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FIXED POINTS AND ITS APPLICATIONS IN C^*-ALGEBRA VALUED PARTIAL METRIC SPACE

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Abstract. We familiarise with the concepts of contractiveness and expansiveness in a C^*-algebra valued partial metric space and create an environment for the existence of fixed point in it. We solve an integral equation and an operator type equation as an application of main result. Further we give some examples to elaborate C^*-algebra valued partial metric space and show that there exist situations when a partial metric result can be applied, while the standard metric one cannot.

Keywords: C^*-algebra valued partial metric space, completeness, contraction, expansion.

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1. Introduction

A C^*-algebra is frequently used to explain a physical system in quantum field theory and statistical mechanics and consequently has emerged as an important area of research (see for instance [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [17] and references there in). Recently Chandok et al. [5] familiarised with the idea of C^*-algebra valued partial metric space combining the notions of partial metric spaces introduced by Matthews [11] and C^*-algebra valued metric spaces introduced by Ma et al. [10]. Motivated by the fact that the expansion of metric fixed point theory essentially rely on improving the existing contractive conditions or obtaining some variant of a metric space, we introduce the concepts of contractiveness and expansiveness in a C^*-algebra valued partial metric space like: C^*-algebra valued contractive map, C^*-algebra valued expansive map, C^*- algebra valued Chatterjea-type contractive map and C^*-algebra valued Kannan-type contractive map and obtain some interesting fixed point results. The basic idea comprises in utilising the set of all positive elements of a unital C^*-algebra as an alternative to set of real number. Our outcomes are magnificent improvements and extensions of the existing fixed

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point results in metric spaces over the set of reals. We also present some examples to elaborate C^*-algebra valued partial metric space and to validate our results. Applications to integral and operator type equations concludes the paper.

2. Prelimaries

Definition 2.1. [5] *A C^*-algebra valued partial metric is a function $p : \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{A}$ on a non-empty set $\mathcal{X}$ if:*

(i) $\theta \preceq p(u, v)$ *and* $p(u, u) = p(v, v) = p(u, v)$ *iff* $u = v$, *where* θ *is zero element of* $\mathbb{A}$;
(ii) $p(u, u) \preceq p(u, v)$;
(iii) $p(u, v) = p(v, u)$;
(iv) $p(u, v) \preceq p(u, w) + p(w, v) - p(w, w)$, $u, v, w \in \mathcal{X}$.

In this case $(\mathcal{X}, \mathbb{A}, p)$ *is a* C^*-*algebra valued partial metric space.*

For details on C^*-*algebra one may refer to* [13] *and* [17]. *Now we give an example of* C^*-*algebra valued partial metric space.*

Example 2.1. *Let* $F(\mathcal{X})$ *be collection of balls such that* $N_\rho(u) = \{v : d(u, v) \leq \rho, \rho > 0, u, v \in \mathcal{X}\}$ *and* $\mathbb{A} = M_n(\mathbb{C})$ *be the* C^*-*algebra of complex matrices.*
If $A = [a_{ij}] \in \mathbb{A}$ *then* $A^* = [\bar{a_{ji}}]$ *is non-zero element of* $\mathbb{A}$.
Norm is defined as, $\|A\| = \sup \{\|A\alpha\|_2 : \alpha \in \mathbb{C}^n, \|\alpha\|_2 \leq 1\}$, *where* $\|.\|_2$ *is the usual* l^2-*norm on* $\mathbb{C}^n$.
Define $p : F(\mathcal{X}) \times F(\mathcal{X}) \longrightarrow \mathbb{A}$ *by* $p[N_\rho(u), N_\sigma(v)] = |u - v|AA^* + \max\{\rho, \sigma\} I$.
Then p *is a* C^*-*algebra valued partial metric but is neither* C^*-*algebra valued metric nor standard partial metric, since* $p[N_\rho(u), N_\rho(u)] = \rho \neq \theta$ *and*

$$\begin{aligned} p[N_\rho(u), N_\tau(v)] &= |u - v|AA^* + \max\{\rho, \tau\} I \\ &\preceq [|u - w| + |w - v|]AA^* + [\max\{\rho, \sigma\} + \max\{\sigma, \tau\} - \sigma]I \\ &= p[N_\rho(u), N_\sigma(w)] + p[N_\sigma(w), N_\tau(v)] - p[N_\sigma(w), N_\sigma(w)]. \end{aligned}$$

If $\mathbb{A} = \mathbb{R}$, then the C^*-algebra valued partial metric reduces to the standard partial metric.

Lemma 2.1. [10] *Let* $\mathbb{A}$ *be a unital* C^*-*algebra with a unit* I.
(i) *If* $\alpha \in \mathbb{A}^+$ *and* $\|\alpha\| \leq \frac{1}{2}$ *then* $I - \alpha$ *is invertible and* $\left\|\alpha(I - \alpha)^{-1}\right\| \leq 1$.
(ii) *If* $\alpha, \beta \in \mathbb{A}$, $\alpha, \beta \succeq \theta$ *and* $\alpha\beta = \beta\alpha$, *then* $\alpha\beta \succeq \theta$.

Remark 2.1. [8] *It is worth mentioning that* $u \preceq v$ *implies* $\|u\| \leq \|v\|, \forall u, v \in \mathbb{A}^+$.

Definition 2.2. [5] *Let* $\mathbb{A}$ *be a unital* C^*-*algebra, then a continuous function* $\Omega : \mathbb{A}^+ \times \mathbb{A}^+ \longrightarrow \mathbb{A}$ *is a* C^*-*class function if:*
(i) $\Omega(\xi, \varphi) \preceq \xi$;
(ii) $\Omega(\xi, \varphi) = \xi$ *implies either* $\xi = \theta$ *or* $\varphi = \theta, \xi, \varphi \in \mathbb{A}$.

Let $\boldsymbol{\psi}$ *denote the set of all continuous functions* $\psi : \mathbb{A}^+ \longrightarrow \mathbb{A}^+$ *such that:*
(i) ψ *is continuous and non decreasing;*
(ii) $\psi(T) = \theta$ *iff* $T = \theta$.

3. Main Results

Following Ma et al. [10], first we familiarize with the notions of contractiveness and expansiveness in C^*-algebra valued partial metric space $(\mathcal{X}, \mathbb{A}, p)$ and then utilize these notions to establish unique fixed point. In the following section $\mathbb{A}^*$ denotes set of non-zero operator of $\mathbb{A}$.

