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CONTINUOUS K-G-FUSION FRAMES IN HILBERT SPACES

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Abstract. This paper aims at introducing the concept of c-K-g-fusion frames, which are generalizations of K-g-fusion frames, proving some new results on c-K-g-fusion frames in Hilbert spaces, defining duality of c-K-g-fusion frames and characterizing the kinds of the duals, and discussing the perturbation of c-K-g-fusion frames.

Keywords: c-K-g frame; K-fusion frame; K-g-fusion frame; c-g-fusion frame; Q-dual.

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1. Introduction

Discrete frames were introduced by Duffin and Schaeffer in 1952 [10] for studying some profound problems in nonharmonic Fourier series. Discrete and continuous frames appear in many applications in both pure and applied mathematics, particularly in the frame theory, which has been extensively used in numerous fields such as filter bank theory, signal and image processing, coding and communications [26].

Over the years, various extensions of the frames have been investigated. Some of these are contained as special cases of the elegant theory for g-frames introduced by W. Sun in [27]. Examples are bounded quasi-projectors, fusion frames, pseudo-frames, oblique frames, and outer frames.

In quantum mechanics, specifically in the theory of coherent states [1, 2], this notion of frames was generalized to a family of vectors indexed by a locally compact space endowed with a positive Radon measure. They have been introduced originally by Ali, Gazeau and Antoine [1, 2] and also, independently, by Kaiser [23]. Since then, several papers dealt with various aspects of the concept, see for instance [12, 13] or [24]. The continuous wavelet transformation and short time Fourier transformation are two well known examples of

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continuous frames. Some aspects of continuous frames on coherent states and specially on wave packet systems studied in the series of papers [17, 18, 19, 20, 21, 22].

Traditionally, frames were studied for the whole space or for a closed subspace. Gavruta in [14] gave another generalization of frames namely K-frames, which allows to reconstruct elements from the range of a linear and bounded operator in a Hilbert space.

K-g-frames have been introduced in [6, 16], and some properties and characterizations of them have been identified (for more information on K-g-frames, the reader can check [16, 28]). Extending the above-mentioned notions, the new concept of c-K-g-frames is introduced in [3].

Fusion frames were considered by Casazza, Kutyniok and Li in connection with distributed processing and are related to the construction of global frames [8].The fusion frame theory is in fact more delicate due to complicated relations between the structure of the sequence of weighted subspaces and the local frames in the subspaces and also due to the extreme sensitivity to changes of the weights.

Recently, Arabyani and Arefijamaal have presented K-frames, K-fusion frames and their duals in [4, 5], and c-K-fusion frames have been introduced in [25]; some properties and characterizations of c-K-fusion frames have also been obtained.

In the current paper, we set out to generalize some results of [5] and [25] to c-K-g-frames. Throughout this paper, H, (Ω, μ) and $\{H_\omega\}_{\omega\in\Omega}$ will be a separable Hilbert space, a measure space with positive measure μ and a family of Hilbert spaces, respectively. π_V is the orthogonal projection from H onto a closed subspace V and $B(H, H_\omega)$ is the set of all bounded and linear operators from H to H_ω. If $H = H_\omega$, then $B(H, H)$ will be denoted by $B(H)$. Also, $\mathbb{H}$ will be the collection of all closed subspaces of H, and $v : \Omega \to \mathbb{R}^+$ is a measurable mapping such that $v \neq 0$ a.e.

Definition 1.1. *Let $K \in B(H)$. A sequence $\{f_n\}_{n=1}^{\infty}$ is called a K-frame for H, if there exist constants $A, B > 0$ such that*

$$A\|K^*f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \leq B\|f\|^2, \quad f \in H. \tag{1}$$

We call A, B the lower and the upper frame bounds of K-frame $\{f_n\}_{n=1}^{\infty}$, respectively. If only the right inequality (1) holds, $\{f_n\}_{n=1}^{\infty}$ is called a Bessel sequence. If $K = I$, then it is just the ordinary frame.

