gradient method for switched - mode systems optimization, in: *Proceedings of the 5th IFAC Conference on Analysis and Design of Hybrid Systems*, Atlanta, USA, 2015, .

[8] V. Batanovic, D. Petrovic, R. Petrovic, Fuzzy logic based algorithms for maximum covering location

problems, *Information Sciences*, vol. 179(12), 2009, pp. 120 – 129.

[9] O. Berman, J. Kalcsics, D. Krass, S. Nickel, The ordered gradual covering location problem on a network, *Discrete Applied Mathematics*, vol. 157(18), 2009, pp. 3689 – 3707.

[10] O. Berman, J. Wang, The minmax regret gradual covering location problem on a network with incomplete information of demand weights, *European Journal of Operational Research*, vol. 208(3), 2011, pp. 233 – 238.

[11] D. Bertsekas, *Nonlinear Programming*, Athena Scientific, Belmont, USA, 1995.

[12] R.D. Galvao, L.G. Acosta Espejo, B. Boffey, A comparison of Lagrangean and surrogate relaxations for the maximal covering location problem, *European Journal of Operational Research*, vol. 124(2), 2000, pp. 377 – 389.

[13] M.S. Canbolat, M. von Massow, Planar maximal covering with ellipses, *Computers and Industrial Engineering*, vol. 57(1), 2009, pp. 201 – 208.

[14] R.L. Church, C.S ReVelle, The maximal covering location problem, in *Papers of the Regional Science Association*, vol. 32, 1974, pp. 101 – 118.

[15] G. Ji, S. Han, A strategy analysis in dual-channel supply chain based on effort levels, in: *Proceedings of the 1th International Conference on Service Systems and Service Management*, Beijing, China, 2014, pp. 2161 – 1890.

[16] H. Kellerer, U. Pferschy, D. Pisinger, *Knapsack Problem*, Springer, Berlin, 2004.

[17] A. Mitsos, B. Chachuat and P.I. Barton, McCormic - based relaxation algorithm, *SIAM Journal on Optimization*, vol. 20, 2009, pp. 573 – 601.

[18] G.C. Moore, C.S. ReVelle, The hierarchical service location problem, *Management Science*, vol. 28, 1982, pp. 775 – 780.

[19] E. Polak, *Optimization*, Springer-Verlag, New York, USA, 1997.

[20] C. ReVelle, M. Scholssberg, J. Williams, Solving the maximal covering location problem with heuristic concentration, *Computers and Operations Research*, vol. 35(2), 2008, pp. 427 – 435.

[21] T. Roubicek, *Relaxation in Optimization Theory and Variational Calculus*, De Gruyter, Berlin, 1997.

[22] H. Shavandi, H. Mahlooji, A fuzzy queuing location model with a genetic algorithm for congested systems, *Applied Mathematics and Computation*, vol. 181(1), 2006, pp. 440 – 456.

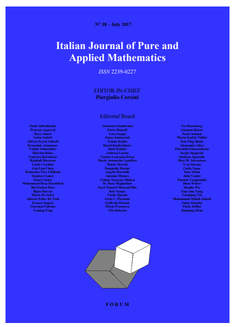
[23] P. Sitek, J. Wikarek, A hybrid approach to modeling and optimization for supply chain management with multimodal transport, in: *Proceedings of the 18th International Conference on Methods and Models in Automation and Robotics*, Miedzyzdroje, Poland, 2013, pp. 777 – 782.

[24] F. Zarandi, A. Haddad Sisakht, S. Davari, Design of a closed-loop supply chain (CLSC) model using an interactive fuzzy goal programming, *The International Journal of Advanced Manufacturing Technology*, vol. 56, 2011, pp. 809 – 821.

[25] G. Ghiani, G. Laporte, R. Musmanno *Introduction to Logistics Systems Planning and Control*, John Wiley & Sons Ltd, Chichester, England, 2004.

2.2. IJPAM Paper

We complete the previous work introducing a discussion of the aspects of a suitable scalarizing procedure. The number of the Journal can be found in `http://ijpam.uniud.it/journal/onl_2017-38.htm`, the web direction http://ijpam.uniud.it/online_issue/201738/57-Azhmyakov-Fernandez-Gutierrez-Pickl.pdf is for the paper.

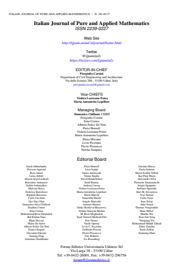

N° 38 – July 2017

Italian Journal of Pure and Applied Mathematics

ISSN 2239-0227

EDITOR-IN-CHIEF
Piergiulio Corsini

FORUM


ITALIAN JOURNAL OF PURE AND APPLIED MATHEMATICS – N. 38-2017

Italian Journal of Pure and Applied Mathematics
ISSN 2239-0227

A SEPARATION METHOD FOR MAXIMAL COVERING LOCATION PROBLEMS WITH FUZZY PARAMETERS

Vadim Azhmyakov[*]
Department of Basic Sciences
Universidad de Medellin
Medellin
Colombia
vazhmyakov@udem.edu.co

Juan Pablo Fernández-Gutiérrez
Department of Basic Sciences
Universidad de Medellin
Medellin
Colombia
jpfernandez@udem.edu.co

Stefan Pickl
Institut für Theoretische Informatik
Mathematik und Operations Research
Fakultät für Informatik
Universität der Bundeswehr München
München
Germany
stefan.pickl@unibw.de

Abstract. Our paper discusses a novel computational approach to the extended Maximal Covering Location Problem (MCLP). We consider a fuzzy-type formulation of the generic MCLP and develop the necessary theoretical and numerical aspects of the proposed Separation Method (SM). A specific structure of the originally given MCLP makes it possible to reduce it to two auxiliary Knapsack-type problems. The equivalent separation we propose reduces essentially the complexity of the resulting computational algorithms. This algorithm also incorporates a conventional relaxation technique and the scalarizing method applied to an auxiliary multiobjective optimization problem. The proposed solution methodology is next applied to Supply Chain optimization in the presence of incomplete information. We study two illustrative examples and give a rigorous analysis of the obtained results.
Keywords: MCLP, integer optimization, numerical optimization

1. Introduction

Optimization of modern technological processes and the corresponding computer oriented methods are nowadays a usual and efficient approach to the practical de-

*. Corresponding author

velopment of several engineering applications (see e.g., [1,5-7,9,10,11,15,18,23]). In our contribution we study an extended MCLP model with an incomplete information and propose a relative simple approach to the effective numerical treatment of this problem. The obtained theoretic and computational results are next applied to the resilient Supply Chain Management System Optimization. The requested optimal design of an optimal management operation can be formalized as a specific MCLP [10]. In that case the information incompleteness mentioned above can be adequately described by an eligibility matrix with the fuzzy structure and the systems "resilience" is related to this incomplete modelling framework.

Let us recall that the conventional and extended MCLP formulations constitute a family of challenging optimization problems with numerous practical applications. It has a decisive role in the success of a Supply Chain management, with several applications including location of industrial plants, landfills, hubs, cross-docks, etc (see e.g., [1,3,8-10,12-15,18,20,22,24]). A well-known MCLP and the related decision making involve the delivery of a manufactured product to the end customer or/and to a warehouse. In a classical MCLP, one seeks the location of a number of facilities on a network in such a way that the covered "population" is maximized [14,24]. MCLP was first introduced by Church and ReVelle [14] on a network, and since then, several extensions to the original problem have been made. A variety of numerical approaches have been proposed to the practical treatment of distinct MCLPs. Recently several heuristical methods are actively used in the practical treatment of the MCLP based models. We refer to [8-10,12-15,18,20,22] for some effective heuristic and metaheuristic algorithms and for further references. Note that heuristics and metaheuristics have usually been employed in order to solve large size MCLPs (see e.g., [3,13,18,20]). A recent interest to MCLPs has arisen out the uncertainty of model parameters, such as demands or/and locations of demand nodes [9,10,24]. The solution procedure (Separation Method) we propose is generally based on an exact optimization procedure. However it also can incorporate some heuristic procedures for solving the obtained auxiliary problems.

This paper is devoted to a further theoretic and numerical development of a newly elaborated solution method for the MCLPs, namely, to the so called Separation Method (see [7]). The optimization approach we follow includes an equivalent transformation (separation) of the original MCLP and solution of two auxiliary Knapsack-type problems (see e.g., [16] and references therein). The proposed SM reduces the complexity of the original problem. Moreover, one can apply various methods to the resulting auxiliary problems. In this paper we use a usual relaxation scheme for the purpose of a concrete computation [12,16]. We also apply the standard scalarizing of an intermediate multiobjective optimization problem we obtain. And, it should be noted already at this point that the MCLP based optimization approach we propose can be effectively implemented (at the prototype stage) in a concrete optimal design of a decision or management system. Concretely, this SM involved approach is applied in our paper to the optimal design of a resilient Supply Chain scheme for a typical manufactures - customers delivery. Finally note that SM we propose in fact involves a suitable (equivalent) decomposition of an initially given MCLP. This fact, namely, the consideration of two resulting auxiliary problems makes it also possible to extend this method to some applied large-scale MCLP (see e.g., [3]).

The remainder of our paper is organized as follows: Section 2 contains an abstract problem formulation and some necessary theoretical concepts and facts. In Section 3 we develop a theoretic basis of the SM. This section also includes a necessary characterization of the obtained auxiliary problems. Section 4 discusses the appropriate numerical schemes in the context of the the initially given and auxiliary optimization problems. We use our main theoretic results and finally propose an implementable and well-determined algorithm for an effective numerical treatment of the originally given MCLP. This algorithm also incorporates the conventional relaxation technique. Section 5 contains two computational examples of an optimal resilient Supply Chain design. These practically oriented examples illustrate the implementability of the resulting computational algorithms and usability of the proposed solution procedure. Section 6 summarizes our contribution.

2. Problem formulation and preliminaries

We start by introducing the main optimization problem with a fuzzy structure. The MCLP we study has the following form:

$$\text{maximize} \quad J(z(y)) := \sum_{j=1}^{n} w_j z_j$$
$$\text{subject to} \quad \begin{cases} \sum_{i=1}^{l} y_i = k \in \mathbb{N},\ l > k, \\ z_j \leq \sum_{i=1}^{l} a_{ij} y_i, \\ z \in \mathbb{B}^n,\ y \in \mathbb{B}^l \end{cases} \tag{1}$$

Here $w_j \in \mathbb{R}_+,\ j = 1, ..., n$ are given nonnegative objective "weights" and variables $z_j,\ j = 1, ..., n$ determine the "facilities to be served". By y_i, where $i = 1, ..., l$, we define the generic decision variables of the problem under consideration and $k \in \mathbb{N}$ in (1) describes the total amount of the facilities to be located. Elements a_{ij}, where

$$1 \geq a_{ij} \geq 0, \sum_{i=1,...,l} a_{ij} \geq 1,$$

are components of the so called "eligibility matrix"

$$A := \big(a_{ij}\big)_{j=1,...,n}^{i=1,...,l}$$

associated with the eligible sites that provide a covering of the demand points indexed by $j = 1, ..., n$. The admissible values of the elements of the matrix A are "distributed" on the interval $[0,1]$. Note that the second index in (1), namely, $i = 1, ..., l$ is related to the given "facilities sites". Finally, the admissible sets $\mathbb{B}^n$ and $\mathbb{B}^l$ in the main problem (1) are defined as follows:

$$\mathbb{B}^n := \{0,1\}^n, \quad \mathbb{B}^l := \{0,1\}^l.$$

Note that the objective functional $J(\cdot)$ from (1) has a linear structure. We use the following vectorial notation

$$z := (z_1, ..., z_n)^T, \quad y := (y_1, ..., y_l)^T.$$

The implicit dependence

$$J(z(y)) = \langle w, z \rangle,$$
$$w := (w_1, ..., w_n)^T$$

of the objective functional J on the vector y is given by the corresponding (componentwise) inequalities constraints

$$z \leq A^T y$$

in (1). By $\langle \cdot, \cdot \rangle$ we denote here the scalar product in the corresponding Euclidean space. A vector pair (z, y) that satisfies all the constraints in (1) is next called an admissible pair for the main problem (1). Note that the objective functional does not depend explicitly on the problem variable y.

The abstract optimization framework (1) provides a constructive and modelling approach for various practically oriented problems (see e.g., [1,9,11,13,18]). Following [14] we next call the main optimization problem (1) a Maximal Covering Location Problem (MCLP). Let us also refer to [24] for a detailed discussion on the applied interpretation of the MCLP (1). The main problem (1) is formulated under the general (non-binary) assumption related to the elements a_{ij} of the eligibility matrix A. This corresponds to a suitable modelling approach under incomplete information (see e.g., [10] and references therein). Roughly speaking every value of an admissible parameter a_{ij} in (1) has a fuzzy nature (similar to [8]). This fuzzy MCLP under consideration provides an adequate formal framework for the resilient Supply Chain Optimization (see Section 5). Let us also observe that the "resilience" concept is understood here as a kind of robustness of the optimization approach we develop. This robustness is considered with respect to a possible incomplete information about the main mathematical model (robustness with respect to uncertainties in the modelling approach). Note that the possible incompleteness of the mathematical model mentioned above and the robustness requirement for a selected optimization approach constitute the common (and adequate) attributes for a realistic Supply Chain optimal design.

The mathematical characterization of (1) can evidently be given in terms of the classic integer programming (see e., g. [11,16,19] for mathematical details). Let us note that (1) possesses an optimal solution (an optimal pair)

$$(z^{opt}, y^{opt}) \in \mathbb{B}^n \otimes \mathbb{B}^l,$$

where

$$z^{opt} := (z_1^{opt}, ..., z_n^{opt})^T,$$
$$y^{opt} := (y_1^{opt}, ..., y_l^{opt})^T.$$

This fact is a direct consequence of the basic results from [11,16,19]. Let us also note that the conventional problem (1) can also be easily extended to the "multi-valued" version, where the admissible sets $\mathbb{B}^n$ and $\mathbb{B}^l$ are replaced by

$$\tilde{\mathbb{B}}^n := \{0, 1, ..., N_n\}^n,$$
$$\tilde{\mathbb{B}}^l := \{0, 1, ..., N_l\}^l,$$

where $N_n,\ N_l \in \mathbb{N}$.

