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ON REFINEMENTS OF SOME INTEGRAL INEQUALITIES

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Abstract. In this study, using Hölder-İşcan integral inequality as a better approach than Hölder integral inequality, Improved power-mean integral inequality as a better approach than power-mean inequality and an identity for differentiable functions, the inequalities for functions whose derivatives in absolute value at certain power are convex are obtained. Some applications to special means of real numbers and some error estimates related to midpoint formula are also given.

Keywords: Convex function, Hermite-Hadamard's inequality, Hölder-İşcan inequality, Improved power-mean inequality.

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1. Introduction

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$f\left(tx + (1-t)y\right) \leq tf\left(x\right) + (1-t)f\left(y\right)$$

valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \emptyset$.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

holds.This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions. See [1, 2, 4, 6, 7, 9], for the results of the generalization, improvement and extention of the famous integral inequality (1).

Theorem 1.1 (Hölder Inequality for Integral [8])**.** *Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are real functions defined on $[a,b]$ and if $|f|^p$, $|g|^q$ are integrable functions on $[a,b]$ then*

$$\int_a^b |f(x)g(x)|\, dx \leq \left(\int_a^b |f(x)|^p\, dx\right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q\, dx\right)^{\frac{1}{q}}, \tag{2}$$

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with equality holding if and only if $A|f(x)|^p = B|g(x)|^q$ *almost everywhere, where* A *and* B *are constants.*

The power-mean integral inequality as a result of the Hölder inequality can be given as follows:

Theorem 1.2 (Power-mean Integral Inequality [8]). *Let* $q \geq 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *If* f *and* g *are real functions defined on* $[a,b]$ *and if* $|f|$, $|f|\,|g|^q$ *are integrable functions on* $[a,b]$ *then*

$$\int_a^b |f(x)g(x)|\,dx \leq \left(\int_a^b |f(x)|\,dx\right)^{1-\frac{1}{q}} \left(\int_a^b |f(x)|\,|g(x)|^q\,dx\right)^{\frac{1}{q}}.$$

In [3], İşcan gave a refinement of the Hölder integral inequality as follows:

Theorem 1.3 (Hölder-İşcan Integral Inequality [3]). *Let* $p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *If* f *and* g *are real functions defined on interval* $[a,b]$ *and if* $|f|^p$, $|g|^q$ *are integrable functions on* $[a,b]$ *then*

$$\begin{aligned}\int_a^b |f(x)g(x)|\,dx \quad \leq \quad & \frac{1}{b-a}\left\{\left(\int_a^b (b-x)\,|f(x)|^p\,dx\right)^{\frac{1}{p}} \left(\int_a^b (b-x)\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right. \\ & \left. + \left(\int_a^b (x-a)\,|f(x)|^p\,dx\right)^{\frac{1}{p}} \left(\int_a^b (x-a)\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right\} \qquad (3)\end{aligned}$$

A refinement of power-mean integral inequality as a different version of the Hölder-İşcan integral inequality can be given as follows:

Theorem 1.4 (Improved power-mean integral inequality [5]). *Let* $q \geq 1$. *If* f *and* g *are real functions defined on interval* $[a,b]$ *and if* $|f|$, $|f|\,|g|^q$ *are integrable functions on* $[a,b]$ *then*

$$\begin{aligned}\int_a^b |f(x)g(x)|\,dx \quad \leq \quad & \frac{1}{b-a}\left\{\left(\int_a^b (b-x)\,|f(x)|\,dx\right)^{1-\frac{1}{q}} \left(\int_a^b (b-x)\,|f(x)|\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right. \\ & \left. + \left(\int_a^b (x-a)\,|f(x)|\,dx\right)^{1-\frac{1}{q}} \left(\int_a^b (x-a)\,|f(x)|\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right\} \qquad (4)\end{aligned}$$

In [4], Kırmacı gave the following Lemma to obtain some midpoint type inequalities differentiable convex functions.

