TWMS J. App. and Eng. Math. V.11, N.2, 2021, pp. 541-551

SOME OPERATIONS OVER INTUITIONISTIC FUZZY MATRICES BASED ON HAMACHER T-NORM AND T-CONORM

I. SILAMBARASAN[1], S. SRIRAM[2], §

Abstract. In this paper, we define the Hamacher scalar multiplication and Hamacher exponentiation operations on Intuitionistic fuzzy matrices and also we construct $n._h A$ and $A^{\wedge_h n}$ of an intuitionistic fuzzy matrix A and studied the algebraic properties of these operations.

Keywords: Intuitionistic fuzzy matrix, Hamacher sum, Hamacher product, Algebraic sum and Algebraic product.

AMS Subject Classification:15B15,15B99.

1. Introduction

The concept of intuitionistic fuzzy sets (IFSs) proposed by Atanassov [1], is an important tool dealing with imperfect and imprecise information. An intuitionistic fuzzy set gives the membership degree and non membership degree to describe the level of an element belonging to a set. Based on a t-norm (T) and t-conorm (T*), a generalized union and a generalized intersection of IFSs were introduced by Deschrijver and Kerre [2]. Huang [5] introduced some operations on the IFSs, such as Hamacher sum, Hamacher product, Hamacher exponentiation, etc., and investigated the multiple attribute decision making problem based on the Hamacher aggregation operators with intuitionistic fuzzy information.

In [12] Pal introduced intuitionistic fuzzy determinant. Khan et al.[8] introduced intuitionistic fuzzy matrices and determined distance between intuitionistic fuzzy matrices (IFMs). Simultaneously, Im et al. [6,7] defined the notion of intuitionistic fuzzy matrix as a generalization of fuzzy matrix. Also they studied the adjoint of square intuitionistic fuzzy matrix and its properties. Mondal and Pal [10] studied the similarity relations , together with invertibility conditions and eigenvalues of IFMs. Emam and Fndh [3] defined some kinds of IFMs, the max-min and min-max composition of IFMs. Also, they derived several important results of these compositions and construct an idempotent IFM from any given one through the min-max composition. Zhang [20] studied intuitionistic fuzzy

[1] Department of Mathematics, Annamalai University,Annamalainagar, 608 002, Tamil Nadu, India.
e-mail: sksimbuking@gmail.com; ORCID: https://orcid.org/0000-0002-7437-4043.
[2] Mathematics Wing (DDE), Annamalai University, Annamalainagar, 608 002, Tamilnadu, India.
e-mail: ssm_3096@yahoo.co.in; ORCID: https://orcid.org/0000-0002-8535-3563 .
§ Manuscript received: May 6, 2019; accepted: September 6, 2019.
TWMS Journal of Applied and Engineering Mathematics, Vol.11, No.2

value and introduced the concept of composition two IFMs. Sriram and Boobalan [17,18] studied the properties of algebraic sum and algebraic product of IFMs and prove that the set of all IFMs form a commutative monoid. Muthuraji et al. [11] obtain a decomposition of an intuitionistic fuzzy matrix by using the new composition operator and modal operators. In[15], we defined Hamacher operations of fuzzy matrices(FMs) and investigated their algebraic properties. Also , they extended Hamacher operations of FMs to IFMs [16].

The paper is organized as follows: In section 2,the basic concepts of IFMs and some existing operations over IFMs. In section 3, we define Hamacher scalar multiplication and Hamacher exponentiation operations on IFMs and also we construct $n._h A$ and $A^{\wedge_h n}$ of an intuitionistic fuzzy matrix A and studied the algebraic properties of these operations.

2. Preliminaries and definitions

In this section, the basic concepts of intuitionistic fuzzy matrices and some existing operations over IFMs.

Definition 2.1. [8] An intuitionistic fuzzy matrix is a matrix of pairs $A = \left(\left\langle a_{ij}, a'_{ij} \right\rangle\right)$, where a_{ij} and a'_{ij} are the membership and non-membership matrix of the ij^{th} element in A satisfying the condition $0 \leq a_{ij} + a'_{ij} \leq 1$ for all i, j.

Khan et al.[8] described some operations over IFMs as follows:

Definition 2.2. Let $A = \left(\left\langle a_{ij}, a'_{ij} \right\rangle\right), B = \left(\left\langle b_{ij}, b'_{ij} \right\rangle\right)$ be two IFMs of same size, then

- $A \vee B = \left(\left\langle \max(a_{ij}, b_{ij}), \min(a'_{ij}, b'_{ij}) \right\rangle\right)$,
- $A \wedge B = \left(\left\langle \min(a_{ij}, b_{ij}), \max(a'_{ij}, b'_{ij}) \right\rangle\right)$,
- $A^C = \left(\left\langle a'_{ij}, a_{ij} \right\rangle\right)$,
- $A \leq B$ iff $a_{ij} \leq b_{ij}$ and $a'_{ij} \leq b'_{ij}$,
- $A \oplus B = \left(\left\langle a_{ij} + b_{ij} - a_{ij}b_{ij}, a'_{ij}b'_{ij} \right\rangle\right)$,
- $A \odot B = \left(\left\langle a_{ij}b_{ij}, a'_{ij} + b'_{ij} - a'_{ij}b'_{ij} \right\rangle\right)$.