Definition 1.2. *Let $K \in B(H)$ and $\Lambda = \{\Lambda_i \in B(H, H_i) : i \in I\}$. We call Λ a K-g-frame for H with respect to $\{H_i\}_{i\in I}$, or simply a K-g-frame for H, if there exist constants $A, B > 0$ such that*

$$A\|K^*f\|^2 \leq \sum_{i\in I} \|\Lambda_i f\|^2 \leq B\|f\|^2, \quad f \in H. \tag{2}$$

The constants A, B are called the lower and upper bounds of K-g-frame, respectively.

Remark 1.1. *Every K-g-frame is also a g-Bessel sequence for H. If $K = I$, K-g-frame is a g-frame.*

Definition 1.3. *Let $W = \{W_j\}_{j\in\mathbb{J}}$ be a family of closed subspaces of H and $v = \{v_j\}_{j\in\mathbb{J}}$ be a family of weights (i.e. $v_j > 0$ for any $j \in \mathbb{J}$). We say that W is a fusion frame with respect to v for H if there exist* $0 < A \leq B < \infty$ *such that for each $h \in H$*

$$A\|h\|^2 \leq \sum_{j\in\mathbb{J}} v_j^2 \|\pi_{W_j}(h)\|^2 \leq B\|h\|^2.$$

The generalized continuous version of fusion frames are defined in [11] as follows:

Definition 1.4. *Let $F : \Omega \to \mathbb{H}$ be such that for each $h \in H$, the mapping $\omega \to \pi_{F(\omega)}(h)$ is measurable (i.e. is weakly measurable) and let $\{H_\omega\}_{\omega\in\Omega}$ be a collection of Hilbert spaces, For each $\omega \in \Omega$, suppose that $\Lambda_\omega \in B(F(\omega), H_\omega)$ and put*

$$\Lambda = \{\Lambda_\omega \in B(F(\omega), H_\omega) : \omega \in \Omega\}.$$

Then (Λ, F, v) is a c-g-fusion frame for H if there exist $0 < A \leq B < \infty$ such that for all $h \in H$

$$A\|h\|^2 \leq \int_\Omega v^2(\omega)\|\Lambda_\omega(\pi_{F(\omega)}(h))\|^2 \, d\mu(\omega) \leq B\|h\|^2. \tag{3}$$

where $\pi_{F(\omega)}$ is the orthogonal projection onto the subspace $F(\omega)$.

(Λ, F, v) is called a tight-c-g-fusion frame for H if $A = B$, and Parseval if $A = B = 1$. (Λ, F, v) is called a Bessel c-g-fusion for H if we only have the upper bound. Let $H_0 = \bigoplus_{\omega\in\Omega} H_\omega$ and $L^2(\Omega, H_0)$ be a collection of all measurable functions $\varphi : \Omega \to H_0$ such that for each $\omega \in \Omega$, $\varphi(\omega) \in H_\omega$ and

$$\int_\Omega \|\varphi(\omega)\|^2 d\mu < \infty.$$

It can be verified that $L^2(\Omega, H_0)$ is a Hilbert space with inner product defined by

$$\langle \varphi, \psi\rangle = \int_\Omega \langle \varphi(\omega), \psi(\omega)\rangle d\mu,$$

for $\varphi, \psi \in L^2(\Omega, H_0)$ and the representation space in this setting is $L^2(\Omega, H_0)$. The continuous version of K-g-frames have been introduced in [3] as following:

Definition 1.5. *Suppose that (Ω, μ) is a measure space with positive measure μ and $K \in B(H)$. A family $\Lambda = \{\Lambda_\omega \in B(H, H_\omega) : \omega \in \Omega\}$, which $\{H_\omega\}_{\omega\in\Omega}$ is a family of Hilbert spaces, is called a continuous K-g-frame, or simply, a c-K-g-frame for H with respect to $\{H_\omega\}_{\omega\in\Omega}$, if*

(i) *for each $f \in H$; $\{\Lambda_\omega f\}_{\omega\in\Omega}$ is strongly measurable,*
(ii) *there exist constants $0 < A \leq B < \infty$ such that*

$$A\|K^* f\|^2 \leq \int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega) \leq B\|f\|^2, \ f \in H. \tag{4}$$

The constants A, B are called lower and upper c-K-g-frame bounds, respectively. If A, B can be chosen such that $A = B$, then $\{\Lambda_\omega\}_{\omega\in\Omega}$ is called a tight c-K-g-frame and if $A = B = 1$, it is called Parseval c-K-g-frame. A family $\{\Lambda_\omega\}_{\omega\in\Omega}$ is called a c-g-Bessel family if the right hand inequality in (4) holds. In this case, B is called the Bessel constant.