Lemma 1.1. [4]. *Let* $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* $I^\circ, a, b \in I^\circ$ *(*I° *is the interior) with* $a < b$. *If* $f' \in L[a,b]$, *then the following identity holds:*

$$\begin{aligned}& \frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \qquad (5) \\ = \quad & (b-a)\left[\int_0^{\frac{1}{2}} tf'(ta+(1-t)b)\,dt + \int_{\frac{1}{2}}^1 (t-1)f'(ta+(1-t)b)\,dt\right].\end{aligned}$$

Note that, if the appropriate variable change in the integrals on the right side of the equation (5) is made, then the following identity is obtained.

$$\begin{aligned} & \frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \qquad (6) \\ = & \frac{b-a}{4}\left[\int_0^1 tf'\left(t\frac{a+b}{2}+(1-t)b\right)dt + \int_0^1 tf'\left(t\frac{a+b}{2}+(1-t)a\right)dt\right]. \end{aligned}$$

Using the equality (5), Kırmacı obtained the following results:

Theorem 1.5. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^\circ, a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{8}\left(\left|f'(a)\right| + \left|f'(b)\right|\right). \qquad (7)$$

In this paper, using Hölder-İşcan integral inequality better approach than Hölder integral inequality and improved power-mean integral inequality better approach than power-mean inequality, a general integral identity for differentiable functions in order to provide inequality for functions whose derivatives in absolute value at certain power are convex are derived. In addition, the obtained results are compared with the previous ones.

2. Main results

Theorem 2.1. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^\circ, a, b \in I^\circ$ with $a < b$ and let $q \geq 1$. If the mapping $|f'|^q$ is convex on the interval $[a,b]$, then the following inequality hold:*

$$\begin{aligned} & \left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \qquad (8) \\ \leq & \frac{b-a}{12}\left\{\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q + |f'(b)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{3\left|f'\left(\frac{a+b}{2}\right)\right|^q + |f'(b)|^q}{4}\right)^{\frac{1}{q}}\right. \\ & \left. + \left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q + |f'(a)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{3\left|f'\left(\frac{a+b}{2}\right)\right|^q + |f'(a)|^q}{4}\right)^{\frac{1}{q}}\right\}. \end{aligned}$$

Proof. From Lemma 1.1, improved power-mean integral inequality and the definition of convexity of the function $|f'|^q$, it is seen that

$$\begin{aligned} & \left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \\ \leq & \frac{b-a}{4}\left[\int_0^1 t\left|f'\left(t\frac{a+b}{2}+(1-t)b\right)\right|dt + \int_0^1 t\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|dt\right] \end{aligned}$$

$$
\begin{aligned}
\leq\ & \frac{b-a}{4}\left\{\left(\int_0^1 (1-t)\,tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)\,t\left|f'\left(t\frac{a+b}{2}+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1 t^2dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^2\left|f'\left(t\frac{a+b}{2}+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}}\right\}\\
&+\frac{b-a}{4}\left\{\left(\int_0^1 (1-t)\,tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)\,t\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1 t^2dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^2\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\right\}\\
\leq\ & \frac{b-a}{4}\left\{\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)\,t\left[t\left|f'\left(\frac{a+b}{2}\right)\right|^q+(1-t)\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left(\int_0^1 t^2\left[t\left|f'\left(\frac{a+b}{2}\right)\right|^q+(1-t)\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right\}\\
&+\frac{b-a}{4}\left\{\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)\,t\left[t\left|f'\left(\frac{a+b}{2}\right)\right|^q+(1-t)\left|f'(a)\right|^q\right]dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left(\int_0^1 t^2\left[t\left|f'\left(\frac{a+b}{2}\right)\right|^q+(1-t)\left|f'(a)\right|^q\right]dt\right)^{\frac{1}{q}}\right\}\\
=\ & \frac{b-a}{12}\left\{\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(a)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(a)\right|^q}{4}\right)^{\frac{1}{q}}\right\},
\end{aligned}
$$

where

$$
\begin{array}{ll}
\int_0^1 (1-t)\,tdt=\frac{1}{6}, & \int_0^1 (1-t)\,t^2dt=\int_0^1 (1-t)^2\,tdt=\frac{1}{12},\\
\int_0^1 t^2dt=\frac{1}{3}, & \int_0^1 t^3dt=\frac{1}{4}.
\end{array}
$$

This completes the proof of the Theorem. □

Corollary 2.1. *Under the conditions of Theorem 2.1, by taking $q=1$ in the inequality (8), the following inequality which is more better than the inequality (7) is obtained.*

$$
\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right|\leq\frac{b-a}{12}\left[2\left|f'\left(\frac{a+b}{2}\right)\right|+\frac{\left|f'(a)\right|+\left|f'(b)\right|}{2}\right]. \tag{9}
$$

Remark 2.1. *In the inequality, by using convexity of $|f'|$, it can be wrote*

$$
\left|f'\left(\frac{a+b}{2}\right)\right|\leq\frac{\left|f'(a)\right|+\left|f'(b)\right|}{2}.
$$

Thus,

$$
\begin{aligned}
&\frac{b-a}{12}\left[2\left|f'\left(\frac{a+b}{2}\right)\right|+\frac{\left|f'(a)\right|+\left|f'(b)\right|}{2}\right]\\
\leq\ &\frac{b-a}{8}\left(\left|f'(a)\right|+\left|f'(b)\right|\right).
\end{aligned}
$$

This inequality shows that the inequality (9) is more better than the inequality (7).

