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MOSTAR INDEX OF BRIDGE GRAPHS

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Abstract. Topological indices are the numerical descriptors of a molecular structure obtained via molecular graph G. Topological indices are used in the structure-property relationship, structure-activity relations, and nanotechnology. Also, they hold us to predict certain physicochemical properties such as boiling point, enthalpy of vaporization, stability, and so on. In this study, it is considered the Mostar index. It is present upper bound for Mostar index of bridge graphs. Moreover, it is given exact expressions for the Mostar index of bridge graphs of the path, star, cycle, and complete graphs.

Keywords: Topological index, mostar index, bridge graph, special graphs.

AMS Subject Classification: 05C12, 05C05

1. Introduction

Graph theory, which is a branch of discrete mathematics started by solving the problem of the bridges of Königsberg by Leonhard Euler in 1736. Graph theory has attracted attention and gained popularity by the publication of the first book on graph theory (1936). Graph theory has been studied in engineering and science such as physics, biology, computer sciences, chemistry, civil engineering, management, and control.

Finding the properties of molecules takes time and money. This problem is solved by Chemical graph theory. The chemical graph theory is focused on finding topological indices. Topological indices are a real number of a molecular structure obtained via molecular graph G whose vertices and edges represent the atoms and the bonds, respectively. They hold us to predict certain physicochemical properties such as boiling point, enthalpy of vaporization, stability, and also are used for studying the properties of molecules such as the structure-property relationship, the structure-activity relationship, and the structural design in chemistry, nanotechnology, and pharmacology.

The first molecular descriptor is the Wiener index, which was introduced by H. Wiener in 1947 in order to calculate the boiling points of paraffin [15]. Over the course of the last seventy years, many topological indices have been defined. These indices can be classified according to the structural characteristics of the graph such as the degree of vertices, the distances between vertices, the matching, and the spectrum and so on. The best-known

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topological indices are the Wiener index which is based on the distance, the Zagreb and the Randic indices which are based on degree, the Estrada index which is based on the spectrum of a graph, the Hosaya index which is based on the matching. Apart from these, it is a bond-additive index, which is a measure of peripherality in graphs.

Doslic et al. defined a new bond-additive topological index which is named Mostar index in 2019. In the same paper, they gave explicit formulas for benzenoid graph, Cartesian product, extremal and unicyclic graphs. Also, they stated several conjectures and open problems [5]. Tepeh proved their conjecture related with bicyclic graph [14]. We encourage reader to references ([1], [3], [4], [10]-[13])

In this study, the Mostar index which is the bond-additive index is studied. It is given upper bound for the Mostar index of bridge graphs. The Mostar index value for bridge graphs of path graph, star graph, cycle graph, complete graph is presented.

2. Preliminaries

Let G be a simple connected graph with a vertex set $V(G)$ and edge set $E(G)$. The number of a vertex set and edge set are defined by n and m, respectively. An edge of G connects the vertices u and v and it writes $e = uv$. The degree of a vertex u is defined by $d(u)$. The distance between vertices u and v is defined by $d(u, v)$. For standard terminology and notations we follow Buckley and Harary [2].

Mostar index is defined as

$$Mo(G) = \sum_{uv\in E(G)} |n_u - n_v| \tag{1}$$

where n_u is the number of vertices of G lying closer to vertex u than to vertex v of the edge uv [5]. Namely,

$$n_u = |N_u = \{x \in V(G) : d(x, u) < d(x, v)\}| . \tag{2}$$

Note that vertices equidistant to u and v not counted. Doslic et. al. presented following results [5]:

Corollary 2.1. *Let K_n be complete graph, C_n be cycle graph and $K_{n,n}$ be complete bipartite graph. Then, $Mo(K_n) = Mo(C_n) = Mo(K_{n,n}) = 0$.*

Corollary 2.2. *Let T_n be tree with n vertices, P_n be path graph and S_n be star graph with n vertices. Then, $\left\lfloor \frac{(n-1)^2}{2} \right\rfloor = Mo(P_n) \leq Mo(T_n) \leq Mo(S_n) = (n-1)(n-2)$ with equality if only if $T_n = S_n$.*

Let $\{G_i\}_i^d$ be a set of finite pair wise disjoint graphs with $|V(G_i)| = n_i$, $|E(G_i)| = m_i$.

