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ANALYTICAL APPROXIMATE SOLUTIONS OF TIME-FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS USING A NEW ITERATIVE TECHNIQUE

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ABSTRACT. In this manuscript, a new iterative technique is proposed to obtain the solutions of linear and nonlinear time-fractional integro-differential equations. The suggested algorithm is a modification of the homotopy analysis method. The deformation equations obtained in this case are easily integrable and the calculations involved in the algorithm are much simpler than the standard homotopy analysis method. The method is illustrated with the help of different numerical test applications. The numerical and graphical results explicitly reveal the potential and accuracy of the proposed technique.

Keywords: Integro-differential equations, Iterative Technique, Caputo fractional derivative, Modified homotopy analysis method.

AMS Subject Classification: 45B05, 34B05, 34B15.

1. INTRODUCTION

During the past two decades, fractional calculus has drawn increasing attention due to its applications in diversified fields of science and engineering. It is an extension of the traditional calculus to non-integer (fractional) order. Thus fractional calculus has the ability to describe different physical phenomena in a more flexible way than the traditional integer-order calculus. The importance of fractional calculus has in turn induced the need for the development of mathematical techniques for the solutions of fractional order differential and integro-differential systems. Various mathematical techniques have been developed for the solutions of fractional order system (*e.g.* see [1]-[3]).
Fractional order integro-differential equations arise in fluid dynamics, biological models and chemical kinetics [4, 5]. Mathematical modeling of heat conduction in materials with memory involves fractional-order Volterra integro-differential equations [6]. Such equations also arise in the combined conduction, convection and radiation problems [7]. The importance and potential of the study of the solutions of integro-differential equations has caught the attention of many researchers during recent years. Arikoglu and Ozkol [8]

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used differential transform method for the solution of fractional order integro-differential equations. Hetmaniok *et al.* [9] used homotpy analysis method for the solutions of integral equations of second kind. Hybrid functions approximation [10] has also been employed successfully for solving such problems.
In this manuscript, a new iterative algorithm is proposed for the solutions of fractional order integro-differential equations. The proposed algorithm is basically a modification of the well-known homotopy analysis method (HAM) with major advantages over the standard method. Homotopy analysis method (HAM) is an approximate analytical technique, built on the concept of homotopy for the solutions of differential equations. The homotopy analysis is highly flexible in many respects so that it might overcome restrictions of perturbation techniques and other non-perturbation methods. The great freedom and flexibility of the HAM has inspired many mathematicians to study HAM in search of better numerical techniques. Optimal homotopy asymptotic method, predictor homotopy analysis method, spectral homotopy analysis method and shooting homotopy analysis method been successfully applied to solve different mathematical problems (*e.g.* see [11]-[15]). Odibat and Bataineh proposed an adaptation of homotopy analysis method introducing homotopy polynomials [16]. Sadaf and Akram [17] proposed an improved adaptation of the method for the solution of higher order boundary value problems. Shaban *et al.* proposed a method based on HAM and the Tau method to study the study a case of magneto-hydrodynamic squeeze flow between two parallel infinite disks [18]. Maitama and Zhao investigated non-differentiable problems on Cantor sets using local fractional homotopy analysis method [20]. Demir *et al.* presented a new technique based on homotopy analysis method to obtain the solutions of space-time fractional differential equations [21].
The paper is organized as follows. Some basic definitions of fractional calculus are stated in Section 2. The basic idea of HAM is briefly described in Section 3. A new iterative technique for the solutions of time-fractional nonlinear integro-differential equations is proposed in Section 4. Convergence analysis of the proposed algorithm is presented in Section 5. Some numerical test applications are illustrated in Section 6.

2. Preliminaries of Fractional Calculus

There are different notions of fractional differential operators but the definition in Caputo sense is most commonly used. The reason for choosing Caputo type fractional derivative is that it is suitable to model real world phenomena.
Definition 1. Caputo's fractional derivative of order α is defined, as

$$D^{\alpha}f(x)=\frac{1}{\Gamma(n-\alpha)}\int_0^x\frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}}ds,\quad n-1<\alpha\leq n,\ n\in\mathbf{N},\tag{1}$$

where α is the order of derivative and n is the smallest integer greater than α.
Definition 2. The Riemann-Liouville fractional integral operator of order α is defined as

$$I^{\alpha}f(x)=\begin{cases}\frac{1}{\Gamma(\alpha)}\int_0^x\frac{f(s)}{(x-s)^{1-\alpha}}ds, & \alpha>0,\\ f(x), & \alpha=0.\end{cases}\tag{2}$$