Theorem 2.2. *Let $f : I^{\circ} \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^{\circ}, a, b \in I^{\circ}$ with $a < b$ and let $q > 1$. If the mapping $|f'|^q$ is convex on the interval $[a, b]$, then the following inequality hold:*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \\
\leq\ & \frac{b-a}{4}\left(\frac{1}{(p+1)(p+2)}\right)^{\frac{1}{p}}\left[\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q + 2\left|f'(b)\right|^q}{6}\right)^{\frac{1}{q}} + \left(\frac{2\left|f'\left(\frac{a+b}{2}\right)\right|^q + \left|f'(a)\right|^q}{6}\right)^{\frac{1}{q}}\right] \\
&+\frac{b-a}{4}\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\left[\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q + 2\left|f'(a)\right|^q}{6}\right)^{\frac{1}{q}} + \left(\frac{2\left|f'\left(\frac{a+b}{2}\right)\right|^q + \left|f'(b)\right|^q}{6}\right)^{\frac{1}{q}}\right], \quad (10)
\end{aligned}$$

where $1/p + 1/q = 1$.

Proof. Using Lemma 1.1, Hölder-İşcan integral inequality and the following inequality

$$\left|f'(ta + (1-t)b)\right|^q \leq t\left|f'(a)\right|^q + (1-t)\left|f'(a)\right|^q,$$

it is easily seen that

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \\
\leq\ & \frac{b-a}{4}\left[\int_0^1 t\left|f'\left(t\frac{a+b}{2} + (1-t)b\right)\right|dt + \int_0^1 t\left|f'\left(t\frac{a+b}{2} + (1-t)a\right)\right|dt\right] \\
\leq\ & \frac{b-a}{4}\left\{\left(\int_0^1 (1-t)\,t^p dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)\left|f'\left(t\frac{a+b}{2} + (1-t)b\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1 t^{p+1}dt\right)^{\frac{1}{p}}\left(\int_0^1 t\left|f'\left(t\frac{a+b}{2} + (1-t)b\right)\right|^q dt\right)^{\frac{1}{q}}\right\} \\
&+\frac{b-a}{4}\left\{\left(\int_0^1 (1-t)^{p+1}\,dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)\left|f'\left(t\frac{a+b}{2} + (1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1 t\,(1-t)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 t\left|f'\left(t\frac{a+b}{2} + (1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

$$\begin{aligned}
\leq\ & \frac{b-a}{4}\left\{\left(\frac{1}{(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)\left[t\left|f'\left(\frac{a+b}{2}\right)\right|^q + (1-t)\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right\} \\
&+\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\left(\int_0^1 t\left[t\left|f'\left(\frac{a+b}{2}\right)\right|^q + (1-t)\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right\} \\
&+\frac{b-a}{4}\left\{\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)\left[t\left|f'\left(\frac{a+b}{2}\right)\right|^q + (1-t)\left|f'(a)\right|^q\right]dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{1}{(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\int_0^1 t\left[t\left|f'\left(\frac{a+b}{2}\right)\right|^q + (1-t)\left|f'(a)\right|^q\right]dt\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

$$= \frac{b-a}{4}\left(\frac{1}{(p+1)(p+2)}\right)^{\frac{1}{p}}\left[\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^{q}+2\left|f'(b)\right|^{q}}{6}\right)^{\frac{1}{q}}+\left(\frac{2\left|f'\left(\frac{a+b}{2}\right)\right|^{q}+\left|f'(a)\right|^{q}}{6}\right)^{\frac{1}{q}}\right]$$

$$+\frac{b-a}{4}\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\left[\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^{q}+2\left|f'(a)\right|^{q}}{6}\right)^{\frac{1}{q}}+\left(\frac{2\left|f'\left(\frac{a+b}{2}\right)\right|^{q}+\left|f'(b)\right|^{q}}{6}\right)^{\frac{1}{q}}\right]. \quad (11)$$

where

$$\begin{array}{ll} \int_0^1 (1-t)\,t^p dt = \int_0^1 t\,|t-1|^p\,dt = \frac{1}{(p+1)(p+2)}, & \int_0^1 (1-t)\,t dt = \frac{1}{6}, \\ \int_0^1 t^{p+1}dt = \int_0^1 (1-t)^{p+1}\,dt = \frac{1}{p+2}, & \int_0^1 t^2 dt = \int_0^1 (1-t)^2\,dt = \frac{1}{3}. \end{array}$$

