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ON SOME PROPERTIES OF IDEAL CONVERGENT DOUBLE SEQUENCES IN FUZZY NORMED SPACES

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Abstract. Recently, Rashid et al. [Rashid, Mohammad HM and Kočinac, Ljubiša DR. Ideal convergence in 2–fuzzy 2–normed spaces, Hacettepe Journal of Mathematics and Statistics, 46(1):149–162, 2017] defined the notion of ideal convergence of single sequences in 2–fuzzy 2–normed linear spaces. The aim of this paper is to generalize this notion to the double sequences in such spaces. For the sake of generalizing we define some concepts that contribute basically to outcomes that we came up with and study some basic properties of these new definitions.

Keywords: 2–fuzzy 2–Normed Spaces, α–2–norms, I–convergence, I–Cauchy, I–bounded.

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1. Introduction

Recall in [7] that a family of sets I of subsets of a non–empty set X is said to be an ideal if and only if (i) $\emptyset \in I$, (ii) for each $A, B \in I$ we have $A \cup B \in I$, (iii) for each $A \in I$ and $B \subset A$ we have $B \in I$. The ideal I is called an admissible in X if $I \neq X$ and $I \supset \{\{x\} : x \in X\}$. A filter on X is a non–empty family of sets $\mathcal{F}$ satisfying (i) for each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$, (ii) for each $A \in \mathcal{F}$ and $B \supset A$ we have $B \in \mathcal{F}$. For each ideal I there is a filter $\mathcal{F}(I)$ which corresponding to I (filter associated with ideal I), that is, $\mathcal{F}(I) = \{K \subseteq X : K^c \in I\}$. Depending on the structure of admissible ideals in the set of natural numbers $\mathbb{N}$, Kostyrko et. al [7] defined the notion of ideal convergence for real sequences as an interesting generalization of the notion of statistical convergence which was introduced by Fast [2] and Steinhaus [15] in 1951. some properties of ideal convergence were studied by Šalát et. al [12]. Kumar and Kumar [8] extended the idea of ideal convergence to be applicable for sequences of fuzzy numbers. Later, Das et. al [1] defined the notion of ideal convergence of double sequences in real line as well as in general metric spaces. The notion of ideal convergence for both single and double sequences has

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been defined and investigated in various setting (see [9, 11, 6, 3, 10, 5, 4, 16, 17]).

In [14] the concept of 2–fuzzy 2–normed linear spaces was introduced and studied some basic properties of the notions of Cauchy and convergent sequences in these spaces. Afterward, Rashid et. al [10] defined the notion of ideal convergence for single sequences in the framework of 2–fuzzy 2–normed linear spaces. In this paper, we define the notions of I–convergence, I–Cauchy and I–bounded of double sequences in 2–fuzzy 2–normed linear spaces. Further, we define some concepts in such spaces that basically related to the notion of ideal convergence and study some topological and algebraic properties of these new notions.

Now, we recall some definitions and Lemmas, which will be used throughout the paper.

Definition 1.1. [14] *Let X be a non–empty set and $F(X)$ be the set of all fuzzy sets in X. If $f \in F(X)$, then $f = \{(x, \mu) : x \in X$ and $\mu \in (0, 1]\}$. Clearly f is a bounded function for $|f(x)| \leq 1$. Then $F(X)$ is a linear space over the field $\mathbb{R}$, where the addition and scalar multiplication are defined by*

$$f + g = \{(x, \mu) + (y, \nu)\} = \{(x + y, \mu \wedge \nu) : (x, \mu) \in f \text{ and } (y, \nu) \in g\},$$

and

$$kf = \{(kx, \mu)\} \text{ such that } (x, \mu) \in f \text{ where } k \in \mathbb{R}.$$

The linear space $F(X)$ is said to be a normed space if for every $f \in F(X)$ there is associated a non–negative real number $\|f\|$ called the norm of f in such a way that

- **(1):** $\|f\| = 0$ *if and only if* $f = 0$,
- **(2):** $\|cf\| = |c|\|f\|$, $c \in \mathbb{R}$,
- **(3):** $\|f + g\| \leq \|f\| + \|g\|$ *for every* $f, g \in F(X)$.

Then $(F(X), \|\cdot\|)$ is a normed linear space.

