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# THE FEKETE-SZEGÖ PROBLEMS FOR A SUBCLASS OF M-FOLD SYMMETRIC BI-UNIVALENT FUNCTIONS

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ABSTRACT. In this paper, we investigate a new subclass $\mathcal{P}^{h,p}_{\Sigma_m}(\lambda,\gamma)$ of m-fold symmetric bi-univalent functions. Moreover, for functions of this subclass, we obtain the coefficient estimates of the Taylor-Maclaurin coefficients $|a_{m+1}|, |a_{2m+1}|$ and Fekete-Szegö problems. The coefficients estimates presented in this paper would generalize and improve those in related works of several earlier authors

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## 1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions $f$ which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : \ |z| < 1\}$, with in the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

we let $\mathcal{S}$ to denote the class of functions $f \in \mathcal{A}$ which are univalent in $\mathbb{U}$ (see details [3, 5]).

Every function $f \in \mathcal{S}$ has an inverse $f^{-1}$, which is defined by

$$f^{-1}(f(z)) = z \ (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \ \left( |w| < r_0(f), \ r_0(f) \geq \frac{1}{4} \right).$$

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In fact, the inverse function $f^{-1}$ is given by

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \quad (2)$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$, if both $f$ and $f^{-1}$ are univalent in $\mathbb{U}$ (see [17]). We denote $\sigma_{\mathcal{B}}$ the class of bi-univalent functions in $\mathbb{U}$ given by (1).

Lewin [9] investigated the class $\sigma_{\mathcal{B}}$ of bi-univalent functions and showed that $|a_2| < 1.51$ for the Taylor-Maclaurin coefficient $|a_2|$ of functions belonging to $\sigma_{\mathcal{B}}$. Subsequently, Brannan et al. [2] conjectured that $|a_2| \leq \sqrt{2}$. However, finding upper bounds of the Taylor-Maclaurin coefficients $|a_n|(n \in \mathbb{N} - \{2, 3\})$ for each $f \in \sigma_{\mathcal{B}}$ is coefficient estimate problem and still an open problem.

For a brief history and interesting examples of functions in the class $\sigma_{\mathcal{B}}$, refer to the papers by Sirvastava et al. [13, 14, 23, 24].

For each function $f \in \mathcal{S}$ function

$$h(z) = \sqrt[m]{f(z^m)} \quad (3)$$

is univalent and maps unit disk $\mathbb{U}$ into a region with m-fold symmetry. A function $f$ is said to be m-fold symmetric (see [8, 10]) if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \ (z \in \mathbb{U}, m \in \mathbb{N}). \quad (4)$$

We denote by $\mathcal{S}_m$ the class of m-fold symmetric univalent functions in $\mathbb{U}$, which are normalized by the series expansion (4). In fact, the functions in class $\mathcal{S}$ are one-fold symmetric.

In [18] Srivastava et al. defined m-fold symmetric bi-univalent functions analogues to the concept of m-fold symmetric univalent functions. They gave some important results, such as each function $f \in \sigma_{\mathcal{B}}$ generates an m-fold symmetric bi-univalent function for each $m \in \mathbb{N}$. Furthermore, for the normalized form of $f$ given by (4), they obtained the series expansion for $f^{-1}$ as follows:

$$\begin{aligned} f^{-1}(w) &= w - a_{m+1}w^{m+1} + [(m+1)a_{m+1}^2 - a_{2m+1}]w^{2m+1} \\ &-[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}]w^{3m+1} + \cdots . \end{aligned} \quad (5)$$

We denote by $\Sigma_m$ the class of m-fold symmetric bi-univalent functions in $\mathbb{U}$. For $m = 1$, formula (5) coincides with formula (2) of the class $\sigma_{\mathcal{B}}$. Some examples of m-fold symmetric bi-univalent functions are given as follows:

$$\left(\frac{z^m}{1 - z^m}\right)^{\frac{1}{m}}, \ \left[\frac{1}{2}\log\left(\frac{1 + z^m}{1 - z^m}\right)^{\frac{1}{m}}\right] \ \textit{and} \ [-\log(1 - z^m)]^{\frac{1}{m}}$$

with the corresponding inverse functions

$$\left(\frac{w^m}{1 + w^m}\right)^{\frac{1}{m}}, \ \left(\frac{e^{2w^m} - 1}{e^{2w^m} + 1}\right)^{\frac{1}{m}} \ \textit{and} \ \left(\frac{e^{w^m} - 1}{e^{w^m}}\right)^{\frac{1}{m}},$$

respectively.