Now, we present some theorems in operator theory which will be needed in next sections.

Lemma 1.1. *([9]). Let $L_1 \in B(H_1, H)$ and $L_2 \in B(H_2, H)$ be on given Hilbert spaces. Then the following assertions are equivalent:*

(1) *$\mathcal{R}(L_1) \subseteq \mathcal{R}(L_2)$;*
(2) *$L_1 L_1^* \leq \lambda^2 L_2 L_2^*$ for some $\lambda > 0$;*
(3) *there exists a mapping $X \in B(H_1, H_2)$ such that $L_1 = L_2 X$.*

Moreover, if those conditions are valid, then there exists a unique operator X so that

(a) $\|X\|^2 = \inf\{\alpha > 0 \mid L_1 L_1^* \leq \alpha L_2 L_2^*\}$;
(b) $\mathcal{N}(L_1) = \mathcal{N}(X)$;
(c) $\mathcal{R}(X) \subseteq \overline{\mathcal{R}(L_2^*)}$.

For the proof of the following lemma, refer to [15].

Lemma 1.2. *Let $V \subseteq H$ be a closed subspace and T be a bounded operator on H. Then*

$$\pi_V T^* = \pi_V T^* \pi_{\overline{TV}}.$$

If T is unitary (i.e. $T^*T = I$), then

$$\pi_{\overline{TV}} T = T\pi_V.$$

2. Continuous $K - g$Fusion Frames

In this section, we introduce the notion of continuous K-g-fusion frames in Hilbert spaces and discuss some of their properties.

Definition 2.1. *Let $F : \Omega \to \mathbb{H}$ be such that for each $h \in H$, the mapping $\omega \to \pi_{F(\omega)}(h)$ is weakly measurable, $K \in B(H)$ and let*

$$\Lambda = \{\Lambda_\omega \in B(F(\omega), H_\omega) : \omega \in \Omega\}.$$

Then (Λ, F, v) is a continuous K-g-fusion frame, or simply a c-K-g-fusion frame for H with respect to v, if there exist $0 < A \leq B < \infty$ such that for all $h \in H$

$$A\|K^*h\|^2 \leq \int_\Omega v^2(\omega)\|\Lambda_\omega \pi_{F(\omega)}(h)\|^2 \, d\mu(\omega) \leq B\|h\|^2. \tag{5}$$

where $\pi_{F(\omega)}$ is the orthogonal projection of H onto the subspace $F(\omega)$.

(Λ, F, v) is called a tight c-K-g-fusion frame for H if $A = B$, and parseval if $A = B = 1$. (Λ, F, v) is called a Bessel c-g-fusion for H if the right-hand inequality in (5) holds.
When $K = I$, a c-K-g-fusion frame is c-g-fusion frame as defined in Definition 1.4.
Since each c-K-g-fusion frame is c-g-fusion Bessel, so the synthesis, analysis and c-K-g-fusion frame operators are defined. Indeed, the synthesis operator is defined weakly as follows (for more details, refer to [11]):

$$T : L^2(\Omega, H_0) \longrightarrow H,$$
$$\langle T(\varphi), h\rangle = \int_\Omega v(\omega)\langle \Lambda_\omega^*(\varphi(\omega)), h\rangle \, d\mu(\omega),$$

where $\varphi \in L^2(\Omega, H_0)$ and $h \in H$. It is obvious that T is linear and by Remark 1.6 in [11], T is a bounded linear operator. Its adjoint, that is called analysis operator

$$T^* : H \longrightarrow L^2(\Omega, H_0),$$
$$T^*(h)(\omega) = v(\omega)\Lambda_\omega \pi_{F(\omega)}(h), \ h \in H.$$

Definition 2.2. *Suppose that (Λ, F, v) is a c-K-g-fusion frame for H with frame bounds A and B. We define $S : H \to H$ by*

$$\langle Sf, g\rangle = \int_\Omega v^2(\omega)\langle \pi_{F(\omega)}\Lambda_\omega^*\Lambda_\omega \pi_{F(\omega)}(f), g\rangle \, d\mu(\omega),$$

and we call it the c-K-g-fusion frame operator.