Our aim is to develop a simple and effective numerical approach to the sophisticated MCLP (1). Facility location has a decisive role in success of Supply Chains with applications in many production and service facilities. It has been a focal center of interest in the last century among practitioners and scholars. For a detailed introduction to location models, one may refer to [15,23,24]. In general the literature of covering models is too diverse to be exhaustively studied in this paper. Although some of known publications in the literature of MCLP are included in this paper, one may refer to valuable reviews for further information.

3. Analytical foundations of the separation method

We next separate the originally given MCLP (1) and introduce two auxiliary optimization problems. These formal constructions provide a necessary basis for the future numerical development. The first optimization problem can be formulated as follows

$$\text{(2)} \qquad \begin{aligned} &\text{maximize} \quad \sum_{j=1}^{n} \mu_j \sum_{i=1}^{l} a_{ij} y_i \\ &\text{subject to} \quad \begin{cases} \sum_{i=1}^{l} y_i = k, \quad y \in \mathbb{B}^l, \\ \mu_j \in [0,1] \quad \forall j = 1, ..., n \end{cases} \end{aligned}$$

The second auxiliary problem has the following specific form:

$$\text{(3)} \qquad \begin{aligned} &\text{maximize} \quad J(z) := \sum_{j=1}^{n} w_j z_j \\ &\text{subject to} \quad \begin{cases} z_j \leq \sum_{i=1}^{l} a_{ij} \hat{y}_i \\ z \in \mathbb{B}^n \end{cases} \end{aligned}$$

where $\hat{y} \in \mathbb{B}^l$ is optimal solution of problem (2). The components of $\hat{y}$ are denoted as $\hat{y}_i,\ i = 1, ..., l$. The existence of an optimal solution for (2) is a direct consequence of the results from [11,19]. The same is also true with respect to the auxiliary problem (3). Let

$$\hat{z} \in \mathbb{B}^n,\ \hat{z} := (\hat{z}_1, ..., \hat{z}_n)^T$$

be an optimal solution to (3). Evidently, problem (3) coincides with the originally given MCLP (1) in a specific case of a fixed variable $y = \hat{y}$. Let us note that in general $\hat{y} \neq y^{opt}$.

The first auxiliary problem, namely, problem (2) can be interpreted as a usual linear scalarization of the following multiobjective optimization problem (vector optimization):

$$\begin{aligned} &\text{maximize} \ \ \{\sum_{i=1}^{l} a_{i1} y_i, ..., \sum_{i=1}^{l} a_{in} y_i\} \\ &\text{subject to} \ \ \begin{cases} \sum_{i=1}^{l} y_i = k, \\ y \in \mathbb{B}^l \end{cases} \end{aligned} \tag{4}$$

Recall that a scalarizing of a multi-objective optimization problem is an adequate numerical approach, which means formulating a single-objective optimization problem such that optimal solutions to the single-objective optimization problem are Pareto optimal solutions to the multi-objective optimization problem. We next assume that the multipliers

$$\mu_j,\ j = 1, ..., n$$

in (2) are chosen by such a way that problems (2) and (4) are equivalent (see e.g., [2,11,19] for necessary details). In this particular case we call (2) an adequate scalarizing of (4). We discuss shortly the adequate scalizing in Section 4.3.

It is easy to see that problems (2) and (3) have a structure of a so-called Knapsack problem (see [16] and references therein). Various efficient numerical algorithms are recently proposed for a generic Knapsack problem. We refer to [16] for a comprehensive overview about the modern implementable numerical approaches to this basic optimization problem.

The relevance and the main motivation of the auxiliary optimization problems (2) and (3) introduced can be stated by the following abstract result.

Theorem 3.1. *Assume* (z^{opt}, y^{opt}) *is an optimal solution of* (1) *and* (2) *is an adequate scalarizing of* (4). *Let* $\hat{y}$ *be an optimal solutions of* (2) *and* $\hat{z}$ *be an optimal solution of the auxiliary problem* (3). *Then* (1) *and* (3) *possess the same optimal values, that is*

$$J(z^{opt}(y^{opt})) = J(\hat{z}). \tag{5}$$

Moreover, in the case problems (1), (2), *and* (3) *possess unique solutions we additionally have*

$$(z^{opt}, y^{opt}) = (\hat{z}, \hat{y}).$$

Proof. Since

$$\sum_{i=1}^{l} \hat{y}_i = k,$$

and

$$\hat{z}_j \leq \sum_{i=1}^{l} a_{ij} \hat{y}_i,$$

we conclude that $(\hat{z}, \hat{y})$ is an admissible pair for the original MCLP (1). Taking into account the definition of an optimal pair for problem (1), we next deduce

$$J(\hat{z}(\hat{y})) \leq J(z^{opt}(y^{opt})). \tag{6}$$

Let

$$\Gamma = \Gamma_z \otimes \Gamma_y \subset \mathbb{B}^n \otimes \mathbb{B}^l$$

be a solutions set (the set of all optimal solutions) for problem (1). We also define the solutions sets

$$\Gamma_{(2)} \subset \mathbb{B}^l, \quad \Gamma_{(3)} \subset \mathbb{B}^n$$

of problems (2) and (3), respectively. From (6) it follows that

$$\Gamma_{(3)} \otimes \Gamma_{(2)} \subset \Gamma. \tag{7}$$

Taking into account the restrictions associated with the variable y in (1) and (2), we next obtain

$$\Gamma_y \equiv \Gamma_{(2)}. \tag{8}$$

Since (2) is an adequate scalarization of the multi-objective maximization problem (4), we deduce

$$z_j \leq \max_{\substack{\sum_{i=1}^{l} y_i = k, \\ y \in \mathbb{B}^l}} \sum_{i=1}^{l} a_{ij} y_i.$$

This fact implies

$$\Gamma_z \subset \Gamma_{(3)}. \tag{9}$$

Inclusions (7), (9) and the basic equivalence (8) now imply the following crucial equivalence

$$\Gamma_{(3)} \otimes \Gamma_{(2)} \equiv \Gamma. \tag{10}$$

Taking into account the same form of the objective functionals in (1) and (2.3), we immediately obtain the basic relation (5). In a specific case of the one point sets Γ, $\Gamma_{(3)}$ and $\Gamma_{(2)}$ the expected relation

$$(z^{opt}, y^{opt}) = (\hat{z}, \hat{y})$$

is a direct consequence of (10). The proof is completed. □

Theorem 3.1 makes it possible to separate (decompose equivalently) the original sophisticated problem (1) into two relative simple optimization problems. It provides a theoretical basis for effective numerical approaches to the abstract MCLPs and to possible applications.

4. Numerical analysis of the auxiliary problems

This section is dedicated to the numerical aspects related to the two optimization problems obtained in Section 3. Our aim is to develop a resulting self-closed algorithm for an effective numerical treatment of the original MCLP (1).

4.1 A combinatorial algorithm for the first auxiliary problem

We first observe that the auxiliary optimization problem (2) has a simple combinatorial structure. It can be easily solved using the following natural scheme:

$$\begin{aligned} &\hat{y}_i = 1 \text{ if } i \in \hat{I}; \\ &\hat{y}_i = 0 \text{ if } i \in \{1, ..., l\} \setminus \hat{I} \end{aligned} \tag{11}$$

where

$$\begin{aligned} &\hat{I} := \{1 \leq i \leq l \mid S_{\mathcal{A}_i} \in \max_k \{S_{\mathcal{A}_1}, ..., S_{\mathcal{A}_l}\}\}, \\ &S_{\mathcal{A}_i} := \sum_{j=1}^{n} \mu_j a_{ij}, \\ &\mathcal{A}_i := (a_{i1}, ..., a_{in})^T. \end{aligned} \tag{12}$$

Here $\mathcal{A}_i$ is a vector of i-row of the eligibility matrix A and operator $\max_k$ determines an array of k-largest numbers from the given array. Evidently, the choice (11)-(12) determines an optimal solution of (2). Roughly speaking the combinatorial algorithm (11)-(12) assigns the maximal value $\hat{y}_i = 1$ for all vectors $\mathcal{A}_i$ which sum of components $S_{\mathcal{A}_i}$ belongs to the array of k-largest sums of components of all vectors

$$\mathcal{A}_i, \; i = 1, ..., l.$$

It is easy to see that for the given eligibility matrix A with the specific elements a_{ij} (determined in Section 2) the sum of components $S_{\mathcal{A}_i}$ constitutes a specific norm of the given vector $\mathcal{A}_i$. The total complexity of the combinatorial algorithm (11)-(12) can be easily calculated and is equal to

$$O(l \times \log k) + O(k).$$

We refer to [16] for the necessary details.

Let us denote

$$c := \sum_{j=1}^{n} \sum_{i=1}^{l} a_{ij} \hat{y}_i.$$

Then the inequality constraints in (3) imply the generic Knapsack-type constraint with uniform weights

$$\sum_{j=1}^{n} z_j \leq c.$$

We now present a fundamental solvability result for the second auxiliary optimization problem, namely, the Knapsack problem (3).

Theorem 4.1. *The* Knapsack *problem* (3) *can be solved in* $O(nc)$ *time and*

$$O(n+c)$$

space.

The formal proof of Theorem 4.1 can be found in [16].

4.2 A relaxation based approach and the resulting computational scheme

The theoretic and numerical results obtained above, namely, Theorem 1 and the combinatorial choice algorithm (11)-(12) provide a theoretic basis for a novel exact solution method for the originally given MCLP (1). We now need to establish an implementable solution procedure for the effective numerical treatment of the second auxiliary problem (3) from the obtained decomposition (2)-(2.3). This optimization problem, which is NP-hard, has been comprehensively studied in the last few decades and several exact algorithms for its solution can be found in the literature (see [16] and the references therein). Constructive algorithms for this Knapsack problems are mainly based on two basic numerical approaches: branch-and-bound and dynamic programming. Let us also mention here the corresponding combined approach.

In this paper we firstly consider the well-known Lagrange relaxation scheme in the context of the second auxiliary problem (problem (3)). "Relaxing a problem" has various meanings in applied mathematics, depending on the areas where it is defined, depending also on what one relaxes (a functional, the underlying space, etc.). We refer to [2,4-7,12, 21] for various relaxation techniques in the modern optimization. Introducing the Lagrange function

$$\mathcal{L}(z,\lambda) := \sum_{j=1}^{n} w_j z_j - \sum_{j=1}^{n} \lambda_j \big(z_j - \sum_{i=1}^{l} a_{ij} \hat{y}_i \big)$$

associated with the Knapsack problem (3), we obtain the following relaxed problem

$$\text{(13)} \qquad \begin{array}{ll} \text{maximize} & \mathcal{L}(z,\lambda) \\ \text{subject to} & z \in \mathbb{B}^n \end{array}$$

The relaxed problem (13) does not contain the originally given unpleasant inequality constraints. These constraints are now included into the objective function $\mathcal{L}(z,\lambda)$ from (13) as a penalty term

$$\sum_{j=1}^{n} \lambda_j \big(z_j - \sum_{i=1}^{l} a_{ij} \hat{y}_i\big).$$

Recall that all feasible solutions to (3) are also feasible solutions to (13). The objective value of feasible solutions to (3) is not larger than the objective value in (13) (see [16] for the necessary proofs). Thus, the optimal solution value to the relaxed problem (13) is an upper bound to the original problem (3) for any vector of nonnegative Lagrange multipliers

$$\lambda := (\lambda_1, ..., \lambda_n)^T, \\ \lambda_j \geq 0, \; j = 1, ..., n.$$

For a concrete numerical solution of the relaxed problem (13) we use here the classic branch-and-bound method (see e.g., [11,16]). In a branch-and-bound algorithm we are interested in achieving the tightest upper bound in (13). Hence, we would like to choose a vector of nonnegative multipliers

$$\hat{\lambda}^{\mathcal{L}} := (\hat{\lambda}_1^{\mathcal{L}}, ..., \hat{\lambda}_n^{\mathcal{L}})^T, \\ \hat{\lambda}_j^{\mathcal{L}} \geq 0, \; j = 1, ..., n$$

such that (13) is minimized. This evidently leads to the generic Lagrangian dual problem

$$\text{(14)} \qquad \begin{aligned} &\text{minimize} \;\; \mathcal{L}(z,\lambda) \\ &\text{subject to} \;\; \lambda \geq 0 \end{aligned}$$

It is well-known that the Lagrangian dual problem (14) yields the least upper bound available from all possible Lagrangian relaxations. The problem of finding an optimal vector of multipliers $\hat{\lambda}^{\mathcal{L}} \geq 0$ in (14) is in fact a linear programming problem [11,19]. In a typical branch-and-bound algorithm one will often be satisfied with a sub-optimal choice of multipliers $\lambda \geq 0$ if only the bound can be derived quickly. In this case sub-gradient optimization techniques can be applied [19]. The following analytic result is an immediate consequence of our main Theorem 1 and of the basic properties of the primal-dual system (13)-(14).

Theorem 4.2. *Let* $(\hat{z}^{\mathcal{L}}, \hat{\lambda}^{\mathcal{L}})$ *be an optimal solution of the primal-dual system* (13)-(14) *associated with the auxiliary problem* (3). *Assume that all conditions of* Theorem 1 *be satisfied. Then*

$$\text{(15)} \qquad J(z^{opt}(y^{opt})) \leq J(\hat{z}^{\mathcal{L}}).$$

The obtained estimation (15) *constitutes a tightest upper bound for the optimal value* $J(z^{opt}(y^{opt}))$.

We are now ready to formulate a complete (conceptual) algorithm for an effective numerical treatment of the basic MCLP (1).

Algorithm 1.

I. *Given an initial* MCLP (1) *separate it into two auxiliary problems* (2) *and* (3);

II. Apply the combinatorial algorithm (11)-(12) and compute $\hat{y}$;

III. Using $\hat{y}$, construct the Lagrange function $\mathcal{L}(z, \lambda)$ and solve the primal-dual system (13)-(14).

The numerical consistency of the proposed Algorithm 1 is an immediate consequence of the obtained main theoretic results, namely, of Theorem 3.1 and Theorem 4.2. Recall that the Lagrange relaxation scheme is usually applied to the original MCLP (1) (see e.g., [12,16]). In that case the resulting (relaxed) problem and the corresponding Lagrangian dual problem possess a higher complexity in comparison with the proposed "partial" Lagrange relaxation (13)-(14) associated with the original MCLP (1). This is a simple consequence of the proposed SM that reduces the initial problem (1) to two (more simple) auxiliary optimization problem (2)-(3). This fact makes it possible to apply the proposed separation methodology to the large-scale MCLPs that are important and realistic mathematical models for many practically oriented (optimal) decision making systems (see e.g., [7,9,10,14,15,18,20,22,23,24]).