In fact that this widely-cited work by Srivastava et al. [18] actually revived the study of m-fold symmetric bi-univalent functions in recent years and that it led to a flood of papers on the subject by (for example) Srivastava et al. [15, 16, 18, 19, 20], and others [6, 7, 11, 12, 21, 22].

The object of the present paper is to introduce new subclass $\mathcal{P}_{\Sigma_m}^{h,p}(\lambda,\gamma)$ of $\Sigma_m$ and obtain estimates on initial coefficients $|a_{m+1}|$, $|a_{2m+1}|$ for functions in subclass and improve some recent works of many authors.

## 2. Subclass $\mathcal{P}_{\Sigma_m}^{h,p}(\lambda,\gamma)$

In this section, we introduce the general subclass $\mathcal{P}_{\Sigma_m}^{h,p}(\lambda,\gamma)$.

**Definition 2.1.** *Let the functions $h,p:\mathbb{U}\to\mathbb{C}$ be so constrained that*

$$\min\{\mathcal{R}e\,((h(z))\,,\mathcal{R}e\,(p(z))\}>0\ (z\in\mathbb{U})\ \text{and}\ h(0)=p(0)=1.$$

*A function $f\in\Sigma_m$ given by (4) is said to be in the subclass $\mathcal{P}_{\Sigma_m}^{h,p}(\lambda,\gamma)$, if the following conditions are satisfied:*

$$1+\frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)}+\lambda\frac{z^2f''(z)}{f(z)}-1\right)\in h(\mathbb{U})\ (0\le\lambda\le1,\gamma\in\mathbb{C}-\{0\},z\in\mathbb{U}) \tag{6}$$

*and*

$$1+\frac{1}{\gamma}\left(\frac{wg'(w)}{g(w)}+\lambda\frac{w^2g''(w)}{g(w)}-1\right)\in p(\mathbb{U})\ (0<\lambda\le1,\gamma\in\mathbb{C}-\{0\},w\in\mathbb{U}), \tag{7}$$

*where $g$ is the extension of $f^{-1}$ to $\mathbb{U}$.*

**Remark 2.1.** *There are many selections of the functions $h(z)$ and $p(z)$ which would provide interesting classes of m-fold symmetric bi-univalent functions $\Sigma_m$. For example, if we let*

$$h(z)=p(z)=\left(\frac{1+z^m}{1-z^m}\right)^{\alpha}=1+2\alpha z^m+2\alpha^2z^{2m}+\cdots\ (0<\alpha\le1),$$

*it is easy to verify that the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 2.1.*

*If $f\in\mathcal{P}_{\Sigma_m}^{h,p}(\lambda,\gamma)$, then*

$$\left|arg\left\{1+\frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)}+\lambda\frac{z^2f''(z)}{f(z)}-1\right)\right\}\right|<\frac{\alpha\pi}{2}$$

*and*

$$\left|arg\left\{1+\frac{1}{\gamma}\left(\frac{wg'(w)}{g(w)}+\lambda\frac{w^2g''(w)}{g(w)}-1\right)\right\}\right|<\frac{\alpha\pi}{2}.$$

*Therefore, for $h(z)=p(z)=\left(\frac{1+z^m}{1-z^m}\right)^{\alpha}$, $\gamma=1$ and $\lambda=0$, the subclass $\mathcal{P}_{\Sigma_m}^{h,p}(\lambda,\gamma)$ reduces to the subclass $\mathcal{S}_{\Sigma_m}^{\alpha}$ which was considered by Altinkaya and Yalcin* [1].

*If we let*

$$h(z)=p(z)=\frac{1+(1-2\beta)z^m}{1-z^m}=1+2(1-\beta)z^m+2(1-\beta)z^{2m}+\cdots\ (0\le\beta<1),$$

*it is easy to verify that the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 2.1.*

*If $f\in\mathcal{P}_{\Sigma_m}^{h,p}(\lambda,\gamma)$, then*

$$\mathcal{R}e\left\{1+\frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)}+\lambda\frac{z^2f''(z)}{f(z)}-1\right)\right\}>\beta$$

*and*

$$\mathcal{R}e\left\{1+\frac{1}{\gamma}\left(\frac{wg'(w)}{g(w)}+\lambda\frac{w^2g''(w)}{g(w)}-1\right)\right\}>\beta.$$

*Therefore, for* $h(z)=p(z)=\frac{1+(1-2\beta)z^m}{1-z^m}$, $\gamma=1$ *and* $\lambda=0$, *the subclass* $\mathcal{P}^{h,p}_{\Sigma_m}(\lambda,\gamma)$ *reduces to the subclass* $\mathcal{S}^{\beta}_{\Sigma_m}$ *which was considered by Altinkaya and Yalcin* [1].