The Caputo derivative and Riemann-Liouville integral satisfy the following property.

$$I^{\alpha}(D^{\alpha}f(x))=f(x)-\sum_{i=0}^{n-1}f^{i}(0^{+})\frac{x^i}{i!}.\tag{3}$$

3. Basic Idea of HAM

Homotopy analysis method is an analytical technique which can be used to compute the solutions of linear and nonlinear differential equations, integral equations and integro-differential equations. For a nonlinear differential equation

$$\mathcal{N}[y(x)] = 0, \; x \in \Theta, \tag{4}$$

where $\mathcal{N}$ is a nonlinear operator, x is an independent variable , $y(x)$ is an unknown function and Θ is the interval of domain, a homotopy $Y(x,p)$ is constructed with an embedding parameter $p \in [0,1]$ by

$$(1-p)\mathcal{L}[Y(x,p) - y_0(x)] - p\hbar H(x)\mathcal{N}[Y(x,p)] = 0, x \in \Theta, \tag{5}$$

where $\hbar$ is auxiliary parameter, $H(x)$ is an auxiliary function and $\mathcal{L}$ is auxiliary linear operator. Application of Taylor's theorem gives the series expansion of $Y(x,p)$, as

$$Y(x,p) = y_0(x) + \sum_{m=1}^{\infty} y_m(x)p^m, \tag{6}$$

where $y_m(x)$ is obtained by dividing the mth-order deformation derivative by $m!$. For suitably chosen $H(x)$, $\hbar$, $\mathcal{L}$ and $y_0(x)$, the series converges to $y(x)$ at $p = 1$. Moreover, $y_m(x), m = 1,2,3,...$ can be calculated using the mth-order deformation equation

$$\begin{aligned} \mathcal{L}[y_m(x) - \chi_m y_{m-1}(x)] - \hbar H(x) R_m(\underline{y}_{m-1}) = 0,& \\ x \in \Theta, p \in [0,1],& \end{aligned} \tag{7}$$

where

$$\begin{aligned} R_m(\underline{y}_{m-1}) &= \frac{1}{(m-1)!}\left.\frac{\partial^{m-1}\mathcal{N}(Y(x,p))}{\partial p^{m-1}}\right|_{p=0}, \\ \underline{y}_{m-1}(x) &= \{y_0(x), y_1(x), ..., y_{m-1}(x)\}, \\ \chi_m &= \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases} \end{aligned} \tag{8}$$

4. New Iterative Technique for Fractional Order Integro-Differential Equations

Although homotopy analysis method gives an efficient way to approximate the solutions to many linear and nonlinear problems, sometimes the higher order deformation equations lead to complicated integrals and tedious calculations. To overcome these difficulties, a new iterative algorithm method is proposed in this section through a modification of HAM. The Fredholm type time-fractional integro-differential equation is considered, as

$$\left.\begin{aligned} &y''(x) + f(x)y'(x) + g(x)D^{\alpha}(y(x)) + h(x)y(x) = r(x) + \textstyle\int_a^b K(x,s)\psi(y(s))ds, \\ &y(a) = 0, y(b) = \delta, \; 0 < \alpha < 1, \end{aligned}\right\} \tag{9}$$

where $\psi(y(x))$ is a function of the unknown function $y(x)$. The fractional order integro-differential equation can be written, as

$$L[y(x)] + N[y(x)] = 0, \; x \in \Theta, \tag{10}$$

where L is a linear differential operator, N is a nonlinear operator, x is independent variable and $y(x)$ is an unknown function. A homotopy $Y(x;p)$ can be constructed with an embedding parameter $p \in [0,1]$ by

$$(1-p)\mathcal{L}[Y(x;p) - y_0(x)] - p\hbar H(L[Y(x;p)] + N[Y(x;p)]) = 0, \; x \in \Theta, \tag{11}$$

where $\hbar$ is auxiliary parameter and $\mathcal{L}$ is auxiliary linear operator. For $p = 0$, Eq.(11) becomes

$$Y_0(x;p) = y_0(x),$$

whereas for $p = 1$, it yields the original nonlinear integro-differential equation (10). Hence the function $Y(x;p)$ varies from the initial guess to the required solution for the variation of p from 0 to 1.
Using the power series expansion

$$Y(x;p) = \sum_{i=0}^{\infty} Y_i(x;p), \tag{12}$$

the nonlinear term in Eq.(11) can be simplified, as

$$N[Y(x;p)] = \sum_{i=0}^{\infty} N_i[Y_0(x;p), Y_1(x;p), ..., Y_i(x;p)]. \tag{13}$$