4.3 A remark on the adequate scalarizing procedure

Let us now make a short remark related to the scalarizing procedure used above (see Section 3, problems (2)-(4)).

It can be shown analytically that the values $S_{\mathcal{A}_i}$ in (12) depend on the multipliers vector μ. This is a consequence of the inclusion (9). Recall that (9) constitutes a useful relation of the SM and for the resulting optimization strategy we propose. Since the obtained multiobjective maximation problem (5) has a linear structure, an adequate scalarizing makes it possible to determine every "non-dominant" points (see [11,19] for mathematical details).

On the other hand, a possible "non-adequate" selection of μ geometrically implies a significant "cutting" (restriction) of the feasible region for problem (3). This feasible region restriction can finally eliminate a true optimal solution. Recall that a scalarizing implemented in the objective function from (2) evidently determines the resulting geometry associated with the basic problem (3). On the other side the geometrical properties of a non-adequately scalarized problem can violate the conceptual condition (9).

5. Optimization of the resilient supply chain management system

This section is devoted to applications of the proposed SM to an optimal resilient Supply Chain Management for a system of manufacturing plants - warehouses. Note that the "resilience" of a Supply Chain Management System is modelled here by a fuzzy-type eligibility matrix A (see Section 2). We use here the notation from Section 4 and denote by $\mathcal{A}_i$ a vector of i-row of the eligibility matrix A $(i = 1, ..., l)$ such that

$$A = (\mathcal{A}_1^T ... \mathcal{A}_l^T)^T.$$

Let us firstly point the common applied meaning of the variables and parameters from the general MCLP (1) in the context of the resilient Supply Chain Management system. The binary variables

$$(z, y) \in \mathbb{B}^n \otimes \mathbb{B}^l$$

constitute the main "decision variables" of the problem under consideration. The vector of weights w can be interpreted as a rentability of the final product. Therefore, the maximization of the cost functional $J(\cdot)$ in (1) expresses the maximization of the total profit (total income) generated by the designed Supply Chain system. The complete "decision resource" associated with the decision variable (vector) y is restricted in (1) by a constant (parameter) $k \in \mathbb{N}$. The eligibility matrix "A" is in fact a useful linear modelling framework that establishes the natural relation between the "producer" decision and "recipient". This relation is formally given by the corresponding elements a_{ij} of the matrix A. Our aim now is to apply the developed SM to two practically oriented examples of the optimal Supply Chain Management design in a classic manufactures - warehouses system.

Example 5.1. The simple Supply Chain system that include $n = 8$ manufacturing plants and $l = 5$ warehouses is indicated on Fig. 1.

We also assume that

$$a_{ij} + a_{i'j} \geq 1, \quad i = 1, ...5 \; j = 1, ...8.$$

Here i' is an index that corresponds to a resilient cover of a demand point. The last condition means that at least two feasible facilities (warehouse) cover a given demand point (the manufacturing plants). The corresponding eligibility matrix A is given as follows:

$$A^T = \begin{pmatrix} 0.81286 & 0.0 & 0.0 & 0.62968 & 0.0 \\ 0.25123 & 0.58108 & 0.32049 & 0.89444 & 0.79300 \\ 0.0 & 0.0 & 0.64850 & 0.91921 & 0.94740 \\ 0.54893 & 0.90309 & 0.74559 & 0.50869 & 0.99279 \\ 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.77105 & 0.27081 & 0.65883 & 0.60434 & 0.23595 \\ 0.0 & 0.51569 & 0.0 & 0.0 & 0.57810 \\ 0.64741 & 0.91733 & 0.60562 & 0.63874 & 0.71511 \end{pmatrix}$$

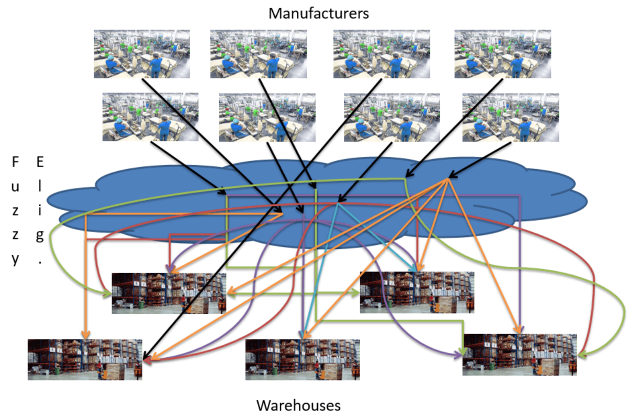


Figure 1: Fuzzy eligibility model

The objective weights

$$w_j \in \mathbb{R}_+, \quad j = 1, ..., 8$$

indicate the service priority and are selected in this example as follows

$$w = (32.0,\ 19.0,\ 41.0,\ 26.0\ 37.0\ 49.0\ 50.0\ 11.0)^T.$$

Note that the fifth demand point in this example has no "resilient" character (only one facility covers this point). We assume that the Supply Chain decision maker is interested opens $k = 2$ facilities. That means

$$\sum_{i=1}^{5} y_i = 2.$$

Moreover, we also define the necessary row vectors (see Section 3) for the combinatorial algorithm (11)-(12):

$$\begin{array}{ll} S_{\mathcal{A}_1} = 8.06295 & S_{\mathcal{A}_2} = 5.86033 \\ S_{\mathcal{A}_3} = 5.30955 & S_{\mathcal{A}_4} = 7.47098 \\ S_{\mathcal{A}_5} = 6.99921 & \end{array}$$

Application of the basic Algorithm 1 leads to the following computational results:

$$\begin{aligned} z^{opt} &= (1,\ 1,\ 0,\ 1,\ 1,\ 1,\ 0,\ 1)^T, \\ y^{opt} &= (1,\ 0,\ 0,\ 1,\ 0)^T, \end{aligned} \tag{16}$$

The corresponding (maximal) value of the objective functional is equal to

$$J(z^{opt}(y^{opt})) = \max_{Problem(1)} J(z(y)) = 174.0$$

Let us also note that the computed scalarizing multiplier μ in the auxiliary problem (2) for the given problem data is equal to

$$\mu = (2.0,\ 2.0,\ 1.0,\ 2.0,\ 2.0,\ 2.0,\ 1.0,\ 2.0)^T.$$

The practical implementation of Algorithm 1 was carried out by using the standard Python package and an author-written program.

For comparison, the given MCLP problem was also solved by a direct application of the standard CPLEX optimization package. We use the concrete problem parameters given above and obtain the same optimal pair as in (16). The CPLEX integer programming solver proceeds with 6 MIP simplex iterations and 0 branch-and-bound nodes for in total 13 binary variables and 9 linear constraints.

Example 5.2. We now consider a formal extension of the previous example (for a double dimension) and put $n = 16,\ l = 10,\ k = 5$. Let

$$w = (29.0,\ 37.0,\ 22.0,\ 42.0,\ 26.0,\ 14.0,\ 27.0,\ 30.0,$$
$$46.0,\ 16.0,\ 10.0,\ 36.0,\ 33.0,\ 39.0,\ 46.0,\ 49.0)^T.$$

The eligibility matrix A is given by rows:

$\mathcal{A}_1 = (0.846109459436,\ 0.0,\ 0.0,\ 0.582693667799, 0.964574511054,\ 0.798899459366,\ 0.0,\ 0.0,\ 1.0,$
$0.300320432977,\ 0.997688107849,\ 0.3335795069, 0.49602683501, 1.0,\ 0.0,\ 0.374671961499),^T$

$\mathcal{A}_2 = (0.0,\ 1.0,\ 0.0,\ 0.0,\ 0.741552391071, 0.537788748272, 0.883796533814,\ 0.585368404373,\ 0.0,$
$0.860903890172, 0.958028639759, 0.0, 0.186896812387, 0.0, 0.968601622008,\ 0.579580096602)^T,$

$\mathcal{A}_3 = (0.407084305512,\ 0.0,\ 0.565187029512, 0.0, 0.420858280659, 0.361836079442,\ 0.472471488805,$
$0.0, 0.0,\ 0.696525107652, 0.436819747759,\ 0.0, 0.587300759229,\ 0.0,\ 0.347864951313,\ 0.0)^T,$

$\mathcal{A}_4 = (0.208102698902,\ 0.0,\ 0.0,\ 0.0,\ 0.0, 0.346461956794, 0.0,\ 0.0,\ 0.0,\ 0.768124612788,$
$0.413970925056, 0.0, 0.97348389961,\ 0.0,\ 0.0,\ 0.0)^T,$

$\mathcal{A}_5 = (0.0,\ 0.0,\ 0.965589029405,\ 0.0,\ 0.0, 0.893792904298,\ 0.0,\ 0.723969499937,\ 0.0,$
$0.562381237935, 0.78216104002,\ 0.557958082269, 0.671624833192, 0.0,\ 0.601221801206,\ 0.0)^T,$

$\mathcal{A}_6 = (0.0,\ 0.0,\ 0.0,\ 0.7732353822,\ 0.0, 0.930557571029, 0.0,\ 0.427721730484,\ 0.0,$
$0.818424694417, 0.795450242494, 0.314453291276, 0.645666417485,\ 0.0, 0.0,\ 0.0)^T,$

$$\mathcal{A}_7 = (0.0,\ 0.0,\ 0.0,\ 0.71613857057, 0.0, 0.573866657173, 0.0,\ 0.692538237821,\ 0.0,\\ 0.296797567788,\ 0.306871729419, 0.334127066948, 0.0, 0.0,\ 0.0,\ 0.976783604764)^T,$$

$$\mathcal{A}_8 = (0.448086601628,\ 0.0,\ 0.888380378484, 0.576276602931, 0.939065250623,\ 0.0,\ 0.0,\\ 0.773234003255,\ 0.0, 0.414398315721, 0.203669220313,\ 0.35600682894, 0.523619957827,\\ 0.0, 0.0,\ 0.527029464076)^T,$$

$$\mathcal{A}_9 = (0.964964029806,\ 0.0,\ 0.0,\ 0.562565185744, 0.0, 0.0, 0.0,\ 0.0,\ 0.0,\ 0.773260049125,\\ 0.468988424786, 0.0, 0.0,\ 0.0,\ 0.0,\ 0.794463270734)^T,$$

$$\mathcal{A}_{10} = (0.0,\ 0.0,\ 0.0,\ 0.545222010668,\ 0.0, 0.0,\ 0.536645142919, 0.212898303253,\ 0.0,\\ 0.197891148706,\ 0.471120100438, 0.0,\ 0.0, 0.0,\ 0.0,\ 0.0)^T.$$

The basic Algorithm 1 was applied to this example. We obtain the following optimal solution:

$$z^{opt} = (1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1, 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1)^T,$$
$$y^{opt} = (1,\ 1,\ 1,\ 0,\ 1, 0,\ 0,\ 1,\ 0,\ 0)^T.$$

The obtained scalarizing multiplier μ in the auxiliary problem (2) for the given problem data is equal to

$$\mu = (2.0,\ 2.0,\ 4.0,\ 4.0,\ 2.0,\ 0.0,\ 8.0,\ 2.0,\ 1.0,\ 2.0,\ 6.0,\ 1.0,\ 0.0,\\ 1.0,\ 1.0,\ 1.0)^T.$$

Finally, the calculated optimal value of the objective functional is equal to

$$J(z^{opt}(y^{opt})) = \max_{Problem(1)} J(z(y)) = 502.$$

Let us note that the successful application of the proposed computationalalgorithm to the above high-dimensional problem indicates a possible usability of this approach in the effective solution procedures of large-scale MCLPs.

Finally let us note that the CPLEX based comparatively analysis and the computational results obtained in Example 5.1 and Example 5.2 illustrate the realisability and effectiveness of the Separation Method developed in our paper.

6. Conclusion

In this contribution, we proposed a conceptually new numerical approach to a wide class of Maximal Covering Location Problems with the fuzzy-type eligibility matrices. This computational algorithm is next applied to the optimal design of a practically motivated Resilient Supply Chain Management System. The developed computational scheme is based on a novel separation approach to the initially given maximization problem. The SM we propose makes it possible to reduce the original sophisticated problem to two Knapsack-type optimization problems. The first one constitutes a generic linear scalarization of a multi-objective optimization problem and the second auxiliary problem is a version of the classic Knapsack formulation. Application of the conventional Lagrange relaxation in combination with a specific combinatorial algorithm leads to an implementable algorithm for the given fuzzy-type Maximal Covering Location Problem.

Theoretical and computational methodologies we present in this contribution can be applied to various generalizations of the basic MCLP. One can combine the elaborated separation scheme with the conventional branch-and-bound method, with the celebrated dynamic programming approach or/and with an alternative exact or heuristic numerical algorithm. Let us finally note that we discussed here only main theoretic aspects of the newly elaborated approach and presented the corresponding conceptual solution procedure. The basic methodology we developed needs further comprehensively numerical examinations that includes solutions of several MCLPs.

References

[1] G. Alexandris, I. Giannikos, *A new model for maximal coverage exploiting GIS capabilities,* European Journal of Operational Research, 202 (2010), 328-338.

[2] K. Atkinson, W. Han, *Theoretical Numerical Analysis*, Springer, New York, 2005.

[3] H. Aytug, C. Saydam, *Solving large-scale maximum expected covering location problems by genetic algorithms: a comparative study,* European Journal of Operational Research, 141 (2002), 480-494.

[4] V. Azhmyakov and W. Schmidt, *Approximations of relaxed optimal control problems,* Journal of Optimization Theory and Applications, 130 (2006), 61-77.

[5] V. Azhmyakov, M. Basin, C. Reincke-Collon, *Optimal LQ-type switched control design for a class of linear systems with piecewise constant inputs,* in: Proceedings of the 19th IFAC World Congress, Cape Town, South Africa, 2014, 6976-6981.

[6] V. Azhmyakov, J. Cabrera, A. Poznyak, *Optimal fixed - levels control for non - linear systems with quadratic cost functionals,* Optimal Control Applications and Methods, 37 (2016), 1035-1055.

[7] V. Azhmyakov, J.P. Fernandez-Gutierrez, St. Pickl, *A novel numerical approach to the resilient MCLP based supply chain optimization,* in: Proceedings of the 12th IFAC Workshop on Intelligent Manufacturing Systems, Austin, USA, 2016, 145-150.

[8] V. Batanovic, D. Petrovic, R. Petrovic, *Fuzzy logic based algorithms for maximum covering location problems,* Information Sciences, 179(2009), 120-129.

