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HERMITE-HADAMARD TYPE INEQUALITIES FOR QUASI-CONVEX FUNCTIONS VIA IMPROVED POWER-MEAN INEQUALITY

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Abstract. In this paper, by using power-mean and improved power-mean integral inequality and an general identity for differentiable functions we can get new estimates on integral inequalities for functions whose derivatives in absolute value at certain power are quasi-convex functions. It is proved that the result obtained improved power-mean integral inequality is better than the result obtained power-mean inequality. Some applications to special means of real numbers are also given.

Keywords: Hermite-Hadamard inequality, improved power-mean inequality, midpoint type inequality, convex function, quasi-convex unctions.

AMS Subject Classification: 26D15, 26E70

1. Introduction

Definition 1.1. *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality*

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$$

valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \emptyset$.

Integral inequalities have played an important role in the development of all branches of Mathematics and the other sciences. The inequalities discovered by Hermite and Hadamard for convex functions are very important in the literature. The classical Hermite-Hadamard integral inequality provides estimates of the mean value of a continuous convex function $f : [a,b] \to \mathbb{R}$. Firstly, let's recall the Hermite-Hadamard integral inequality.

Theorem 1.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \tag{1}$$

holds.

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This double inequality is known in the literature as Hermite-Hadamard integ-ral inequality for convex functions [3]. Note that some of the classical inequalities for means can be derived from (1) for appropriate particular selections of the mapping f. Both inequalities hold in the reversed direction if the function f is concave.

The following inequality is well known in the literature as Simpson's inequality .

Theorem 1.2. *Let $f:[a,b]\to\mathbb{R}$ be a four times continuously differentiable mapping on (a,b) and $\left\|f^{(4)}\right\|_{\infty}=\sup_{x\in(a,b)}\left|f^{(4)}(x)\right|<\infty$. Then the following inequality holds:*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{1}{2880}\left\|f^{(4)}\right\|_{\infty}(b-a)^4.$$

In recent years many authors have studied error estimations for Simpson's inequality; for refinements, counterparts, generalizations and new Simpson's type inequalities, see [7, 8] and therein.

Theorem 1.3 (Power-mean Integral Inequality)**.** *Let $q\geq 1$. If f and g are real functions defined on $[a,b]$ and if $|f|$, $|f|\,|g|^q$ are integrable functions on $[a,b]$ then*

$$\int_a^b|f(x)g(x)|\,dx\leq\left(\int_a^b|f(x)|\,dx\right)^{1-\frac{1}{q}}\left(\int_a^b|f(x)|\,|g(x)|^q\,dx\right)^{\frac{1}{q}}.$$

Recently, in [4], İşcan gave a refinement of the Hölder integral inequality as following:

Theorem 1.4 (Hölder-İşcan Integral Inequality [4])**.** *Let $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$. If f and g are real functions defined on interval $[a,b]$ and if $|f|^p$, $|g|^q$ are integrable functions on $[a,b]$ then*

$$\begin{aligned}&\int_a^b|f(x)g(x)|\,dx\\ \leq\;&\frac{1}{b-a}\left\{\left(\int_a^b(b-x)\,|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b(b-x)\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_a^b(x-a)\,|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b(x-a)\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right\}\end{aligned}$$

An refinement of power-mean integral inequality as a result of the Hölder-İşcan integral inequality can be given as follows:

Theorem 1.5 (Improved power-mean integral inequality [6])**.** *Let $q\geq 1$. If f and g are real functions defined on interval $[a,b]$ and if $|f|$, $|f|\,|g|^q$ are integrable functions on $[a,b]$ then*

$$\begin{aligned}&\int_a^b|f(x)g(x)|\,dx\\ \leq\;&\frac{1}{b-a}\left\{\left(\int_a^b(b-x)\,|f(x)|\,dx\right)^{1-\frac{1}{q}}\left(\int_a^b(b-x)\,|f(x)|\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_a^b(x-a)\,|f(x)|\,dx\right)^{1-\frac{1}{q}}\left(\int_a^b(x-a)\,|f(x)|\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right\}\end{aligned}$$

Definition 1.2 ([2]). *A function $f:[a,b]\to\mathbb{R}$ is said quasi-convex on $[a,b]$ if*

$$f\left(tx+(1-t)y\right)\leq\max\left\{f(x),f(y)\right\}$$

for any $x,y\in[a,b]$ and $t\in[0,1]$.