Using Eq.(12) and Eq.(13) in Eq.(11), the modified higher order deformation equations are obtained, as

$$\mathcal{L}[Y_1(x;p)] = \hbar H(L[Y_0(x;p)] + N_0[Y_0(x;p)]), \tag{14}$$

and for $k = 1, 2, 3, ...$

$$\begin{aligned} \mathcal{L}[Y_{k+1}(x;p)] &= \mathcal{L}[Y_k(x;p)] + \hbar H(L[Y_k(x;p)] \\ &\quad + N_k[Y_0(x;p), Y_1(x;p), ..., Y_k(x;p)]). \end{aligned} \tag{15}$$

Finally, the N-th order approximate solution can be calculated, as

$$y_N(x) = \sum_{i=0}^{N} y_i(x). \tag{16}$$

5. Convergence Analysis

In this section, convergence of the solution series using the proposed technique is discussed.

Theorem 5.1. *If the series $y_0(x) + \sum_{k=1}^{+\infty} y_k(x)$ is convergent, where $y_k(x)$ is governed by Eqns.(14) and (15), it must be an exact solution of problem (9).*

Proof. Convergence of the series $\sum_{k=1}^{\infty} y_k(x)$ implies

$$\lim_{k\to\infty} y_k(x) = 0. \tag{17}$$

Consider the series $\sum_{k=1}^{\infty} \hbar H(L[Y_{k-1}(x;p)] + N_{k-1}[Y_0(x;p), Y_1(x;p), ..., Y_{k-1}(x;p)])$ for $p = 1$. Using Eqns.(15) and (17) yield

$$\begin{aligned}
& \sum_{k=1}^{\infty} \hbar H(L[Y_{k-1}(x;p)] + N_{k-1}[Y_0(x;p), Y_1(x;p), ..., Y_{k-1}(x;p)]) \\
= & \sum_{k=1}^{\infty} (\mathcal{L}[Y_k(x;p)] - \chi_k \mathcal{L}[Y_{k-1}(x;p)]) \\
= & \mathcal{L} \sum_{k=1}^{\infty} [y_k(x) - \chi_k y_{k-1}(x)] \\
= & \mathcal{L}(\lim_{k\to\infty} y_k(x)) \\
= & 0,
\end{aligned} \tag{18}$$

where the linearity of the operator $\mathcal{L}$ is used.
Since $\hbar \neq 0$, $H(x) \neq 0$, so it can be expressed, as

$$\sum_{k=1}^{\infty} (L[Y_{k-1}(x;p)] + N_{k-1}[Y_0(x;p), Y_1(x;p), ..., Y_{k-1}(x;p)]) = 0. \tag{19}$$

Moreover,

$$\begin{aligned}
& \sum_{k=1}^{\infty} (L[Y_{k-1}(x;p)] + N_{k-1}[Y_0(x;p), Y_1(x;p), ..., Y_{k-1}(x;p)]) \\
= & \sum_{k=1}^{\infty} [y''_{k-1}(x) + f(x)y'_{k-1}(x) + g(x)D^{\beta}(y_{k-1}(x)) + h(x)y_{k-1}(x) \\
& - \int_a^b K(x,s)\rho(y_{k-1}(s))ds] + (1 - \chi_m)r(x)]. \\
= & \sum_{k=1}^{\infty} [y''_{k-1}(x) + f(x)y'_{k-1}(x) + g(x)D^{\beta}(y_{k-1}(x)) + h(x)y_{k-1}(x) \\
& - \int_a^b K(x,s)\rho(y_{k-1}(s))ds] + r(x).
\end{aligned} \tag{20}$$

From Eq.(18) and Eq.(20), it can be written as

$$\begin{aligned}
& \sum_{k=1}^{\infty} [y''_{k-1}(x) + f(x)y'_{k-1}(x) + g(x)D^{\beta}(y_{k-1}(x)) + h(x)y_{k-1}(x) \\
& - \int_a^b K(x,s)\rho(y_{k-1}(s))ds] + r(x) = 0,
\end{aligned} \tag{21}$$

which shows that the series solution satisfies the integro-differential Eq.(9). This completes the proof. □

Remark: The valid region of $\hbar$ for convergence of series solution can be determined, although approximately, by plotting the $\hbar$-curves. Let $x_0 \in [a, b]$, then $Y_k(x_0; p)$ is function of $\hbar$. The graph of $Y_k(x_0; p)$ versus $\hbar$ ia a $\hbar$-curve [19].