[9] O. Berman, J. Kalcsics, D. Krass, S. Nickel, *The ordered gradual covering location problem on a network,* Discrete Applied Mathematics, 157 (2009), 3689-3707.

[10] O. Berman, J. Wang, *The minmax regret gradual covering location problem on a network with incomplete information of demand weights,* European Journal of Operational Research, 208 (2011), 233-238.

[11] D. Bertsekas, *Nonlinear Programming*, Athena Scientific, Belmont, USA, 1995.

[12] R.D. Galvao, L.G. Acosta Espejo, B. Boffey, *A comparison of Lagrangean and surrogate relaxations for the maximal covering location problem,* European Journal of Operational Research, 124(2000), 377-389.

[13] M.S. Canbolat, M. von Massow, *Planar maximal covering with ellipses,* Computers and Industrial Engineering, 57(2009), 201-208.

[14] R.L. Church, C.S ReVelle, *The maximal covering location problem,* Papers of the Regional Science Association, 32 (1974), 101-118.

[15] G. Ji, S. Han, *A strategy analysis in dual-channel supply chain based on effort levels,* in: Proceedings of the 1th International Conference on Service Systems and Service Management, Beijing, China, 2014, 2161-1890.

[16] H. Kellerer, U. Pferschy, D. Pisinger, *Knapsack Problem*, Springer, Berlin, 2004.

[17] A. Mitsos, B. Chachuat and P.I. Barton, *McCormic - based relaxation algorithm,* SIAM Journal on Optimization, 20 (2009), 573-601.

[18] G.C. Moore, C.S. ReVelle, *The hierarchical service location problem,* Management Science, 28 (1982), 775-780.

[19] E. Polak, *Optimization*, Springer-Verlag, New York, USA, 1997.

[20] C. ReVelle, M. Scholssberg, J. Williams, *Solving the maximal covering location problem with heuristic concentration,* Computers and Operations Research, 35 (2008), 427-435.

[21] T. Roubicek, *Relaxation in Optimization Theory and Variational Calculus*, De Gruyter, Berlin, 1997.

[22] H. Shavandi, H. Mahlooji, *A fuzzy queuing location model with a genetic algorithm for congested systems,* Applied Mathematics and Computation, 181 (2006), 440-456.

[23] P. Sitek, J. Wikarek, *A hybrid approach to modeling and optimization for supply chain management with multimodal transport,* in: Proceedings of the 18th International Conference on Methods and Models in Automation and Robotics, Miedzyzdroje, Poland, 2013, 777-782.

[24] F. Zarandi, A. Haddad Sisakht, S. Davari, *Design of a closed-loop supply chain (CLSC) model using an interactive fuzzy goal programming,* The International Journal of Advanced Manufacturing Technology, 56 (2011), 809-821.

Accepted: 28.06.2017

CHAPTER 3

Paper on Dynamic Model

This submitted paper [41] introduce a switched dynamic structure to a specific class of Maximal Covering Location Problems, namely DCMLP, and propose a newly developed computational optimization approach for it. It dues to the complexity of the switched-type dynamic constraints, the "generalization" of Separation Method (SM) mentioned below constitutes a challenging theoretic and computational problem. Most of the results obtained for the MCLP address the "static" case where an optimal decision is determined on a fixed time-period. The SM uses a generic geometrical structure of the constraints under consideration makes it possible to separate the originally given dynamic optimization problem and reduce it to a specific family of relative simple auxiliary problems. The generalized (SM) for the DMCLP with a switched structure leads to a computational solution scheme. This extension constitutes a conceptually new solution approach and includes a necessary formal characterization of the resulting optimization problems obtained during the proposed separation procedure, and the computational aspects. The proposed SM based algorithm is applied to a concrete application of the proposed extension of the classic MCLP, namely, the dynamic version of the SM to a problem from the telecommunication engineering, as well as to find an optimal design of a (dynamic) mobile network configuration.

Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

doi:10.3934/xx.xx.xx.xx

pp. **X–XX**

A SEPARATION BASED OPTIMIZATION APPROACH TO DYNAMIC MAXIMAL COVERING LOCATION PROBLEMS WITH SWITCHED STRUCTURE

VADIM AZHMYAKOV*

Department of Basic Science,
Universidad de Medellín, Medellín, Colombia

JUAN P. FERNÁNDEZ-GUTIÉRREZ

Department of Basic Science,
Universidad de Medellín, Medellín, Colombia

ERIK I. VERRIEST

School of Electrical and Computer Engineering,
Georgia Institute of Technology, Atlanta, USA

STEFAN W. PICKL

Department of Computer Science,
Universität der Bundeswehr München, München, Germany

(Communicated by the associate editor name)

ABSTRACT. This paper extends a newly developed computational optimization approach to a specific class of Maximal Covering Location Problems (MCLPs) with a switched dynamic structure. Most of the results obtained for the conventional MCLP address the "static" case where an optimal decision is determined on a fixed time-period. In our contribution we consider a dynamic MCLP based optimal decision making and propose an effective computational method for the numerical treatment of the switched-type Dynamic Maximal Covering Location Problem (DMCLP). A generic geometrical structure of the constraints under consideration makes it possible to separate the originally given dynamic optimization problem and reduce it to a specific family of relative simple auxiliary problems. The generalized Separation Method (SM) for the DMCLP with a switched structure finally leads to a computational solution scheme. The resulting numerical algorithm also includes the classic Lagrange relaxation. We present a rigorous formal analysis of the DMCLP optimization methodology and also discuss computational aspects. The proposed SM based algorithm is finally applied to a practically oriented example, namely, to an optimal design of a (dynamic) mobile network configuration.

1. **Introduction.** Applications of diverse methods from the modern Mathematical Optimization Theory and the corresponding numerical techniques are nowadays a usual and efficient approach to the development of engineering applications (see e.g., [2, 3, 4, 5, 7, 10, 11, 23, 26, 31, 32]). Optimal facility location methodology, amongst others, plays an important role in a success of Supply Chains and provides an important analytic tool for many real-world manufacturing and service problems [6, 8, 14, 16, 18, 20, 22, 25, 30, 34].

2010 *Mathematics Subject Classification.* Primary: 90C10, 49M30; Secondary: 90C30.
Key words and phrases. dynamic MCLP, optimization of switched systems, dynamic integer programming, separation method.

* Corresponding author: V. Azhmyakov.

Let us recall that the conventional Maximal Covering Location Problem (MCLP) gives an optimal solution to cover a set of demands such that an objective is to be maximized. The basic MCLP was introduced by Church and ReVelle [14] and thereafter numerous practically important applications, theoretical and computational extensions to the classical MCLP have been developed. Let us mention here some known applications of the conventional MCLP for optimal location of industrial plants, landfills, hubs, cross-docks, networks, etc (see e.g., [8, 9, 13, 16, 17, 18, 19, 20, 22, 25, 29, 30, 34]). We also refer to [6] and references therein.

In our contribution, we propose a switched-type dynamic extension of the MCLP model with an incomplete information. We study a specific MCLP-type optimization problem with dynamic constraints. These constraints have a switched structure (depend on the switching time intervals) for some given switching times. The incomplete information of the Dynamic Maximal Covering Location Problem (DMCLP) under consideration is modelled by including the fuzzy-type eligibility matrices into the problem formulation. These two conceptual modifications of the generic MCLP involve more usability of the resulting DMCLP at the modelling stage and make it possible to incorporate the "resilience" or (and) "fuzzy" properties into the modelling approach. We give a self-closed and mathematically rigorous introduction to the new class of MCLP-type optimization problem, namely, to the DMCLPs and also develop a relative simple and implementable computational approach. In fact, the proposed methodology generalise the newly elaborated approach to the classic MCLP (see e.g. [6]).

Recall that a variety of computer oriented approaches have been proposed for an effective computational treatment of distinct classes of "static" MCLPs. Recently several heuristical methods are actively used in the practical numerical treatment of the MCLPs. We refer to [8, 17, 27, 29] for some effective heuristic and meta-heuristic algorithms and for further references. Note that heuristics and meta-heuristics have usually been employed in order to solve large size MCLPs (see e.g., [18, 19, 27, 29, 30]). However, the exact and various heuristic methods for the conventional MCLP are not sufficiently extended to a class of dynamic MCLP-type problems with the switched structure. The solution procedure we propose, namely, the generalized version of the Separation Method (SM) is based on an exact optimization procedure. The analytic method we propose can also be easily incorporated (as an independent exact solution tool) into some heuristic procedures.

The optimization approach we follow in our paper includes an equivalent transformation (called "separation") of the originally given DMCLP and further consideration of two auxiliary dynamic Knapsack-type problems (see e.g., [23] and references therein). The proposed dynamic version of the SM reduces the complexity of the original optimization problem in the presence of dynamic constraints. In this paper, we additionally use the celebrated Lagrange relaxation scheme for the purpose of a concrete computation [6, 12]. Let us also refer to [2, 7, 24, 28] for some advanced relaxation schemes of dynamic optimization.

It is necessary to stress that due to the extreme complexity of the general switched-type dynamic constraints and in particular to the complexity of the resulting dynamics the "generalization" of SM mentioned above constitutes a challenging theoretic and computational problem and cannot be considered as a simple "theory / facts transfer" from the conventional MCLP theory. Theoretic and numerical results obtained in this paper are next applied to a practically motivated example from the area of telecommunication engineering. The optimization problem we study in this example constitutes a (simplified) special case of the general Restricted Covering Problem (RCP) from the theory of mobile communication networks. Due to the dynamic nature of the communicative processes we try to maximize an average covering for a given system of the radio base stations. The requested optimal covering "design" and the resulting optimal management (decision making) can be formalized here as a specific switched DMCLP. In that case the natural information incompleteness of the used model can be adequately described by an eligibility matrix with a fuzzy structure.

The remainder of our paper is organized as follows: Section 2 contains a mathematically rigorous DMCLP problem formulation and some necessary theoretical concepts and facts. In Section 3 we extend the existing (static) SM to the switched-type dynamic MCLP. This extension constitutes a conceptually new solution approach and includes a necessary formal characterization of the resulting optimization problems obtained during the proposed separation procedure. Section 4 contains the concrete numerical schemes for the auxiliary optimization problems. Using the equivalence between the initially given and auxiliary problems (established in the previous sections), we finally develop a new implementable algorithm for a consistent numerical treatment of the initial DMCLP. The algorithm we consider also incorporates the Lagrange relaxation technique. Section 5 contains a concrete application of the proposed theoretic and numerical extensions of the classic MCLP techniques, namely, the dynamic version of the SM to a problem from the telecommunication engineering. We study a problem of the optimal covering of a cellular (mobile) communication network. This engineering examples shows the practical usability of the new solution approach we propose. It also illustrate the implementability and effectiveness of the resulting computational algorithm. Section 6 summarizes our contribution.

2. **Problem Formulation and Preliminaries.** We study a specific case of the integer programming problem with some dynamic variables and parameters

$$\text{maximize } \ J(z(\cdot)) := \frac{1}{(t_f - t_0)} \int_{t_0}^{t_f} \sum_{j=1}^{n} w_j(t) z_j(t) dt$$

subject to

$$\begin{cases} \sum_{i=1}^{l} y_i(t) = k^s, \ t \in (t_{s-1}, t_s], \\ z_j(t) \leq \sum_{i=1}^{l} a_{ij}^s y_i(t), \ t \in (t_{s-1}, t_s], \\ z(t) \in \mathbb{B}^n, \ y(t) \in \mathbb{B}^l, \ t \in [t_0, t_f], \\ s = 1, ..., S \in \mathbb{N}, \\ j = 1, ..., n \in \mathbb{N}, \ i = 1, ..., l \in \mathbb{N}. \end{cases} \tag{1}$$

Here $w_j(t) \in \mathbb{R}_+, \ j = 1, ..., n$ are the given dynamic weights of a demand node j for $t \in [t_0, t_f]$ and $z_j(t)$ is a binary "state" variable which is equal to 1 if a j-demand node is covered by at least one facility at the time t, otherwise $z_j(t) = 0$. By $y_i(t)$ we denote a binary "decision" variable which is equal to 1 if a i-facility is opened at the time t, otherwise $y_i(t) = 0$. We next assume that the complete operational time interval $[t_0, t_f]$ for the given model is divided into S adjoint intervals

$$(t_{s-1}, t_s], \ s = 1, ..., S$$

by switching times:

$$t_0 < t_1 ... t_{S-1} < t_S = t_f.$$

Additionally $k^s \in \mathbb{N}$ for $s = 1, ..., S$ in (1) describes a total number of facilities to be located on every time-interval $(t_{s-1}, t_s]$. Evidently, $k^s \leq l$ for every $s = 1, ..., S$. Coefficients

$$a_{ij}^s, \ s = 1, ..., S,$$

where

$$1 \geq a_{ij}^s \geq 0, \ \sum_{i=1}^{l} a_{ij}^s \geq 1,$$

are constant on every interval $(t_{s-1}, t_s]$ and constitute components of the "eligibility matrix" associated with the optimization problem (1)

$$A^s := (a_{ij}^s)_{j=1,...,n}^{i=1,...,l}.$$

This matrix describes a "resilient" (or fuzzy) covering of the demand nodes indexed by $j = 1, ..., n$ for every specific time-interval

$$(t_{s-1}, t_s], \ s = 1, ..., S.$$

Let us note that the index $i = 1, ..., l$ in (1) is related to the given facilities. The admissible sets $\mathbb{B}^n$ (sometimes called "state space") and $\mathbb{B}^l$ ("decision space") in (1) are defined as follows:

$$\mathbb{B}^n := \{0,1\}^n, \ \mathbb{B}^l := \{0,1\}^l.$$

We use here the natural notation

$$z := (z_1, ..., z_n)^T, \ y := (y_1, ..., y_l)^T.$$

Motivated from practical applications we next assume that all dynamic components in the basic problem (1) have a structure of piecewise-constant functions determined on the full time interval $[t_0, t_f]$:

$$w_j(t) = c_j^s > 0 \ \forall t \in (t_{s-1}, t_s],$$
$$z_j(t), \ y_i(t) = 0 \text{ or } 1 \ \forall t \in (t_{s-1}, t_s], \ s = 1, ..., S$$

where $j = 1, ..., n$ and $i = 1, ..., l$. In this specific case we evidently have

$$J(z(\cdot)) := \sum_{s=1}^{S} \frac{(t_s - t_{s-1})}{(t_f - t_0)} \sum_{j=1}^{n} (w_j(t) z_j(t))_{|_{t \in (t_{s-1}, t_s]}} =$$
$$\sum_{s=1}^{S} \frac{(t_s - t_{s-1})}{(t_f - t_0)} \langle w(t), z(t) \rangle_{|_{t \in (t_{s-1}, t_s]}}.$$

Here $w := (w_1, ..., w_n)^T$. By $\langle \cdot, \cdot \rangle$ we denote here the scalar product in the corresponding Euclidean space. As we can see the objective functional $J(\cdot)$ in (1) has a linear structure and can be interpreted as optimal average costs over the complete operating time-interval. Note that the implicit dependence of $z(\cdot)$ on the decision function $y(\cdot)$ is given by the inequalities constraints

$$z(t) \leq (A^s)^T y(t),$$
$$t \in (t_{s-1}, t_s], \ s = 1, ..., S$$

in (1). A pair $(z(\cdot), y(\cdot))$ of piecewise-constant functions that satisfies all the constraints in (1) is next called an admissible pair of this problem.