It is clear that every nonnegative convex function is also quasi-convex function.
In order to prove our main theorems, we need the following lemmas:

Lemma 1.1 ([5]). *Let $f:I^{\circ}\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on $I^{\circ}, a,b\in I^{\circ}$ (I° is the interior) with $a<b$. If $f'\in L[a,b]$, then we have*

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right) \qquad (2)\\ =\ &(b-a)\left[\int_0^{\frac{1}{2}} tf'\left(ta+(1-t)b\right)dt+\int_{\frac{1}{2}}^1 (t-1)f'\left(ta+(1-t)b\right)dt\right].\end{aligned}$$

We state that if we make the appropriate variable change in the integrals on the right side of the equation (2), then we get the following identity:

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\\ =\ &\frac{b-a}{4}\left[\int_0^1 tf'\left(t\frac{a+b}{2}+(1-t)b\right)dt+\int_0^1 tf'\left(t\frac{a+b}{2}+(1-t)a\right)dt\right].\end{aligned}$$

Lemma 1.2 ([1]). *Let the function $f:I^{\circ}\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^{\circ}$ with $a<b$. If $f'\in L[a,b]$, then the following equality holds:*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx=\frac{b-a}{2}\int_0^1 (1-2t)\,f'\left(ta+(1-t)\,b\right)dt.$$

Our aim is to obtain new type inequalities for the quasi-convex function using the power-mean and improved power-mean integral inequalities and Lemma 1.1 and Lemma 1.2.

Throught this paper, we will use the following notation for shortness

$$\begin{aligned}M &= \left(\max\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right)^{1/q}=\max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}\\ M_x &= \left(\max\left\{\left|f'(\frac{a+b}{2})\right|^q,\left|f'(x)\right|^q\right\}\right)^{1/q}=\max\left\{\left|f'(\frac{a+b}{2})\right|,\left|f'(x)\right|\right\}\end{aligned}$$

2. Main result

One of the biggest goals of the researchers working on inequalities is to find the best approach. So, we obtained new results with different applications of power-mean and improved power-mean integral inequalities by using right and left side Hermite-Hadamard type identities for differentiable functions. Moreover, it is proved that the result obtained improved power-mean integral inequality is better than the result obtained power-mean integral inequality.

2.1. **New inequalities related to the left hand side of Hermite-Hadamard.** Using Lemma 1.1 we shall give some results for quasi-convex functions.

Theorem 2.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is quasi-convex function on the interval $[a,b]$, $q \geq 1$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{4}\left(M_a + M_b\right)\left(\frac{1}{q+1}\right)^{\frac{1}{q}}. \tag{3}$$

Proof. Using the Lemma 1.1, power-mean integral inequality and the quasi-convexity of the function $|f'|^q$, we obtain

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \\
\leq\ & \frac{b-a}{4}\left[\int_0^1 |t|\left|f'\left(t\frac{a+b}{2}+(1-t)b\right)\right|dt + \int_0^1 |t|\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|dt\right] \\
\leq\ & \frac{b-a}{4}\left\{\left(\int_0^1 dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |t|^q\left|f'\left(t\frac{a+b}{2}+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\int_0^1 dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |t|^q\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\right\} \\
\leq\ & \frac{b-a}{4}\left\{M_a\left(\int_0^1 |t|^q\,dt\right)^{\frac{1}{q}} + M_b\left(\int_0^1 |t|^q\,dt\right)^{\frac{1}{q}}\right\} \\
=\ & \frac{b-a}{4}\left(M_a+M_b\right)\left(\frac{1}{q+1}\right)^{\frac{1}{q}}
\end{aligned}$$

□

Corollary 2.1. *If we take $q = 1$ in the Theorem 2.1, we get the following inequality:*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{4}A\left(M_a, M_b\right),$$

where A is the arithmetic mean.