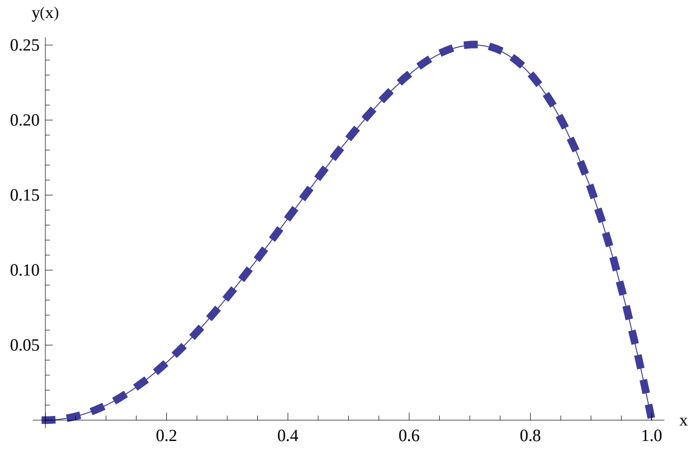

FIGURE 1. Comparison of exact solution (solid line) and approximate solution (dashed line) for Example 1

6. Test Applications

Application 1

The time-fractional linear integro-differential equation is considered, as

$$y''(x) - x^2 D^{0.3} y(x) + xu(x) = r(x) + \int_0^1 K(x,t) y(t) dt \tag{22}$$

subject to the constraints $y(0) = 0$ and $y(1) = 0$, where $r(x) = -x^5 + x^3 - 12x^2 + \frac{4x}{15} + \frac{68}{35} + \frac{24}{\Gamma(4.7)} x^{5.7} - \frac{2}{\Gamma(2.7)} x^{3.7}$ and $K(x,t) = t^2 - 2x$. The analytic solution of this integro-differential system is

$$y(x) = x^2(1 - x^2).$$

The initial approximation is calculated using the standard HAM, as $y_0(x) = 0$. The auxiliary function $H(x)$ is taken as $H(x) = 1$. Using the first and second order deformation equations, the second order approximation to the exact solution is calculated, as

$$y(x) = y_0(x) + y_1(x) + y_2(x), \tag{23}$$

where the value of $\hbar$ is taken to be $\hbar = -1$. The approximate solution values and corresponding absolute errors are summarized in Table 1. The results are graphically represented by Figure 1.

Table 1: Approximate solution values and absolute errors for Example 1

x	Exact solution	Approximate solution	Absolute error
0.0	0.000000	0.000000	0.000000
0.1	0.009900	0.009913	1.306720×10^{-5}
0.2	0.038400	0.038448	4.821870×10^{-5}
0.3	0.081900	0.081200	9.979060×10^{-5}
0.4	0.134400	0.134562	1.623850×10^{-4}
0.5	0.187500	0.187730	2.304120×10^{-4}
0.6	0.230400	0.230698	2.977360×10^{-4}
0.7	0.249900	0.250257	3.565400×10^{-4}
0.8	0.230400	0.230792	3.918480×10^{-4}
0.9	0.153900	0.154263	3.632010×10^{-4}
1.0	0.000000	0.000157	1.568040×10^{-4}

Application 2
The time-fractional linear integro-differential equation is considered, as

$$y''(x) + xD^{0.7}y(x) + (x+2)y'(x) = r(x) + \int_0^1 K(x,t)y(t)dt \tag{24}$$

subject to the constraints $y(0) = 0$ and $y(1) = 0$, where

$$\begin{aligned} r(x) \quad = \quad & -12x^2 + 20x^3 + \frac{(240 - 36\pi^2 + \pi^4)(1 + 6x^2)}{\pi^6} + (2+x)(-4x^3 + 5x^4) \\ & -\frac{\Gamma(5)}{\Gamma(4.3)}x^{4.3} + \frac{\Gamma(6)}{\Gamma(5.3)}x^{5.3} \end{aligned} \tag{25}$$

and $K(x,t) = (1+6x^2)\cos \pi t$. The analytic solution of this integro-differential system is

$$y(x) = x^4(x-1).$$

The initial approximation is calculated according to standard HAM, as $y_0(x) = 0$. The auxiliary function $H(x)$ is taken as $H(x) = 1$. Moreover, value of $\hbar$ is chosen as $\hbar = -0.65$. The second order approximation to the exact solution is calculated using the proposed method and the numerical results are summarized in Table 2. The results are expressed graphically in Figure 2.