Lemma 2.1. *Let (Λ, F, v) be a c-g-fusion Bessel for H. Then (Λ, F, v) is a c-K-g-fusion frame for H if only if there exists $A > 0$ such that $S \geq AKK^*$ where S is c-K-g-fusion frame operator.*

Proof. We have for each $h \in H$,

$$\begin{aligned}\langle Sh, h\rangle = \|T^*(h)\|^2 &= \int_\Omega v^2(\omega)\langle \pi_{F(\omega)}\Lambda_\omega^*\Lambda_\omega\pi_{F(\omega)}(h), h\rangle d\mu(\omega)\\ &= \int_\Omega v^2(\omega)\|\Lambda_\omega\pi_{F(\omega)}(h)\|^2\, d\mu(\omega).\end{aligned}$$

So, (Λ, F, v) is a c-K-g-fusion frame for H with bounds A and B if and only if

$$A\|K^*h\|^2 \leq \int_\Omega v^2(\omega)\|\Lambda_\omega(\pi_{F(\omega)}(h)\|^2\, d\mu(\omega) \leq B\|h\|^2,\ h \in H.$$

That is,

$$A\|K^*h\|^2 \leq \langle Sh, h\rangle \leq B\|h\|^2,\ h \in H.$$

Therefore,

$$AKK^* \leq S \leq B. \tag{6}$$

□

Remark 2.1. *In c-K-g-fusion frame, like c-K-g-frames and c-K-fusion frames, the c-K-g-fusion frame operator is not invertible. But if $K \in B(H)$ has closed range, then the operator S is an invertible operator on the subspace $\mathcal{R}(K) \subseteq H$. Indeed, suppose that $f \in \mathcal{R}(K)$, then*

$$\|f\|^2 = \|(K^\dagger|_{\mathcal{R}(K)})^*K^*f\|^2 \leq \|K^\dagger\|^2\|K^*f\|^2.$$

Thus, we have

$$A\|K^\dagger\|^{-2}\|f\|^2 \leq \langle Sf, f\rangle \leq B\|f\|^2, \tag{7}$$

which implies that $S : \mathcal{R}(K) \to S(\mathcal{R}(K))$ is a homeomorphism. Furthermore, for each $f \in S(\mathcal{R}(K))$ we have

$$B^{-1}\|f\|^2 \leq \langle (S|_{\mathcal{R}(K)})^{-1}f, f\rangle \leq A^{-1}\|K^\dagger\|^2\|f\|^2, \quad f \in H. \tag{8}$$

Remark 2.2. *By Lemma 2.1, $S \in B(H)$ is positive and self-adjoint. Since $B(H)$ is a C^*-algebra, then*

$$(S^{-1})^* = (S^*)^{-1} = S^{-1},$$

Thus, S^{-1} is self-adjoint and positive too whenever $K \in B(H)$ is surjective. Hence, for each $f \in S(\mathcal{R}(K))$, we can write

$$\begin{aligned}\langle Kf, f\rangle &= \langle Kf, SS^{-1}f\rangle\\ &= \langle S(Kf), S^{-1}f\rangle\\ &= \int_\Omega v^2(\omega)\langle \pi_{F(\omega)}\Lambda_\omega^*\Lambda_\omega\pi_{F(\omega)}(Kf), S^{-1}f\rangle d\mu(\omega)\\ &= \int_\Omega v^2(\omega)\langle S^{-1}\pi_{F(\omega)}\Lambda_\omega^*\Lambda_\omega\pi_{F(\omega)}(Kf), f\rangle\, d\mu(\omega).\end{aligned}$$

Theorem 2.1. *Let $U \in B(H)$ be an invertible operator on H and (Λ, F, v) be a c-K-g-fusion frame for H with bounds A and B. Then (Γ, G, v) is a c-UK-g-fusion frame for H where $\Gamma = \{\Gamma_\omega\}_{\omega\in\Omega} = \{\Lambda_\omega\pi_{F(\omega)}U^* \in B(H, H_\omega); \omega \in \Omega\}$ and $G(\omega) = UF(\omega)$.*