Definition 3.1. *A self-map T of $(\mathcal{X}, \mathbb{A}, p)$ is called C^*-algebra valued contractive map if there exists an $A \in \mathbb{A}^*$, $\|A\| < 1$, satisfying $p(Tu, Tv) \preceq A^*p(u, v)A$ for all $u, v \in \mathcal{X}$.*

Example 3.1. *Let $\mathcal{X} = \mathbb{C}$ and $\mathbb{A} = M_2(\mathbb{C})$ with $\|A\| = \max_i \left\{\Sigma_j |a_{ij}|^2\right\}, i, j = 1, 2.$ Define $T : \mathcal{X} \to \mathcal{X}$ and a C^*-algebra valued partial metric as:*

$Tu = \frac{u}{4}$ *and* $p(u, v) = \begin{bmatrix} \max\{|u|, |v|\} & 0 \\ 0 & \max\{|u|, |v|\} \end{bmatrix}, u, v \in \mathcal{X}.$

Now

$$
\begin{aligned}
p(Tu, Tv) &= p(\tfrac{u}{4}, \tfrac{v}{4}) \\
&= \begin{bmatrix} \max\{|\frac{u}{4}|, |\frac{v}{4}|\} & 0 \\ 0 & \max\{|\frac{u}{4}|, |\frac{v}{4}|\} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \max\{|u|, |v|\} & 0 \\ 0 & \max\{|u|, |v|\} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \\
&= A^*p(u, v)A,
\end{aligned}
$$

where $A = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ *and $\|A\| = \frac{1}{4} < 1$. Hence T is a C^*-algebra valued contractive map.*

Definition 3.2. *A self-map T of $(\mathcal{X}, \mathbb{A}, p)$ is called C^*-algebra valued expansive map if there exists an invertible element $A \in \mathbb{A}^*$, satisfying*

(i) $T(\mathcal{X}) = \mathcal{X}$.

(ii) $p(Tu, Tv) \succeq A^*p(u, v)A$, for all $u, v \in \mathcal{X}$ and $\left\|A^{-1}\right\| < 1$.

Example 3.2. *Let $\mathcal{X} = \mathbb{C}$ and $\mathbb{A} = M_2(\mathbb{C})$ with $\|A\| = \max\{|a_{11}, |, |a_{12}|, |a_{21}|, |a_{22}|\}$. Define $T : \mathcal{X} \to \mathcal{X}$ and a C^*-algebra valued partial metric as:*

$Tu = 4u$ *and* $p(u, v) = \begin{bmatrix} |u - v - 1| & 0 \\ 0 & |u - v - 1| \end{bmatrix}, u, v \in \mathcal{X}.$

Now

$$
\begin{aligned}
p(Tu, Tv) &= p(4u, 4v) \\
&= \begin{bmatrix} |4(u - v) - 1| & 0 \\ 0 & |4(u - v) - 1| \end{bmatrix} \\
&= 4 \begin{bmatrix} |u - v - \frac{1}{4}| & 0 \\ 0 & |u - v - \frac{1}{4}| \end{bmatrix} \\
&\succeq 4 \begin{bmatrix} |u - v - 1| & 0 \\ 0 & |u - v - 1| \end{bmatrix} \\
&= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} |u - v - 1| & 0 \\ 0 & |u - v - 1| \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \\
&= A^*p(u, v)A,
\end{aligned}
$$

where $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ *and $\left\|A^{-1}\right\| = \frac{1}{2} < 1$. Hence T is a C^*-algebra valued expansive map.*

Definition 3.3. *A self-map T of $(\mathcal{X}, \mathbb{A}, p)$ is called C^*-algebra valued Chatterjea-type contractive map if there exists an $A \in \mathbb{A}^*$, $K \in \mathbb{A}^+$ and $\|K\| < \frac{1}{2}$ satisfying $p(Tu, Tv) \preceq K(p(Tu, v) + p(Tv, u))$ for all $u, v \in \mathcal{X}$.*

Example 3.3. *Let $\mathcal{X} = [0, \infty)$ and $B(\mathcal{X})$ is set of all bounded operators, $f \in B(\mathcal{X})$ then $B(\mathcal{X})$ becomes a C^*-algebra with $\|f(u)\| = \sup_{u \in \mathcal{X}} |f(u)|$. Define $T : \mathcal{X} \to \mathcal{X}$ and a C^*-algebra valued partial metric as:*

$Tu = \frac{u}{2}$ and $p(u, v) = \alpha(u + v)f$, $0 \leq \alpha < \frac{1}{2}, u, v \in \mathcal{X}$.

Let f be any constatnt function. Now

$$
\begin{aligned}
p(Tu, Tv) &= p(\tfrac{u}{2}, \tfrac{v}{2}) \\
&= \alpha(\tfrac{u}{2} + \tfrac{v}{2})f
\end{aligned}
$$

$\preceq \alpha(\frac{u}{2} + v + \frac{v}{2} + u)f$
$= \alpha(p(Tu, v) + p(Tv, u)).$
Hence T is C^*-algebra valued Chatterjea-type contractive map.

Definition 3.4. *A self-map T of $(\mathcal{X}, \mathbb{A}, p)$ is called C^*-algebra valued Kanan-type contractive map if there exists an $A \in \mathbb{A}^*$, $K \in \mathbb{A}^+$ and $\|K\| < \frac{1}{2}$ satisfying $p(Tu, Tv) \preceq K(p(Tu, u) + p(Tv, v))$ for all $u, v \in \mathcal{X}$.*

Example 3.4. *Let $\mathcal{X} = [0, \infty)$ and $M_2(\mathcal{X})$ with $\|A\| = \sigma_{\max}(A)$, where $\sigma_{\max}(A)$ represents largest singular value of A. Define $T : \mathcal{X} \to \mathcal{X}$ and a C^*-algebra valued partial metric as: $Tu = \frac{u}{4}$ and $p(u, v) = \beta \begin{bmatrix} u+v & 0 \\ 0 & u+v \end{bmatrix}$, $0 \leq \beta < \frac{1}{2}, u, v \in \mathcal{X}$.*
Now

$$\begin{aligned} p(Tu, Tv) &= p(\tfrac{u}{4}, \tfrac{v}{4}) \\ &= \beta \begin{bmatrix} \frac{u}{4} + \frac{v}{4} & 0 \\ 0 & \frac{u}{4} + \frac{v}{4} \end{bmatrix} \\ &\preceq \beta \begin{bmatrix} \frac{u}{4} + u & 0 \\ 0 & \frac{u}{4} + u \end{bmatrix} + \beta \begin{bmatrix} \frac{v}{4} + v & 0 \\ 0 & \frac{v}{4} + v \end{bmatrix} \\ &= \beta(p(Tu, u) + p(Tv, v)). \end{aligned}$$

Hence T is C^*-algebra valued Kanan-type contractive map.