Sriram and Boobalan [17], further defined the following operations over IFMs.

Definition 2.3. The scalar multiplication operation over an IFM A is denoted by nA and is defined by

$$nA = \left(\left\langle 1 - (1 - a_{ij})^n, \left(a'_{ij}\right)^n \right\rangle\right)$$

Where n is any positive integer.

Definition 2.4. The exponentiation operation over an IFM A is denoted by A^n and is defined by

$$A^n = \left(\left\langle \left(a'_{ij}\right)^n, 1 - (1 - a_{ij})^n \right\rangle\right)$$

Where n is any positive integer.

Hamacher operations: T-norm and t-conorm are important notions in fuzzy set theory, Hamacher operations [4] include the Hamacher product and the Hamacher sum, they are defined as follows:
Hamacher product $\odot$ is a $t - norm$ and Hamacher sum $\oplus$ is a $t - conorm$, where

$$T(a,b) = a \odot b = \frac{ab}{\gamma + (1-\gamma)(a+b-ab)},\ \gamma \geq 0$$
$$T^*(a,b) = a \oplus b = \frac{a+b-ab-(1-\gamma)ab}{1-(1-\gamma)ab},\ \gamma \geq 0$$

In particular for $\gamma = 1$, Hamacher $t-norm$ and $t-conorm$ will be reduced to the algebraic $t-norm$ and $t-conorm$

$$T(a,b) = a \odot b = ab$$
$$T^*(a,b) = a \oplus b = a+b-ab$$

respectively, using these operations Shyamal and Pal [14] defined the algebraic operations on fuzzy matrices.
When $\gamma = 2$, then Hamacher $t-norm$ and $t-conorm$ will be reduced to the Einstein $t-norm$ and $t-conorm$

$$T(a,b) = a \odot b = \frac{ab}{1+(1-a)(1-b)}$$
$$T^*(a,b) = a \oplus b = \frac{a+b}{1+ab}$$

respectively, using these operations Selvarajan et al. [13] defined Einstein operations of fuzzy matrices.

We have defined Hamacher operations of Intuitionistic fuzzy matrices as follows[16].

Definition 2.5. Let $A = \left(\left\langle a_{ij}, a'_{ij}\right\rangle\right), B = \left(\left\langle b_{ij}, b'_{ij}\right\rangle\right)$ be any two IFMs of same size, then

(i) The Hamacher sum of A and B is defined by
$A \oplus_H B = c_{ij}$,

$$\text{where } c_{ij} = \begin{cases} \langle 1,0\rangle,\ if\ \left\langle a_{ij}, a'_{ij}\right\rangle = \langle 1,0\rangle, \left\langle b_{ij}, b'_{ij}\right\rangle = \langle 1,0\rangle \\ \left\langle \dfrac{a_{ij}+b_{ij}-2a_{ij}b_{ij}}{1-a_{ij}b_{ij}}, \dfrac{a'_{ij}b'_{ij}}{a'_{ij}+b'_{ij}-a'_{ij}b'_{ij}} \right\rangle, otherwise \end{cases}$$

for all i, j.

(ii) The Hamacher product of A and B is defined by
$A \odot_H B = c_{ij}$,

$$\text{where } c_{ij} = \begin{cases} \langle 0,1\rangle,\ if \left\langle a_{ij}, a'_{ij}\right\rangle = \langle 0,1\rangle, \left\langle b_{ij}, b'_{ij}\right\rangle = \langle 0,1\rangle \\ \left\langle \dfrac{a_{ij}b_{ij}}{a_{ij}+b_{ij}-a_{ij}b_{ij}}, \dfrac{a'_{ij}+b'_{ij}-2a'_{ij}b'_{ij}}{1-a'_{ij}b'_{ij}} \right\rangle, otherwise \end{cases}$$

for all i, j.

3. Some Operations over Intuitionistic fuzzy matrix Based on Hamacher t-norm and t-conorm

In this section, we define Hamacher scalar multiplication and Hamacher exponentiation operations on IFMs and also we construct $n._h A$ and $A^{\wedge_h n}$ of an intuitionistic fuzzy matrix A and studied the algebraic properties of these operations.

Based on the Definition 2.5 and Hamacher operations of IFMs, we can get the following results.

Theorem 3.1. *Let* $A = \left(\left\langle a_{ij}, a'_{ij}\right\rangle\right), B = \left(\left\langle b_{ij}, b'_{ij}\right\rangle\right)$ *be any two IFMs of same size, then* $A \oplus_H B$ *and* $A \odot_H B$ *calculated are also IFMs.*

Proof. For two IFMs $A = \left(\left\langle a_{ij}, a'_{ij} \right\rangle\right)$ and $B = \left(\left\langle b_{ij}, b'_{ij} \right\rangle\right)$, the following relations are evident:

$$0 \le a_{ij} \le 1, \quad 0 \le a'_{ij} \le 1, \quad 0 \le a_{ij} + a'_{ij} \le 1 :$$
$$0 \le b_{ij} \le 1, \quad 0 \le b'_{ij} \le 1, \quad 0 \le b_{ij} + b'_{ij} \le 1.$$