Definition 2.1. *For given vertices $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$, a link or bridge of two graphs G_1 and G_2 is defined as the graph $G_1 \sim G_2\,(v_1, v_2)$ obtained by joining v_1 and v_2 by an edge (see figure 1). For simply we show the bridge (link) of two graphs G_1 and G_2 by $G_1 \sim G_2$.*

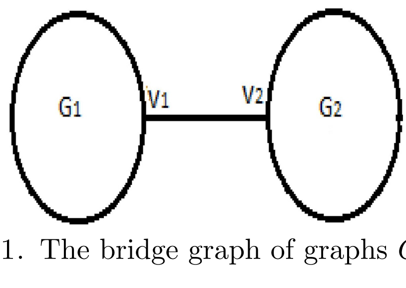

Figure 1. The bridge graph of graphs G_1 and G_2

Definition 2.2. *The bridge of graphs $G_1, G_2, ..., G_d$ obtained joining a vertex of $V(G_i)$ with a vertex of $V(G_{i+1})$ and a vertex of $V(G_{i+2})$ with same vertex of $V(G_{i+1})$, $i = 1, ..., d-2$ (fig. 2) is $G_1 \sim G_2 \sim ... \sim G_d$. If $G_1 = G_2 = ... = G_d = G$ then we use of the notation $G \overset{d}{\sim} G$.*

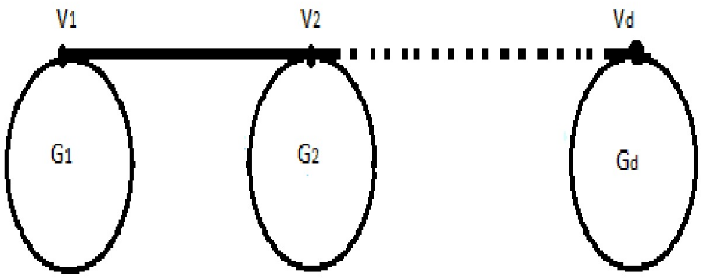

FIGURE 2. The bridge of graphs $G_1, G_2, ..., G_d$

Definition 2.3. *The bridge of graphs $G_1, G_2, ..., G_d$ obtained joining a vertex of $V(G_i)$ with a vertex of $V(G_{i+1})$ $(v_{i+1} \in V(G_{i+1}))$ by an edge and different a vertex of $V(G_{i+1})$ $(x_{i+1} \neq v_{i+1} \in V(G_{i+1}))$ with a vertex of $V(G_{i+2})$ by an edge, $i = 1, ..., d-2$ (fig. 3) is defined as $G_1 \overset{l}{\sim} G_2 \overset{l}{\sim} . .. \overset{l}{\sim} G_d$. If $G_1 = G_2 = ... = G_d = G$ then we use of the notation $G \overset{l_d}{\sim} G$.*

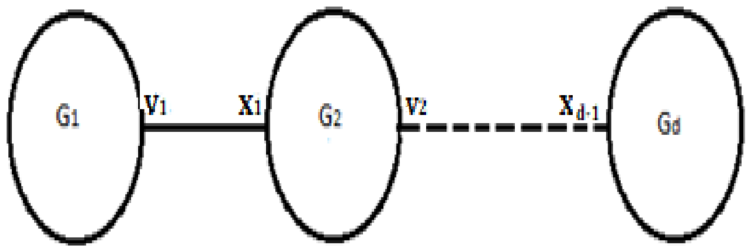

FIGURE 3. The bridge of graphs $G_1, G_2, ..., G_d$

3. Mostar Index of Bridge Graphs

In this section, it is given formulas for the mostar indices of bridge graphs. Note that $d(v_i, v_i) = d(u_i, u_i) = 0$.