Remark 3.1. *In view of Examples 3.1, 3.2, 3.3 and 3.4, we point out that any contactive or expansive map in C^*-algebra valued partial metric space is not a contractive or expansive map ([1], [2], [3] and others existing in literature) in a standard metric space, partial metric space or C^*-algebra valued metric space.*

Now we discuss the convergence of the sequence when it converges to zero element of $(\mathcal{X}, \mathbb{A}, p)$ and introduce definitions related to it.

Definition 3.5. *(i) A sequence $\{u_n\}_{n\in\mathbb{N}}$ is called a Cauchy sequence in $(\mathcal{X}, \mathbb{A}, p)$ if $\lim_{n,m\longrightarrow\infty} p(x_n, x_m)$ exists with respect to $\mathbb{A}$ and is finite.*

(ii) $(\mathcal{X}, \mathbb{A}, p)$ is complete if every Cauchy sequence $\{u_n\}_{n\in\mathbb{N}}$ in X converges, with respect to $\mathbb{A}$, to a point $u \in \mathcal{X}$ and satisfy

$$\lim_{n,m\to\infty} p(u_n, u_m) = \lim_{n\to\infty} p(u_n, u_n) = p(u, u).$$

(iii) The sequence $\{u_n\}_{n\in\mathbb{N}}$ in $(\mathcal{X}, \mathbb{A}, p)$ $\theta-$converges to a point $u \in \mathcal{X}$ if $\lim_{n\to\infty} p(u_n, u) = \lim_{n\to\infty} p(u_n, u_n) = p(u, u) = \theta$.

(iv) A sequence $\{u_n\}_{n\in\mathbb{N}}$ is $\theta-$cauchy if $\lim_{n,m\to\infty} p(u_m, u_n) = \theta$.

(v) $(\mathcal{X}, \mathbb{A}, p)$ is said to be $\theta-$ complete if every $\theta-$Cauchy sequence converges to a point $u \in \mathcal{X}$ and $p(u, u) = \theta$.

Remark 3.2. *It is worth mentioning here that if a sequence $\theta-$converges to some point then its self-distance as well as the self-distance of that point is equal to zero element of C^*-algebra valued partial metric space .*

Example 3.5. *Define $p(u, v) = I$, if $u = v$ and $p(u, v) = 2I$, otherwise.*
If $\mathcal{X}$ is a Hausdorff space and $B(\mathcal{X})$ is the set of all bounded functions , then $B(\mathcal{X})$ becomes a C^*-algebra with $\|f(u)\| = \sup_{u\in\mathcal{X}} |f(u)|$.
Here the sequence $\{u_n\} = a$, $n \geq 1$ is not $\theta-$Cauchy as it converges to a. However $\{u_n\}$ is a Cauchy sequence.
Implying there by that every $\theta-$Cauchy sequences in $(\mathcal{X}, \mathbb{A}, p)$ is a Cauchy sequence however the reverse implication is not necessarily true.

Theorem 3.1. *If T is a C^*-algebra valued contractive map of a $\theta-$complete C^*-algebra valued partial metric space $(\mathcal{X}, \mathbb{A}, p)$ then T has a unique fixed point $u^* \in \mathcal{X}$ and Picard sequence of iterates $\{T^n u_0\}$ converge, for $u_0 \in \mathcal{X}$, to u^* and $p(u^*, u^*) = \theta$.*

Proof. If $A = \theta$, T maps the $\mathcal{X}$ into a single point. So let $A \neq \theta$. Choose $u_0 \in \mathcal{X}$ and define $u_{n+1} = Tu_n = T^{n+1}u_0$, $n = 1, 2, \ldots$ and $B = p(u_1, u_0)$.
Since

$$
\begin{aligned}
p(u_{n+1}, u_n) = p(Tu_n, Tu_{n-1}) &\preceq A^* p(u_n, u_{n-1})A \\
&\preceq (A^*)^2 p(u_{n-1}, u_{n-2})(A^2) \\
&\preceq \quad \ldots \\
&\preceq (A^*)^n p(u_1, u_0)A^n \\
&= (A^*)^n B A^n.
\end{aligned}
$$

Now

$$
\begin{aligned}
p(u_n, u_m) \preceq\ & p(u_n, u_{n-1}) + p(u_{n-1}, u_{n-2}) + \ldots + p(u_{m+1}, u_m) - p(u_{n-1}, u_{n-1}) - p(u_{n-2}.u_{n-2}) - \\
& \cdots - p(u_{m+1}, u_{m+1}) \\
\preceq\ & p(u_n, u_{n-1}) + p(u_{n-1}, u_{n-2}) + \ldots + p(u_{m+1}, u_m) \\
\preceq\ & (A^*)^{n-1} B(A^{n-1}) + (A^*)^{n-2} B(A^{n-2}) + \ldots + (A^*)^m B(A)^m \\
=\ & \textstyle\sum_{k=m}^{n-1} (A^*)^k B(A)^k \\
=\ & \textstyle\sum_{k=m}^{n-1} (A^*)^k B^{\frac{1}{2}} B^{\frac{1}{2}} (A)^k \\
=\ & \textstyle\sum_{k=m}^{n-1} (B^{\frac{1}{2}} A)^* (B^{\frac{1}{2}} A) \\
=\ & \textstyle\sum_{k=m}^{n-1} \left| (B^{\frac{1}{2}}) A^k \right|^2 \\
\preceq\ & \left\| \textstyle\sum_{k=m}^{n-1} \left| (B^{\frac{1}{2}}) A^k \right|^2 \right\| I \\
\preceq\ & \textstyle\sum_{k=m}^{n-1} \left\| B^{\frac{1}{2}} \right\|^2 \left\| A^k \right\|^2 I \\
\preceq\ & \left\| B^{\frac{1}{2}} \right\|^2 \textstyle\sum_{k=m}^{n-1} \|A\|^{2k} I \\
\preceq\ & \left\| B^{\frac{1}{2}} \right\|^2 \frac{\|A\|^{2m}}{1 - \|A\|} I \to \theta \text{ as } m \to \infty.
\end{aligned}
$$

i.e., $\lim_{n\to\infty} p(u_n, u_m) = \theta$.
Therefore $\{u_n\}$ is a $\theta-$ Cauchy sequence with respect to $\mathbb{A}$.
Since, $(\mathcal{X}, \mathbb{A}, p)$ is $\theta-$ complete. So there exists $u^* \in \mathcal{X}$, satisfying $\lim_{n,m\to\infty} p(u_n, u_m) = \lim_{n\to\infty} p(u_n, u_n) = \lim_{n\to\infty} p(u_n, u^*) = p(u^*, u^*) = \theta$(since $\theta \preceq p(u_n, u_n) \preceq p(u_n, u_m)$, so $\lim_{n\to\infty} p(u_n, u_n) = \theta$).
Next, we assert that $p(u^*, Tu^*) = \theta$.
Now,

$$
\begin{aligned}
\theta \preceq p(u^*, Tu^*) &\preceq p(u^*, Tu_n) + p(Tu_n, Tu^*) - p(Tu_n, Tu_n) \\
&\preceq p(u^*, u_{n+1}) + p(Tu_n, Tu^*) \\
&\preceq p(u^*, u_{n+1}) + A^* p(u_n, u^*)A.
\end{aligned}
$$