Definition 1.2. [14] *A 2–fuzzy set on X is a fuzzy set on $F(X)$.*

Definition 1.3. [14] *Let $F(X)$ be a linear space over $\mathbb{R}$. A fuzzy subset N of $F(X) \times F(X) \times \mathbb{R}$ is called a 2–fuzzy 2–norm on X (or fuzzy 2–norm on $F(X)$) if and only if*

- **(N1):** *for all* $t \in \mathbb{R}$, *with* $t \leq 0$, $N(f_1, f_2, t) = 0$,
- **(N2):** *for all* $t \in \mathbb{R}$, *with* $t > 0$, $N(f_1, f_2, t) = 1$ *if and only if* f_1 *and* f_2 *are linearly dependent,*
- **(N3):** $N(f_1, f_2, t)$ *is invariant under any permutation of* f_1, f_2,
- **(N4):** *for all* $t \in \mathbb{R}$, *with* $t > 0$, $N(f_1, cf_2, t) = N(f_1, f_2, t/|c|)$ *if* $c \neq 0$, $c \in \mathbb{R}$,
- **(N5):** *for all* $s, t \in \mathbb{R}$, $N(f_1, f_2 + f_3, s + t) \geq \min\{N(f_1, f_2, s), N(f_1, f_3, t)\}$,
- **(N6):** $N(f_1, f_2, \cdot) : (0, \infty) \to [0, 1]$ *is continuous,*
- **(N7):** $\lim_{t \to \infty} N(f_1, f_2, t) = 1$.

Then (X, N) is a 2–fuzzy 2–normed linear space or $(F(X), N)$ is a fuzzy 2–normed linear space .

Lemma 1.1 ([14], Theorem 3.2)**.** *Let $(F(X), N)$ be a fuzzy 2–normed linear space. Assume that*

(N8): $N(f_1, f_2, t) > 0$ *for all* $t > 0$ *implies* f_1 *and* f_2 *are linearly dependent, define*

$$\|f_1, f_2\|_\alpha = \inf\{t : N(f_1, f_2, t) \geq \alpha, \alpha \in (0, 1)\}.$$

Then $\{\|\cdot, \cdot\|_\alpha : \alpha \in (0, 1)\}$ is an ascending family of 2–norms on $F(X)$. These 2–norms are called α–2–norms on $F(X)$ corresponding to the fuzzy 2–norms.

Definition 1.4. [14] *A sequence* $\{f_n\}$ *in a fuzzy* 2*–normed linear space* $(F(X), N)$ *is said to converges to* f *with respect to* α*–*2*–norm if* $\lim_{n\to\infty} \|f_n - f, g\|_\alpha = 0$ *for all* $g \in F(X)$.

Definition 1.5. [14] *A sequence* $\{f_n\}$ *in a fuzzy* 2*–normed linear space* $(F(X), N)$ *is called a Cauchy sequence with respect to* α*–*2*–norm if there exist* $g, h \in F(X)$ *which are linearly independent such that* $\lim_{n\to\infty} \|f_n - f_m, g\|_\alpha = 0$ *and* $\lim_{n\to\infty} \|f_n - f_m, h\|_\alpha = 0$.

Definition 1.6. [15] *If* $K = \{k \in K : k \le n\}$ *is a subset of* $\mathbb{N}$*, then the asymptotic density of the set* K *is given by*

$$d(K) = \lim_{n\to\infty} \frac{1}{n}|K|$$

whenever the limit exists, here $|B|$ *denotes the cardinality of the set* B.

Definition 1.7. [2, 15] *A number sequence* $x = (x_k)$ *is said to be statistically convergent to the number* ℓ *if, for each* $\epsilon > 0$*, the set* $A(\epsilon) = \{k \le n : |x_k - \ell| \ge \epsilon\}$ *has asymptotic density zero, i.e.,*

$$\lim_n \frac{1}{n}|\{k \le n : |x_k - \ell| \ge \epsilon\}| = 0.$$

In this case we write st–$\lim x = \ell$.

Definition 1.8. [2, 15] *A number sequence* $x = (x_k)$ *is said to be statistically Cauchy if, for each* $\epsilon > 0$*, there exists a number* $N = N(\epsilon)$ *such that*

$$\lim_n \frac{1}{n}|\{k \le n : |x_k - x_N| \ge \epsilon\}| = 0.$$

Definition 1.9. [7] *A sequence* $x = (x_k) \in \omega$ *is said to be* I*–convergent to a number* $\ell \in \mathbb{R}$ *if, for every* $\epsilon > 0$*, the set* $\{k \in \mathbb{N} : |x_k - \ell| \ge \epsilon\} \in I$*. And we write* I*–*$\lim x = \ell$*. In case* $\ell = 0$ *then* $(x_k) \in \omega$ *is said to be* I*–null.*

Definition 1.10. [12] *A sequence* $x = (x_k) \in \omega$ *is said to be* I*–Cauchy if, for every* $\epsilon > 0$*, there exists a number* $N = N(\epsilon)$ *such that the set* $\{k \in \mathbb{N} : |x_k - x_N| \ge \epsilon\} \in I$.