Theorem 3.1. *Consider the graph G_1 and G_2. The Mostar index of $G_1 \sim G_2$ is*

$$Mo(G_1 \sim G_2) \leq Mo(G_1) + Mo(G_2) + m_2 n_1 + m_1 n_2 + |n_1 - n_2|.$$

Proof. From Definition 2.1, we see that $G_1 \sim G_2$ consist of vertex set

$$V(G_1 \sim G_2) = V_1 \cup V_2$$

and edge set

$$E(G_1 \sim G_2) = E_1 \cup E_2 \cup E^{'} \tag{3}$$

where

$$E' = \{v_1 v_2 \in E(G_1 \sim G_2) \, | v_1 \in V_1, v_2 \in V_2\}$$

From Eq.(1) and Eq. (3), we have:

$$Mo(G_1 \sim G_2) = \sum_{uv\in E_1} |n_u - n_v| + \sum_{uv\in E_2} |n_u - n_v| + \sum_{uv\in E'} |n_u - n_v|. \quad (4)$$

Let $v_1v_2 \in E(G_1 \sim G_2)$ where $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$. We discuss the following three cases from Eq. (4):

Case 1. For $vu \in E_1$,

i. When $d(u, v_1) < d(v, v_1)$,then we easy see that

$$n_u = |N_u(G_1), V_2| = n_u(G_1) + n_2$$

$$n_v = |N_v(G_1)| = n_v(G_1)$$

Thus we have:

$$\sum_{uv\in E_1} |n_u - n_v| = \sum_{uv\in E_1} |n_u(G_1) + n_2 - n_v(G_1)| \leq \sum_{uv\in E_1} |n_u(G_1) - n_v(G_1)| + \sum_{uv\in E_1} |n_2|$$

From Eq. (1), it is obtained

$$\sum_{uv\in E_1} |n_u - n_v| \leq Mo(G_1) + m_1n_2$$

ii. When $d(v, v_1) = d(u, v_1)$. We can write

$$\sum_{uv\in E_1} |n_u - n_v| = \sum_{uv\in E_1} |n_u(G_1) - n_v(G_1)| = Mo(G_1)$$

From (i) and (ii), we see that

$$\sum_{uv\in E_1} |n_u - n_v| \leq Mo(G_1) + m_1n_2$$

Case 2. If $vu \in E_2$,then we obtain the following equation by similar to the Case 1

$$\sum_{uv\in E_2} |n_u - n_v| \leq Mo(G_2) + m_2n_1$$

Case 3. If $vu \in E'$,then $n_u = |\{V_1\}| = n_1$ and $n_v = |\{V_2\}| = n_2$. Thus, we have

$$\sum_{uv\in E'} |n_u - n_v| = |n_1 - n_2|.$$

By summing up the Cases 1, 2 and 3 the proof is completed. □

From Theorem 3.1 , The following results are easy to obtain

Corollary 3.1. *If $d(u, v_1) = d(v, v_1)$ for all $uv \in E(G_1)$ and $d(x, v_2) = d(y, v_2)$ for all $xy \in E(G_2)$, where $v_1v_2 \in E(G_1 \sim G_2)$, $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$, then*

$$Mo(G_1 \sim G_2) = Mo(G_1) + Mo(G_2) + |n_1 - n_2|$$

Corollary 3.2. *Let G be with n vertices and m edges. The Mostar index of $G \sim G$ is*

$$Mo(G \sim G) \leq 2\,(Mo(G) + mn)$$

Theorem 3.2. *The mostar index of $G_1 \sim G_2 \sim ... \sim G_d$ is*

$$Mo(G_1 \sim G_2 \sim ... \sim G_d) \leq \sum_{i=1}^{d} Mo\,(G_i) + \sum_{i=1}^{d} m_i \left(\sum_{j=1}^{d} n_j - n_i\right) + \sum_{i=1}^{d-1} \left(\left|\sum_{j=1}^{i} n_j - \sum_{j=i+1}^{d} n_j\right|\right)$$

Proof. From Definition 2.2, it has vertex set

$$V(G_1 \sim G_2 \sim ... \sim G_d) = V_1 \cup V_2 \cup ... \cup V_d$$

and edge set

$$E(G_1 \sim G_2 \sim ... \sim G_d) = E_1 \cup E_2 \cup ... \cup E_d \cup E^{'} \tag{5}$$

where

$$E' = \{v_i v_{i+1} \in E(G_1 \sim G_2 \sim ... \sim G_d) \, | v_i \in V_i, v_{i+1} \in V_{i+1}\} \, .$$