Proof. By applying Lemma 1.2, and the fact that U is invertible, for each $f \in H$, we have

$$\begin{aligned}\int_\Omega v^2(\omega)\|\Lambda_\omega \pi_{F(\omega)} U^* \pi_{UF(\omega)} f\|^2 \, d\mu &= \int_\Omega v^2(\omega)\|\Lambda_\omega \pi_{F(\omega)} U^* f\|^2 \, d\mu \\ &\leq B\|U^* f\|^2 \\ &\leq B\|U\|^2\|f\|^2.\end{aligned}$$

So, (Γ, G, v) is a c-g-fusion Bessel sequence for H, on the other hand,

$$\begin{aligned}\int_\Omega v^2(\omega)\|\Lambda_\omega \pi_{F(\omega)} U^* \pi_{UF(\omega)} f\|^2 \, d\mu &= \int_\Omega v^2(\omega)\|\Lambda_\omega \pi_{F(\omega)} U^* f\|^2 \, d\mu \\ &\geq A\|K^* U^* f\|^2 \\ &= A\|(UK)^* f\|^2\end{aligned}$$

Therefore, (Γ, G, v) is a c-UK-g-fusion frame for H . □

Corollary 2.1. *Let $U \in B(H)$ be an invertible operator on H and (Λ, F, v) is a c-K-g-fusion frame for H with bounds A and B and $UK = KU$. Then (Γ, G, v) is a c-K-g-fusion frame for H with bounds $A\|U^{-1}\|^{-2}$ and $B\|U\|^2$ where $\Gamma = \{\Gamma_\omega\}_{\omega\in\Omega} = \{\Lambda_\omega \pi_{F(\omega)} U^* \in B(H, H_\omega); \omega \in \Omega\}$ and $G(\omega) = UF(\omega)$.*

Proof. We have for each $f \in H$,

$$\|K^* f\|^2 = \|(U^{-1})^* U^* K^* f\|^2 \leq \|U^{-1}\|^2\|K^* U^* f\|^2.$$

So,

$$A\|U^{-1}\|^{-2}\|K^* f\|^2 \leq \|K^* U^* f\|^2$$

and by Theorem 2.1 the proof is completed. □

Theorem 2.2. *Let $U \in B(H)$ be a unitary operator on H and (Λ, F, v) be a c-K-g-fusion frame for H with bounds A and B. Then $(\Lambda_\omega U^{-1}, UF, v)$ is a c- $(U^{-1})^* K$-g-fusion frame for H.*

Proof. By Lemma 1.2, we can write for any $f \in H$,

$$\begin{aligned}A\|((U^{-1})^* K)^* f\|^2 = A\|K^* U^{-1} f\|^2 &\leq \int_\Omega v^2(\omega)\|\Lambda_\omega U^{-1} \pi_{UF(\omega)} f\|^2 \, d\mu \\ &= \int_\Omega v^2(\omega)\|\Lambda_\omega \pi_{F(\omega)} U^{-1} f\|^2 \, d\mu \\ &\leq B\|U^{-1}\|^2\|f\|^2.\end{aligned}$$

□

Corollary 2.2. *Let $U \in B(H)$ be a unitary operator on H and (Λ, F, v) be a c-K-g-fusion frame for H with bounds A and B and $K^* U = UK^*$. Then $(\Lambda_\omega U^{-1}, UF(\omega), v)$ is a c-K-g-fusion frame for H.*

Proof. We can write for any $f \in H$,

$$\|K^* f\|^2 = \|UU^{-1} K^* f\|^2 = \|UK^* U^{-1} f\|^2 \leq \|U\|^2\|K^* U^{-1} f\|^2.$$

So,

$$A\|U\|^{-2}\|K^* f\|^2 \leq A\|K^* U^{-1} f\|^2.$$

By the proof of Theorem 2.2, we conclude the result. □

Proposition 2.3. *Let $U \in B(H)$, (Λ, F, v) be a c-K-g-fusion frame for H with bounds A, B and $\mathcal{R}(U) \subseteq \mathcal{R}(K)$. Then (Λ, F, v) is a c-U-g-fusion frame for H.*