Letting $n \to \infty$, $p(u^*, Tu^*) \preceq \theta$, a contradiction. So $p(u^*, Tu^*) = \theta$.
Now $\theta \preceq p(Tu^*, Tu^*) \preceq p(u^*, Tu^*)$ implies $p(Tu^*, Tu^*) = \theta$.
Hence $p(u^*, Tu^*) = p(Tu^*, Tu^*) = p(u^*, u^*) = \theta$ implies $Tu^* = u^*$.
To conclude the proof, let u^* and z are two distinct fixed points of T. So $p(u^*, w) = p(Tu^*, Tw) \preceq A^* p(u^*, w)A$, a contradiction. Hence $p(u^*, w) = \theta$.
Thus, $p(u^*, u^*) = p(u^*, w) = p(w, w) = \theta, i.e., u^* = w$.
Now $\lim_{n\to\infty} T^n u_0 = \lim_{n\to\infty} Tu_{n-1} = \lim_{n\to\infty} u_n = u^*$, i.e., Picard sequence of iterates $\{T^n u_0\}$ coverges to u^*. □

Example 3.6. *Let I be a collection of bounded intervals in $\mathbb{R}$ and $I = \{[a, b] : a \preceq b\}$. For $[a, b], [c, d] \in I$, let a C^*-algebra valued partial metric $p : I \times I \longrightarrow \mathbb{A}$ be defined as*

$p([a,b],[c,d]) = [\max\{b,d\} - \min\{a,c\}]KK^*$, where $K \in \mathbb{A}^*$ and $T : I \longrightarrow I$ be defined as $T(a,b) = \frac{1}{2}(a,b)$.
Clearly T is a C^*-algebra valued contractive map with contractive constant $\|A\| = \frac{1}{\sqrt{2}}$.
On taking $\{u_n\} = \left\{\frac{1}{n}, \frac{1}{2n}\right\}, \lim_{n\to\infty} T^n u_0 = \lim_{n\to\infty} u_n = (0,0)$, for all $u_0 \in I$ and $(I, \mathbb{A}, p)$ is $\theta-$complete, where $\mathbb{A}$ is defined as in Example 2.1. Hence all the hypothesis of Theorem 3.1 are verified and $(0,0)$ is unique fixed point of T and $p((0,0),(0,0)) = (0,0)$. ===+

Remark 3.3. *On taking $T(a,b) = (a,b)$, T satisfies a contractive condition for $\|A\| = 1$ and has infinite fixed points. If $T(a,b) = (1,1) - (a,b)$ then also T satisfies a contractive condition for $\|A\| = 1$, however in this case T has unique fixed points. This shows that the condition $\|A\| < 1$ is optimum to ensure uniqueness of fixed point, otherwise map may or may not have a uique fixed point.*

Theorem 3.2. *Let T be a C^*-algebra valued expansive map on a $\theta-$ complete C^*-algebra valued partial metric space $(\mathcal{X}, \mathbb{A}, p)$. If T is surjective then T has a unique fixed point $u^* \in X$ and Picard sequence of iterates $\{T^n u_0\}$ converges for every $u_0 \in \mathcal{X}$, to u^* and $p(u^*, u^*) = \theta$.*

Proof. Suppose that for any $u, v \in \mathcal{X}$ and $u \neq v$, if $Tu = Tv$, then $T(\mathcal{X}) \subseteq \mathcal{X}$, a contradiction, since T is surjective. So $Tu \neq Tv$. Thus T is injective.
Since T is invertible, substituting $u = T^{-1}u$ and $v = T^{-1}v$ in C^*-algebra valued expansion map

$$p(u,v) \succeq A^* p(T^{-1}u, T^{-1}v)A,$$

i.e., $(A^*)^{-1} p(u,v) A^{-1} \succeq p(T^{-1}u, T^{-1}v)$,
i.e., $\quad p(T^{-1}u, T^{-1}v) \preceq (A^*)^{-1} p(u,v) A^{-1}$.

Using Theorem 3.2., u^* is unique fixed point of T^{-1}. So $T^{-1}u^* = u^*$ or $Tu^* = u^*$, *i.e.*, u^* is unique fixed point of T. Also Picard sequence of iterates $\{T^n u_0\}$ coverges to u^* and $p(u^*, u^*) = \theta$.

□

Example 3.7. *Let $\mathcal{X}$ be a set of all points of unit circle. Define $p : \mathcal{X} \times \mathcal{X} \to \mathbb{A}$ and $T : \mathcal{X} \to \mathcal{X}$ be defined as:*

$$p(u,v) = |u - v - 1|\, KK^* \text{ and } Tu = T(\exp(i\theta)) = \begin{cases} \exp(i\theta + 2), & \theta \neq (4n+1)\frac{\pi}{2} \\ i, & \theta = (4n+1)\frac{\pi}{2} \end{cases}, u, v \in \mathcal{X}, \theta \in \mathbb{R}.$$

Taking $u = e^{i\theta}$, $v = e^{i\phi}, \theta, \phi \neq (4n+1)\frac{\pi}{2}$,

$$\begin{aligned} p(Tu, Tv) &= \left|e^2(e^{i\theta} - e^{i\phi}) - 1\right| KK^* \\ &= e^2 \left|e^{i\theta} - e^{i\phi} - \tfrac{1}{e^2}\right| KK^* \\ &\succeq e^2 \left|e^{i\theta} - e^{i\phi} - 1\right| KK^* \\ &= A^* p(u,v) A. \end{aligned}$$

Clearly T is a C^*-algebra valued expansive map with $\left\|A^{-1}\right\| = \frac{1}{e} < 1$. Hence all the hypothesis of Theorem 3.2 are verified and i is unique fixed point of T.