**Remark 2.2.** *For one-fold symmetric bi-univalent functions, we denote the subclass* $\mathcal{P}^{h,p}_{\Sigma_1}(\lambda,\gamma)=\mathcal{P}^{h,p}_{\Sigma}(\lambda,\gamma)$. *Special cases of this subclass illustrated below:*

- *By putting* $h(z)=p(z)=\left(\frac{1+z^m}{1-z^m}\right)^{\alpha}$, $\gamma=1$ *and* $\lambda=0$, *then the subclass* $\mathcal{P}^{h,p}_{\Sigma}(\lambda,\gamma)$ *reduces to the subclass* $\mathcal{S}^*_{\Sigma}[\alpha]$ *of strongly bi-starlike functions of order* $\alpha(0<\alpha\le 1)$.
- *By putting* $h(z)=p(z)=\frac{1+(1-2\beta)z^m}{1-z^m}$, $\gamma=1$ *and* $\lambda=0$, *then the subclass* $\mathcal{P}^{h,p}_{\Sigma}(\lambda,\gamma)$ *reduces to the subclass* $\mathcal{S}^*_{\Sigma}(\beta)$ *of bi-starlike functions of order* $\beta(0\le\beta<1)$.

**Theorem 2.1.** *Let* $f(z)$ *given by* (4) *be in subclass* $\mathcal{P}^{h,p}_{\Sigma_m}(\lambda,\gamma)$ $(0\le\lambda<1,\gamma\in\mathbb{C}-\{0\})$. *Then*

$$|a_{m+1}|\le\min\left\{\frac{|\gamma||h_m|}{m[(1+\lambda(m+1)]},\sqrt{\frac{|\gamma|(|h_{2m}|+|p_{2m}|)}{2m^2[1+2\lambda(m+1)]}}\right\}$$

*and*

$$|a_{2m+1}|\le\min\left\{\frac{|\gamma|(|h_{2m}|+|p_{2m}|)}{4m[1+\lambda(2m+1)]}+\frac{|\gamma|(m+1)(|h_m|^2+|p_m|^2)}{4m^2[1+\lambda(m+1)]^2},\right.$$
$$\left.\frac{|\gamma|\left[(2m+1)+\lambda(m+1)(4m+1)\right]|h_{2m}|+|\gamma|\left[\lambda(m+1)+1\right]|p_{2m}|}{4m^2\left(1+\lambda(2m+1)\right)\left(1+2\lambda(m+1)\right)}\right\}.$$

*Proof.* First of all, we write the argument inequalities in (6) and (7) in their equivalent forms as follows:

$$1+\frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)}+\lambda\frac{z^2f''(z)}{f(z)}-1\right)=h(z)\ (z\in\mathbb{U}) \tag{8}$$

and

$$1+\frac{1}{\gamma}\left(\frac{wg'(w)}{g(w)}+\lambda\frac{w^2g''(w)}{g(w)}-1\right)=p(w)\ (w\in\mathbb{U}), \tag{9}$$

respectively, where functions $h(z)$ and $p(w)$ satisfy the conditions of Definition 2.1.

Furtheremore, the functions $h(z)$ and $p(w)$ have the forms:

$$h(z)=1+h_mz^m+h_{2m}z^{2m}+h_{3m}z^{3m}+\cdots \tag{10}$$

and

$$p(w)=1+p_mw^m+p_{2m}w^{2m}+p_{3m}w^{3m}+\cdots, \tag{11}$$

respectively.