This completes the proof of the Theorem. □

3. Some applications for special means

Let us recall the following special means of two nonnegative number a, b with $b > a$:

(1) The arithmetic mean

$$A = A(a,b) := \frac{a+b}{2}.$$

(2) The geometric mean

$$G = G(a,b) := \sqrt{ab}.$$

(3) The harmonic mean

$$H = H(a,b) := A^{-1}\left(a^{-1}, b^{-1}\right) = \frac{2ab}{a+b}.$$

(4) The Logarithmic mean

$$L = L(a,b) := \frac{b-a}{\ln b - \ln a}.$$

(5) The p-Logarithmic mean

$$L_p = L_p(a,b) := \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}, \quad p \in \mathbb{R}\backslash\{-1,0\}.$$

(6) The Identric mean

$$I = I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}.$$

These means are often used in numerical approximation and in other areas. However, the following simple relationships are known in the literature:

$$H \le G \le L \le I \le A.$$

It is also known that L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$.

Proposition 3.1. *Let $0 < a < b$, $q > 1$ and $1/p + 1/q = 1$. Then, the following inequality holds:*

$$\begin{aligned} & \left|L_2^2 - A^2\right| \\ \leq\ & (b-a)\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\left[\left(\frac{1}{p+1}\right)^{\frac{1}{p}} A\left(\left(\frac{A^q + 2b^q}{6}\right)^{1/q}, \left(\frac{2A^q + a^q}{6}\right)^{1/q}\right)\right. \\ & \left. + A\left(\left(\frac{A^q + 2a^q}{6}\right)^{1/q}, \left(\frac{2A^q + b^q}{6}\right)^{1/q}\right)\right]. \end{aligned}$$

Proof. The assertion follows from the inequality (10) in Theorem 2.2, for $f : (0, \infty) \to \mathbb{R}$, $f(x) = \frac{x^2}{2}$. □

Proposition 3.2. *Let $0 < a < b$, $q > 1$ and $1/p + 1/q = 1$. Then, the following inequality holds:*

$$\begin{aligned} & \left|L^{-1} - A^{-1}\right| \\ \leq\ & \frac{b-a}{2}\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\left[\left(\frac{1}{p+1}\right)^{\frac{1}{p}} A\left(\left(\frac{A^{-2q} + 2b^{-2q}}{6}\right)^{1/q}, \left(\frac{2A^{-2q} + a^{-2q}}{6}\right)^{1/q}\right)\right. \\ & \left. + A\left(\left(\frac{A^{-2q} + 2a^{-2q}}{6}\right)^{1/q}, \left(\frac{2A^{-2q} + b^{-2q}}{6}\right)^{1/q}\right)\right]. \end{aligned}$$

Proof. The assertion follows from the inequality (10) in Theorem 2.2, for $f : (0, \infty) \to \mathbb{R}$, $f(x) = 1/x$. □

Proposition 3.3. *Let $0 < a < b$ and $q \geq 1$. Then, the following inequality holds:*

$$\begin{aligned} & \left|L_2^2 - A^2\right| \\ \leq\ & \frac{b-a}{6}\left\{2^{\frac{1}{q}} A\left(\left(\frac{A^q + b^q}{4}\right)^{1/q}, \left(\frac{A^q + a^q}{4}\right)^{1/q}\right)\right. \\ & \left. + 2A\left(\left(\frac{3A^q + b^q}{4}\right)^{1/q}, \left(\frac{3A^q + a^q}{4}\right)^{1/q}\right)\right\}. \end{aligned}$$

Proof. The assertion follows from the inequality (8) in Theorem 2.1, for $f : (0, \infty) \to \mathbb{R}$, $f(x) = f(x) = \frac{x^2}{2}$. □