Now we establish our next result for C^*-algebra valued Chatterjea-type contractive map via C^*-class function that covers an extensive class of contractive conditions.

Theorem 3.3. *If T is a self-map of a $\theta-$ complete C^*-algebra valued partial metric space $(\mathcal{X}, \mathbb{A}, p)$ satisfying*

$$\psi(p(Tu,Tv)) \preceq \Omega(\psi K[p(u,Tv) + p(v,Tu)], \phi K[p(u,Tv) + p(v,Tu)]) \tag{1}$$

where $u, v \in \mathcal{X}$, $K \in \mathbb{A}^+$, $\|K\| < \frac{1}{2}$, ϕ, $\psi \in \boldsymbol{\psi}$ and Ω is a C^*−class function. Then T has a unique fixed point and Picard sequence of iterates $\{T^n u_0\}$ converges, for every $u_0 \in \mathcal{X}$, to u^* and $p(u^*, u^*) = \theta$.

Proof. Using $\psi(p(Tu, Tv)) \preceq \Omega(\psi(p(u, Tv) + p(v, Tu))), \phi(K(p(u, Tv) + p(v, Tu)))$
$\preceq \psi(K(p(u, Tv) + p(v, Tu)).$
Since ψ is non-decreasing function, so

$$p(Tu, Tv) \preceq K(p(u, Tv) + p(v, Tu)). \tag{2}$$

If $K = \theta$, T maps the $\mathcal{X}$ into a single point. So, let $K \neq \theta$. Take an arbitrary $u_0 \in \mathcal{X}$. Define
$u_{n+1} = Tu_n = T^{n+1}u_0, n = 0, 1, 2, \ldots$
If $u_n = u_{n+1}$, then u_n is a fixed point of T. Hence proof is complete.
Let $K \neq \theta$, then $K(d(Tu, v) + d(Tv, u))$ is also a positive element since $K \in \mathbb{A}^+$. Suppose that $u_n \neq u_{n+1}$, $n = 0, 1, 2, \ldots$ Taking $u = u_{n+1}$ and $v = u_{n+2}$ in (2) we obtain

$$\begin{aligned} p(u_{n+1}, u_{n+2}) &= p(Tu_n, Tu_{n+1}) \\ &\preceq K[p(u_n, Tu_{n+1}) + p(u_{n+1}, Tu_n)] \\ &\preceq K[p(Tu_{n-1}, Tu_{n+1}) + p(Tu_n, Tu_n)] \\ &\preceq K[p(Tu_{n-1}, Tu_n) + p(Tu_n, Tu_{n+1}) - p(Tu_n, Tu_n) + p(Tu_n, Tu_n)] \\ &\preceq K[p(Tu_{n-1}, Tu_n) + p(Tu_n, Tu_{n+1})] \\ &\preceq K[p(u_n, u_{n+1}) + p(u_{n+1}, u_{n+2})], \end{aligned}$$

$$i.e., (I - K)p(u_{n+1}, u_{n+2}) \preceq Kp(u_n, u_{n+1}). \tag{3}$$

Since $K \in \mathbb{A}, \|K\| < \frac{1}{2}$, then $I - K$ is an invertible operator and $(I - K)^{-1} \in \mathbb{A}^+$.
Furthermore $(I - K)^{-1}K \in \mathbb{A}^+$ and $\left\|(I - K)^{-1}K\right\| < 1$.
By (3), $p(u_{n+1}, u_{n+2}) \preceq hp(u_n, u_{n+1})$, where $h = (I - K)^{-1}K$.
Now

$$\begin{aligned} p(u_n, u_m) &\preceq p(u_n, u_{n-1}) + p(u_{n-1}, u_{n-2}) + \cdots + p(u_{m+1}, u_m) - p(u_{n-1}, u_{n-1}) - p(u_{n-2}, u_{n-2}) - \cdots - p(u_{m+1}, u_{m+1}) \\ &\preceq p(u_n, u_{n-1}) + p(u_{n-1}, u_{n-2}) + \cdots + p(u_{m+1}, u_m) \\ &\preceq (h^{n-1} + h^{n-2} + \cdots + h^m)p(u_1, u_0) \\ &= \textstyle\sum_{i=m}^{n-1} h^i B \\ &= \textstyle\sum_{i=m}^{n-1} h^{\frac{i}{2}} h^{\frac{i}{2}} B^{\frac{1}{2}} B^{\frac{1}{2}} \\ &= \textstyle\sum_{i=m}^{n-1} (B^{\frac{1}{2}} h^{\frac{i}{2}})^* (B^{\frac{1}{2}} h^{\frac{i}{2}}) \\ &= \textstyle\sum_{i=m}^{n-1} \left|B^{\frac{1}{2}} h^{\frac{i}{2}}\right|^2 \\ &\preceq \left\|\textstyle\sum_{i=m}^{n-1} \left|B^{\frac{1}{2}} h^{\frac{i}{2}}\right|^2\right\| I \\ &\preceq \textstyle\sum_{i=m}^{n-1} \left\|B^{\frac{1}{2}}\right\|^2 \left\|h^{\frac{i}{2}}\right\|^2 I \\ &\preceq \|B\| \textstyle\sum_{i=m}^{n-1} \|h\|^i I \\ &= \|B\| \frac{\|h\|^m}{1 - \|h\|} I \longrightarrow \theta \text{ as } m \to \infty. \end{aligned}$$

Therefore $\{u_n\}$is a $\theta-$ Cauchy sequence in $\mathcal{X}$ with respect to $\mathbb{A}$. Since $(\mathcal{X}, \mathbb{A}, p)$ is $\theta-$ complete, there exists $u^* \in \mathcal{X}$, satisfying
$\lim_{n,m\to\infty} p(u_n, u_m) = \lim_{n\to\infty} p(u_n, u_n) = \lim_{n\to\infty}(u_n, u^*) = p(u^*, u^*) = \theta.$
Now, we assert that $p(u^*, Tu^*) = \theta$.
So,

$$\begin{aligned} p(Tu^*, u^*) &\preceq p(Tu^*, Tu_n) + p(Tu_n, u^*) - p(Tu_n, Tu_n) \\ &\preceq K[p(u^*, Tu_n) + p(u_n, Tu^*)] + p(Tu_n, u^*) - p(Tu_n, Tu_n) \\ &\preceq K[p(u^*, Tu_n) + p(u_n, u^*) + p(u^*, Tu^*) - p(u^*, u^*)] + p(Tu_n, u^*) - p(Tu_n, Tu_n) \end{aligned}$$