Now, upon substituting from (10) and (11) into (8) and (9), respectively, and equating the coefficients, we get

$$\frac{m[1+\lambda(m+1)]a_{m+1}}{\gamma}=h_m, \tag{12}$$

$$\frac{2m[1+\lambda(2m+1)]}{\gamma}a_{2m+1}-\frac{m[1+\lambda(m+1)]}{\gamma}a_{m+1}^2=h_{2m}, \tag{13}$$

$$-\frac{m[1+\lambda(m+1)]}{\gamma}a_{m+1}=p_m, \tag{14}$$

and

$$-\frac{2m[1+\lambda(2m+1)]}{\gamma}a_{2m+1}+\frac{m[(2m+1)+\lambda(m+1)(4m+1)]}{\gamma}a_{m+1}^2=p_{2m}. \tag{15}$$

From (12) and (14), we get

$$h_m=-p_m \tag{16}$$

and

$$a_{m+1}^2=\frac{\gamma^2(h_m^2+p_m^2)}{2m^2[1+\lambda(m+1)]^2}. \tag{17}$$

Adding (13) and (15), we get

$$a_{m+1}^2=\frac{\gamma(h_{2m}+p_{2m})}{2m^2[1+2\lambda(m+1)]}. \tag{18}$$

Therefore, we find from the equations (16), (17) and (18) that

$$|a_{m+1}|\le\frac{|\gamma||h_m|}{m[(1+\lambda(m+1)]},$$

$$|a_{m+1}|\le\sqrt{\frac{|\gamma|(|h_{2m}|+|p_{2m}|)}{2m^2[1+2\lambda(m+1)]}}.$$

respectively. So we get the desired estimate on the coefficient $|a_{m+1}|$.

Next, in order to find the bound on the coefficient $|a_{2m+1}|$, we subtract (15) from (13), we get

$$a_{2m+1}=\frac{\gamma(h_{2m}-p_{2m})}{4m[1+\lambda(2m+1)]}+\frac{(m+1)}{2}a_{m+1}^2. \tag{19}$$

Therefore, we find from (17) and (19) that

$$a_{2m+1}=\frac{\gamma(h_{2m}-p_{2m})}{4m[1+\lambda(2m+1)]}+\frac{\gamma^2(m+1)(h_m^2+p_m^2)}{4m^2[1+\lambda(m+1)]^2}. \tag{20}$$

Also, from (18) and (19), we have

$$a_{2m+1}=\frac{\gamma\left[(2m+1)+\lambda(m+1)(4m+1)\right]h_{2m}+\gamma\left[\lambda(m+1)+1\right]p_{2m}}{4m^2\left(1+\lambda(2m+1)\right)\left(1+2\lambda(m+1)\right)}. \tag{21}$$

So, from the equations (20) and (21), we get

$$|a_{2m+1}|\le\frac{|\gamma|(|h_{2m}|+|p_{2m}|)}{4m[1+\lambda(2m+1)]}+\frac{|\gamma|(m+1)(|h_m|^2+|p_m|^2)}{4m^2[1+\lambda(m+1)]^2}$$

and

$$|a_{2m+1}|\le\frac{|\gamma|\left[(2m+1)+\lambda(m+1)(4m+1)\right]|h_{2m}|+|\gamma|\left[\lambda(m+1)+1\right]|p_{2m}|}{4m^2\left(1+\lambda(2m+1)\right)\left(1+2\lambda(m+1)\right)}.$$

□

**Theorem 2.2.** *Let $f(z)$ given by (4) be in subclass $\mathcal{P}^{h,p}_{\Sigma_m}(\lambda,\gamma)$ $(0 \le \lambda < 1, \gamma \in \mathbb{C} - \{0\})$. Also let $\rho$ be real number. Then*

$$|a_{2m+1} - \rho a_{m+1}^2| \le \begin{cases} \frac{|\gamma|}{4m(1+\lambda(2m+1))}\left\{(1+T(\rho))\,|h_{2m}| + (1-T(\rho))\,|p_{2m}|\right\}; & |T(\rho)| \le 1 \\ \frac{|\gamma|}{4m(1+\lambda(2m+1))}\left\{\left|1+T(\rho)\right||h_{2m}| + \left|T(\rho)-1\right||p_{2m}|\right\}; & |T(\rho)| \ge 1. \end{cases}$$

*where*

$$T(\rho) = \frac{(m-2\rho+1)\,(1+\lambda(2m+1))}{m\,(1+2\lambda(m+1))}.$$

*Proof.* From the equation (19), we get

$$a_{2m+1} - \rho a_{m+1}^2 = \frac{\gamma(h_{2m}-p_{2m})}{4m[1+\lambda(2m+1)]} + \frac{m-2\rho+1}{2}a_{m+1}^2. \tag{22}$$