Then it follows that:

$$a_{ij} \ge a_{ij}b_{ij}, \quad b_{ij} \ge a_{ij}b_{ij}, \quad 1 \ge a_{ij}b_{ij} \ge 0.$$
$$1 \ge a'_{ij}b'_{ij} \ge 0, \quad a'_{ij} \ge a'_{ij}b'_{ij}, \quad b'_{ij} \ge a'_{ij}b'_{ij}.$$

Which indicates that

$$a_{ij} + b_{ij} - 2a_{ij}b_{ij} = a_{ij} - a_{ij}b_{ij} + b_{ij} - a_{ij}a_{ij} \ge 0,$$
$$1 - a_{ij}b_{ij} \ge 0, \quad a'_{ij} + b'_{ij} - a'_{ij}b'_{ij} \ge 0.$$

Thus

$$\frac{a_{ij} + b_{ij} - 2a_{ij}b_{ij}}{1 - a_{ij}b_{ij}} \ge 0, \frac{a'_{ij}b'_{ij}}{a'_{ij} + b'_{ij} - a'_{ij}b'_{ij}} \ge 0.$$

Since $1 - a_{ij}b_{ij} - (a_{ij} + b_{ij} - 2a_{ij}b_{ij}) = 1 - a_{ij} - b_{ij} + a_{ij}b_{ij} = (1 - a_{ij})(1 - b_{ij}) \ge 0$, and $a'_{ij} + b'_{ij} - a'_{ij}b'_{ij} - a'_{ij}b'_{ij} = a'_{ij} - a'_{ij}b'_{ij} + b'_{ij} - a'_{ij}b'_{ij} \ge 0$, then we have

$$\frac{a_{ij} + b_{ij} - 2a_{ij}b_{ij}}{1 - a_{ij}b_{ij}} \le 1, \frac{a'_{ij}b'_{ij}}{a'_{ij} + b'_{ij} - a'_{ij}b'_{ij}} \le 1.$$

It is obvious that $0 \le a'_{ij} \le 1 - a_{ij}, 0 \le b'_{ij} \le 1 - b_{ij}$, so we can get:

$$\frac{a'_{ij}b'_{ij}}{a'_{ij} + b'_{ij} - a'_{ij}b'_{ij}} = \frac{(1 - a_{ij})(1 - b_{ij})}{1 - a_{ij}b_{ij}}.$$

Then it follows:

$$\frac{a_{ij} + b_{ij} - 2a_{ij}b_{ij}}{1 - a_{ij}b_{ij}} + \frac{a'_{ij}b'_{ij}}{a'_{ij} + b'_{ij} - a'_{ij}b'_{ij}}$$
$$\le \frac{a_{ij} + b_{ij} - 2a_{ij}b_{ij}}{1 - a_{ij}b_{ij}} + \frac{(1 - a_{ij})(1 - b_{ij})}{1 - a_{ij}b_{ij}} = \frac{1 - a_{ij}b_{ij}}{1 - a_{ij}b_{ij}} = 1$$

Hence

$$0 \le \frac{a_{ij} + b_{ij} - 2a_{ij}b_{ij}}{1 - a_{ij}b_{ij}} \le 1, \quad 0 \le \frac{a'_{ij}b'_{ij}}{a'_{ij} + b'_{ij} - a'_{ij}b'_{ij}} \le 1$$
$$0 \le \frac{a_{ij} + b_{ij} - 2a_{ij}b_{ij}}{1 - a_{ij}b_{ij}} + \frac{a'_{ij}b'_{ij}}{a'_{ij} + b'_{ij} - a'_{ij}b'_{ij}} \le 1.$$

Similarly, we can also get:

$$0 \le \frac{a'_{ij} + b'_{ij} - 2a'_{ij}b'_{ij}}{1 - a'_{ij}b'_{ij}} \le 1, \quad 0 \le \frac{a_{ij}b_{ij}}{a_{ij} + b_{ij} - a_{ij}b_{ij}} \le 1,$$
$$0 \le \frac{a_{ij}b_{ij}}{a_{ij} + b_{ij} - a_{ij}b'_{ij}} + \frac{a'_{ij} + b'_{ij} - 2a'_{ij}b'_{ij}}{1 - a'_{ij}b'_{ij}} \le 1.$$

Therefore, both of $A \oplus_H B$ and $A \odot_H B$ are IFMs. □

Theorem 3.2. *If n is any positive integer and A is an IFM, then the Hamacher scalar multiplication operation is*

$$n._h A = \underbrace{A \oplus_h ... \oplus_h A}_{n} = \left(\left\langle \frac{na_{ij}}{1 + (n-1)a_{ij}}, \frac{a'_{ij}}{n - (n-1)a'_{ij}} \right\rangle\right) \tag{3.1}$$