The basic optimization framework (1) provides a useful modelling approach to the variety of real-world applications (see e.g., [8, 9, 13, 16, 17, 18, 19, 20, 22, 25, 29, 30, 34]). Following [18] we next call the main optimization problem (1) a Dynamic Maximal Covering Location Problem (DMCLP). Evidently, the given DMCLP has a specific switched dynamic structure. Let us also refer to [6, 17] for a detailed discussion on the conventional (static) MCLPs and some possible generalizations. The main DMCLP (1) is formulated here using the general (non-binary) values of the elements a_{ij}^s of matrix A^s. This fact is motivated by a possible incomplete ”eligibility” information in the practical optimal design (see e.g., [8, 9] and references therein). The resulting abstract framework can also be interpreted as a ”resilience” modelling approach. In that case an admissible value of a parameter a^s_{ij} has a fuzzy nature (see e.g., [6, 29]). The fuzzy DMCLP (1) provides an adequate formal model for many applied engineering problems, for example, for the optimal mobile networking design and for the optimization of Resilient Supply Chain Management Systems (RSCMSs). We refer to [6, 30] for the necessary technical details and some interesting example.

The DMCLP under consideration has a structure of an integer programming problem (see e.g., [10, 14, 26] for mathematical details). Let us note that (1) possesses an optimal solution (an optimal pair) $(z^{opt}(\cdot), y^{opt}(\cdot))$

$$(z^{opt}(t), y^{opt}(t)) \in \mathbb{B}^n \bigotimes \mathbb{B}^l,$$

where

$$z^{opt} := (z_1^{opt}, ..., z_n^{opt})^T, \; y^{opt} := (y_1^{opt}, ..., y_l^{opt})^T.$$

The main optimization problem (1) has an evident switched structure related to the given time-intervals $(t_{s-1}, t_s], \; s = 1, ..., S$. For the objective functional $J(\cdot)$ in (1) we obtain the natural decomposition

$$J(z(\cdot)) = \sum_{s=1}^{S} J_s(z(\cdot)), \tag{2}$$

where

$$J_s(z(\cdot), y(\cdot)) := \frac{(t_s - t_{s-1})}{(t_f - t_0)} \langle w(t), z(t) \rangle_{|_{t \in (t_{s-1}, t_s]}}.$$

The following theorem uses the additivity property (2) of the objective functional and is in fact a compilation of the celebrated Bellman Optimality Principle from Optimal Control (see e.g., [7, 26]).

Theorem 2.1. *The restriction $(z^{opt}(\cdot), y^{opt}(\cdot))_s$ of an optimal solution $(z^{opt}(\cdot), y^{opt}(\cdot))$ on the time interval $(t_{s-1}, t_s], \; s = 1, ..., S$ is an optimal solution to the following particular* MCLP

$$\begin{aligned} &\text{maximize } \; J_s(z(\cdot)) \\ &\text{subject to} \\ &\begin{cases} \sum_{i=1}^{l} y_i(t) = k^s, \; t \in (t_{s-1}, t_s], \\ z_j(t) \leq \sum_{i=1}^{l} a^s_{ij} y_i(t), \; t \in (t_{s-1}, t_s], \\ z(t) \in \mathbb{B}^n, \; y(t) \in \mathbb{B}^l, \; t \in [t_{s-1}, t_s], \\ s \in \mathbb{N}, \; s \leq S, \\ j = 1, ..., n \in \mathbb{N}, \; i = 1, ..., l \in \mathbb{N}. \end{cases} \end{aligned} \tag{3}$$

The presented results means that the total optimal solution $(z^{opt}(\cdot), y^{opt}(\cdot))$ can be found sequentially. This optimal solution constitutes a formal union of optimal solutions for the particular MCLPs of the type (3) determined on intervals

$$(t_{s-1}, t_s], \; s = 1, ..., S.$$

This fact is an immediate consequence of the ”independence” of the particular problems of the type (3): an optimal solution of problem (3) for $s+1$ does not depend on the previous solution to MCLP (3) with the index s.

We refer to [7, 33] for some general theoretic and computational results related to switched dynamic optimization problems. The aim of our contribution is to propose an

10 Dec 2018 20:30:57 PST
Version 1 - Submitted to J. Ind. Manag. Optim.

effective approach for the numerical treatment of the sophisticated DMCLP (1). We generalize here the newly elaborated ("static") separation method for this purpose (see [6]) and next combine it with the Lagrange relaxation scheme (see e.g., [23]).

3. **Theoretical Foundations of the General Separation Method.** Let us now introduce a sequence Pr^S of the following auxiliary problems (indicated by $s = 1,...,S$):

$$\begin{aligned} &\text{maximize} \quad \sum_{j=1}^{n} \mu_j^s \sum_{i=1}^{l} a_{ij}^s y_i(t),\ t \in (t_{s-1}, t_s], \\ &\text{subject to} \\ &\begin{cases} \sum_{i=1}^{l} y_i(t) = k^s,\ t \in (t_{s-1}, t_s], \\ y(t) \in \mathbb{B}^l,\ t \in (t_{s-1}, t_s], \\ \mu_j^s \in [0,1],\ j = 1,...,n, \end{cases} \end{aligned} \tag{4}$$

Assume $\hat{y}^s(\cdot)$, where $\hat{y}^s(t) \in \mathbb{B}^l,\ s = 1,...,S$ are optimal solutions to the given problems (4). The components of $\hat{y}^s(t)$ are denoted by $\hat{y}_i^s(t),\ i = 1,...,l$. In parallel to (4) consider the single auxiliary problem

$$\begin{aligned} &\text{maximize} \quad J(z(\cdot)) \\ &\text{subject to} \\ &\begin{cases} z_j(t) \le \sum_{i=1}^{l} a_{ij}^s \hat{y}_i^s(t),\ t \in (t_{s-1}, t_s], \\ z(t) \in \mathbb{B}^n,\ t \in [t_0, t_f] \\ s = 1,...,S,\ j = 1,...,n. \end{cases} \end{aligned} \tag{5}$$

Note that the existence of optimal solutions to the auxiliary problems (4) and (5) is a direct consequence of the general results from [10, 23, 26]. Let us also underline here that every problem from Pr^S can be solved independently from problem (5). Therefore, we next define a total optimal vector $\hat{y}(t) \in \mathbb{B}^l$ for Pr^S determined by the natural composition

$$\hat{y}(t) := \hat{y}^s(t)\ \forall t \in [t_0, t_f],\ s = 1,...,S. \tag{6}$$

By $\hat{z}(\cdot)$, where $\hat{z}(t) \in \mathbb{B}^n$ and $\hat{z}(\cdot) := (\hat{z}_1(\cdot),...,\hat{z}_n(\cdot))^T$, we next denote an optimal solution to the auxiliary problem (5). It is easy to see that problem (5) coincides with the originally given DMCLP (1) for the specific case

$$y(\cdot) = \hat{y}(\cdot).$$

In the general case we evidently have $\hat{y}(\cdot) \neq y^{opt}(\cdot)$ and hence

$$(z^{opt}(\cdot), y^{opt}(\cdot)) \neq (\hat{z}(\cdot), \hat{y}(\cdot)).$$

We next call the pair $(\hat{z}(\cdot), \hat{y}(\cdot))$ an optimal solution of the family of auxiliary problems (4)-(5).

Every problem (for a fixed index s) from the family Pr^S of problems (4) was obtained by a standard linear scalarization of the following multiobjective optimization problem (sometimes also called "vector optimization"):

$$\begin{aligned} &\text{maximize} \quad \{\sum_{i=1}^{l} a_{i1}^s y_i(t), ..., \sum_{i=1}^{l} a_{in}^s y_i(t)\} \\ &\text{subject to} \\ &\begin{cases} \sum_{i=1}^{l} y_i(t) = k^s, \\ y(t) \in \mathbb{B}^l,\ t \in (t_{s-1}, t_s], \end{cases} \end{aligned} \tag{7}$$

where $s = 1,...,S$. Let us recall that a suitable scalarizing of a multi-objective optimization problem is an adequate theoretic and computational approach to an initially given vector optimization problem. We refer to [15, 21] for the corresponding technical details and algorithms. From the general results of the vector optimization it follows that there exist vectors μ^s, $s = 1,...,S$ of multipliers μ^s_j, $j = 1,...,n$ in (4) such that an optimal solution to the scalarized optimization problem (4) is a Pareto optimal solution for the originally given multi-objective optimization problem. In this case problems (4) and (2.2a) are called equivalent (see e.g., [10] for the necessary mathematical foundations).

We now assume that the multipliers μ^s_j, $j = 1,...,n$ in (4) are chosen such that problems (4) and (7) are equivalent for every $s = 1,...,S$. In this case we call (4) an adequate scalarizing of the auxiliary problem (7). We now observe that every problem (4) from the family Pr^S and problem (5) have a structure of the celebrated Knapsack problem (see [23] and references therein). Let us recall that many powerful computational algorithms are recently proposed for an effective numerical treatment of the generic Knapsack problem. We refer to [16] for a comprehensive overview of the theory and modern numerical schemes for this celebrated optimization problem.

The importance of the family Pr^S and of the auxiliary optimization problem (5) introduced above can be recognized from the following theoretic result.

Theorem 3.1. *Assume* (4) *is an adequate scalarizing of problem* (7)*. Then an optimal solution* $(\hat{z}(\cdot), \hat{y}(\cdot))$ *of the family of auxiliary problems* $\{Pr^S$*, (5)*$\}$ *is also an optimal solution to the originally given* DMCLP (1) *and*

$$J(z^{opt}(\cdot)) = J(\hat{z}). \tag{8}$$

If additionally all problems (1), (4) *and* (5) *have unique optimal solutions, then*

$$(z^{opt}(\cdot), y^{opt}(\cdot)) = (\hat{z}(\cdot), \hat{y}(\cdot)). \tag{9}$$

Proof. Since

$$\hat{y}^s(\cdot),\ s = 1,...,S$$

and $\hat{z}(\cdot)$ are admissible solutions for (4) and (5), we have

$$\sum_{i=1}^{l} \hat{y}^s_i(t) = k^s,$$

$$\hat{z}_j(t) \leq \sum_{i=1}^{l} a^s_{ij} \hat{y}_i(t)$$

for every $t \in (t_{s-1}, t_s]$ and $s = 1,...,S$. Therefore, $(\hat{z}, \hat{y})$ is also an admissible pair for the originally given DMCLP (1). By the property of an optimal pair

$$(z^{opt}(\cdot), y^{opt}(\cdot))$$

we next obtain

$$J(\hat{z}(\cdot)) \leq J(z^{opt}(\cdot)). \tag{10}$$

Let us introduce the set of all optimal solutions (solution set) associated with the original problem (1)

$$\mathscr{F} := \mathscr{F}_z \bigotimes \mathscr{F}_y$$

Evidently,

$$z^{opt}(\cdot) \in \mathscr{F}_z,$$
$$y^{opt}(\cdot) \in \mathscr{F}_y.$$

The particular solution sets of problems (4) and (5) denote by

$$\mathscr{F}^s_{(2.2)},\ s = 1, \dots S$$

and $\mathscr{F}_{(2.3)}$, respectively. From (10) it follows that

$$\mathscr{F}_{(2.3)} \bigotimes \{ \bigcup_{s=1,\dots,S} \mathscr{F}^s_{(2.2)} \} \subset \mathscr{F}. \tag{11}$$

Observe that the constraints for the decision variable $y(\cdot)$ in (1) and in all problems (4) from Pr^S are the same. This fact implies

$$\mathscr{F}_y \equiv \{ \bigcup_{s=1,\dots,S} \mathscr{F}^s_{(2.2)} \}. \tag{12}$$

Since (4) is an adequate scalarization of the multi-objective maximization problem (7) and taking into consideration the inequality constraints for the state variable $z(\cdot)$ in (1), we next deduce

$$z_j^{opt}(t) \leq \max_{\begin{cases} \sum_{i=1}^l y_i(t) = k^s, \\ y(t) \in \mathbb{B}^l \end{cases}} \sum_{i=1}^l a^s_{ij} y_i(t) \equiv \sum_{i=1}^l a^s_{ij} \hat{y}^s_i(t). \tag{13}$$

for every

$$t \in (t_{s-1}, t_s],\ s = 1, \dots, S.$$

Here

$$z_j^{opt}(\cdot),\ j = 1, \dots, n$$

is the j-component of $z^{opt}(\cdot)$. The obtained condition (13) implies that $z^{opt}(\cdot)$ is an admissible solution of the auxiliary problem (5). Since $\hat{z}$ is an optimal solution of (5), we get

$$J(\hat{z}(\cdot)) \geq J(z^{opt}(\cdot)) \tag{14}$$

for the admissible solution $z^{opt}(\cdot)$ in (5). The obtained inequalities (10) and (15) for the objective functional $J(\cdot)$ imply the expected result (8).