From the equation (18) and (22), we have

$$a_{2m+1} - \rho a_{m+1}^2 = \frac{|\gamma|}{4m\,(1+\lambda(2m+1))}\left\{\left[1 + \frac{(m-2\rho+1)\,(1+\lambda(2m+1))}{m\,(1+2\lambda(m+1))}\right]h_{2m} + \left[\frac{(m-2\rho+1)\,(1+\lambda(2m+1))}{m\,(1+2\lambda(m+1))} - 1\right]p_{2m}\right\}.$$

Next, taking the absolute values we obtain

$$|a_{2m+1} - \rho a_{m+1}^2| \le \frac{|\gamma|}{4m\,(1+\lambda(2m+1))}\left\{\left|1 + \frac{(m-2\rho+1)\,(1+\lambda(2m+1))}{m\,(1+2\lambda(m+1))}\right||h_{2m}| + \left|\frac{(m-2\rho+1)\,(1+\lambda(2m+1))}{m\,(1+2\lambda(m+1))} - 1\right||p_{2m}|\right\}.$$

Then, we conclude that

$$|a_{2m+1} - \rho a_{m+1}^2| \le \begin{cases} \frac{|\gamma|}{4m(1+\lambda(2m+1))}\left\{(1+T(\rho))\,|h_{2m}| + (1-T(\rho))\,|p_{2m}|\right\}; & |T(\rho)| \le 1 \\ \frac{|\gamma|}{4m(1+\lambda(2m+1))}\left\{\left|1+T(\rho)\right||h_{2m}| + \left|T(\rho)-1\right||p_{2m}|\right\}; & |T(\rho)| \ge 1. \end{cases}$$

□

## 3. Conclusions

By putting

$$h(z) = p(z) = \left(\frac{1+z^m}{1-z^m}\right)^{\alpha} = 1 + 2\alpha z^m + 2\alpha^2 z^{2m} + \cdots \ (0 < \alpha \le 1, z \in \mathbb{U}),$$

$\lambda = 0$ and $\gamma = 1$ in Theorems 2.1 and 2.2, we conclude the following results.

**Corollary 3.1.** *Let $f$ given by (4) be in subclass $\mathcal{S}^{\alpha}_{\Sigma_m}(0 < \alpha \le 1, m \in \mathbb{N})$. Then*

$$|a_{m+1}| \le \min\left\{\frac{2\alpha}{m}, \frac{\sqrt{2}\alpha}{m}\right\} = \frac{\sqrt{2}\alpha}{m}$$

*and*

$$|a_{2m+1}| \le \min\left\{\frac{\alpha^2}{m} + \frac{2(m+1)\alpha^2}{m^2}, \frac{(m+1)\alpha^2}{m^2}\right\} = \frac{(m+1)\alpha^2}{m^2}.$$

**Corollary 3.2.** *Let $f$ given by (4) be in subclass $\mathcal{S}^{\alpha}_{\Sigma_m}(0<\alpha\leq 1, m\in\mathbb{N})$. Also let $\rho$ be real number. Then*

$$|a_{2m+1}-\rho a_{m+1}^2|\leq\begin{cases}\frac{\alpha^2}{m};\ \frac{|m-2\rho+1|}{m}\leq 1\\ \frac{|m-2\rho+1|\alpha}{m^2};\ \frac{|m-2\rho+1|}{m}\geq 1.\end{cases}$$

**Remark 3.1.** *The bounds on $|a_{m+1}|$ and $|a_{2m+1}|$ given in Corollary 3.1 are better than those given in* [1, Corollary 6]*. Because*

$$\frac{\sqrt{2}\alpha}{m}\leq\frac{2\alpha}{m\sqrt{\alpha+1}}$$

*and*

$$\frac{(m+1)\alpha^2}{m^2}\leq\frac{\alpha^2}{m}+\frac{2(m+1)\alpha^2}{m^2}\leq\frac{\alpha}{m}+\frac{2(m+1)\alpha^2}{m^2}.$$

By putting

$$h(z)=p(z)=\frac{1+(1-2\beta)z^m}{1-z^m}=1+2(1-\beta)z^m+2(1-\beta)z^{2m}+\cdots\ (0\leq\beta<1, z\in\mathbb{U}),$$

$\lambda=0$ and $\gamma=1$ in Theorems 2.1 and 2.2, we conclude the following results.