Proof. Via Lemma 1.1, there exists $\lambda > 0$ such that $UU^* \leq \lambda^2 KK^*$. Thus, for each $f \in H$ we have

$$\|U^*f\|^2 = \langle UU^*f, f\rangle \leq \lambda^2 \langle KK^*f, f\rangle = \lambda^2 \|K^*f\|^2.$$

It follows that

$$\frac{A}{\lambda^2}\|U^*f\|^2 = A\|K^*f\|^2 \leq \int_\Omega v^2(\omega)\|\Lambda_\omega \pi_{F(\omega)} f\|^2\, d\mu.$$

□

Theorem 2.4. *Let $K \in B(H)$ be closed range, (Λ, F, v) be a c-K-g-fusion frame for H with bounds A , B and $U \in B(H)$ with $\mathcal{R}(U^*) \subseteq \mathcal{R}(K)$. Then $(\Lambda_\omega \pi_{F(\omega)} U^*, \overline{UF(\omega)}, v)$ is a c-K-g-fusion frame for H if and only if there exists a constant $\delta > 0$ such that for every $f \in H$,*

$$\|U^*f\| \geq \delta \|K^*f\|.$$

Proof. Let $f \in H$, $U \in B(H)$ and $(\Lambda_\omega \pi_{F(\omega)} U^*, \overline{UF(\omega)}, v)$ be a c-K-g-fusion frame for H with the lower bound C . So, by Lemma 1.2, we obtain

$$C\|K^*f\|^2 \leq \int_\Omega v^2(\omega)\|\Lambda_\omega \pi_{F(\omega)} U^* \pi_{\overline{UF(\omega)}} f\|^2\, d\mu = \int_\Omega v^2(\omega)\|\Lambda_\omega \pi_{F(\omega)} U^* f\|^2\, d\mu.$$

On the other hand, we have

$$\int_\Omega v^2(\omega)\|\Lambda_\omega \pi_{F(\omega)} U^* f\|^2\, d\mu \leq B\|U^*f\|^2,$$

therefore, $\sqrt{\frac{C}{B}}\|K^*f\| \leq \|U^*f\|$.
For the opposite implication, we can write for each $f \in H$,

$$\|U^*f\| = \|(K^\dagger)^* K^* U^* f\| \leq \|K^\dagger\|.\|K^*U^*f\|.$$

Thus,

$$\begin{aligned}
A\delta^2\|K^\dagger\|^{-2}\|K^*f\|^2 &\leq A\|K^\dagger\|^{-2}\|U^*f\|^2 \\
&\leq A\|K^*U^*f\|^2 \\
&\leq \int_\Omega v^2(\omega)\|\Lambda_\omega \pi_{F(\omega)} U^* f\|^2\, d\mu \\
&= \int_\Omega v^2(\omega)\|\Lambda_\omega \pi_{F(\omega)} U^* \pi_{\overline{UF(\omega)}} f\|^2\, d\mu \\
&\leq B\|U\|^2\|f\|^2.
\end{aligned}$$

So, $(\Lambda_\omega \pi_{F(\omega)} U^*, \overline{UF((\omega)}, v)$ is a c-K-g-fusion frame for H. □

3. Duality of Continuous $K - g$Fusion Frames

In this section, we present some descriptions for duality of c-K-g-fusion frames. Then, we try to characterize and identity duals of c-K-g-fusion frames.

Definition 3.1. *Let (Λ, F, v) be a c-K-g-fusion frame for H. A c-g-fusion Bessel sequence $(\widetilde{\Lambda}, \widetilde{F}, \widetilde{v})$ is called Q-dual c-K-g-fusion frame (or cQKg-dual) for (Λ, F, v) if there exists a bounded linear operator $Q : L^2(\Omega, H_0) \to L^2(\Omega, H_0)$ such that*

$$T_\Lambda Q^* T^*_{\widetilde{\Lambda}} = K. \tag{9}$$

The following theorem presents equivalent conditions of the above definition:

Proposition 3.1. *Let $(\widetilde{\Lambda}, \widetilde{F}, \widetilde{v})$ be a cQKg-dual for (Λ, F, v). The following conditions are equivalent:*