i.e., $(I-K)p(Tu^*,u^*) \preceq K[p(u^*,u_{n+1})+p(u_n,u^*)-p(u^*,u^*)]+p(u_{n+1},u^*)-p(u_{n+1},x_{n+1})$
i.e., $p(Tu^*,u^*) \preceq K(I-K)^{-1}[p(u^*,u_{n+1})+p(u_n,u^*)-p(u^*,u^*)]+(I-K)^{-1}[p(u_{n+1},u^*)-p(u_{n+1},u_{n+1})]$,
i.e., $\|p(Tu^*,u^*)\| < \|p(u^*,u_{n+1})+p(u_n,u^*)-p(u^*,u^*)\|+\frac{1}{2}\|p(u_{n+1},u^*)-p(u_{n+1},u_{n+1})\|$.
Now as $n\to\infty$, $\|p(Tu^*,u^*)\| < \|p(u^*,u^*)\|$,
i.e., $\|p(Tu^*,u^*)\| = \|p(u^*,u^*)\|$
i.e., $p(Tu^*,u^*) = p(u^*,u^*) = \theta$.
So
$p(Tu^*,Tu^*) \preceq K[p(u^*,Tu^*)+p(Tu^*,u^*)] = \theta$ implies $p(Tu^*,Tu^*) = \theta$.
Hence $P(u^*,Tu^*) = p(Tu^*,Tu^*) = p(u^*,u^*) = \theta$ implies $Tu^* = u^*$.
To conclude the proof, let u^*, v are two distinct fixed point of T. Now

$$\begin{aligned} p(u^*,v) = p(Tu^*,Tv) &\preceq K[p(u^*,Tv)+p(v,Tu^*)] \\ &\preceq K[p(u^*,v)+p(v,u^*)] \\ &\preceq 2Kp(u^*,v), \end{aligned}$$

which gives $\|K\| > \frac{1}{2}$, a contradiction. Therefore $u^* = v$.
Also $\lim_{n\to\infty}(T^n u_0) = \lim_{n\to\infty} Tu_{n-1} = \lim_{n\to\infty} u_n = u^*$, i.e., Picard sequence of iterates $\{T^n u_0\}$ coverges to u^* and $p(u^*,u^*) = \theta$. □

Example 3.8. *If H is a complex Hilbert space with inner product $\langle .,. \rangle$. The collection of bounded linear operators $B(H)$ is a C^*-algebra with usual operator norm and $p : \mathbb{R}\times\mathbb{R} \longrightarrow B(H)$ defined by $p(u,v) = \max\{\|u\|,\|v\|\} I, u, v \in \mathcal{X}$ is a C^*-algebra valued partial metric.*
Let $T : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by-

$$Tu = \begin{cases} \frac{u}{2}, & u \geq 0 \\ 0, & u < 0. \end{cases}$$

For any $U, V \in B(H)^+$, define $\Omega : B(H)^+ \times B(H)^+ \longrightarrow B(H)$ by $\Omega(U,V) = U - V$ and $\phi, \psi : B(H)^+ \longrightarrow B(H)^+$ are continuous and non-decreasing functions such that $\phi(U) = \frac{U}{4}$ and $\psi(U) = \frac{U}{2}$, $U \in B(H)^+$. Clearly, T satisfies condition (1), with $\frac{I}{4} \preceq K \preceq \frac{I}{2}$ and $\frac{1}{4} < \|K\| < \frac{1}{2}$.
On taking $\{u_n\} = \left\{\frac{1}{n+1}\right\}$, $\lim_{n\to\infty} T^n u_0 = \lim_{n\to\infty} u_n = 0$. So $(\mathbb{R}, B(H), p)$ is a $\theta-$complete C^*-algebra valued partial metric space. All the hypothesis of Theorem 3.3 are verified. Consequently, T has unique fixed point 0 and $p(u^*,u^*) = \theta$.

Remark 3.4. *It is interesting to see that following similar arguments Theorem 3.3 remains true even if we replace C^*-algebra valued Chatterjea-type contractive map by C^*-algebra valued Kanan-type contractive map.*

Remark 3.5. *Here we point out that Examples 3.6, 3.7 and 3.8 are not covered by existing Theorems in a standard metric space, partial metric space or C^*-algebra valued metric space. Moreover T is discontinuous map in Example 3.8.*

Here it is merits noticing that on varying the elements of C^*-class function reasonably, extended / improved versions of variety of contractions existing in the literature can be found. In particular Theorem 3.4. remains true even if one replaces inequality (1) by any one of the following:
(i) $\psi(p(Tu,Tv) \preceq \psi(K[p(u,Tv)+p(v,Tu)] - \phi(K[p(u,Tv)+p(v,Tu)]$, taking $\Omega(A,B) = A - B$.
(ii) $\psi(p(Tu,Tv) \preceq \frac{\phi(K[p(u,Tv)+p(v,Tu)]-\psi(K[p(u,Tv)+p(v,Tu)]}{I+\psi(K[p(u,Tv)+p(v,Tu)]}$, taking $\Omega(A,B) = \frac{B-A}{I+A}$.
(iii) $\psi(p(Tu,Tv) \preceq \frac{\psi(K[p(u,Tv)+p(v,Tu)]}{I+\phi(K[p(u,Tv)+p(v,Tu)]}$, taking $\Omega(A,B) = \frac{A}{I+B}$.

(iv) $\psi(p(Tu,Tv) \preceq \log\frac{\phi(K[p(u,Tv)+p(v,Tu)]+M^{\psi(K[p(u,Tv)+p(v,Tu)]}}{I+\psi(K[p(u,Tv)+p(v,Tu)]}$, taking $\Omega(A,B)=\log\frac{B+M^A}{I+B}$, $\|M\|>1$.
(v) $\psi(p(Tu,Tv)) \preceq (\psi(K[p(u,Tv)+p(v,Tu)]+I)^{\frac{I}{I+\phi(K[p(u,Tv)+p(v,Tu)]}}$, taking $\Omega(A,B)=(A+I)^{\frac{I}{I+B}}$.
(vi) $\psi(p(Tu,Tv)) \preceq \psi(K[p(u,Tv)+p(v,Tu)]\log_{M+\phi(K[p(u,Tv)+p(v,Tu)]}M$, taking $\Omega(A,B)=A\log_{M+B}M$, $\|M\|>1$.
(vii) $\psi(p(Tu,Tv)) \preceq K\psi(K[p(u,Tv)+p(v,Tu)]$, taking $\Omega(A,B)=KA$, $0<\|K\|<1$.