Theorem 2.2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is quasi-convex function on the interval $[a,b]$, $q \geq 1$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \\
\leq\ & \frac{b-a}{4}\left(M_a^{\frac{1}{q}} + M_b^{\frac{1}{q}}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{q+2}\right)^{\frac{1}{q}}\left[\left(\frac{1}{q+1}\right)^{\frac{1}{q}}+1\right].
\end{aligned} \tag{4}$$

Proof. Using the Lemma 1.1, improwed power-mean integral inequality and the quasi-convexity of the function $|f'|^q$, we get

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \\
\leq\ & \frac{b-a}{4}\left[\int_0^1 |t|\left|f'\left(t\frac{a+b}{2}+(1-t)b\right)\right|dt + \int_0^1 |t|\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|dt\right]
\end{aligned}$$

$$
\begin{aligned}
\leq\ & \frac{b-a}{4}\left\{\left(\int_0^1 (1-t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)\left|t\right|^q\left|f'\left(t\frac{a+b}{2}+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
& +\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|t\right|^q\left|f'\left(t\frac{a+b}{2}+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}} \\
& +\left(\int_0^1 (1-t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)\left|t\right|^q\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}} \\
& \left.+\left(\int_0^1 (1-t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)\left|t\right|^q\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\right\} \\
\leq\ & \frac{b-a}{4}\left\{M_b\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{(q+1)(q+2)}\right)^{\frac{1}{q}}+M_b\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{q+2}\right)^{\frac{1}{q}}\right. \\
& \left. M_a\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{(q+1)(q+2)}\right)^{\frac{1}{q}}+M_a\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{q+2}\right)^{\frac{1}{q}}\right\} \\
=\ & \frac{b-a}{4}\left(M_a+M_b\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{q+2}\right)^{\frac{1}{q}}\left[\left(\frac{1}{q+1}\right)^{\frac{1}{q}}+1\right],
\end{aligned}
$$

where

$$
\begin{aligned}
\int_0^1 (1-t)dt &= \int_0^1 tdt = \frac{1}{2} \\
\int_0^1 (1-t)\left|t\right|^q dt &= \frac{1}{(q+1)(q+2)} \\
\int_0^1 t\left|t\right|^q dt &= \frac{1}{q+2}
\end{aligned}
$$

□

Remark 2.1. *The inequality (4) is better approach than the inequality (3). Really, by using concavity of the function* $h:[0,\infty)\to\mathbb{R}, h(x)=x^{\lambda}, 0<\lambda\leq 1$, *we obtain*

$$
\begin{aligned}
\frac{\alpha(p)}{\beta(p)} &= \frac{\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{q+2}\right)^{\frac{1}{q}}\left[\left(\frac{1}{q+1}\right)^{\frac{1}{q}}+1\right]}{\left(\frac{1}{q+1}\right)^{\frac{1}{q}}} = \left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{q+1}{q+2}\right)^{\frac{1}{q}}\left[\left(\frac{1}{q+2}\right)^{\frac{1}{q}}+1\right] \\
&= 2\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\frac{1}{2}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}+\frac{1}{2}\left(\frac{q+1}{q+2}\right)^{\frac{1}{q}}\right] \\
&\leq 2\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\frac{\frac{1}{q+1}+\frac{q+1}{q+2}}{2}\right]^{\frac{1}{q}} \\
&= 2\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{2}\right)^{\frac{1}{q}} \\
&= 1.
\end{aligned}
$$

Therefore $\alpha(p)\leq\beta(p)$.

Corollary 2.2. *If we take $q = 1$ in the Theorem 2.2, we get the following inequality:*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{4}A\left(M_a, M_b\right),$$

where A is the arithmetic mean.