Definition 1.11. [6] *A sequence* $x = (x_k) \in \omega$ *is said to be* I*–bounded if there exists* $K > 0$*, such that, the set* $\{k \in \mathbb{N} : |x_k| \ge K\} \in I$.

Definition 1.12. [12] *Let* $x = (x_k)$ *and* $z = (z_n)$ *be two sequences. We say that* $x = z$ *for almost all* n *relative to* I *(in short a.a.k.r.I) if the set* $\{k \in \mathbb{N} : x \ne z\} \in I$.

Lemma 1.2. [13] *Let* $K \in \mathcal{F}(I)$ *and* $M \subseteq \mathbb{N}$*. If* $M \notin I$*, then* $M \cap K \notin I$.

2. Mean Results

In this section, we define and study the concepts of convergence, Cauchy, *st*–convergence, *st*–Cauchy, I–convergence, I–Cauchy and I–bounded for double sequences in fuzzy 2–normed spaces $(F(X), N)$ with respect to α–2–norms. Throughout the paper, we assume that I is an admissible ideal in $\mathbb{N} \times \mathbb{N}$.

Definition 2.1. *A double sequence* $\{f_{nk}\}$ *in a fuzzy* 2*–normed linear space* $(F(X), N)$ *is said to be convergent to* $f \in F(X)$ *with respect to* α*–*2*–norm on* $F(X)$ *if for every* $\epsilon > 0$ *and* $g \in F(X)$*, there exists a positive integer* $n_0 = n_0(\epsilon)$ *such that* $\|f_{nk} - f, g\|_\alpha < \epsilon$ *for all* $n, k \ge n_0$*. In this case we write* $\lim_{n,k\to\infty} \|f_{nk} - f, g\|_\alpha = 0$*. The element* f *is called the limit of* $\{f_{nk}\}$ *in* $F(X)$.

Definition 2.2. *A double sequence* $\{f_{nk}\}$ *in a fuzzy* 2*–normed space* $(F(X), N)$ *is said to be Cauchy with respect to* α*–*2*–norm on* $F(X)$ *if for every* $\epsilon > 0$ *and* $g, h \in F(X)$ *which are linearly independent there exists a positive integer* $n_0 = n_0(\epsilon)$ *such that* $\|f_{nk} - f_{st}, g\|_\alpha < \epsilon$ *and* $\|f_{nk} - f_{st}, h\|_\alpha < \epsilon$ *for all* $n, k, s, t \ge n_0$.

Definition 2.3. *A double sequence* $\{f_{nk}\}$ *in a fuzzy* 2*–normed linear space* $(F(X), N)$ *is said to be statistically convergent to* $f \in F(X)$ *with respect to* α*–2–norm if for every* $\epsilon > 0$ *and each* $g \in F(X)$*, the set* $\{(n, k) \in \mathbb{N} \times \mathbb{N} : \|f_{nk} - f, g\|_\alpha \geq \epsilon\}$ *has double natural density zero. In this case we write* $st\text{–}\lim \|f_{nk} - f, g\|_\alpha = 0$*. The element* f *is called the st–limit of* $\{f_{nk}\}$ *in* $F(X)$*.*

Definition 2.4. *A double sequence* $\{f_{nk}\}$ *in a fuzzy* 2*–normed space* $(F(X), N)$ *is said to be statistically Cauchy with respect to* α*–2–norm on* $F(X)$ *if for every* $\epsilon > 0$ *and* $g, h \in F(X)$ *which are linearly independent, there exist two positive integers* $s = s(\epsilon)$*,* $t = t(\epsilon)$*, the sets* $\{n, k \in \mathbb{N} : \|f_{nk} - f_{st}, g\|_\alpha \geq \epsilon\}$ *and* $\{n, k \in \mathbb{N} : \|f_{nk} - f_{st}, h\|_\alpha \geq \epsilon\}$ *have double natural density zero.*

Definition 2.5. *A double sequence* $\{f_{nk}\}$ *in a fuzzy* 2*–normed linear space* $(F(X), N)$ *is said to be* I*–Cauchy with respect to* α*–2–norm on* $F(X)$ *if for every* $\epsilon > 0$ *and* $g, h \in F(X)$*, there exist two positive integers* $s = s(\epsilon)$*,* $t = t(\epsilon)$ *such that the sets* $\{n, k \in \mathbb{N} : \|f_{nk} - f_{st}, g\|_\alpha \geq \epsilon\}$ *and* $\{n, k \in \mathbb{N} : \|f_{nk} - f_{st}, h\|_\alpha \geq \epsilon\}$ *belong to* I*.*