From Eq. (1) and Eq. (5), we can write the following equation

$$Mo(G_1 \sim G_2 \sim ... \sim G_d) = \sum_{i=1}^{d} \sum_{uv \in E_i} |n_u - n_v| + \sum_{uv \in E'} |n_u - n_v| \, . \tag{6}$$

We discuss the following two cases from Eq. (6):

Case 1. Let $u_i v_i \in E(G_1 \sim G_2 \sim ... \sim G_d)$ where $v_i, u_i \in V(G_i)$ for $i = \overline{1,d}$ and $x_i x_{i+1} \in E'$ where $x_i \in V(G_i), x_{i+1} \in V(G_{i+1})$ for $i = \overline{1, d-1}$.

i. If $d(u_i, x_i) < d(v_i, x_i)$, then it is written similar to the Case 1 (i) of Theorem 3.1:

$$n_{u_i} = \left|N_{u_i}(G_i), V_j - V_i\right|_{j=\overline{1,d}} = \left|N_{u_i}(G_i)\right| + \sum_{j=1}^{d} |V_j| - |V_i|$$

$$n_{v_i} = \left|N_{v_i}(G_i)\right|$$

Thus, we get

$$\sum_{i=1}^{d} \sum_{u_i v_i \in E_i} \left| n_{u_i}(G_i) + \sum_{j=1}^{d} n_j - n_i - n_{v_i}(G_i) \right| \leq \sum_{i=1}^{d} \sum_{u_i v_i \in E_i} \left| n_{u_i}(G_i) - n_{v_i}(G_i) \right| \; + \sum_{i=1}^{d} \sum_{u_i v_i \in E_i} \sum_{j=1}^{d} n_j - n_i$$

From Eq. (1), we obtain

$$\sum_{i=1}^{d} \sum_{u_i v_i \in E_i} \left| n_{u_i} - n_{v_i} \right| \leq \sum_{i=1}^{d} Mo\,(G_i) + \sum_{i=1}^{d} m_i \left(\sum_{j=1}^{d} n_j - n_i \right) \tag{7}$$

ii. We known that If $d(u_i, x_i) = d(v_i, x_i)$, then $n_{u_i} = \left|N_{u_i}(G_i)\right|$ and $n_{v_i} = \left|N_{v_i}(G_i)\right|$.

$$\sum_{i=1}^{d} \sum_{u_i v_i \in E_i} \left| n_{u_i} - n_{v_i} \right| = \sum_{i=1}^{d} Mo\,(G_i)$$

Case 2. Let $u_i v_i \in E'$ where $u_i \in V(G_i), v_i \in V(G_{i+1})$ for $i = \overline{1, d-1}$.

$$n_{u_i} = \sum_{j=1}^{i} |V_j| \quad \text{and} \quad n_{v_i} = \sum_{j=i+1}^{d} |V_j|$$

Thus, we get

$$\sum_{uv \in E'} |n_u - n_v| = \sum_{i=1}^{d-1} \sum_{u_i v_i \in E'} |n_{u_i} - n_{v_i}| = \sum_{i=1}^{d-1} \left(\left| \sum_{j=1}^{i} n_j - \sum_{j=i+1}^{d} n_j \right| \right)$$

By summing up the Cases 1 and 2, it is clear that

$$Mo(G_1 \sim G_2 \sim ... \sim G_d) \leq \sum_{i=1}^{d} Mo\left(G_i\right) + \sum_{i=1}^{d} m_i \left(\sum_{j=1}^{d} n_j - n_i \right) + \sum_{i=1}^{d-1} \left| \sum_{j=1}^{i} n_j - \sum_{j=i+1}^{d} n_j \right|$$

□

Corollary 3.3. *Let G be graph with n vertices and m edges. The Mostar index of $G \overset{d}{\sim} G$ is*

$$Mo(G \overset{d}{\sim} G) \leq dMo\left(G\right) + d\left(d-1\right)nm + 2n \left\lfloor \frac{d-1}{2} \right\rfloor \left(d - 1 - \left\lfloor \frac{d-1}{2} \right\rfloor \right)$$