Proof. Mathematical induction can be used to prove that the above equation (3.1) holds for all positive integer n. The equation (3.1) is called P(n).
when n=2, we have,

$$A.{}_hA = \left(\left\langle \frac{a_{ij}+a_{ij}-2a_{ij}a_{ij}}{1-a_{ij}a_{ij}}, \frac{a'_{ij}a'_{ij}}{a'_{ij}+a'_{ij}-a'_{ij}a'_{ij}} \right\rangle\right)$$

$$2.{}_hA = \left(\left\langle \frac{2a_{ij}(1-a_{ij})}{1-a^2_{ij}}, \frac{a^2_{ij}}{a_{ij}(2-a_{ij})} \right\rangle\right)$$

$$= \left(\left\langle \frac{2a_{ij}}{1+a_{ij}}, \frac{a_{ij}}{2-a_{ij}} \right\rangle\right)$$

$$= \left(\left\langle \frac{2a_{ij}}{1+(2-1)a_{ij}}, \frac{a'_{ij}}{2-(2-1)a'_{ij}} \right\rangle\right)$$

P(n) holds when n=2.
Suppose that equation (3.1) holds for $n = m$,

(i.e) $m.{}_hA = \underbrace{A \oplus_h ... \oplus_h A}_{m} = \left(\left\langle \frac{ma_{ij}}{1+(m-1)a_{ij}}, \frac{a'_{ij}}{m-(m-1)a'_{ij}} \right\rangle\right)$,

then we have:

$$(m+1).{}_hA = ((m.{}_hA).{}_hA) = \left(\left\langle \frac{a_{ij}(m+1)(1-a_{ij})}{(1+ma_{ij})(1-a_{ij})}, \frac{(a'_{ij})^2}{a'_{ij}(m+1-ma'_{ij})} \right\rangle\right)$$

$$= \left(\left\langle \frac{(m+1)a_{ij}}{1+ma_{ij}}, \frac{a'_{ij}}{m+1-ma'_{ij}} \right\rangle\right)$$

$$= \left(\left\langle \frac{(m+1)a_{ij}}{[1+(m+1)-1]a_{ij}}, \frac{a'_{ij}}{m+1-[(m+1)-1]a'_{ij}} \right\rangle\right)$$

So, when $n = m+1, n.{}_hA = \underbrace{A \oplus_h ... \oplus_h A}_{n} = \left(\left\langle \frac{na_{ij}}{1+(n-1)a_{ij}}, \frac{a'_{ij}}{n-(n-1)a'_{ij}} \right\rangle\right)$,

also holds. Using the induction hypothesis that $P(n)$ holds for any positive integer n. □

Similarly, we can prove the following theorem.

Theorem 3.3. *If n is any positive integer and A is an IFM, then the Hamacher exponentiation operation is*

$$A^{\wedge_h n} = \underbrace{A \odot_h ... \odot_h A}_{n} = \left(\left\langle \frac{a_{ij}}{n-(n-1)a_{ij}}, \frac{na'_{ij}}{1+(n-1)a'_{ij}} \right\rangle\right) \tag{3.2}$$

Next, we prove the result of $n.{}_hA$ and $A^{\wedge_h n}$ are also IFMs.

Theorem 3.4. *Let n be any positive integer and A be an IFM, then $n.{}_hA$ and $A^{\wedge_h n}$ are also IFMs.*

Proof. Since $0 \le a_{ij} \le 1, 0 \le a'_{ij} \le 1, 0 \le a_{ij} + a'_{ij} \le 1$, and $n > 1$,
we have:

$$(n-1)a_{ij} > -1, \quad 1+(n-1)a_{ij} > 0,$$
$$n-(n-1)a'_{ij} = (1-a'_{ij})n + a'_{ij} > a'_{ij} \le 0.$$

Then it is easy to get that $\frac{na_{ij}}{1+(n-1)a_{ij}} \ge 0, \frac{a'_{ij}}{n-(n-1)a'_{ij}} \ge 0.$

Considering that $1+(n-1)a_{ij} = na_{ij}+1-a_{ij} \ge na_{ij}$
and $n-(n-1)a'_{ij} = a'_{ij}+n(1-a'_{ij}) \ge a'_{ij}$, we get:

$$\frac{na_{ij}}{1+(n-1)a_{ij}} \leq 1, \frac{a'_{ij}}{n-(n-1)a'_{ij}} \leq 1.$$

For $a_{ij}+a'_{ij} \leq 1, 0 \leq a'_{ij} \leq 1-a_{ij}$, it can be got that:

$$\frac{na_{ij}}{1+(n-1)a_{ij}} + \frac{a'_{ij}}{n-(n-1)a'_{ij}} = \frac{na_{ij}}{1+(n-1)a_{ij}} + \frac{1}{\frac{n}{a'_{ij}}-(n-1)} = 1$$

Thus

$$0 \leq \frac{na_{ij}}{1+(n-1)a_{ij}} \leq 1, \qquad 0 \leq \frac{a'_{ij}}{n-(n-1)a'_{ij}} \leq 1$$

$$0 \leq \frac{na_{ij}}{1+(n-1)a_{ij}} + \frac{a'_{ij}}{n-(n-1)a'_{ij}} \leq 1.$$

Similarly, we can also get,

$$0 \leq \frac{a_{ij}}{n-(n-1)a_{ij}} \leq 1, \qquad 0 \leq \frac{na'_{ij}}{1+(n-1)a'_{ij}} \leq 1.$$

$$0 \leq \frac{a_{ij}}{n-(n-1)a_{ij}} + \frac{na'_{ij}}{1+(n-1)a'_{ij}} \leq 1.$$

Therefore , $n._h A$ and $A^{\wedge_h n}$ are IFMs. □

It can be easily prove that the following Theorems.