Moreover, inequality (15) also implies the following inclusion

$$\mathscr{F}_z \subset \mathscr{F}_{(2.3)}. \tag{15}$$

From (11), (12) and (15) we immediately deduce the crucial consequence

$$\mathscr{F}_{(2.3)} \bigotimes \{ \bigcup_{s=1,\dots,S} \mathscr{F}^s_{(2.2)} \} \equiv \mathscr{F}. \tag{16}$$

If the solutions sets $\mathscr{F}, \mathscr{F}_{(2.3)}$ and $\mathscr{F}^s_{(2.2)},\ s = 1, \dots, S$ are one-point-sets, we deduce the expected result (9) as a direct consequence of the obtained equivalence (16). The proof is completed. ■ □

As we can see Theorem 3.1 separates equivalently the originally given sophisticated DMCLP (1) into two relatively simple optimization problems. It provides the theoretical foundation of the Separation Method we propose and generates effective numerical approaches to the dynamic MCLPs of the type (1). In fact, Theorem 3.1 reduce a dynamic optimization problem (1) with a switched structure to a family of specific Knapsack problems. This complexity reduction is a direct consequence of the geometrical structure of constraints in the originally given DMCLP (1).

4. **The Separation Based Computational Approach to DMCLP.** We now consider a particular problem from the family Pr^S, namely, problem (4) and introduce the following notation

$$\hat{I}^s := \{1 \leq i \leq l \mid \mathscr{S}_{\mathscr{A}_i^s} \in \max_k \{\mathscr{S}_{\mathscr{A}_1^s}, ..., \mathscr{S}_{\mathscr{A}_l^s}\}\},$$
$$\mathscr{S}_{\mathscr{A}_i^s} := \sum_{j=1}^{n} \mu_j a_{ij}, \tag{17}$$
$$\mathscr{A}_i^s := (a_{i1}^s, ..., a_{in}^s)^T.$$

Here $\mathscr{A}_i^s$ is a vector of i-row of the eligibility matrix A^s and operator $\max_{k_s}\{\cdot\}$ determines an array of k_s-largest numbers from the given array. Since every problem (4) has a trivial combinatorial structure it can be easily solved by the naturally combinatorial algorithm.

Algorithm 1.

$$\hat{y}_i^s = 1 \text{ if } i \in \hat{I}^s;$$
$$\hat{y}_i^s = 0 \text{ if } i \in \{1, ..., l\} \setminus \hat{I}^s, \tag{18}$$

The following result establishes the consistency of the obtained Algorithm 1.

Theorem 4.1. *Algorithm 1 and the corresponding optimal choice (18) determines an optimal solution of problem (4) for every $s = 1, ..., S$.*

Proof. The finite selection algorithm (18) assigns the maximal (from admissible) value $\hat{y}_i^s = 1$ for all vectors $\mathscr{A}_i^s$ such that the weighted sum of components $\mathscr{S}_{\mathscr{A}_i^s}$ of this vector belongs to the k^s-dimensional array of largest sums of components of all vectors

$$\mathscr{A}_\rangle^\int, \; i = 1, ..., l.$$

Hence the resulting sum

$$\sum_{j=1}^{n} \mu_j^s \sum_{i=1}^{l} a_{ij}^s \hat{y}_i^s(t)$$

for $t \in (t_{s-1}, t_s]$ is maximal. The proof is completed. ■ □

Let us also note that for the given eligibility matrix A^s with the specific positive elements a_{ij}^s (as determined in Section 2) the sum of components $\mathscr{S}_{\mathscr{A}_i^s}$ constitutes a specific norm of the row-vector $\mathscr{A}_i^s$. The total complexity of the proposed combinatorial Algorithm 1 for a fixed index s can be calculated as follows (see e.g., [23] for details)

$$O(l \times \log k^s) + O(k^s).$$

We now turn back to the second auxiliary problem (5) and define

$$c^s := \sum_{j=1}^{n} \sum_{i=1}^{l} a_{ij}^s \hat{y}_i^s,$$
$$c := \max_{s=1,...,S} \{c^s\}$$

Then the inequality constraints in (5) imply the generic Knapsack-type constraint with uniform weights for every index $s = 1, ..., S$

$$\sum_{j=1}^{n} z_j \leq c^s.$$

Let us present a fundamental solvability result for the second auxiliary optimization problem, namely, the "switched-type" Knapsack problem (5).

Theorem 4.2. *The* Knapsack *problem* (5) *can be solved in maximum* $O(nc)$ *time and*

$$O(n+c)$$

space.

A formal proof of Theorem 4.2 can be found in [23].

We now need to determine an adequate implementable numerical procedure for problem (5). This auxiliary optimization problem, which is $\mathcal{NP}$-hard, has been comprehensively studied in the last few decades and several exact algorithms for its solution can be found in the literature (see [6, 23] and the references therein). Computational algorithms for Knapsack problems are mostly based on two basic approaches: the celebrated branch-and-bound methods and dynamic programming techniques. Moreover, one can combine these two basic numerical approaches. These two main solution procedures can also be used in combination with a relaxation scheme applied to an initial model.

"Relaxing a problem" has various meanings in applied mathematics, depending on the areas where it is defined, depending also on what one relaxes (a functional, the underlying space, etc.). We refer to [2, 5, 7, 12, 23, 24, 28] for various implementable relaxation techniques. In this contribution we apply the celebrated Lagrange relaxation scheme to the second auxiliary problem, namely, to the Knapsack-type problem (5). Let us introduce the Lagrange function for (5)

$$\mathscr{L}(z(\cdot),\lambda(\cdot)) := \sum_{s=1}^{S} \frac{(t_s - t_{s-1})}{(t_f - t_0)} \langle w(t), z(t)\rangle_{|_{t\in(t_{s-1},t_s]}} -$$
$$\sum_{j=1}^{n} \lambda_j(t)\big(z_j - \sum_{i=1}^{l} a_{ij}^s \hat{y}_i^s\big)$$

where $\lambda_j(\cdot),\ j=1,...,n$ are piecewise constant on the time intervals

$$(t_{s-1},t_s],\ s=1,...,S$$

functions. Let us denote

$$\lambda(t) := (\lambda_1(t),...,\lambda_n(t))^T,\ t\in[t_0,t_f].$$

The following problem is called Lagrange relaxation of the auxiliary optimization problem (5)

$$\begin{aligned}&\text{maximize } \mathscr{L}(z(\cdot),\lambda(\cdot))\\ &\text{subject to}\\ &z(t)\in\mathbb{B}^n\ t\in[t_0,t_f].\end{aligned} \tag{19}$$

Let us refer to [7, 12, 23] for the foundations of the Lagrange relaxation in optimization. Note that the relaxed problem (19) does not contain the unpleasant inequality constraints which are included in the objective (Lagrange) function $\mathscr{L}(\cdot,\cdot)$ as a penalty term. All feasible solutions to (5) are also feasible solutions for problem (19). Moreover, the objective value calculated for the feasible solutions to (5) is not larger than the objective value obtained using a solution of (19) (see [1, 23] for the necessary proofs). This fact implies that the optimal solution value to the relaxed problem (19) provides an upper bound to the original problem (5) for any vector of non-negative Lagrange multipliers

$$\lambda(\cdot),\ \lambda_j(t)\geq 0,\ t\in[t_0,t_f].$$

Applying the branch-and-bound algorithm to the relaxed problem we are interested in achieving the tightest upper bound for the objective functional in (19). Hence, we would like to choose a vector of non-negative Lagrange multipliers (piecewise constant functions)

$$\hat{\lambda}^{\mathscr{L}}(t) := (\hat{\lambda}_1^{\mathscr{L}}(t), ..., \hat{\lambda}_n^{\mathscr{L}}(t))^T,$$
$$\hat{\lambda}_j^{\mathscr{L}} \geq 0$$

such that $L(\cdot,\cdot)$ in (19) is minimized. This consideration strongly motivates the celebrated concept of a Lagrangian dual problem (see e.g., [23] for details)

$$\begin{aligned} &\text{minimize } \mathscr{L}(z(\cdot), \lambda(\cdot)) \\ &\text{subject to} \\ &\lambda(t) \geq 0 \; t \in [t_0, t_f] \end{aligned} \tag{20}$$

By $(\hat{z}^{\mathscr{L}}(\cdot), \hat{\lambda}^{\mathscr{L}}(\cdot))$ we next denote the optimal solution pair to the primal-dual system (19)-(20). It is well-known that the Lagrangian dual problem (20) yields the least upper bound for (5) available from all possible Lagrangian relaxations. Note that the dual problem (20) is in fact a linear (integer) programming problem [6, 10]. The following basic result is a direct consequence of the basic properties of the above primal-dual system (19)-(20) and of our main theoretic result, namely, of Theorem 3.1.

Theorem 4.3. *Let $(\hat{z}^{\mathscr{L}}(\cdot), \hat{\lambda}^{\mathscr{L}}(\cdot))$ be an optimal solution of the primal-dual system (19)-(20) associated with the auxiliary problem (5). Assume that all conditions of* Theorem 3.1 *are satisfied. Then*

$$J(z^{opt}(\cdot)) = J(\hat{z}(\cdot)) \leq J(\hat{z}^{\mathscr{L}}(\cdot)). \tag{21}$$

where $(\hat{z}(\cdot), \hat{y}(\cdot))$ is an optimal solution of the family of auxiliary problems (4)-(5). Moreover, inequality (21) constitutes a tightest upper bound available from all possible Lagrangian *relaxations (19) of the optimization problem (5).*

Theorem 4.3 can be proved by a direct calculation using the corresponding result from [23].

5. **Numerical Aspects.** In this section we apply the generalized SM developed in Section 3 and Section 4 and study a specific real-world example from the area of telecommunication. Our aim is to optimize a cellular mobile communication network, namely, to find an optimal solution to a specific restricted covering problem (see [9, 16]). Note that the mobile communication network constitutes a (dynamic) switching system. Our objective leads to the definition of a transmitter location problem as a locating problem that does not require the coverage of all demand nodes. The coverage model of a mobile communication network includes a limited budget as a constraint on the number of facilities to be located. The optimization model we implement has a fuzzy nature and includes a fixed number

$$k^s, \; s = 1, ..., S \in \mathbb{N}$$

of the totally $l \in \mathbb{N}$ base stations such that a specific average covering (considered on a fixed time interval) is maximized.

Example 1. *Consider a mathematical abstraction of the cellular mobile communication network in the form of the generic* DMCLP (1). *Assume a study region split into $n = 15$ demand nodes. In this particular technical problem a concrete demand node represents the center of an area that contains a quantum of demand (from a traffic viewpoint) accounted in*

10 Dec 2018 20:30:57 PST
Version 1 - Submitted to J. Ind. Manag. Optim.

a fixed number of call requests per time unit. We next assume to have $l = 5$ totally feasible locations for the base stations. As mentioned above for every interval $(t_{s-1}, t_s]$ we assume

$$k^s,\ s = 1, ..., S \in \mathbb{N}$$

"active" base stations (the stations in service). In our example we put $t_0 = 0$ and $t_f = 2$ and the (unique) switching time $t_1 = 1$. Moreover, let

$$S = 2,\ k^1 = 2,\ k^2 = 4.$$

Let us now specify the further variables and parameters from the general DMCLP *framework* (1)*: the weights vectors associated with the corresponding time intervals are selected as follow*

$$w_1 = (27, 42, 11, 29, 43, 16, 14, 19, 45, 30, 0, 0, 16, 42, 0)^T,$$
$$w_2 = (27, 26, 0, 48, 0, 35, 49, 10, 49, 48, 16, 14, 41, 28, 0)^T.$$

The eligibility matrices

$$A^s \in \mathbb{R}^{5\times 15},\ s = 1, 2$$

associated with the corresponding time intervals are selected as follows (given by rows $\mathscr{A}_i$ of A^s for $i = 1, ..., 5$)

$$\mathscr{A}_1 = (0.0, 0.959, 0.832, 0.965, 0.0, 0.0, 0.9380, 1.0, 0.0, 0.0, 0.816,$$
$$0.528, 0.378, 0.0, 0.998),$$
$$\mathscr{A}_2 = (0.5069, 0.846, 0.388, 0.624, 0.970, 0.0, 0.0, 0.0, 0.653, 0.779, 0.701,$$
$$0.511, 0.876, 0.566, 0.963),$$
$$\mathscr{A}_3 = (0.340, 0.222, 0.211, 0.745, 0.0, 0.0, 0.0, 0.0, 0.0, 0.542, 0.0,$$
$$0.0, 0.730, 0.832, 0.0),$$
$$\mathscr{A}_4 = (0.0, 0.478, 0.593, 0.0, 0.0, 0.939, 0.0, 0.0, 0.736, 0.0, 0.0,$$
$$0.0, 0.794, 0.406, 0.0),$$
$$\mathscr{A}_5 = (0.0, 0.437, 0.0, 0.0, 0.959, 0.596, 0.619, 0.0, 0.0, 0.819, 0.0,$$
$$0.0, 0.611, 0.0, 0.806),$$
$$\text{for } s = 1$$

and

$$\mathscr{A}_1 = (0.531, 0.283, 0.0, 1.0, 0.975, 0.0, 0.383, 1.0, 0.479, 0.829, 0.0,$$
$$0.0, 0.707, 0.688, 0.0),$$
$$\mathscr{A}_2 = (0.558, 0.512, 1.0, 0.0, 0.691, 0.949, 0.929, 0.0, 0.650, 0.979, 0.438,$$
$$0.0, 0.517, 0.0, 0.0),$$
$$\mathscr{A}_3 = (0.0, 0.563, 0.0, 0.0, 0.530, 0.532, 0.0, 0.0, 0.0, 0.843, 0.0,$$
$$0.0, 0.0, 0.605, 1.0),$$
$$\mathscr{A}_4 = (0.0, 0.375, 0.0, 0.0, 0.241, 0.0, 0.0, 0.0, 0.0, 0.0, 0.360,$$
$$0.739, 0.928, 0.0, 0.0),$$
$$\mathscr{A}_5 = (0.559, 0.0, 0.0, 0.0, 0.615, 0.606, 0.969, 0.0, 0.0, 0.0, 0.0,$$
$$0.432, 0.470, 0.686, 0.0),$$
$$\text{for } s = 2.$$

Both the eligibility matrices have a fuzzy structure and express the technical reliability of the communication network under consideration.