Theorem 3.3. *Consider graphs G_1, G_2 ,..., G_d. The Mostar index of $G_1 \overset{l}{\sim} G_2 \overset{l}{\sim} ... \overset{l}{\sim} G_d = G^*$ is*

$$Mo(G^*) \leq \sum_{i=1}^{d} Mo(G_i) + \sum_{i=1}^{d-1} \left| \sum_{j=1}^{i} n_j - \sum_{j=i+1}^{d} n_j \right| + \sum_{i=1}^{d} m_i \begin{cases} \left| \sum\limits_{j=i}^{d} n_j - n_i \right| & \text{if (i) and (iii)} \\ \sum\limits_{j=1}^{i-1} n_j & \text{if (i) and (iv)} \\ \left| \sum\limits_{j=1}^{i-1} n_j - \sum\limits_{j=i+1}^{d} n_j \right| & \text{if (i) and } \overline{\text{(iii)}} \\ \sum\limits_{j=i+1}^{d} n_j & \text{if (ii) and (iii)} \\ 0 & \text{if (ii) and (iv)} \end{cases}$$

where $u_i y_i \in E_i$ for $i = \overline{1, d}$ and $v_i x_i \in E'$, $v_i, x_{i-1} \in V(G_i)$, $x_i \in V(G_{i+1})$ for $i = \overline{1, d-1}$. Let (i) be $d(u_i, x_{i-1}) < d(y_i, x_{i-1})$, $\overline{(i)}$ be $d(u_i, x_{i-1}) > d(y_i, x_{i-1})$, (ii) be $d(u_i, x_{i-1}) = d(y_i, x_{i-1})$, (iii) be $d(u_i, v_i) < d(y_i, v_i)$, $\overline{(iii)}$ be $d(u_i, v_i) > d(y_i, v_i)$, and (iv) be $d(u_i, v_i) = d(y_i, v_i)$.

Proof. From Definition 2.3, we partition the vertices and the edges of $G_1 \overset{l}{\sim} G_2 \overset{l}{\sim} ... \overset{l}{\sim} G_d$ as follows:

$$V(G_1 \overset{l}{\sim} G_2 \overset{l}{\sim} ... \overset{l}{\sim} G_d) = V_1 \cup V_2 \cup ... \cup V_d$$

and

$$E(G_1 \overset{l}{\sim} G_2 \overset{l}{\sim} ... \overset{l}{\sim} G_d) = E_1 \cup E_2 \cup ... \cup E_d \cup E' \tag{8}$$

where

$$E' = \left\{ v_i x_i \in E(G_1 \overset{l}{\sim} G_2 \overset{l}{\sim} ... \overset{l}{\sim} G_d) \, \middle| v_i \in V_i, x_i \in V_{i+1}, i = \overline{1, d-1} \right\}.$$

From Eq. (1) and Eq. (8), we get

$$Mo(G_1 \overset{l}{\sim} G_2 \overset{l}{\sim} ... \overset{l}{\sim} G_d) = \sum_{uv \in E_1} |n_u - n_v| + ... + \sum_{uv \in E_d} |n_u - n_v| + \sum_{uv \in E'} |n_u - n_v|.$$

We discuss the following two cases from the above equation:

Let $u_i y_i \in E_i$ for $i = \overline{1, d}$ and $v_i x_i \in E'$, $v_i, x_{i-1} \in V(G_i)$, $x_i \in V(G_{i+1})$ for $i = \overline{1, d-1}$.

Case 1. When $u_i y_i \in E_i$ for $i = \overline{1, d}$. Let (i) be $d(u_i, x_{i-1}) < d(y_i, x_{i-1})$, $\overline{(i)}$ be $d(u_i, x_{i-1}) > d(y_i, x_{i-1})$, (ii) be $d(u_i, x_{i-1}) = d(y_i, x_{i-1})$, (iii) $d(u_i, v_i) < d(y_i, v_i)$, $\overline{(iii)}$ $d(u_i, v_i) > d(y_i, v_i)$, and (iv) $d(u_i, v_i) = d(y_i, v_i)$. Then,

a) When (i) and (iii), we have for $u_iy_i \in E_i$, $i = \overline{1,d}$

$$\begin{aligned}\sum_{uv\in E_i}|n_u - n_v| &= \sum_{u_iy_i\in E_i}\left|\left|N_{u_i}(G_i)\right| + \sum_{j=1}^{i-1}|V_j| + \sum_{j=i+1}^{d}|V_j| - \left|N_{y_i}(G_i)\right|\right| \\ &\leq \sum_{u_iy_i\in E_i}|n_{u_i}(G_i) - n_{y_i}(G_i)| + \sum_{u_iy_i\in E_i}\left|\sum_{j=i}^{d}n_j - n_i\right|\end{aligned}$$