Theorem 3.5. *Let* $A = \left(\left\langle a_{ij}, a'_{ij}\right\rangle\right), B = \left(\left\langle b_{ij}, b'_{ij}\right\rangle\right)$ *be any two IFMs of same size, then*
(i) $A \odot_H B = (A^C \oplus_H B^C)^C$,
(ii)$A \oplus_H B = (A^C \odot_H B^C)^C$.

Theorem 3.6. *For any IFM A and for any positive integer n, then*
$A^{\wedge_h n} = (n._h A^C)^C, n._h A = ((A^C)^{\wedge_h n})^C$.

Theorem 3.7. *Let* $A = \left(\left\langle a_{ij}, a'_{ij}\right\rangle\right), B = \left(\left\langle b_{ij}, b'_{ij}\right\rangle\right)$ *be any two IFMs of same size, and* n, n_1, n_2 *be positive integers, then*
(i)$n_1._h A \oplus_H n_2._h A = (n_1+n_2)._h A$,
(ii)$(n._h A) \oplus_H (n._h B) = n._h(A \oplus_H B)$,
(iii)$A^{\wedge_h n_1} \odot_H A^{\wedge_h n_2} = A^{\wedge_h (n_1+n_2)}$,
(iv)$A^{\wedge_h n} \odot_H B^{\wedge_h n} = (A \odot_H B)^{\wedge_h n}$,
(v)$n_2._h(n_1._h A) = (n_1 n_2)._h A$,
(vi)$(A^{\wedge_h n_1})^{\wedge_h n_2} = A^{\wedge_h (n_1 n_2)}$.

Proof. By Definition 2.5, Theorem 3.2 and Theorem 3.3, we have:

(i) $$n_1._h A = \left(\left\langle \frac{n_1 a_{ij}}{1+(n_1-1)a_{ij}}, \frac{a'_{ij}}{n_1-(n_1-1)a'_{ij}} \right\rangle\right) = \left(\left\langle b_{ij}, b'_{ij}\right\rangle\right)$$

$$n_2._h A = \left(\left\langle \frac{n_2 a_{ij}}{1+(n_2-1)a_{ij}}, \frac{a'_{ij}}{n_2-(n_2-1)a'_{ij}} \right\rangle\right) = \left(\left\langle c_{ij}, c'_{ij}\right\rangle\right)$$

$$B \oplus_H C = \left(\left\langle \frac{b_{ij}+c_{ij}-2b_{ij}c_{ij}}{1-b_{ij}c_{ij}}, \frac{b'_{ij}c'_{ij}}{b'_{ij}+c'_{ij}-b'_{ij}c'_{ij}} \right\rangle\right)$$

We can further get:

$$\frac{b_{ij}+c_{ij}-2b_{ij}c_{ij}}{1-b_{ij}c_{ij}}$$

$$= \frac{\frac{n_1 a_{ij}}{1+(n_1-1)a_{ij}} + \frac{n_2 a_{ij}}{1+(n_2-1)a_{ij}} - 2\frac{n_1 a_{ij}}{1+(n_1-1)a_{ij}}\frac{n_2 a_{ij}}{1+(n_2-1)a_{ij}}}{1 - \frac{n_1 a_{ij}}{1+(n_1-1)a_{ij}}\frac{n_2 a_{ij}}{1+(n_2-1)a_{ij}}}$$

$$= \frac{(n_1+n_2)a_{ij}}{1+(n_1+n_2-1)a_{ij}}$$

and

$$\frac{b'_{ij}c'_{ij}}{b'_{ij}+c'_{ij}-b'_{ij}c'_{ij}}$$

$$= \frac{\frac{a'_{ij}}{n_1-(n_1-1)a'_{ij}}\frac{a'_{ij}}{n_2-(n_2-1)a'_{ij}}}{\frac{a'_{ij}}{n_1-(n_1-1)a'_{ij}} + \frac{a'_{ij}}{n_2-(n_2-1)a'_{ij}} - \frac{a'_{ij}}{n_1-(n_1-1)a'_{ij}}\frac{a'_{ij}}{n_2-(n_2-1)a'_{ij}}}$$

$$= \frac{a'_{ij}}{(n_1+n_2)-(n_1+n_2-1)a'_{ij}}$$

Since $(n_1+n_2).{}_hA = \left(\left\langle \frac{(n_1+n_2)a_{ij}}{1+(n_1+n_2-1)a_{ij}}, \frac{a'_{ij}}{(n_1+n_2)-(n_1+n_2-1)a'_{ij}} \right\rangle\right)$,
we can finally get $(n_1.{}_hA) \oplus_H (n_2.{}_hA) = (n_1+n_2).{}_hA$.