Proposition 3.4. *Let $0 < a < b$ and $q \geq 1$. Then, the following inequality holds:*

$$\begin{aligned} \left|L^{-1} - A^{-1}\right| \leq\ & \frac{b-a}{12}\left\{2^{\frac{1}{q}} A\left(\left(\frac{A^{-2q} + b^{-2q}}{4}\right)^{1/q}, \left(\frac{A^{-2q} + a^{-2q}}{4}\right)^{1/q}\right)\right. \\ & \left. + 2A\left(\left(\frac{3A^{-2q} + b^{-2q}}{4}\right)^{1/q}, \left(\frac{3A^{-2q} + a^{-2q}}{4}\right)^{1/q}\right)\right\}. \end{aligned}$$

Proof. The assertion follows from the inequality (8) in Theorem 2.1, for $f : (0, \infty) \to \mathbb{R}$, $f(x) = 1/x$. □

4. The midpoint formula

Let d be a division $a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ of the interval $[a, b]$ and consider the quadrature formula

$$\int_a^b f(x)dx = T(f, d) + E(f, d) \tag{12}$$

where

$$T(f, d) = \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right)(x_{i+1} - x_i)$$

is the midpoint version and $E(f, d)$ denotes the approximation error. Here, some error estimates for midpoint formula are derived.

Proposition 4.1. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^\circ, a, b \in I^\circ$ with $a < b$ and let $q \geq 1$. If the mapping $|f'|^q$ is convex on the interval $[a, b]$, then in (12), for every division d of $[a, b]$ the following inequality holds*

$$|E(f, d)| \leq \frac{1}{12} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2$$

$$\times \left\{ \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left(\frac{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q + |f'(x_{i+1})|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{3\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q + |f'(x_{i+1})|^q}{4}\right)^{\frac{1}{q}}\right.$$

$$\left. + \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left(\frac{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q + |f'(x_i)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{3\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q + |f'(x_i)|^q}{4}\right)^{\frac{1}{q}} \right\}.$$

Proof. By applying Theorem 2.1 on the subinterval $[x_i, x_{i+1}], i = 0, 1, ..., n-1$, of the division d, it is easily seen that

$$\left|\int_{x_i}^{x_{i+1}} f(x)dx - f\left(\frac{x_i + x_{i+1}}{2}\right)(x_{i+1} - x_i)\right| \leq \frac{(x_{i+1} - x_i)^2}{12}$$

$$\times \left\{ \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left(\frac{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q + |f'(x_{i+1})|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{3\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q + |f'(x_{i+1})|^q}{4}\right)^{\frac{1}{q}}\right.$$

$$\left. + \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left(\frac{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q + |f'(x_i)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{3\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q + |f'(x_i)|^q}{4}\right)^{\frac{1}{q}} \right\}.$$

Hence,

$$\begin{aligned}
&\left|\int_a^b f(x)dx - T(f,d)\right| \\
= &\left|\sum_{i=0}^{n-1}\left\{\int_{x_i}^{x_{i+1}} f(x)dx - f\left(\frac{x_i+x_{i+1}}{2}\right)(x_{i+1}-x_i)\right\}\right| \\
\le &\sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}} f(x)dx - f\left(\frac{x_i+x_{i+1}}{2}\right)(x_{i+1}-x_i)\right| \\
\le &\frac{1}{12}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2 \\
&\times\left\{\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q+|f'(x_{i+1})|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q+|f'(x_{i+1})|^q}{4}\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q+|f'(x_i)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q+|f'(x_i)|^q}{4}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

□

Proposition 4.2. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^\circ, a, b \in I^\circ$ with $a < b$ and let $q > 1$ so that $1/p + 1/q = 1$. If the mapping $|f'|^q$ is convex on the interval $[a,b]$, then in (12), for every division d of $[a,b]$ the following inequality holds*

$$|E(f,d)| \le \frac{1}{4}\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2$$

$$\times\left\{\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q+2|f'(x_{i+1})|^q}{6}\right)^{\frac{1}{q}}+\left(\frac{2\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q+|f'(x_i)|^q}{6}\right)^{\frac{1}{q}}\right]\right.$$

$$\left.+\left[\left(\frac{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q+2|f'(x_i)|^q}{6}\right)^{\frac{1}{q}}+\left(\frac{2\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q+|f'(x_{i+1})|^q}{6}\right)^{\frac{1}{q}}\right]\right\}.$$

Proof. The proof is done similarly to Proposition 4.2 by using Theorem 2.2. □

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