4. Applications

Now we solve an integral equation and an operator equation to demonstrate the applicability of C^*-algebra valued contractive map.

Theorem 4.1. *Consider the integral equation* $u(t)=\int_E K(t,y,u(y))dy+q(t)$, *where* $t,y\in E$, *a Lebesgue measurable set,* $K: E\times E\times\mathbb{R}\longrightarrow\mathbb{R}$ *and* $q\in L^\infty(E)$ *such that there exist a continuous function* $\phi: E\times E\longrightarrow\mathbb{R}$ *and* $\eta\in[0,1)$ *satisfying* $|K(t,y,u(y))-K(t,y,v(y))|<k\,|\phi(t,y)(u-v)|$, $u,v\in\mathbb{R}$ *and* $\sup_{t\in E}\int_E\|\phi(t,y)\|\,dy\ <1$. *Then the integral equation has a unique solution* $u^*\in L^\infty(E)$.

Proof. Let $L^\infty(E)$ be the set of bounded measurable functions on E and $H=L^2(E)$, be a Hilbert space. The set of bounded linear operators $L(H)$ is a C^*-algebra with the usual operator norm.
Define $p: L^\infty(E)\times L^\infty(E)\longrightarrow L(H)$ by $p(u,v)=\pi_{|u-v|+I}$, where $\pi_h: H\longrightarrow H$ is the multiplication operator defined by $\pi_h(\phi)=h.\phi$, $\phi\in H$. Then $(L^\infty(E),L(H),p)$ is a complete C^*-algebra valued partial metric space.
Let $T: L^\infty(E)\longrightarrow L^\infty(E)$ be defind as $Tu(t)=\int_E(K(t,y,u(y))dy+q(t), t\in E$.
Now the solution of integral equation is equivalent to finding fixed point of T.
Suppose, $A=\eta I$, then $A\in L(H)$ and $\|A\|=\eta<1$. For $h\in H$,

$$\begin{aligned}
\|p(Tu,Tv)\| &= \sup_{\|h\|=1}(\pi_{[|Tu-Tv|+I]h},h)\\
&= \sup_{\|h\|=1}\int_E([|Tu-Tv|+I])h(t)\overline{h(t)}dt\\
&= \sup_{\|h\|=1}\int_E\int_E(|K(t,y,u(y))-K(t,y,v(y))|+I)dyh(t)\overline{h(t)}dt\\
&< \sup_{\|h(t)\|=1}\int_E\int_E|K(t,y,u(y))-K(t,y,v(y))|\,dy\,\|h(t)\|^2\,dt+\|I\|\,sup_{\|h(t)\|=1}\int_E\int_E dy\,\|h(t)\|^2\,dt\\
&< \sup_{\|h(t)\|=1}\int_E\int_E\eta\,|\phi(t,y)(u(y)-v(y))|\,dy\,\|h(t)\|^2\,dt+1\\
&< \eta\,\|u-v\|_\infty\sup_{t\in E}\int_E\|\phi(t,y)\|\,dy+1\\
&< \eta(\|u-v\|_\infty+1)\\
&= \|A\|\,.\,\|p(u,v)\|.
\end{aligned}$$

Hence all the hypothesis of a Theorem 3.1 are verified and consequently, the integral equation has a unique solution $u^*\in L^\infty(E)$. □

Example 4.1. *Consider the nonlinear functional integral equation:*

$$u(t)=\frac{e^{-t^2}sinu(t)}{3+|sinu(t)|}+\int_0^1\frac{e^{-(y+3)}}{7}|cosu(y+3)|dy. \tag{4}$$

The above integral equation is a special case of Theorem 4.1 with $q(t)=\frac{e^{-t^2}sinu(t)}{3+|sinu(t)|}$ *and* $K(t,y,u(y))=\frac{e^{-(y+3)}}{7}|cosu(y+3)|$.
The function $q(t)$ *is continuous and bounded such that* $|q(t)|=|\frac{e^{-t^2}sinu(t)}{3+|sinu(t)|}|\leq 1$.
Again, we see that $K(t,y,u(y))$ *is continuous and*

$$\begin{aligned}|K(t,y,u(y))-K(t,y,v(y))| &= |\tfrac{e^{-(y+3)}}{7}|cosu(s+3)| - \tfrac{e^{-(y+3)}}{7}|cosv(y+3)|| \\ &\leq \tfrac{1}{7}(e^{-(y+3)}|cosu(y+3) - cosv(y+3)|), \\ &\leq \tfrac{1}{7}(e^{-(y+3)}|u(y)-v(y)|).\end{aligned}$$

Here $\eta = \frac{1}{7} < 1$ *and* $\phi(t,y) = e^{-(y+3)}$ *such that* $\sup_{t\in E}\int_E \|\phi(t,y)\|\, dy < 1$.
Hence all the conditions of Theorem 4.1 are satisfied and hence the nonlinear integral equation has a unique solution in $L^\infty(E)$.

Example 4.2. *Consider the nonlinear functional integral equation:*

$$u(t) = \frac{e^{-t}sinu(t)}{5+|cosu(t)|} + \int_0^1 \frac{e^{-(y+1)}}{10}|arctanu(y+1)|dy. \tag{5}$$

The above integral equation is a special case of Theorem 4.1 with $q(t) = \frac{e^{-t}sinu(t)}{5+|cosu(t)|}$ and $K(t,y,u(y)) = \frac{e^{-(y+1)}}{10}|arctanu(y+1)|$.
The function $q(t)$ is continuous and bounded such that $|q(t)| = |\frac{e^{-t}sinu(t)}{5+|cosu(t)|}| \leq 1$.
Again, we see that $K(t,y,u(y))$ is continuous and

$$\begin{aligned}|K(t,y,u(y))-K(t,y,v(y))| &= |\tfrac{e^{-(y+1)}}{10}|arctanu(y+1)| - \tfrac{e^{-(y+1)}}{10}|arctanv(y+1)|| \\ &\leq \tfrac{1}{10}(e^{-(y+1)}|arctanu(y+1) - arctanv(y+1)|) \\ &\leq \tfrac{1}{10}(e^{-(y+1)}|u(y)-v(y)|).\end{aligned}$$

Here $\eta = \frac{1}{10} < 1$ and $\phi(t,y) = e^{-(y+1)}$ such that $\sup_{t\in E}\int_E \|\phi(t,y)\|\, dy < 1$.
Hence all the conditions of Theorem 4.1 are satisfied and hence the nonlinear integral equation has a unique solution in $L^\infty(E)$.