Definition 2.6. *A double sequence* $\{f_{nk}\}$ *in a fuzzy* 2*–normed linear space* $(F(X), N)$ *is said to be* I*–convergent to* $f \in F(X)$ *with respect to* α*–2–norm if for every* $\epsilon > 0$ *and each* $g \in F(X)$*, the set* $\{(n, k) \in \mathbb{N} \times \mathbb{N} : \|f_{nk} - f, g\|_\alpha \geq \epsilon\} \in I$*. In this case we write* $I\text{–}\lim \|f_{nk} - f, g\|_\alpha = 0$*. The element* f *is called the* I*–limit of* $\{f_{nk}\}$ *in* $F(X)$*. If* $f = 0$ *then* $\{f_{nk}\} \in \omega$ *is said to be* I*–null.*

Definition 2.7. *A double sequence* $\{f_{nk}\}$ *in a fuzzy* 2*–normed linear space* $(F(X), N)$ *is said to be* I*–bounded with respect to* α*–2–norm if there exists* $K > 0$*, such that, the set* $\{(n, k) \in \mathbb{N} \times \mathbb{N} : \|f_{nk} - f, g\|_\alpha \geq K\}$ *belongs to* I*.*

Example 2.1. *Let* I *be the class* I_f *of all finite subsets of* $\mathbb{N} \times \mathbb{N}$*. Then* I_f *is an admissible ideal and* I_f*–convergence coincides with the usual convergence of double sequences with respect to* α*–2–norm on* $F(X)$*.*

Example 2.2. *Let* $I_\delta = \{A \subset \mathbb{N} \times \mathbb{N} : \delta(A) = 0\}$*. Then* I_δ *is an admissible ideal in* $\mathbb{N} \times \mathbb{N}$*. Then* I_δ*–convergence coincides with the statistical convergence of double sequences with respect to* α*–2–norm on* $F(X)$*.*

Remark 2.1. *Every convergent double sequence in a fuzzy 2–normed spaces is obviously statistically convergent since all finite subsets of the natural numbers have natural density zero. However, the converse is not true. For example, define the double sequence* (f_{nk}) *in* $(F(X), N)$ *such that*

$$f_{nk} = \begin{cases} (1, 0), & \text{if } nk \text{ is square,} \\ (0, 0), & \text{otherwise.} \end{cases}$$

Let $f = (0, 0)$*. Then for given* $\epsilon > 0$ *and every* $g \in F(X)$*, we have*

$$\{(n, k) \in \mathbb{N} \times \mathbb{N} : \|f_{nk}(x) - f, g\|_\alpha \geq \epsilon\} \subseteq \{1, 4, 9, 16, \dots\}.$$

Since the set of square natural numbers has natural density zero, we get

$$\delta(\{(n, k) \in \mathbb{N} \times \mathbb{N} : \|f_{nk} - f, g\|_\alpha \geq \epsilon\}) = 0.$$

This implies that (f_{nk}) *is statistically convergent double sequence, but* (f_{nk}) *does not converges .*

Theorem 2.1. *Let* $(F(X), N)$ *be a fuzzy* 2*–normed linear space and* I *be an admissible ideal in* $\mathbb{N} \times \mathbb{N}$*. If a double sequence* $\{f_{nk}\}$ *in* $F(X)$ *is* I*–converges with respect to the* α*–2–norm on* $F(X)$*, then* I*–limit of* $\{f_{nk}\}$ *is unique.*

Proof. Suppose that I–$\lim \|f_{nk} - f, g\|_\alpha = 0$ and I–$\lim \|f_{nk} - h, g\|_\alpha = 0$ be such that $f \neq h$ for each $g \in F(X)$. Choose

$$\epsilon \in (0, \frac{\|f - h, g\|_\alpha}{2}), \tag{1}$$

we have

$$A = \{(n,k) \in \mathbb{N} \times \mathbb{N} : \|f_{nk} - f, g\|_\alpha \geq \epsilon\},$$

and

$$B = \{(n,k) \in \mathbb{N} \times \mathbb{N} : \|f_{nk} - h, g\|_\alpha \geq \epsilon\}$$

belong to I. Then by assumption and definition of filter associated with ideal, the sets $A^c, B^c \in \mathcal{F}(I)$. But then the set $A^c \cap B^c \in \mathcal{F}(I)$, too. Hence there is $(s,t) \in \mathbb{N} \times \mathbb{N}$ such that

$$\|f_{st} - f, g\|_\alpha < \epsilon \text{ and } \|f_{st} - h, g\|_\alpha < \epsilon.$$