From Eq. (1), we obtain

$$\sum_{uv\in E_i}|n_u - n_v| \leq Mo(G_i) + m_i\left|\sum_{j=i}^{d}n_j - n_i\right|$$

b) When (i) and (iv) for $u_iy_i \in E_i$, $i = \overline{1,d}$ then, we have

$$\sum_{uv\in E_i}|n_u - n_v| = \sum_{u_iy_i\in E_i}\left|\left|N_{u_i}(G_i)\right| + \sum_{j=1}^{i-1}|V_j| - \left|N_{y_i}(G_i)\right|\right|$$

From Eq. (1), we obtain

$$\sum_{uv\in E_i}|n_u - n_v| \leq Mo(G_i) + m_i\sum_{j=1}^{i-1}n_j$$

c) When (i) and $\overline{(iii)}$ for $u_iy_i \in E_i$, $i = \overline{1,d}$ then, we have

$$\begin{aligned}\sum_{uv\in E_i}|n_u - n_v| &= \sum_{uv\in E_i}\left|\left|N_{u_i}(G_i)\right| + \sum_{j=1}^{i-1}|V_j| - \left|N_{y_i}(G_i)\right| - \sum_{j=i+1}^{d}|V_j|\right| \\ &\leq Mo(G_i) + m_i\left|\sum_{j=1}^{i-1}n_j - \sum_{j=i+1}^{d}n_j\right|\end{aligned}$$

d) When (ii) and (iii) for $u_iy_i \in E_i$, $i = \overline{1,d}$ then, we have

$$\sum_{uv\in E_i}|n_u - n_v| \leq Mo(G_i) + \sum_{uv\in E_i}\sum_{j=i+1}^{d}n_j$$

e) When (ii) and (iv) for $u_iy_i \in E_i$, $i = \overline{1,d}$ then, we have

$$\sum_{uv\in E_i}|n_u - n_v| \leq Mo(G_i)$$

Case 2. When $v_ix_i \in E'$ for $i = \overline{1,d-1}$. It is easy obtained to

$$n_{v_i} = \sum_{j=1}^{i}|V_j| \text{ and } n_{x_i} = \sum_{j=i+1}^{d}|V_j|$$

Then, we have: $\sum\limits_{v_ix_i\in E'}|n_{v_i} - n_{x_i}| = \left|\sum\limits_{j=1}^{i}n_j - \sum\limits_{j=i+1}^{d}n_j\right|$ for $i = \overline{1,d-1}$.

By summiting the Cases 1 and 2, the proof is completed. □

4. Mostar index of bridge graphs of some special graph

In this section, path, star, cycle and complete graphs are considered. The Mostar indices for bridge graphs of this considered graphs are computed.

Theorem 4.1. *Consider P_{n_1} with n_1 vertices and P_{n_2} with n_2 vertices be path graphs. Let $P_{n_1} \sim P_{n_2} = P^*$. Then,*

$$Mo(P_{n_1+n_2}) \leq Mo(P^*) \leq Mo(P_{n_1}) + Mo(P_{n_2}) + (n_2-1)n_1 + (n_1-1)n_2 + |n_1 - n_2|$$

Proof. The upper bound is easily obtained from Theorem 3.1. Bridge graph of path graph is a path graph with $n_1 + n_2$ or a tree graph with $n_1 + n_2$ vertices. So, the lower bound is obtained from Corollary 2.2. □

Corollary 4.1. *Consider P_n with n vertices. Then,*

$$\left\lfloor \frac{(2n-1)^2}{2} \right\rfloor = Mo(P_{2n}) \leq Mo(P_n \sim P_n) \leq 2Mo(P_n) + 2n(n-1)$$