Remark 2.2. *In case of $q = 1$, the inequality (3) coincides with the inequality (3)*

2.2. **New inequalities related to the right hand side of Hermite-Hadamard.** Using Lemma 1.2 we shall give some results for quasi-convex functions.

Theorem 2.3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is quasi-convex function on the interval $[a, b]$, $q \geq 1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2}M\left(\frac{1}{q+1}\right)^{\frac{1}{q}}. \tag{5}$$

Proof. Using the Lemma 1.2, power-mean integral inequality and the quasi-convexity of the function $|f'|^q$, we obtain

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{b-a}{2}\int_0^1 |1-2t|\left|f'\left(ta+(1-t)b\right)\right| dt. \\
\leq\ & \frac{b-a}{2}\left(\int_0^1 dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-2t|^q\left|f'\left(ta+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{b-a}{2}M\left(\int_0^1 |1-2t|^q dt\right)^{\frac{1}{q}} \\
=\ & \frac{b-a}{2}M\left(\frac{1}{q+1}\right)^{\frac{1}{q}}
\end{aligned}$$

By simple computation

$$\int_0^1 |1-2t|^q dt = \frac{1}{q+1}$$

Thus, we obtain the inequality (5). This completes the proof. □

Theorem 2.4. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is quasi-convex function on the interval $[a, b]$, $q \geq 1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}M. \tag{6}$$

Proof. Using the Lemma 1.2, improved power-mean integral inequality and the quasi-convexity of the function $|f'|^q$, we obtain

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{b-a}{2}\int_0^1 |1-2t|\left|f'\left(ta+(1-t)b\right)\right|dt. \\
\leq\ & \frac{b-a}{2}\left(\int_0^1 (1-t)\,|1-2t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)\,|1-2t|\left|f'\left(ta+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}} \\
&+\frac{b-a}{2}\left(\int_0^1 t\,|1-2t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\,|1-2t|\left|f'\left(ta+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{b-a}{2}M\left(\int_0^1 (1-t)\,|1-2t|\,dt+\int_0^1 t\,|1-2t|\,dt\right) \\
=\ & \frac{b-a}{4}M
\end{aligned}$$

By simple computation

$$\int_0^1 (1-t)\,|(1-2t)|\,dt=\int_0^1 t\,|1-2t|\,dt=\frac{1}{4}$$

Thus, we obtain the inequality (6). This completes the proof. □

Remark 2.3. *The inequality (6) is better approach than the inequality (5). Really,*

$$\frac{R(p)}{P(p)}=\frac{\frac{1}{4}}{\frac{1}{2}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}}=\frac{(q+1)^{\frac{1}{q}}}{2}\leq 1.$$

Therefore $R(p)\leq P(p)$.

3. Applications to Special Means

We now consider the applications of the following special means:
i. The arithmetic mean: $A=A(a,b):=\frac{a+b}{2}, a,b>0$
ii. The harmonic mean: $H=H(a,b):=\frac{2ab}{a+b}, a,b>0$
iii. The logarithmic mean:

$$L=L(a,b):=\begin{cases} a, & a=b \\ \frac{b-a}{\ln b-\ln a}, & a\neq b \end{cases},\ a,b>0$$

iv. The p-logarithmic mean:

$$L_p=L_p(a,b):=\begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & a=b \\ a, & a\neq b \end{cases},\ p\in\mathbb{R}\setminus\{-1,0\}\,;a,b>0.$$

vi. The identric mean

$$I=I\left(a,b\right)=\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}.$$

These means are often used in numerical approximation and in other areas. However, the following simple relationships are known in the literature:

$$H\leq G\leq L\leq I\leq A.$$

It is also known that L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$.