From this, for $g \in F(X)$ we have

$$\|f - h, g\|_\alpha < \|f_{nk} - f, g\|_\alpha + \|f_{nk} - h, g\|_\alpha < 2\epsilon,$$

which is a contradiction to (1). □

Theorem 2.2. *Let I be an admissible ideal in $\mathbb{N} \times \mathbb{N}$. If a double sequence $\{f_{nk}\}$ is convergent to f in a fuzzy 2–normed space $(F(X), N)$ with respect to α–2–norm, then it is I–convergent to the same limit. But the converse is not true.*

Proof. Let $\{f_{nk}\}$ is convergent to f in a fuzzy 2–normed space $(F(X), N)$ with respect to α–2–norm, then for every $\epsilon > 0$ and each $g \in F(X)$, there is a positive integer $n_0 = n_0(\epsilon)$ such that

$$\|f_{nk} - f, g\|_\alpha < \epsilon \quad \text{for all } n, k \geq n_0.$$

Since

$$A = \{(n,k) \in \mathbb{N} \times \mathbb{N} : \|f_{nk} - f, g\|_\alpha \geq \epsilon\} \subseteq \{1, 2, 3, \ldots, n_0 - 1\} \times \{1, 2, 3, \ldots, n_0 - 1\}$$

and the ideal I is admissible, we have $A \in I$. This shows that I–$\lim_{n,k\to\infty} \|f_{nk} - f, g\|_\alpha = 0$.

The following example shows that the converse is not true.

Example 2.3. *Let $I = I_\delta$. Define a double sequence $\{f_{nk}\}$ in a fuzzy 2–normed linear space $(F(X), N)$ by*

$$f_{nk} = \begin{cases} (nk, 0), & \text{if } k \text{ is square}, \\ (0,0), & \text{otherwise}. \end{cases}$$

Let $f = (0,0)$. Then, for every $\epsilon > 0$ and every $g \in F(X)$, we have

$$\{k \in \mathbb{N} \times \mathbb{N} : \|f_{nk} - f, g\|_\alpha > \epsilon\} \subseteq \{1, 4, 9, \ldots, (nk)^2, \ldots\}.$$

Since the set of squares of natural numbers has asymptotic density zero, we have

$$\delta(\{k \in \mathbb{N} : \|f_k - f, g\|_\alpha > \epsilon\}) = 0,$$

for every $\epsilon > 0$ and every $g \in F(X)$. This implies that the double sequence $\{f_{nk}\}$ is I–converges in $(F(X), N)$, but $\{f_{nk}\}$ does not converges.

□

Theorem 2.3. *Let $(F(X), N)$ be a fuzzy 2–normed spaces. Let I be an admissible ideal in $\mathbb{N} \times \mathbb{N}$. Let $\{f_{nk}\}$ and $\{h_{nk}\}$ are two double sequences in $F(X)$ such that I–$\lim \|f_{nk} - f, g\|_\alpha = 0$ and I–$\lim \|h_{nk} - h, g\|_\alpha = 0$, where $f, h \in F(X)$. Then for every $g \in F(X)$*

(i): I–$\lim \|f_{nk} + h_{nk} - (f + h), g\|_\alpha = 0$.

(ii): I–lim $\|c(f_{nk} - f), g\|_\alpha = 0$, *for all scalar* $c \in \mathbb{R}$.
(iii): I–lim $\|f_{nk}h_{nk} - fh, g\|_\alpha = 0$.

Proof. **(1):** Let $\epsilon > 0$ and for each $g \in F(X)$ we sets

$$A_1 := \{(n,k) \in \mathbb{N} \times \mathbb{N} : \|f_{nk} - f, g\|_\alpha \geq \frac{\epsilon}{2}\} \in I,$$

and

$$A_2 := \{(n,k) \in \mathbb{N} \times \mathbb{N} : \|h_{nk} - h, g\|_\alpha \geq \frac{\epsilon}{2}\} \in I.$$

Let

$$A := \{(n,k) \in \mathbb{N} \times \mathbb{N} : \|(f_{nk} + h_{nk}) - (f + h), g\|_\alpha \geq \epsilon\}.$$

Then the inclusion $A \subset A_1 \cup A_2$ holds and the statement follows.