Theorem 4.2. *Consider S_{n_1} star graph with n_1+1 vertices and S_{n_2} star graph with n_2+1 vertices. Let bridge vertices be $x \in S_{n_1}, y \in S_{n_2}$. Let $S_{n_1} \sim S_{n_2} = S^*$. Then,*

$$Mo(S^*) = \begin{cases} (n_1+n_2-1)(n_1+n_2) + |n_1-n_2+2| + |n_1-n_2| & \text{if } x = v_c, y \neq v_c \\ (n_1+n_2)^2 + |n_1-n_2| & \text{if } x = v_c, y = v_c \\ |n_2-n_1+2| + |n_1-n_2+2| + |n_1-n_2| + (n_1+n_2-2)(n_1+n_2) & \text{if } x, y \neq v_c \end{cases}$$

where v_c is center vertex of star graph.

Proof. Let $xy \in E', x \in S_{n_1}, y \in S_{n_2}$. We discuss the following three cases:

Case 1. If $x = v_c, y \neq v_c$ then

i. For $vu \in E_1$ $(v = v_c = x)$, $n_u = 1, n_v = n_1 + (n_2+1)$ from Eq. (2). Thus,

$$\sum_{uv \in E_1} |n_u - n_v| = n_1 (n_1 + n_2)$$

ii. For $vu \in E_2$,

a.if $v = y$ and $u = v_c$, then, $n_u = n_2$, $n_v = 1 + (n_1+1)$. Thus, $|n_u - n_v| = |n_1 + 2 - n_2|$

b. If $v \neq y$ and $u = v_c$, then, $n_u = n_2 + (n_1+1)$, $n_v = 1$. Thus, $\sum\limits_{uv \in E_2 - \{yv_c\}} |n_u - n_v| = (n_2-1)(n_1+n_2)$.

Thus,

$$\sum_{uv \in E_2} |n_u - n_v| = |n_1 + 2 - n_2| + (n_2-1)(n_1+n_2) .$$

iii. The result follows from Theorem 3.1 (Case 3).

By summiting (i), (ii) and (iii), for $x = v_c, y \neq v_c$ it is clear that

$$Mo(S^*) = n_1(n_1+n_2) + |n_1+2-n_2| + (n_2-1)(n_1+n_2) + |n_1-n_2|$$

Case 2. If $x = v_c, y = v_c$ then

i. For $vu \in E_1$, we easy see that $\sum\limits_{uv \in E_1} |n_u - n_v| = n_1(n_1+n_2)$

ii. For $vu \in E_2$, we easy see that $\sum\limits_{uv \in E_2} |n_u - n_v| = n_2(n_1+n_2)$

iii. The result follows from Theorem 3.1 (Case 3).

Thus, for $x = v_c, y = v$, it is obtained

$$Mo(S^*) = (n_1+n_2)(n_1+n_2) + |n_1-n_2|$$

Case 3. If $x \neq v_c, y \neq v_c$ then by using the Case 1 (ii) for $vu \in E_1, E_2$ and Theorem 3.1 (Case 3) for $vu \in E'$, the proof of this case is completed. □

Corollary 4.2. *Let S_n be star graph with $n+1$ vertices. Then,*

$$Mo(S_n \sim S_n) = \begin{cases} 2n\,(2n-1)+2 & \text{if } x = v_c, y \neq v_c \\ 4n^2 & \text{if } x = v_c, y = v_c \\ 4n\,(n-1)+4 & \text{if } x, y \neq v_c \end{cases}$$

Theorem 4.3. *If $C_{n_1}(V_1, E_1)$ with n_1 vertices and $C_{n_2}(V_2, E_2)$ with n_2 vertices then,*

$$Mo(C_{n_1} \sim C_{n_2}) = \begin{cases} 2n_1n_2 + |n_1 - n_2| & \text{if } n_1 \text{ and } n_2 \text{ are even} \\ 2n_1n_2 - (n_1+n_2) + |n_1 - n_2| & \text{if } n_1 \text{ and } n_2 \text{ are odd} \\ 2n_1n_2 - n_1 + |n_1 - n_2| & \text{if } n_1 \text{ is even and } n_2 \text{ is odd} \end{cases}$$

Proof. From Eq.(1) and Eq. (3), we have Eq. (4). We discuss the following three cases according to even or odd of vertices. Let $xy \in E', x \in V\ (C_{n_1}), y \in V\ (C_{n_2})$.