(iii) Considering Theorem 3.5 and Theorem 3.6 and the property $(n_1.{}_hA) \oplus_H (n_2.{}_hA) = (n_1+n_2).{}_hA$, we get the result $A^{\wedge_h n_1} \odot_H A^{\wedge_h n_2} = A^{\wedge_h (n_1+n_2)}$.

(v) By Theorem 3.2 and Theorem 3.3, we have:

$$n_1.{}_hA = \left(\left\langle \frac{na_{ij}}{1+(n-1)a_{ij}}, \frac{a'_{ij}}{n-(n-1)a'_{ij}} \right\rangle\right) = \left(\left\langle b_{ij}, b'_{ij} \right\rangle\right)$$

$$n_2.{}_h(n_1.{}_hA) = \left(\left\langle \frac{n_2 b_{ij}}{1+(n_2-1)b_{ij}}, \frac{b'_{ij}}{n_2-(n_2-1)b'_{ij}} \right\rangle\right)$$

We can further get:

$$\frac{n_2 b_{ij}}{1+(n_2-1)b_{ij}} = \frac{n_2\frac{na_{ij}}{1+(n-1)a_{ij}}}{1+(n_2-1)\frac{na_{ij}}{1+(n-1)a_{ij}}} = \frac{n_1 n_2 a_{ij}}{1+(n_1 n_2-1)a_{ij}}$$

and

$$\frac{b'_{ij}}{n_2-(n_2-1)b'_{ij}} = \frac{\frac{a'_{ij}}{n-(n-1)a'_{ij}}}{n_2-(n_2-1)\frac{a'_{ij}}{n-(n-1)a'_{ij}}} = \frac{a'_{ij}}{n_1 n_2-(n_1 n_2-1)a'_{ij}}$$

Since $(n_1 n_2).{}_hA = \left(\left\langle \frac{n_1 n_2 a_{ij}}{1+(n_1 n_2-1)a_{ij}}, \frac{a'_{ij}}{n_1 n_2-(n_1 n_2-1)a'_{ij}} \right\rangle\right)$

Hence, we get $n_2.{}_h(n_1.{}_hA) = (n_1 n_2).{}_hA$. Proofs of (ii),(iv),(vi) and similar to those of (i),(iii),(v). □

Theorem 3.8. *Let* $A = \left(\left\langle a_{ij}, a'_{ij} \right\rangle\right)$, $B = \left(\left\langle b_{ij}, b'_{ij} \right\rangle\right)$ *be any two IFMs of same size and for any positive integer* n, *then*

$(i) n._h(A \wedge B) = (n._hA) \wedge (n._hB)$,
$(ii) n._h(A \vee B) = (n._hA) \vee (n._hB)$,
$(iii) (A \wedge B)^{\wedge_h n} = A^{\wedge_h n} \wedge B^{\wedge_h n}$,
$(iv) (A \vee B)^{\wedge_h n} = A^{\wedge_h n} \vee B^{\wedge_h n}$.

Proof. Since $(A \wedge B) = \left(\left\langle \min\{a_{ij}, b_{ij}\}, \max\left\{a'_{ij}, b'_{ij}\right\}\right\rangle\right)$,then

(i) $n._h(A \wedge B) = \left(\left\langle c_{ij}, c'_{ij}\right\rangle\right), n._hA = \left(\left\langle d_{ij}, d'_{ij}\right\rangle\right), n._hB = \left(\left\langle e_{ij}, e'_{ij}\right\rangle\right)$
where

$$c_{ij} = \frac{n(\min\{a_{ij}, b_{ij}\})}{1 + (n-1)(\min\{a_{ij}, b_{ij}\})} \text{ and } c'_{ij} = \frac{(\max\left\{a'_{ij}, b'_{ij}\right\})}{n - (n-1)(\max\left\{a'_{ij}, b'_{ij}\right\})},$$

we have

$$\begin{aligned} c_{ij} &= \frac{n(\min\{a_{ij}, b_{ij}\})}{1 + (n-1)(\min\{a_{ij}, b_{ij}\})}, \\ &= \min\left\{\frac{na_{ij}}{1 + (n-1)a_{ij}}, \frac{nb_{ij}}{1 + (n-1)b_{ij}}\right\}, \\ &= \min\{d_{ij}, e_{ij}\}, \end{aligned}$$

and

$$\begin{aligned} c'_{ij} &= \frac{(\max\left\{a'_{ij}, b'_{ij}\right\})}{n - (n-1)(\max\left\{a'_{ij}, b'_{ij}\right\})}, \\ &= \max\left\{\frac{a'_{ij}}{n - (n-1)a'_{ij}}, \frac{b'_{ij}}{n - (n-1)b'_{ij}}\right\}, \\ &= \max\left\{d'_{ij}, e'_{ij}\right\}. \end{aligned}$$

Thus, $n._h(A \wedge B) = (n._hA) \wedge (n._hB)$.