(2): It is trivial if $c = 0$. Let $c(\neq 0) \in \mathbb{R}$, since $I-\lim\limits_{n,k\to\infty} \|f_{nk} - f, g\|_\alpha = 0$, then for every $\epsilon > 0$, $g \in F(X)$, we have

$$\{(n,k) \in \mathbb{N} \times \mathbb{N} : \|f_{nk} - f, g\|_\alpha \geq \frac{\epsilon}{|c|}\} \in I.$$

Since $\|\cdot,\cdot\|_\alpha$ is an α–2–norm, then the inclusion

$$\{(n,k) \in \mathbb{N} \times \mathbb{N} : \|cf_{nk} - cf, g\|_\alpha \geq \epsilon\} \subseteq \{(n,k) \in \mathbb{N} \times \mathbb{N} : \|f_{nk} - f, g\|_\alpha \geq \frac{\epsilon}{|c|}\} \quad (2)$$

can be easily verified. The set on the right hand side of (2) belongs to I. By definition if ideal the set on the left hand side of (2) belongs to I, too. This implies that I–lim $\|c(f_{nk} - f), g\|_\alpha = 0$, for all scalar $c \in \mathbb{R}$ and every $g \in F(X)$.

(3): Since I–lim $\|f_{nk} - f, g\|_\alpha = 0$ for every $g \in F(X)$, we have

$$A = \{(n,k) \in \mathbb{N} \times \mathbb{N} : \|f_{nk} - f, g\|_\alpha < 1\} \in \mathcal{F}(I),$$

for $n, k \in A$, we get

$$\|f_{nk}, g\|_\alpha < \|f, g\|_\alpha + 1. \quad (3)$$

Now for every $g \in F(X)$, we have

$$\|f_{nk}h_{nk} - fh, g\|_\alpha \leq \|f_{nk}, g\|_\alpha \|h_{nk} - h, g\|_\alpha + \|h, g\|_\alpha \|f_{nk} - f, g\|_\alpha.$$

From inequality (3), we get

$$\|f_{nk}h_{nk} - fh, g\|_\alpha \leq (\|f, g\|_\alpha + 1)\|h_{nk} - h, g\|_\alpha + \|h, g\|_\alpha \|f_{nk} - f, g\|_\alpha. \quad (4)$$

Now let $\epsilon > 0$ be given and for every $g \in F(X)$. Choose $\eta > 0$ such that

$$0 < \eta < \frac{\epsilon}{\|f, g\|_\alpha + \|h, g\|_\alpha + 1}. \quad (5)$$

Since I–lim $\|f_{nk} - f, g\|_\alpha = 0$ and I–lim $\|h_{nk} - h, g\|_\alpha = 0$, the sets

$$B = \{(n,k) \in \mathbb{N} \times \mathbb{N} : \|f_{nk} - f, g\|_\alpha < \eta\} \in \mathcal{F}(I),$$

and

$$C = \{(n,k) \in \mathbb{N} \times \mathbb{N} : \|h_{nk} - h, g\|_\alpha < \eta\} \in \mathcal{F}(I).$$

Thus we have $B \cap C \in \mathcal{F}(I)$. Obviously, $A \cap B \cap C \in \mathcal{F}(I)$ and for each $(n,k) \in \mathbb{N} \times \mathbb{N}$, from the equations (4) and (5), it follows that

$$\begin{aligned} \|f_{nk}h_{nk} - fh, g\|_\alpha &\leq (\|f,g\|_\alpha + 1)\eta + (\|h,g\|_\alpha)\eta \\ &< \frac{(\|f,g\|_\alpha + 1)\epsilon}{\|f,g\|_\alpha + \|h,g\|_\alpha + 1} + \frac{(\|h,g\|_\alpha)\epsilon}{\|f,g\|_\alpha + \|h,g\|_\alpha + 1} \\ &< \frac{(\|f,g\|_\alpha + \|h,g\|_\alpha + 1)\epsilon}{\|f,g\|_\alpha + \|h,g\|_\alpha + 1} \\ &< \epsilon. \end{aligned}$$

This implies that

$$\{(n,k) \in \mathbb{N} \times \mathbb{N} : \|f_{nk}h_{nk} - fh, g\|_\alpha \geq \epsilon\} \in I.$$

That is, I–$\lim \|f_{nk}h_{nk} - fh, g\|_\alpha = 0$.