Case 1. Let n_1 and n_2 are even.

i. If $vu \in E_1$ and $d(u,x) < d(v,x)$ then

$$n_u = |N_u(C_{n_1}), \{V_2\}| = n_u(C_{n_1}) + n_2$$

$$n_v = |N_v(C_{n_1})| = n_v(C_{n_1})$$

We known that $Mo(C_n) = 0$ that is $\sum_{uv \in E} |n_u - n_v| = 0, n_u = n_v$. And also, $|E(C_{n_1})| = n_1$. Thus,

$$\sum_{uv \in E_1} |n_u - n_v| = \sum_{uv \in E_1} |n_u(C_{n_1}) + n_2 - n_v(C_{n_1})| = n_1n_2$$

ii. If $vu \in E_2$, and $d(u,x) < d(v,x)$ then the result is obtained by similar to the Case-1(i) of this proof.

iii. The result follows from Theorem 3.1 (Case-3).

From i, ii and iii, it is clear that $Mo(C_{n_1} \sim C_{n_2}) = 2n_1n_2 + |n_1 - n_2|$.

Case 2. Let n_1 and n_2 are odd.

i. Let $uv \in E_1$. Only one edge has $d(u,x) = d(v,x)$.Therefore, by similar to the Case-1(i) of this proof we have:

$$\sum_{uv \in E_1} |n_u - n_v| = \sum_{uv \in E_1} |n_u(C_{n_1}) + n_2 - n_v(C_{n_1})| = (m_1 - 1)\,n_2 = (n_1 - 1)\,n_2$$

ii. for $vu \in E_2$, the result is obtained by similar to the Case-2 (i) of this proof.

iii. The result follows from Theorem 3.1 (Case-3).

By summiting the Cases i, ii and iii, the proof is completed when n_1 and n_2 are odd.

Case 3. Let n_1 is even and n_2 is odd.Then, from the Case-1 (i) and the Case-2 (ii) of this proof, we easy see that

$$Mo(C_{n_1} \sim C_{n_2}) = 2n_1n_2 - n_1 + |n_1 - n_2|$$

□

Corollary 4.3. *Let C_n be cycle graph with n vertices. Then,*

$$Mo(C_{n_1} \sim C_{n_2}) = \begin{cases} 2n^2 & \text{if } n \text{ are even} \\ 2n^2 - 2n & \text{if } n \text{ are odd} \end{cases}$$

Theorem 4.4. *Consider K_{n_i} complete graph with n_i, $i = 1, 2$. Then,*

$$Mo(K_{n_1} \sim K_{n_2}) = 2n_1n_2 - (n_1+n_2) + |n_1 - n_2|$$

Proof. Let $xy \in E'$. From Eq. (4), we discuss the following three cases

Case 1. For $uv \in E_1$,

i. If $v, u \neq x$ then all the vertices are equidistant and $\sum_{E_1 - E_k} |n_u - n_v| = 0$, where $E_k = \{v_1 x \in E, v_1, x \in V_1\}$,

ii. If $u = x$ then $\sum_{E_k} |(1 + n_2) - 1| = (n_1 - 1) n_2$.

Then,

$$\sum_{uv \in E_1} |n_u - n_v| = (n_1 - 1) n_2$$

Case 2. The result is obtaned by similar to Case 1 of this proof.

Case 3. The result follows from Theorem 3.1 (Case-3).

By summiting the Case 1, 2 and 3, the proof is completed. □

Corollary 4.4. *Let K_n be complete graph with n vertices. Then,*

$$Mo(K_n \sim K_n) = 2n^2 - 2n$$

5. Conclusions

In this paper, the Mostar index was studied. Upper bounds for the Mostar indices of bridge graphs were obtained. It is given exact expressions of the mostar indices for bridge graphs of special graphs such as path, star, cycle, complete graph. We known that $Mo(C_n) = Mo(K_n) = 0$. The Mostar index value for the bridge graphs of C_n and K_n were different from zero.

The results will help to prove the conjectures and the open problems of Declic. Also, the Mostar index value for the bridge graphs of the considered graphs will help us understand chemically the behavior.

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