(iii) Since $(A \wedge B) = \left(\left\langle \min\{a_{ij}, b_{ij}\}, \max\left\{a'_{ij}, b'_{ij}\right\}\right\rangle\right)$,then
$(A \wedge B)^{\wedge_h n} = \left(\left\langle c_{ij}, c'_{ij}\right\rangle\right), A^{\wedge_h n} = \left(\left\langle d_{ij}, d'_{ij}\right\rangle\right), B^{\wedge_h n} = \left(\left\langle e_{ij}, e'_{ij}\right\rangle\right)$
where

$$c_{ij} = \frac{(\min\{a_{ij}, b_{ij}\})}{n - (n-1)(\min\{a_{ij}, b_{ij}\})} \text{ and } c'_{ij} = \frac{n(\max\left\{a'_{ij}, b'_{ij}\right\})}{1 + (n-1)(\max\left\{a'_{ij}, b'_{ij}\right\})},$$

we have

$$\begin{aligned} c_{ij} &= \frac{(\min\{a_{ij}, b_{ij}\})}{n - (n-1)(\min\{a_{ij}, b_{ij}\})}, \\ &= \min\left\{\frac{a_{ij}}{n - (n-1)a_{ij}}, \frac{b_{ij}}{n - (n-1)b_{ij}}\right\}, \\ &= \min\{d_{ij}, e_{ij}\}. \end{aligned}$$

and

$$\begin{aligned} c'_{ij} &= \frac{n(\max\left\{a'_{ij}, b'_{ij}\right\})}{1 + (n-1)(\max\left\{a'_{ij}, b'_{ij}\right\})}, \\ &= \max\left\{\frac{na'_{ij}}{1 + (n-1)a'_{ij}}, \frac{nb'_{ij}}{1 + (n-1)b'_{ij}}\right\}, \\ &= \max\left\{d'_{ij}, e'_{ij}\right\}. \end{aligned}$$

Thus, $(A \wedge B)^{\wedge_h n} = A^{\wedge_h n} \wedge B^{\wedge_h n}$,
Proofs of (ii),(iv) are similar to those of (i) and (iii). □

Theorem 3.9. *Let $A = \left(\left\langle a_{ij}, a'_{ij} \right\rangle\right), B = \left(\left\langle b_{ij}, b'_{ij} \right\rangle\right)$ be any two IFMs of same size, then $n > 0$ be an integer. The operation $n._h A$ is monotonically increasing with respect to A and n. The operation $A^{\wedge_h n}$ is monotonically increasing with respect to A and decreasing with respect to n.*

Proof. Considering two functions $p(x,y) = \dfrac{xy}{1+(x-1)y}$ and $q(x,y) = \dfrac{x}{x-(x-1)y}$, with $x > 0$, $0 \le y \le 1$, we compute their derivatives as:

$$\frac{\partial p}{\partial x} = \frac{y(1-y^2)}{(1+(x-1)y)^2} \ge 0, \quad \frac{\partial p}{\partial y} = \frac{x(1+y^2)}{(1+(x-1)y)^2} \ge 0,$$
$$\frac{\partial q}{\partial x} = \frac{y-1}{(x-(x-1)y)^2} \le 0, \quad \frac{\partial q}{\partial y} = \frac{x}{(x-(x-1)y)^2} \ge 0.$$

So $p(x,y)$ is increasing with respect to x and y.
$q(x,y)$ is decreasing with x and increasing with y.
For two IFMs $A = \left(\left\langle a_{ij}, a'_{ij} \right\rangle\right)$ and $B = \left(\left\langle b_{ij}, b'_{ij} \right\rangle\right)$, $A \le B \Rightarrow a_{ij} \le b_{ij}$ and $a'_{ij} \ge b'_{ij}$.
Then we can get:

$$p(n, a_{ij}) = \frac{na_{ij}}{1+(n-1)a_{ij}} \le \frac{nb_{ij}}{1+(n-1)b_{ij}} = p(n, b_{ij}),$$
$$q(n, a'_{ij}) = \frac{a'_{ij}}{n-(n-1)a'_{ij}} \ge \frac{b'_{ij}}{n-(n-1)b'_{ij}} = q(n, b'_{ij}),$$
$$q(n, a_{ij}) = \frac{a_{ij}}{n-(n-1)a_{ij}} \le \frac{b_{ij}}{n-(n-1)b_{ij}} = q(n, b_{ij}),$$
$$p(n, a'_{ij}) = \frac{na'_{ij}}{1+(n-1)a'_{ij}} \ge \frac{nb'_{ij}}{1+(n-1)b'_{ij}} = p(n, b'_{ij}).$$

Thus, $n._h A \le n._h B, A^{\wedge_h n} \le B^{\wedge_h n}$.
Given two integers $0 < n_1 \le n_2$, we have

$$p(n_1, a_{ij}) = \frac{n_1 a_{ij}}{1+(n_1-1)a_{ij}} \le \frac{n_2 b_{ij}}{1+(n_2-1)b_{ij}} = p(n_2, b_{ij}),$$
$$q(n_1, a'_{ij}) = \frac{a'_{ij}}{n_1-(n_1-1)a'_{ij}} \ge \frac{b'_{ij}}{n_2-(n_2-1)b'_{ij}} = q(n_2, b'_{ij}),$$
$$q(n_1, a_{ij}) = \frac{a_{ij}}{n_1-(n_1-1)a_{ij}} \le \frac{b_{ij}}{n_2-(n_2-1)b_{ij}} = q(n_2, b_{ij}),$$
$$p(n_1, a'_{ij}) = \frac{n_1 a'_{ij}}{1+(n_1-1)a'_{ij}} \ge \frac{n_2 b'_{ij}}{1+(n_2-1)b'_{ij}} = p(n_2, b'_{ij}).$$