Recall that the decision variable from (1), namely,

$$y_i(t),\ i = 1,...5$$

represents here the "activating" of the i *- base station at the time instant* $t \in [0,2]$*. Additionally the specific state variable*

$$z_j(t),\ j = 1,...,15$$

from the general model (1) describes here the binary state of the j *- demand node. Recall that it is equal to* 1 *if the* j *- demand node is covered by at least one base station (facility). Otherwise*

$$z_j(t) = 0.$$

Application of the numerical Algorithm 1 *leads to the computational results for the optimal variables* $z^{opt}(t)$ *and* $y^{opt}(t)$*, where*

$$t \in [0,1] \text{ for } s = 1$$

and

$$t \in [1,2] \text{ for } s = 2.$$

We have obtained the following optimal pairs $(z^{opt}(t),\ y^{opt}(t))$*:*

$$\begin{aligned} z^{opt}(t) &= (0,1,0,1,0,0,0,0,0,1,0,0,1,1,0)^T, \\ y^{opt}(t) &= (0,1,1,0,0)^T, \end{aligned} \tag{22}$$

for $s = 1,\ t \in [0,1]$ *and*

$$\begin{aligned} z^{opt}(t) &= (1,1,0,1,0,1,1,1,1,1,0,1,1,1,0)^T, \\ y^{opt}(t) &= (1,1,0,1,1)^T, \end{aligned} \tag{23}$$

where

$$s = 2,\ t \in [1,2].$$

The optimal value $\max\{J(z(\cdot))\}$ *of the objective functional* $J(\cdot)$ *in (1) is as follow:*

$$\max\{J(z(\cdot))\} = 534.$$

The dynamic behavior of the optimal decision variable $y^{opt}(t)$ *is also presented in* Fig. 1.

Let us now give a practical interpretation of the obtained computational (optimal) results (22)-(23)*. From* (22) *we deduce that the active base stations are the stations* no.2 *and* no.3*. Moreover, the following dynamic demand nodes* no.2, no.4*, and* no.10 *are covered. The optimal pair in* (23) *indicates that the active base stations are the stations* no.1, no.2, no.4*, and* no.5*. In that case the demand nodes excluding the following nodes:* { no.3, no.5, no.11, no.15 } *are covered.*

Finally, note that the implementation code of the computational Algorithm 1 *was carried out by using the Python optimization packages and an author-written program.*

In the above illustrative Example 1 we have considered some given eligibility matrices A^s. In the telecommunication engineering this technical parameter has a specific structure and is usually a function of the required cover range of the existing base stations.

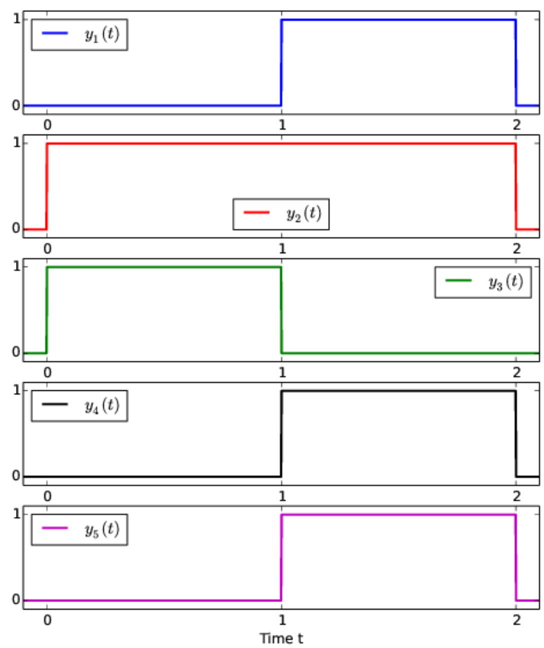


FIGURE 1. Optimal dynamics of the switched decision variables $y^{opt}(t)$

6. **Conclusion.** This paper proposes advanced theoretic and computational approaches to a new class of the Dynamic Maximal Covering Location Problems with a switched structure. In our contribution, we have theoretically extended the classic static Maximal Covering Location Problems and also implemented some additional novel modelling approaches. First of all we have proposed a switched type dynamical generalization of the conventional MCLP problem statement. This new approach makes it possible to incorporate a wide range of real-world application and engineering effects into the proposed framework. Evidently, switched dynamic processes constitute in general a more adequate modelling approach to the technical processes and systems. Moreover, we have considered the DMCLP optimization with the fuzzy-type eligibility matrices. This fact also involves more flexibility of the resulting optimization model that finally includes some necessary ”resilience” or (and) ”fuzzy” interpretations. In this paper, we have followed a mathematically rigorous considerations and developed a consistent numerical solution approach.

The obtained computational algorithm was next applied to an engineering motivated problem of optimal covering in a cellular communication network. Application of the proposed algorithm makes it possible to optimize a specific RCP type problem (see Section

5) and maximize an average covering for a given configuration of the base stations. The SM and the corresponding numerical scheme we developed reduce the originally given sophisticated optimization problem to two Knapsack-type auxiliary problems. The first one is in fact a linear scalarization of a specific multiobjective program. The second Knapsack-type model in the framework of SM constitutes a classic problem. We have studied the proposed dynamic generalization of the SM in the context of the conventional Lagrange relaxation scheme and in combination with an additional combinatorial algorithm. The obtained theoretic results finally lead to an implementable computational algorithm for the switched type DMCLPs with a fuzzy-type eligibility matrix.

The analytic and numeric methodologies we propose in our paper can be applied to various further generalizations of the generic MCLP. Moreover, the developed SM for a class of DMCLP under consideration can be combined with the celebrated branch-and-bound method and with the dynamic programming approach. It can also be included (as an auxiliary computational tool) into the modern heuristic solution approaches. Let us note that our paper mainly discussed the theoretic foundations of the newly elaborated SM for DMCLPs. The novel solution methodology we proposed needs further comprehensively numerical examinations and computer based simulations of several switched-type DMCLPs.

REFERENCES

[1] K. Atkinson, W. Han, *Theoretical Numerical Analysis*. Springer, New York 2005.

[2] V. Azhmyakov and W. Schmidt, Approximations of relaxed optimal control problems, *Journal of Optimization Theory and Applications*, vol. 130, 2006, pp. 61 -– 77.

[3] V. Azhmyakov, M.V. Basin, J. Raisch, A Proximal point based approach to optimal control of affine switched systems, *Discrete Event Dynamic Systems*, vol. 22, 2012, pp. 61 – 81.

[4] V. Azhmyakov, M. Basin, C. Reincke-Collon, Optimal LQ-type switched control design for a class of linear systems with piecewise constant inputs, in: *Proceedings of the 19th IFAC World Congress*, Cape Town, South Africa, 2014, pp. 6976 – 6981.

[5] V. Azhmyakov, J. Cabrera, A. Poznyak, Optimal fixed - levels control for non - linear systems with quadratic cost functionals, *Optimal Control Applications and Methods*, vol. 37, 2016, pp. 1035 – 1055.

[6] V. Azhmyakov, J.P. Fernandez-Gutierrez, S.K. Gadi, St. Pickl, A novel numerical approach to the resilient MCLP based supply chain optimization, in: *Proceedings of the 12th IFAC Workshop on Intelligent Manufacturing Systems*, Austin, USA, 2016, pp. 145 – 150.

[7] V. Azhmyakov, *A Relaxation Based Approach to Optimal Control of Hybrid and Switched Systems*, Elsevier, USA, 2018.

[8] V. Batanovic, D. Petrovic, R. Petrovic, Fuzzy logic based algorithms for maximum covering location problems, *Information Sciences*, vol. 179(1–2), 2009, pp. 120 – 129.

[9] O. Berman, J. Kalcsics, D. Krass, S. Nickel, The ordered gradual covering location problem on a network, *Discrete Applied Mathematics*, vol. 157(18), 2009, pp. 3689 – 3707.

[10] D. Bertsekas, *Nonlinear Programming*, Athena Scientific, Belmont, USA, 1995.

[11] M. Boccadoro, Y. Wardi, M. Egerstedt, E.I. Verriest, Optimal Control of Switching Surfaces in Hybrid Dynamical Systems *Discrete Event Dynamic Systems*, vol. 15, 2005, pp. 433 – 448.

[12] R.D. Galvao, L.G. Acosta Espejo, B. Boffey, A comparison of Lagrangian and surrogate relaxations for the maximal covering location problem, *European Journal of Operational Research*, vol. 124(2), 2000, pp. 377 – 389.

[13] M.S. Canbolat, M. von Massow, Planar maximal covering with ellipses, *Computers and Industrial Engineering*, vol. 57(1), 2009, pp. 201 – 208.

[14] R.L. Church, C.S ReVelle, The maximal covering location problem, in *Papers of the Regional Science Association*, vol. 32, 1974, pp. 101 – 118.

[15] C. A. C. Coello, G. B. Lamont, D. A. Van Veldhuizen, *Evolutionary Algorithms for Solving Multi-Objective Problems*, Kluwer Academic, Mass, USA, 2002.

[16] P. Dell Olmo, N. Ricciardi, A. Sgalambro, A Multiperiod maximal covering location model for the optimal location of intersection safety cameras on an urban traffic network, *Procedia - Social and Behavioral Sciences*, vol 108, 2014, pp. 106 – 117.

[17] L. Dupont, M. Lauras, C. Yugma, Generalized covering location problem with multiple-coverage: Exact and heuristic method, *IFAC Proceedings Volumes*, vol. 46(9), 2013, pp. 442 – 447.
[18] M.H. Fazel Zarandi, S. Davari, S.A. Haddad Sisakht, The large-scale dynamic maximal covering location problem. *Mathematical and Computer Modelling*, vol. 57, 2013, pp. 710 – 719.
[19] M. Gendreau, G. Laporte, F. Semet, A dynamic model and parallel tabu search heuristic for real-time ambulance relocation, *Parallel Computing*, vol. 27(12), 2001, pp. 1641 -– 1653.
[20] J. Gu, Y. Zhou, A. Das, I. Moon, G. M. Lee, Medical relief shelter location problem with patient severity under a limited relief budget, *Computers and Industrial Engineering*, to appear in 2018,
[21] J. Jahn, *Vector Optimization*, Springer, Berlin, 2011.
[22] G. Ji, S. Han, A strategy analysis in dual-channel supply chain based on effort levels, in: *Proceedings of the 1th International Conference on Service Systems and Service Management*, Beijing, China, 2014, pp. 2161 – 1890.
[23] H. Kellerer, U. Pferschy, D. Pisinger, *Knapsack Problem*, Springer, Berlin, 2004.
[24] A. Mitsos, B. Chachuat and P.I. Barton, McCormic - based relaxation algorithm, *SIAM Journal on Optimization*, vol. 20, 2009, pp. 573 – 601.
[25] G.C. Moore, C.S. ReVelle, The hierarchical service location problem, *Management Science*, vol. 28, 1982, pp. 775 – 780.
[26] E. Polak, *Optimization*, Springer-Verlag, New York, USA, 1997.
[27] C. ReVelle, M. Scholssberg, J. Williams, Solving the maximal covering location problem with heuristic concentration, *Computers and Operations Research*, vol. 35(2), 2008, pp. 427 – 435.
[28] T. Roubicek, *Relaxation in Optimization Theory and Variational Calculus*, De Gruyter, Berlin, 1997.
[29] H. Shavandi, H. Mahlooji, A fuzzy queuing location model with a genetic algorithm for congested systems, *Applied Mathematics and Computation*, vol. 181(1), 2006, pp. 440 – 456.
[30] P. Sitek, J. Wikarek, A hybrid approach to modeling and optimization for supply chain management with multimodal transport, in: *Proceedings of the 18th International Conference on Methods and Models in Automation and Robotics*, Miedzyzdroje, Poland, 2013, pp. 777 – 782.
[31] E.I. Verriest, Pseudo-continuous multi-dimensional multi-mode systems, *Discrete Event Dynamic Systems*, vol. 22, 2012, pp. 27 – 59.
[32] E.I. Verriest, V. Azhmyakov, Advances in optimal control of differential systems with state suprema, in: *Proceedings of the 56th Conference on Decision and Control*, Melbourne, Australia, 2017.
[33] Y. Wardi, Switched-mode systems: gradient-descent algorithms with Armijo step sizes, *Discrete Event Dynamic Systems*, vol. 25(4), 2015, pp. 571 – 599.
[34] F. Zarandi, A. Haddad Sisakht, S. Davari, Design of a closed-loop supply chain (CLSC) model using an interactive fuzzy goal programming, *The International Journal of Advanced Manufacturing Technology*, vol. 56, 2011, pp. 809 – 821.

Received xxxx 20xx; revised xxxx 20xx.

E-mail address: vazhmyakov@udem.edu.co
E-mail address: jpfernandez@udem.edu.co
E-mail address: erik.verriest@ece.gatech.edu
E-mail address: stefan.pickl@unibw.de

CHAPTER 4

Future work

We are shown in this article-based thesis two lines of work on the papers. First, we did a shift in the traditional static model, when included fuzzy concept inside the cover constraint for allowing to include disruptions in the facility. Second, we extended the static model to the dynamic model without complex dynamic laws, which allow us to connect this model with the static model through Bellman Principle, and extend the conceptual algorithm to this case. In both cases, we provided a conceptual algorithm for solving it. This in mind, we can introduce the next ideas to future work.

4.1. Modeling Approach

Let's begin to come back to the static point of view. In the traditional static model, we assume that the eligibility matrix comes from a covering radius, and assuming the availability of the facility is always true, instead of a family of statistical distribution. This family can model in a better way realistic assumption of the functionality of the facility, and avoid the sensitivity of the configuration of the selected facility with respect to the service radius. Furthermore, we assume this set must be bound, closed, and convex set, then we will be able to calculate maximin concepts in this model, but we need to provide a suitable family for it and prove some theorems on their relations and how to calculate the minimization with respect this family. Lastly, we proceed to calculate the optimal selection of the facilities to get the maximum cover for the customer.

For the dynamic case, we will have to consider more complex dynamic relations time-spatial. For the third paper, we considered the case when a telecommunication service has to become widespread from one location, and then we move to another location -not spatially related- when we make a decision over the new spatial distribution on a different time until we arrive at the last location. Now, we can think in a car, ship, or airplane, goes to one location to a final location, and we need to provide a service during all the trip as much as we can. We need to model what kind of spatial-time relation there exist for the specific type of service that we need to provide. Include dynamical growth of population, when the service is for a population, to name a few possible instances.

Finally, MCLP tries to get the better service to a customer including the constraint for this service, but if we include the expectation of the customer, we can think of providing the better service, minimizing the dissatisfaction of the customer including the technical constraint to attend the customers. This is known as Bi-level optimization, and now, the scientific community has developed algorithms for this kind of optimization, it is an interesting field of research.

4.2. Computational Techniques

For the proposed algorithm, we need to study the building of the initial point, the computational time performance for different large instances. Likewise, it needs to compare the performance of the implementation with other currently implementation through a specific measure of computational effectiveness and efficiency.