Table 2: Approximate solution values and absolute errors for Example 2

x	Exact solution	Approximate solution	Absolute error
0.0	0.000000	0.000000	0.000000
0.1	−0.000090	−0.000071	1.938800×10^{-5}
0.2	−0.001280	−0.001119	1.607010×10^{-4}
0.3	−0.005670	−0.005158	5.121850×10^{-4}
0.4	−0.015360	−0.014380	9.809290×10^{-4}
0.5	−0.031250	−0.029999	1.251410×10^{-3}
0.6	−0.051840	−0.050938	9.023600×10^{-4}
0.7	−0.072030	−0.072357	3.267400×10^{-4}
0.8	−0.081920	−0.084078	2.157550×10^{-3}
0.9	−0.065610	−0.068967	3.357210×10^{-3}
1.0	0.000000	−0.001403	1.403230×10^{-3}

Application 3
The time-fractional nonlinear integro-differential equation is considered, as

$$y''(x) + D^{0.9}y(x) + xy(x) = r(x) + \int_0^1 K(x,t)y^2(t)dt \tag{26}$$

subject to the constraints $y(0) = 0$ and $y(1) = 1$, where

$$r(x) = \frac{13}{6} + \frac{x}{5} + \frac{\Gamma(3)}{\Gamma(2.1)}x^{1.1} + x^3 \tag{27}$$

and $K(x,t) = -(x+t)$. The analytic solution of this integro-differential system is

$$y(x) = x^2.$$

The initial approximation is taken as $y_0(x) = 0$. The auxiliary function $H(x)$ is taken as $H(x) = 1$. Moreover, value of $\hbar$ is chosen as $\hbar = -0.6$. The second order approximation to the exact solution is calculated using the proposed method and the numerical results are summarized in Table 3. The results are expressed graphically in Figure 3.

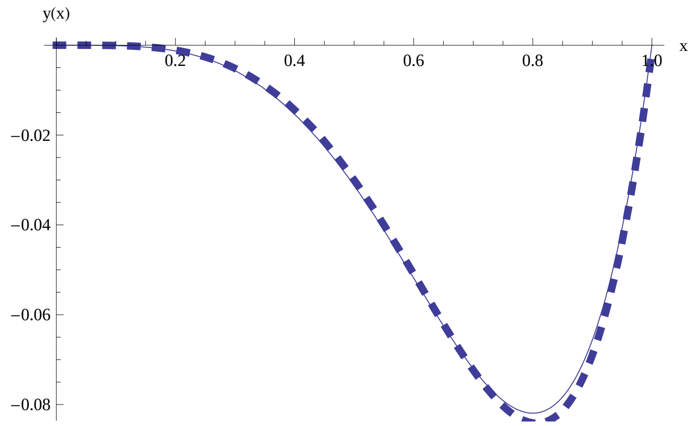

FIGURE 2. Comparison of exact solution (solid line) and approximate solution (dashed line) for Example 2

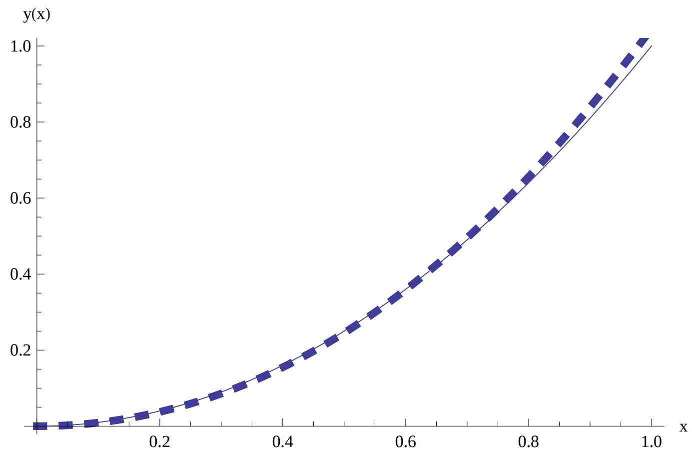

FIGURE 3. Comparison of exact solution (solid line) and approximate solution (dashed line) for Example 3