Thus, $n_1 ._h A \le n_2 ._h B, A^{\wedge_h n_1} \le B^{\wedge_h n_2}$.
Therefore, we conclude that $n._h A$ is monoyonically increasing with respect to A and n, and the operation $A^{\wedge_h n}$ is monotonically increasing with respect to A but decreasing with respect to n. □

Theorem 3.10. *Let $A = \left(\left\langle a_{ij}, a'_{ij} \right\rangle\right), B = \left(\left\langle b_{ij}, b'_{ij} \right\rangle\right)$ be any two IFMs of same size, then $A \odot B \le A \oplus_H B \le A \oplus B, A \odot B \le A \odot_H B \le A \oplus B$.*

Proof. Let $f(x,y), g(x,y), h(x,y)$ and $k(x,y)$ be four functions defined in $[0,1] \times [0,1]$ as:

$$f(x,y)=\frac{x+y-2xy}{1-xy}, g(x,y)=\frac{xy}{x+y-xy}$$

$h(x,y)=xy,\quad k(x,y)=x+y-xy$, then $f(x,y)-h(x,y)\geq 0$,
$f(x,y)-h(x,y)\leq 0, g(x,y)-h(x,y)\geq 0, k(x,y)-g(x,y)\geq 0$.
From all above we can conclude that $h(x,y)\leq f(x,y)\leq k(x,y)$,
$h(x,y)\leq g(x,y)\leq k(x,y)$,
Hence,

$$\begin{aligned}h(a_{ij},b_{ij})&\leq f(a_{ij},b_{ij})\leq k(a_{ij},b_{ij})\\ a_{ij}b_{ij}&\leq\frac{a_{ij}+b_{ij}-2a_{ij}b_{ij}}{1-a_{ij}b_{ij}}\leq a_{ij}+b_{ij}-a_{ij}b_{ij}.\\ h(a'_{ij},b'_{ij})&\leq g(a'_{ij},b'_{ij})\leq k(a'_{ij},b'_{ij})\\ a'_{ij}b'_{ij}&=\frac{a'_{ij}b'_{ij}}{a'_{ij}+b'_{ij}-a'_{ij}b'_{ij}}\leq a'_{ij}+b'_{ij}-a'_{ij}b'_{ij}.\\ h(a_{ij},b_{ij})&\leq g(a_{ij},b_{ij})\leq k(a_{ij},b_{ij})\\ a_{ij}b_{ij}&=\frac{a_{ij}b_{ij}}{a_{ij}+b_{ij}-a_{ij}b_{ij}}\leq a_{ij}+b_{ij}-a_{ij}b_{ij}.\\ h(a'_{ij},b'_{ij})&\leq f(a'_{ij},b'_{ij})\leq k(a'_{ij},b'_{ij})\\ a'_{ij}b'_{ij}&\leq\frac{a'_{ij}+b'_{ij}-2a'_{ij}b'_{ij}}{1-a'_{ij}b'_{ij}}\leq a'_{ij}+b'_{ij}-a'_{ij}b'_{ij}.\end{aligned}$$

Based on the partial order, we have
$A\odot B\leq A\oplus_H B\leq A\oplus B, A\odot B\leq A\odot_H B\leq A\oplus B$.
Theorem 3.10 tells the truth that our developed addition and multiplication operations are neutral between two operations $A\oplus B$ and $A\odot B$ defined in Definition 2.2 □

We can prove that the following theorem establish the relation between nA and $n._hA$, then A^n and $A^{\wedge_h n}$.

Theorem 3.11. *For any IFM A and for any positive integer n, then we have:*

$$nA\leq n._hA\leq A^n,\quad nA\leq A^{\wedge_h n}\leq A^n, \textit{ for } 0<n\geq 1:$$
$$A^n\leq n._hA\leq nA,\quad A^n\leq A^{\wedge_h n}\leq nA, \textit{ for } n\geq 1.$$

4. Conclusions

In this paper, we have defined Hamacher scalar multiplication and Hamacher exponentiation operations on IFMs and also we constructed $n._hA$ and $A^{\wedge_h n}$ of an intuitionistic fuzzy matrix A and studied the algebraic properties of these operations.It is worth to point out that the proposed Hamacher operations over IFMs will be applied to aggregating Pythagorean fuzzy information in the future.

Acknowledgement. The authors wish to thank the referee, for the careful reading of the paper and for the helpful suggestions and comments.

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I.SILAMBARASAN is a Ph.D student in the Department of Mathematics, Annamalai University, Tamilnadu, India. He has published 10 research articles in various international journals (scopus and web of science). He has presented 3 research papers in international conferences in India. His research interests is Fuzzy Matrix theory. .

S. SRIRAM is a professor of Annamalai University and his research interests is Fuzzy Matrix theory. He has published more than 60 research papers in various international and national journals(scopus and web of science). He has supervised 10 master theses and 7 doctoral theses. .