Theorem 4.2. *Let* $L(H)$ *be the set of linear bounded operators on a Hilbert space* H. *Let* $B_1, B_2, B_3, \ldots, B_n \in L(H)$ *satisfy* $\sum_{n=1}^{\infty}\|B_n\|^2 < 1$. *Then the operator equation*

$$U - \sum_{n=1}^{\infty} B_n^* U B_n = -Q,\ U \in L(H) \text{ and } Q \in L(H)^+$$

has a unique solution in $L(H)$.

Proof. Let $\xi = \sum_{n=1}^{\infty}\|B_n\|^2$. Then $\xi = 0$ implies $B_n = \theta$ for all $n \in \mathbb{N}$ and the equation has unique solution in $L(H)$.
Now, let $\xi > 0$ and $A \in L(H)$ be a positive operator. Define

$$p(U,V) = \max\{\|U\|, \|V\|\}AA^*, U, V \in L(H).$$

Then $(L(H), L(H), p)$ is a complete C^*-algebra valued partial metric with respect to usual operator norm. Define a map $T : L(H) \longrightarrow L(H)$ by

$$T(U) = \sum_{n=1}^{\infty} B_n^* U B_n - Q.$$

$$\begin{aligned}\text{Then } p(T(U),T(V)) &= \max\{\|T(U)\|, \|T(V)\|\}AA^* \\ &= \max(\|\textstyle\sum_{n=1}^{\infty} B_n^*UB_n - Q\|, \|\sum_{n=1}^{\infty} B_n^*VB_n - Q\|)AA^* \\ &= \textstyle\sum_{n=1}^{\infty}\|B_n^*UB_n - Q\|AA^* \text{ (say)} \\ &\preceq \|\textstyle\sum_{n=1}^{\infty} B_n^*UB_n\|AA^* \\ &\preceq \|U\|\textstyle\sum_{n=1}^{\infty}\|B_n\|^2 AA^* \\ &\preceq \xi\|U\|AA^* = \xi\max\{\|U\|,\|V\|\}AA^* \\ &\preceq (\xi^{\frac{1}{2}}I)^* p(U,V)(\xi^{\frac{1}{2}}I), \qquad \text{where } \left\|\xi^{\frac{1}{2}}I\right\| < 1.\end{aligned}$$

Hence all the hypothesis of a Theorem 3.1 are verified and consequently, the operator equation has a unique solution $U \in L(H)$. □

Example 4.3. *Consider the following matrix equation*

$$U - \sum_{n=1}^{m} (B_n)^* U B_n = -Q, \tag{6}$$

where Q is a positive definite matrix and $B_1, B_2, B_3, \ldots, B_m$ are arbitrary $n \times n$ matrices. Using Theorem 4.2, matrix equation (6) has a unique solution.

5. Conclusions

Acknowledging the notion of C^*-algebra valued partial metric space, we introduced contractiveness and expansiveness to elicit the fixed point theorems in the most generalized enviroment. In the sequel we also demonstrated the applicability of C^*-algebra valued partial metric space for a significant C^*-class functions introduced initially by Ansari [4] (also see Chandok et al. [5] and Tomar et al. [16]) that cover a large class of contractive conditions. Our results generalize, improve and unify several existing results. In the end, we utilised obtained results to solve integral and operator equations.

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References

[1] Agarwal, P., Mohamed, J. and Samet, B., (2018), Banach Contraction Principle and Applications, (2018), Fixed Point Theory in Metric Spaces, pp. 1-23.

[2] Agarwal, P., Mohamed, J. and Samet, B., (2018), The Class of $(\alpha-\psi)$-Contractions and Related Fixed Point Theorems, Fixed Point Theory in Metric Spaces, pp. 45-66.

[3] Agarwal, P., Mohamed, J. and Samet, B., (2018), On fixed points to the zero set of a certain function, Fixed Point Theory in Metric Spaces, pp. 101-122.

[4] Ansari, A. H., (2014), Note on "$(\phi-\psi)-$contractive type mappings and related fixed point", The 2nd Regional Conference on Mathematics and Applications, PNU, Sept., pp. 377-380.

[5] Chandok, S., Kumar, D. and Park, C., (2019), C^*-algebra valued partial metric space and fixed point theorems, Proc. Math. Sci. 129, (3): 37.

[6] Douglas, R., (1998), Banach Algebra Techniques in Operator Theory, Springer, Berlin.

[7] Dung, N. V., Hang, V. T. L. and D-Djekić, D., (2017), An equivalence of results in C^*-algebra valued $b-$metric and $b-$metric spaces, Appl. Gen. Topol. 18, (2), pp. 241-253.

[8] Kadelburg, Z. , Radenović, S., (2016), Critical remarks on some recent fixed points results in C^*-algebra-valued metric spaces, Fixed Point Theory and Applications 2016: 53.

[9] Kirk, W. and Shahzad, N., (2014), Fixed Point Theory in Distance Spaces, Springer International Publishing Switzerland.

[10] Ma, Z., Jiang, L. and Sun, H., (2014), C^*-algebra-valued metric spaces and related fixed point theorems, Fixed Point Theory Appl., 206.

[11] Matthews, S. G., (1994), Partial metric topology, Proc. 8th Summer Conference on General Topology and Applications, Queens College, 1992. Ann. New York Acad. Sci. 728, pp. 183-197.

[12] Mondal, S., (2017), A. Chanda and S. Karmakar, Common fixed point and best proximity point theorems in C^*-algebra-valued metric spaces, Int. J. Pure Appl. Math., 115, (3), pp. 477-496.

[13] Murphy, Gerard J., (1990), C^*-Algebra and Operator Theory, Academic Press, London.

[14] Radenović, S., Vetro, P., Nastasi, A. and Quan, L. T., (2017), Coupled fixed point theorems in C^*-algebra-valued $b-$metric spaces, Scientific publications of the state University of Novi Pazar, Ser. A: Appl. Math. Inform. and Mech., 9, (1), pp. 81-90.

[15] Segal, I., (1947), Irreducible representations of operator algebras, Bull. Amer. Math. Soc. (N.S.), 53, (2), pp. 73-88.

[16] Tomar, A., Sharma, R. and Ansari, A. H., (2019), Strict coincidence and common strict fixed point of a faintly compatible hybrid pair of maps via $C-$class function and applications, Palestine J. Math. in press.

[17] Xu, Q. H., Bieke, T. E. D. and Chen, Z. Q., (2010), Introduction to Operator Algebras and Non commutative $Lp-$Spaces, Science Press, Beijing, (In Chinese).

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