□

Theorem 2.4. *Every I–convergent double sequence in a fuzzy 2–normed linear space $(F(X), N)$ is I–bounded but the converse is not true.*

Proof. Let $\{f_{nk}\}$ be a double sequence in $(F(X), N)$ converging to f. Then for given $\epsilon > 0$ and every $g \in F(X)$, we have

$$\{(n,k) \in \mathbb{N} \times \mathbb{N} : \|f_{nk} - f, g\|_\alpha < \epsilon\} \in \mathcal{F}(I).$$

Therefore,

$$\|f_{nk}, g\|_\alpha = \|f_{nk} - f + f, g\|_\alpha \leq \|f_{nk} - f, g\|_\alpha + \|f, g\|_\alpha.$$

From this it easily follows that the double sequence $\{f_{nk}\}$ is I–bounded. Further, we show the converse part by constructing the following example.

Example 2.4. *Consider the double sequence $\{f_{nk}\}$ in $(F(X), N)$ such that*

$$f_{nk} = \begin{cases} (\sqrt{nk}, 0) & \text{if } nk \text{ is square} \\ (1, 0) & \text{if } nk \text{ is odd non–square} \\ (0, 0) & \text{if } nk \text{ is even non–square.} \end{cases}$$

Then the double sequence $\{f_{nk}\}$ is I–bounded, but not I–converges in $(F(X), N)$.

□

Theorem 2.5. *Let I be an admissible ideal. Then the following are equivalent:*

(1): *I–$\lim \|f_{nk} - f, g\|_\alpha = 0$;*

(2): *There exists a convergent double sequence $\{h_{nk}\}$ in $(F(X), N)$ such that $f_{nk} = h_{nk}$, for a.a.$nk, r.I$;*

(3): *There exists a convergent double sequence $\{h_{nk}\}$ in $(F(X), N)$ and I–null double sequence $\{g_{nk}\}$ in $(F(X), N)$ such that $f_{nk} = h_{nk} + g_{nk}$ for all $(n,k) \in \mathbb{N} \times \mathbb{N}$ and $\{(n,k) \in \mathbb{N} \times \mathbb{N} : \|f_{nk} - f, g\|_\alpha \geq \epsilon\} \in I$;*

(4): *There exists a subset $K = \{(n_i, k_i) : i \in \mathbb{N}, n_1 < n_2 < \dots$ and $k_1 < k_2 < \dots\}$ of $\mathbb{N} \times \mathbb{N}$, such that $K \in \mathcal{F}(I)$ and $\lim_{i \to \infty} \|f_{n_i k_i} - f, g\|_\alpha = 0$.*

Proof. **(1) implies (2):** . Let I–$\lim \|f_{nk} - f, g\|_\alpha = 0$, then for given $\epsilon > 0$ and every $g \in F(X)$, we have

$$\{(n,k) \in \mathbb{N} \times \mathbb{N} : \|f_{nk} - f, g\|_\alpha \geq \epsilon\} \in I.$$

Let (m_{st}) be an increasing double sequence with $m_{st} \in \mathbb{N} \times \mathbb{N}$ such that

$$\left\{(n,k) \leq m_t : \|f_{nk} - f, g\|_\alpha \geq (st)^{-1}\right\} \in I.$$

Define a double sequence $\{h_{nk}\}$ as $h_{nk} = f_{nk}$ for all $(n,k) \leq m_{11}$ and, for $m_{st} < nk < m_{s+1,t+1}$, $(s,t) \in \mathbb{N} \times \mathbb{N}$ as

$$h_{nk} = \begin{cases} f_{nk}, & \text{if } \|f_{nk} - f, g\|_\alpha < (st)^{-1} \\ f, & \text{otherwise.} \end{cases}$$

Then $\{h_{nk}\}$ is convergent double sequence in $(F(X), N)$ and from the inclusion

$$\{(n,k) \leq m_{st} : f_{nk} \neq h_{nk}\} \subseteq \{(n,k) \in \mathbb{N} \times \mathbb{N} : \|f_{nk} - f, g\|_\alpha \geq \epsilon\} \in I$$

we get $f_{nk} = h_{nk}$ for $a.a.nk.r.I$.

(2) implies (3): . For I–convergent double sequence (f_{nk}) there is a double sequence (h_{nk}) such that $f_{nk} = h_{nk}$ for a, a, nk, r, I. Let $A = \{k \in \mathbb{N} : f_k \neq h_k\}$, then $K \in I$. Define a double sequence $\{g_{nk}\}$ as

$$g_{nk} = \begin{cases} f_{nk} - h_{nk}, & \text{if } (n,k) \in A \\ 0, & \text{otherwise.} \end{cases}$$

Then $\{g_{nk}\}$ is I–null double sequence and the double sequence $\{h_{nk}\}$ is converges.