**Corollary 3.3.** *Let $f$ given by (4) be in subclass $\mathcal{S}^{\beta}_{\Sigma_m}(0\leq\beta<1, m\in\mathbb{N})$. Then*

$$|a_{m+1}|\leq\begin{cases}\frac{\sqrt{2(1-\beta)}}{m};\ 0\leq\beta\leq\frac{1}{2}\\ \frac{2(1-\beta)}{m};\ \frac{1}{2}\leq\beta<1\end{cases}$$

*and*

$$|a_{2m+1}|\leq\begin{cases}\frac{(m+1)(1-\beta)}{m^2};\ 0\leq\beta\leq\frac{1+2m}{2(1+m)}\\ \frac{2(m+1)(1-\beta)^2}{m^2}+\frac{1-\beta}{m};\ \frac{1+2m}{2(1+m)}\leq\beta<1.\end{cases}$$

**Corollary 3.4.** *Let $f$ given by (4) be in subclass $\mathcal{S}^{\beta}_{\Sigma_m}(0\leq\beta<1, m\in\mathbb{N})$ . Also let $\rho$ be real number. Then*

$$|a_{2m+1}-\rho a_{m+1}^2|\leq\begin{cases}\frac{(1-\beta)}{m};\ \frac{|m-2\rho+1|}{m}\leq 1\\ \frac{(1-\beta)|m-2\rho+1|}{m^2};\ \frac{|m-2\rho+1|}{m}\geq 1.\end{cases}$$

**Remark 3.2.** *The bounds on $|a_{m+1}|$ and $|a_{2m+1}|$ given in Corollary 3.3 are better than those given in* [1, Corolary 7].

By setting $m=1$ in Corollary 3.1, we conclude the following result.

**Corollary 3.5.** *Let $f$ given by (1) be in subclass $\mathcal{S}^{*}_{\Sigma}[\alpha]$ of strongly bi-starlike functions of order $\alpha(0<\alpha\leq 1)$. Then*

$$|a_2|\leq\min\left\{2\alpha,\sqrt{2}\alpha\right\}=\sqrt{2}\alpha$$

*and*

$$|a_3|\leq\min\left\{5\alpha^2,2\alpha^2\right\}=2\alpha^2.$$

**Remark 3.3.** *The bounds on $|a_2|$ and $|a_3|$ given in Corollary 3.5 are better than those given in* [4, Corolary 2.5].

By setting $m = 1$ in Corollary 3.2, we conclude the following result.

**Corollary 3.6.** *Let $f$ given by (1) be in subclass $\mathcal{S}^*_\Sigma[\alpha]$ of strongly bi-starlike functions of order $\alpha(0 < \alpha \leq 1)$. Also let $\rho$ be real number. Then*

$$|a_3 - \rho a_2^2| \leq \begin{cases} \alpha^2 \ ; \ |1-\rho| \leq \frac{1}{2} \\ \\ 2|1-\rho|\alpha; \ |1-\rho| \geq \frac{1}{2}. \end{cases}$$

By setting $m = 1$ in Corollary 3.3, we conclude the following result.

**Corollary 3.7.** *Let $f$ given by (1) be in subclass $\mathcal{S}^*_\Sigma(\beta)$ of bi-starlike functions of order $\beta(0 \leq \beta < 1)$. Then*

$$|a_2| \leq \begin{cases} \sqrt{2(1-\beta)}; \ 0 \leq \beta \leq \frac{1}{2} \\ \\ 2(1-\beta) \ ; \ \frac{1}{2} \leq \beta < 1 \end{cases}$$

*and*

$$|a_3| \leq \begin{cases} 2(1-\beta) \ ; \ 0 \leq \beta \leq \frac{3}{4} \\ \\ 4(1-\beta)^2 + (1-\beta); \ \frac{3}{4} \leq \beta < 1. \end{cases}$$

**Remark 3.4.** *The bound on $|a_2|$ given in Corollary 3.7 is better than that given in* [4, Corolary 3.5].

By setting $m = 1$ in Corollary 3.4, we conclude the following result.

**Corollary 3.8.** *Let $f$ given by (1) be in subclass $\mathcal{S}^*_\Sigma(\beta)$ of bi-starlike functions of order $\beta(0 \leq \beta < 1)$. Also let $\rho$ be real number. Then*

$$|a_3 - \rho a_2^2| \leq \begin{cases} 1-\beta \ ; \ |1-\rho| \leq \frac{1}{2} \\ \\ 2(1-\beta)|1-\rho|; \ |1-\rho| \geq \frac{1}{2}. \end{cases}$$

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