Table 3: Approximate solution values and absolute errors for Example 3

x	Exact solution	Approximate solution	Absolute error
0.0	0.000000	0.000000	0.000000
0.1	0.010000	0.009232	7.684470×10^{-4}
0.2	0.040000	0.037501	2.498910×10^{-3}
0.3	0.090000	0.085704	4.295280×10^{-3}
0.4	0.160000	0.154750	5.249810×10^{-3}
0.5	0.250000	0.245527	4.473170×10^{-3}
0.6	0.360000	0.358858	1.142350×10^{-3}
0.7	0.490000	0.495416	5.416150×10^{-3}
0.8	0.640000	0.655591	1.559150×10^{-2}
0.9	0.810000	0.839282	2.928200×10^{-2}
1.0	1.000000	1.045590	4.559110×10^{-2}

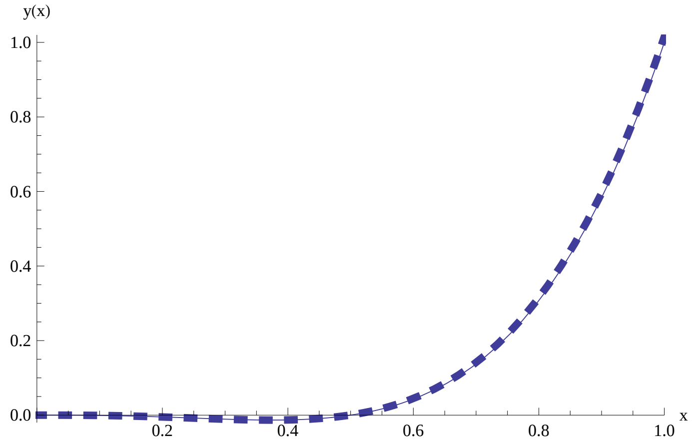

FIGURE 4. Comparison of exact solution (solid line) and approximate solution (dashed line) for Example 4

Application 4

The time-fractional nonlinear integro-differential equation is considered, as

$$y''(x) + x^2 D^{0.7} y(x) + xy(x) = r(x) + \int_0^1 K(x,t) y(t) y'(t) dt \tag{28}$$

subject to the constraints $y(0) = 0$ and $y(1) = 1$, where

$$r(x) = \frac{115}{252} + x + 12x^2 + (6x + x^4)(-1 + 2x) + 2\frac{\Gamma(5)}{\Gamma(4.3)} x^{3.3} - \frac{\Gamma(4)}{\Gamma(3.3)} x^{4.3} \tag{29}$$

and $K(x,t) = -(2x + t)$. The analytic solution of this integro-differential system is

$$y(x) = x^3(2x - 1).$$

Taking $y_0(x) = 0$, $H(x) = 1$ and $\hbar = -0.735$, the second order approximation to the exact solution is calculated using the proposed method and the numerical results are summarized in Table 4. The results are graphically represented by Figure 4.

Table 4: Approximate solution values and absolute errors for Example 4

x	Exact solution	Approximate solution	Absolute error
0.0	0.000000	0.000000	0.000000
0.1	−0.000080	−0.000855	5.494240×10^{-5}
0.2	−0.000480	−0.004915	1.149590×10^{-4}
0.3	−0.010800	−0.010925	1.248700×10^{-4}
0.4	−0.012800	−0.012819	1.858790×10^{-5}
0.5	0.000000	0.0004225	4.222490×10^{-4}
0.6	0.043200	0.0448090	1.609010×10^{-3}
0.7	0.137200	0.1412260	4.025870×10^{-3}
0.8	0.307200	0.3150820	7.881910×10^{-3}
0.9	0.583200	0.5956600	1.246040×10^{-2}
1.0	1.000000	1.0150800	1.507960×10^{-2}

7. Conclusions

The purpose of this study is to introduce a reliable iterative algorithm to approximate the solutions of time-fractional integro-differential equations of the form (9). The fractional derivative is considered in Caputo sense. The proposed algorithm is a modification of the well-known homotopy analysis method which has two major advantages over the standard homotopy analysis method. Firstly, it involves fewer terms in each iteration. Secondly, the integrals involved on each iteration step are easier to manipulate. The algorithm is illustrated using different test applications. The numerical and graphical representations of the results show explicitly the accuracy and generality of the suggested method.

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