4.2.1. Complexity

This kind of NP-*hard* problem is quite normal to produce metaheuristic or matheuristic approaches, because of the significant subproblems thereof. We need to remember that Knapsack problem is an NP-*hard* problem itself, and from the outlook of algorithms and codes that integrate (meta)heuristics and MIP strategies and software, it will be successful to lead new methodologies.

4.2.2. Scalarazing

Let us now make a short remark related to the scalarizing procedure used above (see Sections 2.1, 2.2 and Chapter 3).

It can be shown analytically that the values $S_{\mathcal{A}_i}$ in sections 2.1 and 2.2 depend on the multipliers vector μ, and the values $S_{\mathcal{A}_i^s}$ in Chapter 3 depend on the multipliers vector μ^s. This is a consequence of the inclusion constitutes a useful relation of the SM in both cases, and for the resulting optimization strategy we propose. Since the obtained multiobjective maximation problem relate has a linear structure, an adequate scalarizing makes it possible to determine every "non-dominant" points (see [25, 26] for mathematical details).

On the other hand, this is a limitation of the proposed SM, because a possible "non-adequate" selection of μ or μ^s geometrically implies a significant "cutting"(restriction) of the feasible region for respective problems. This feasible region restriction can finally eliminate a true optimal solution. Recall that a scalarizing implemented in the objective function evidently determines the resulting geometry associated with the basic key problem. On the other side, the geometrical properties of a non-adequately scalarized problem can violate the conceptual condition $\Gamma_z \subset \Gamma_{(3)}$ in the static case or $\mathcal{F}_z \subset \mathcal{F}_{(2,3)}$ for the dynamic case.

4.3. Practical Implementation

For the Resilient Supply Chain Management System creates a good measure of the resilient concept to be implemented is a hard task, and that makes the difference between the applied practical work with the theoretical work as this. The building of the eligibility matrix in the real context is not studied here, even the toy examples were creating under this assumption that the value a_{ji} is known, and the synthetic data parameters satisfy the condition explained in the introduction of the model in static as well as in dynamic case.

As disruption is constituted by different topic, create a measure (Fuzzy or Stochastic) that can make the role of the measure of ji eligibility parameter is another interesting research field that allows going from theoretical optimization into a realistic application of this model.

Conclusions

On published paper in section 2.1 and 2.2

In both contributions, we proposed a conceptually new numerical approach to a wide class of Maximal Covering Location Problems with the fuzzy-type eligibility matrices. The paper in section 2,2 is a suitable extension from the paper in section 2,1 according to the publication law, where complete some elements that was no cover in the previous one, and allow us to have two different publications.
In general term. In one hand, the eligibility location matrix in the traditional MCLP model is a binary selection when we choose its components (j, i) as 1 (cover), 0 (not cover) values. Fuzzy parameters, numbers between $[0, 1]$, normally represents the weight of a cut in the membership function, which can represent a grade of preference. In this way, any component (j, i) of the eligibility location matrix can represent the grade of coverage of the facility j to customer i as an "natural" extension of the prior selection concept, and generalize the binary coverage assumption, which could be unrealistic. That introduces a class of preference without include assignment model for evaluating the disruption of a facility, which implies we must mix disruption in facility model in location constraint as well in the objective function.

In another hand, the sum of several fuzzy parameters of facility selection must be more than one, imply there exist at least two facilities that cover the customer, and we can avoid to include a specification of how many facilities we need to attend this particular customer, and the relationship to expected (minimax) transportation and lost-sales costs. Transportation and lost-sales costs are difficult to estimate because the transportation cost depends of the failed open facilities, and lost-sales depend on demand distribution, both elements have much variability inside. Fuzzy parameters allow staying only in location model, but the open facilities are able to introduce in the next assignment problem a new resilient coverage to tackle the service to the customer. This methodology does not include, as other methods either, competitive aspect present in retail facilities. It moves away of facility fortification, and the risk measure (expected cost and minimax cost).

The resilient definition can be approaching in a different way, first combining the distribution of the service inside the location, second using the P-median model to lead central distribution service, and third coming back to the traditional path, location first, assignment after. The two first method involves to have an immediate evaluation of the disruption in the system, the latter one imply to solve the assignment of service to get this kind of report (we do not work in this case), but we only take in account the optimal open facilities in the before step.

Theoretical and computational methodologies we present in both contribution can be applied to various generalizations and extensions of the basic MCLP and also to several optimization problems associated with the Resilient Supply Chain Management System (RSCMS) design. One can combine the elaborated separation scheme with the conventional branch-and-bound method, with the celebrated dynamic programming approach or/and with an alternative exact or heuristic numerical algorithm. Let us finally note that we discussed here only theoretic aspects of the newly elaborated approach and presented the corresponding conceptual solution procedure. The basic methodology we developed needs a comprehensively numerical examination that includes solutions of several MCLPs and simulations of the corresponding optimal RSCMSs.

On submitted paper in chapter 3.

This paper proposes advanced theoretic and computational approaches to a new class of the Dynamic Maximal Covering Location Problems with a switched structure. In our contribution, we have theoretically extended the classic static Maximal Covering Location Problems and also implemented some additional novel modeling approaches. First of all, we have proposed a switched type dynamical generalization of the conventional MCLP problem statement. This new approach makes it possible to incorporate a wide range of real-world application and engineering effects into the proposed framework. Evidently, switched dynamic processes constitute in general a more adequate modeling approach to the technical processes and systems. Moreover, we have considered the DMCLP optimization with the fuzzy-type eligibility matrices. This fact also involves more flexibility of the resulting optimization model that finally includes some necessary "resilience" or (and) "fuzzy" interpretations. In this paper, we have followed mathematically rigorous considerations and developed a consistent numerical solution approach.

The obtained computational algorithm was next applied to an engineering motivated problem of optimal covering in a cellular communication network. Application of the proposed algorithm makes it possible to optimize a specific RCP type problem and maximize an average covering for a given configuration of the base stations. The SM and the corresponding numerical scheme we developed reduce the originally given sophisticated optimization problem to two Knapsack-type auxiliary problems. The first one is, in fact, a linear scalarization of a specific multiobjective program. The second Knapsack-type model in the framework of SM constitutes a classic problem. We have studied the proposed dynamic generalization of the SM in the context of the conventional Lagrange relaxation scheme and in combination with an additional combinatorial algorithm. The obtained theoretic results finally lead to an implementable computational algorithm for the switched type DMCLPs with a fuzzy-type eligibility matrix.

Bibliografía

[1] G. Alexandris, I. Giannikos, A new model for maximal coverage exploiting GIS capabilities, *European Journal of Operational Research*, vol. 202(2), 2010, pp. 328 – 338.

[2] D. Romero, F. Vernadat, *Enterprise information systems state of the art: Past, present and future trends*. Computer in Industry, vol. 79, June 2016, pp. 3 – 13.

[3] H. Aytug, C. Saydam, Solving large-scale maximum expected covering location problems by genetic algorithms: a comparative study, *European Journal of Operational Research*, vol. 141(3), 2002, pp. 480 – 494.

[4] V. Azhmyakov and W. Schmidt, Approximations of relaxed optimal control problems, *Journal of Optimization Theory and Applications*, vol. 130, 2006, pp. 61 – 77.

[5] V. Azhmyakov, M. Basin, C. Reincke-Collon, Optimal LQ-type switched control design for a class of linear systems with piecewise constant inputs, in: *Proceedings of the 19th IFAC World Congress*, Cape Town, South Africa, 2014, pp. 6976 – 6981.

[6] V. Azhmyakov, J. Cabrera, A. Poznyak, Optimal fixed - levels control for non - linear systems with quadratic cost functionals, *Optimal Control Applications and Methods*, to appear in 2016.

[7] V. Azhmyakov, R. Juarez, On the projected gradient method for switched - mode systems optimization, in: *Proceedings of the 5th IFAC Conference on Analysis and Design of Hybrid Systems*, Atlanta, USA, 2015, .

[8] V. Batanovic, D. Petrovic, R. Petrovic, Fuzzy logic based algorithms for maximum covering location problems, *Information Sciences*, vol. 179(12), 2009, pp. 120 – 129.

[9] O. Berman, J. Kalcsics, D. Krass, S. Nickel, The ordered gradual covering location problem on a network, *Discrete Applied Mathematics*, vol. 157(18), 2009, pp. 3689 – 3707.

[10] O. Berman, J. Wang, The minmax regret gradual covering location problem on a network with incomplete information of demand weights, *European Journal of Operational Research*, vol. 208(3), 2011, pp. 233 – 238.

[11] R.D. Galvao, L.G. Acosta Espejo, B. Boffey, A comparison of Lagrangean and surrogate relaxations for the maximal covering location problem, *European Journal of Operational Research*, vol. 124(2), 2000, pp. 377 – 389.

[12] M.S. Canbolat, M. von Massow, Planar maximal covering with ellipses, *Computers and Industrial Engineering*, vol. 57(1), 2009, pp. 201 – 208.

[13] R.L. Church, C.S ReVelle, The maximal covering location problem, in *Papers of the Regional Science Association*, vol. 32, 1974, pp. 101 – 118.

[14] G. Ji, S. Han, A strategy analysis in dual-channel supply chain based on effort levels, in: *Proceedings of the 1th International Conference on Service Systems and Service Management*, Beijing, China, 2014, pp. 2161 – 1890.

[15] H. Kellerer, U. Pferschy, D. Pisinger, *Knapsack Problem*, Springer, Berlin, 2004.

[16] A. Giret, E. Garcia, V. Botti An engineering framework for Service-Oriented Intelligent Manufacturing Systems, *Computers in Industry*, vol. 81, September 2016, pp. 116 – 127.

[17] G.C. Moore, C.S. ReVelle, The hierarchical service location problem, *Management Science*, vol. 28, 1982, pp. 775 – 780.

[18] B. Neef, M. Plank, G. Posselt, S. Thiede, C. Herrmann, Demand-oriented Selection and Combination of Industrial Bus Systems for Advanced Energy Management Purposes, *Pocedia CIRP*, vol. 48, 2016, pp. 224 – 229.

[19] C. ReVelle, M. Scholssberg, J. Williams, Solving the maximal covering location problem with heuristic concentration, *Computers and Operations Research*, vol. 35(2), 2008, pp. 427 – 435.

[20] T. Roubicek, *Relaxation in Optimization Theory and Variational Calculus*, De Gruyter, Berlin, 1997.

[21] H. Shavandi, H. Mahlooji, A fuzzy queuing location model with a genetic algorithm for congested systems, *Applied Mathematics and Computation*, vol. 181(1), 2006, pp. 440 – 456.

[22] P. Sitek, J. Wikarek, A hybrid approach to modeling and optimization for supply chain management with multimodal transport, in: *Proceedings of the 18th International Conference on Methods and Models in Automation and Robotics*, Miedzyzdroje, Poland, 2013, pp. 777 – 782.

[23] F. Zarandi, A. Haddad Sisakht, S. Davari, Design of a closed-loop supply chain (CLSC) model using an interactive fuzzy goal programming, *The International Journal of Advanced Manufacturing Technology*, vol. 56, 2011, pp. 809 – 821.

[24] G. Ghiani, G. Laporte, R. Musmanno *Introduction to Logistics Systems Planning and Control*, John Wiley & Sons Ltd, Chichester, England, 2004.

[25] E. Polak, *Optimization*, Springer-Verlag, New York, USA, 1997.

[26] D. Bertsekas, *Nonlinear Programming*, Athena Scientific, Belmont, USA, 1995.

[27] A. Mitsos, B. Chachuat and P.I. Barton, McCormic - based relaxation algorithm, *SIAM Journal on Optimization*, vol. 20, 2009, pp. 573 – 601.

[28] G. B. Dantzig, Discrete Variable Extremum Problems, *Operations Research*, vol. 5, 1957, pp. 266 – 277.

[29] L. V. Snyder, M. S. Daskin, Models for Realible Supply Chain Network Design, *unpublishing paper*, Lehigh University, Bethlehem, PA, email: larry.snyder@lehigh.edu.

[30] O. Berman, D. Krass, The generalized maximal covering location problem, *Computers & Operations Research*, vol. 29, 2002, pp. 563 – 581.

[31] O. Karasakal, E. K. Karasakal, A maximal covering location model in the presence of partial coverage, *Computers & Operations Research*, vol. 31, 2004, pp. 1515 – 1526.

[32] L. V. Snyder, P. M. Scaparra, M. S. Daskin, R. L. Church, Planning for Disruptions in Supply Chain Networks, *Tutorials in Operations Research*, Informs, 2005, pp. 1 – 23.

[33] S.-D. Lee, W.-T. Chang, On solving the discrete location problems when the facilities are prone to failure, *Applied Mathematical Modelling*, vol. 31, 2007, pp. 817 – 831.

[34] J. Nocedal, S. J. Wright, *Numerical optimization*, Springer-Verlag New York, Inc, USA, 1999.

[35] J. F. Bonnas, J. C. Gilbert, C. Lemaréchal, C. A. Sagastizábal *Numerical optimization, theoretical and practical Aspects*, Springer-Verlag Berlin Heidelberg, Germany, 2003.

[36] P. E. Gill, W. Murray, M. H. Wright *Practical optimization*, Springer-Verlag Berlin Heidelberg, Germany, 2003.

[37] P. Spellucci *Numerische Verfahren der nichtlinearen Optimierung*, Birkhäuser Verlag, Basel, Schweiz, 1993.

[38] Z. Han, D. Niyato, W. Saad, T. Başar, A. Hjørungnes *Game Theory in Wireless and Communication Networks: Theory, Models, and Applications*, Cambridge University Press, UK, 2012.

[39] V. Azhmyakov, J.P. Fernandez-Gutierrez, S. K. Gadi, St. Pickl, A novel numerical approach to the resilient MCLP based supply chain optimization, in: *Proceedings of the 12th IFAC Workshop on Intelligent Manufacturing Systems*, Austin, USA, 2016, pp. 145 – 150.

[40] V. Azhmyakov, J.P. Fernandez-Gutierrez, St. Pickl, A Separation Method for Maximal Covering Location Problems with Fuzzy Parameters, in: *Italian Journal of Pure and Appied Mathematics*, N. 38, 2017, pp. 653 – 670.

[41] V. Azhmyakov, J.P. Fernandez-Gutierrez, E.I. Verriest, St. Pickl, A Separation based Optimization approach to Dynamic Maximal Covering Location Problems with Switched Structure, submitted to : *Journal of Industrial and Management Optimization*, N. XXX, 2018, pp. XXX – XXX.