Proposition 3.1. *Let $a, b \in \mathbb{R}$, $0 < a < b$. Then, for all $q \geq 1$, we have the following inequality:*

$$\left|L^{-1} - A^{-1}\right| \leq \frac{b-a}{2}\left(\frac{2}{q+1}\right)^{\frac{1}{q}} A\left(a^{-2}, A^{-2}\right) A\left(\left(\frac{q+1}{q+2}\right)^{\frac{1}{q}}, \left(\frac{1}{q+2}\right)^{\frac{1}{q}}\right).$$

Proof. The assertion follows from the Theorem 2.2 applied to the quasi-convex function $f(x) = \frac{1}{x}, x \in [a, b]$. □

Proposition 3.2. *Let $0 < a < b$ and $q \geq 1$. Then we have the following inequality*

$$\left|L_2^2 - A^2\right| \leq (b-a)\left(\frac{2}{q+1}\right)^{\frac{1}{q}} A(A, b)\, A\left(\left(\frac{q+1}{q+2}\right)^{\frac{1}{q}}, \left(\frac{1}{q+2}\right)^{\frac{1}{q}}\right).$$

Proof. The assertion follows from the inequality (4) in Theorem 2.2, for $f : (0, \infty) \to \mathbb{R}$, $f(x) = \frac{x^2}{2}$. □

Proposition 3.3. *Let $a, b \in \mathbb{R}$, $0 < a < b$ and $n \in \mathbb{N}, n \geq 2$. Then, for all $q \geq 1$, we have the following inequality:*

$$\left|L_n^n(a, b) - A^n(a, b)\right| \leq n\frac{b-a}{2}\left(\frac{2}{q+1}\right)^{\frac{1}{q}} A\left(A^{n-1}, b^{n-1}\right) A\left(\left(\frac{q+1}{q+2}\right)^{\frac{1}{q}}, \left(\frac{1}{q+2}\right)^{\frac{1}{q}}\right).$$

Proof. The assertion follows from the Theorem 2.2 applied to the quasi-convex function $f(x) = x^n, x \in [a, b]$ and $n \in \mathbb{N}$. □

Proposition 3.4. *Let $0 < a < b$ and $q \geq 1$. Then we have the following inequality*

$$\frac{A}{I} \leq \exp\left\{\frac{b-a}{2}\left(\frac{2}{q+1}\right)^{\frac{1}{q}} A(\ln A, \ln b)\, A\left(\left(\frac{q+1}{q+2}\right)^{\frac{1}{q}}, \left(\frac{1}{q+2}\right)^{\frac{1}{q}}\right)\right\}.$$

Proof. The assertion follows from the inequality (4) in Theorem 2.2, for $f : (0, \infty) \to \mathbb{R}$, $f(x) = \ln x$. □

Proposition 3.5. *Let $a, b \in \mathbb{R}$, $0 < a < b$. Then, for all $q \geq 1$, we have the following inequality:*

$$\left|H^{-1} - L^{-1}\right| \leq \frac{b-a}{4a}.$$

Proof. The assertion follows from the Theorem 2.4 applied to the quasi-convex function $f(x) = \frac{1}{x}, x \in [a, b]$. □

Proposition 3.6. *Let $0 < a < b$ and $q \geq 1$. Then we have the following inequality*

$$\left|L_2^2 - A^2\right| \leq \frac{b-a}{2}b.$$

Proof. The assertion follows from the inequality (6) in Theorem 2.4, for $f : (0, \infty) \to \mathbb{R}$, $f(x) = \frac{x^2}{2}$. □

Proposition 3.7. *Let $a, b \in \mathbb{R}$, $0 < a < b$ and $n \in \mathbb{N}, n \geq 2$. Then, for all $q \geq 1$, we have the following inequality:*

$$\left|L_n^n(a, b) - A(a^n, b^n)\right| \leq n\frac{b-a}{4}b^{n-1}.$$

Proof. The assertion follows from the Theorem 2.4 applied to the quasi-convex function $f(x) = x^n, x \in [a, b]$ and $n \in \mathbb{N}$. □

Proposition 3.8. *Let $0 < a < b$ and $q \geq 1$. Then we have the following inequality*

$$\frac{A}{I} \leq \exp\left\{\frac{b-a}{4a}\right\}.$$

Proof. The assertion follows from the inequality (6) in Theorem 2.4, for $f : (0, \infty) \to \mathbb{R}$, $f(x) = \ln x$. □

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