(3) implies (4): . Let $P = \{(n,k) \in \mathbb{N} \times \mathbb{N} : \|f_{nk} - f, g\|_\alpha \geq \epsilon\} \in I$, and

$$K = P^c = \{(n_i, k_i) \in \mathbb{N} \times \mathbb{N} : i \in \mathbb{N}, n_1 < n_2 < \dots \text{ and } k_1 < k_2 < \dots\} \in \mathcal{F}(I).$$

Then we have

$$\lim_{i\to\infty} \|f_{n_i k_i} - f, g\|_\alpha = 0.$$

(4) implies (1): . Suppose that (3) holds. Then for any $\epsilon > 0$, and given $g \in F(X)$ and by Lemma 1.2 we have

$$\{(n,k) \in \mathbb{N} \times \mathbb{N} : \|f_{nk} - f, g\|_\alpha \geq \epsilon\} \subseteq K^c \cup \{(n,k) \in K : \|f_{nk} - f, g\|_\alpha \geq \epsilon\}.$$

Thus I–$\lim \|f_{nk} - f, g\|_\alpha = 0$.

□

Theorem 2.6. *Let I be an admissible ideal in $\mathbb{N} \times \mathbb{N}$ and $(F(X), N)$ be a fuzzy 2–normed space. A double sequence $\{f_{nk}\}$ in $F(X)$ is I–converges with respect to α–2–norm, if and only if for all $g \in F(X)$ and every $\epsilon > 0$ there exist $s(\epsilon), t(\epsilon) \in \mathbb{N}$ such that*

$$\{(n,k) \in \mathbb{N} \times \mathbb{N} : \|f_{nk} - f_{st}, g\|_\alpha < \epsilon\} \in \mathcal{F}(I). \quad (6)$$

Proof. Suppose that the double sequence $\{f_{nk}\}$ is I–convergent to f in $F(X)$. Then for all $g \in F(X)$ and given $\epsilon > 0$, we have

$$A = \left\{(n,k) \in \mathbb{N} \times \mathbb{N} : \|f_{nk} - f, g\|_\alpha < \frac{\epsilon}{2}\right\} \in \mathcal{F}(I).$$

Fix an integers $s = s(\epsilon)$ and $t = t(\epsilon)$ in A. Then the following holds for all $(n,k) \in A$.

$$\|f_{nk} - f_{st}, g\|_\alpha \leq \|f_{nk} - f, g\|_\alpha + \|f - f_{st}, g\|_\alpha < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence (6) holds.

Conversely, suppose that (6) holds for all $g \in F(X)$ and every $\epsilon > 0$. Then the set

$$B_\epsilon = \{(n,k) \in \mathbb{N} \times \mathbb{N} : f_{nk} \in [f_{st} - \epsilon, f_{st} + \epsilon]\} \in \mathcal{F}(I), \text{ for all } \epsilon > 0.$$

Let $J_\epsilon = [f_{st} - \epsilon, f_{st} + \epsilon]$. Fixing $\epsilon > 0$, we have $B_\epsilon \in \mathcal{F}(I)$ and $B_{\frac{\epsilon}{2}} \in \mathcal{F}(I)$. Hence $B_\epsilon \cap B_{\frac{\epsilon}{2}} \in \mathcal{F}(I)$. This implies that

$$J = J_\epsilon \cap J_{\frac{\epsilon}{2}} \neq \emptyset.$$

That is,

$$\{(n,k) \in \mathbb{N} \times \mathbb{N} : f_{nk} \in J\} \in \mathcal{F}(I).$$

And thus

$$\text{diam } J \leq \frac{1}{2} \text{ diam } J_\epsilon,$$

where the diam of J denotes the length of interval J. Proceeding in this way, by induction, we get a sequence of closed intervals

$$J_\epsilon = I_0 \supseteq I_1 \supseteq \cdots \supseteq I_m \supseteq \cdots$$

with the property

$$\text{diam } (I_m) \leq \frac{1}{2} \text{ diam } (I_{m-1}), \text{ for } m = 2, 3, \ldots$$

and

$$\{(n,k) \in \mathbb{N} \times \mathbb{N} : f_{nk} \in I_m\} \in \mathcal{F}(I).$$

Then there exists $h \in \bigcap_{m \in \mathbb{N}} I_m$ and it is a routine work to verify that $I-\lim \|f_{nk} - h, g\|_\alpha = 0$. □

3. Conclusions

In this present paper we have defined the notions of convergence, Cauchy, st–convergence, st–Cauchy, I–convergence, I–Cauchy and I–bounded for double sequences in 2–fuzzy 2–normed spaces with respect to α–2–norms, and some basic results related to these notions are investigated. This definitions and results provide new tools to deal with the convergence problems for double sequences in the fuzzy settings, occurring in many branches of science and engineering.

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