(1) $T_\Lambda Q^* T^*_{\widetilde{\Lambda}} = K$;
(2) $T_{\widetilde{\Lambda}} Q T^*_\Lambda = K^*$;
(3) *for each $f, f' \in H$, we have*

$$\langle Kf, f' \rangle = \langle T_\Lambda Q^* T^*_{\widetilde{\Lambda}}(f), f' \rangle = \langle T^*_{\widetilde{\Lambda}}(f), QT^*_\Lambda(f') \rangle = \langle Q^* T^*_{\widetilde{\Lambda}}(f), T^*_\Lambda(f') \rangle.$$

Proof. By an easy calculation, the proof is clear. □

Proposition 3.2. *If $(\widetilde{\Lambda}, \widetilde{F}, \widetilde{v})$ is a cQKg-dual for c-K-g-fusion frame (Λ, F, v). Then $(\widetilde{\Lambda}, \widetilde{F}, \widetilde{v})$ is a c-K^*-g-fusion frame for H.*

Proof. We can write,

$$\begin{aligned}
\|Kh\|^4 &= |\langle Kh, Kh \rangle|^2 \\
&= |\langle T_\Lambda Q^* T^*_{\widetilde{\Lambda}}(h), Kh \rangle|^2 \\
&= |\langle T^*_{\widetilde{\Lambda}}(h), QT^*_\Lambda(Kh) \rangle|^2 \\
&\leq \|T^*_{\widetilde{\Lambda}}(h)\|^2 \|Q\|^2 B \|Kh\|^2 \\
&= \|Q\|^2 B \|Kh\|^2 \int_\Omega \widetilde{v}^2(\omega) \|\widetilde{\Lambda}_\omega \pi_{\widetilde{F}(\omega)} h\|^2 \, d\mu.
\end{aligned}$$

for every $f \in H$, where B is an upper bound of (Λ, F, v). Therefore by definition, this completes the proof. □

Suppose that (Λ, F, v) is a c-K-g-fusion frame for H. Since $S \geq AKK^*$, then by Lemma 1.1, there exists an operator $V \in B(H, L^2(\Omega, H_0))$ such that

$$T_\Lambda V = K. \tag{10}$$

By this operator, we can construct some $cQKg$-fusion duals for (Λ, F, v).

Theorem 3.3. *Let (Λ, F, v) be a c-K-g-fusion frame for H. If V be an operator as in (10) and $(\widetilde{\Lambda}, \widetilde{F}, \widetilde{v})$ is a c-g-fusion frame where $\widetilde{\Lambda} = \Lambda V^* V$ and $\widetilde{F} = V^* V F$. Then $(\widetilde{\Lambda}, \widetilde{F}, \widetilde{v})$ is a cQKg-dual for (Λ, F, v).*

Proof. Define the mapping

$$\begin{aligned}
\varphi : \mathcal{R}(T^*_{\widetilde{\Lambda}}) &\to \mathscr{L}^2(\Omega, H_0), \\
\varphi(T^*_{\widetilde{\Lambda}} f) &= Vf.
\end{aligned}$$

Since $\widetilde{\Lambda}$ is a c-g-fusion frame, so it is clear that φ is well-defined, bounded and linear. Therefore, it has a unique linear extension to $\overline{\mathcal{R}(T^*_{\widetilde{\Lambda}})}$. Define ψ on $L^2(\Omega, H_0)$ by setting

$$\psi = \begin{cases} \varphi, & \text{on } \overline{\mathcal{R}(T^*_{\widetilde{\Lambda}})}, \\ 0, & \text{on } \overline{\mathcal{R}(T^*_{\widetilde{\Lambda}})}^\perp \end{cases}$$

and let $Q = \psi^*$. This implies that $Q^* \in B(L^2(\Omega, H_0), L^2(\Omega, H_0))$ and

$$T_\Lambda Q^* T^*_{\widetilde{\Lambda}} = T_\Lambda \psi T^*_{\widetilde{\Lambda}} = T_\Lambda V = K.$$

□

4. Perturbation of c-Kg-Fusion Frames

Perturbation of discrete frames and frames associated with measurable spaces (c-frame) have been discussed in [7] and [13], respectively. Stability and perturbation of K-g-frames and c-K-g-frames have been investigated in [3, 16]; also perturbations of K-fusion frames and gc-fusion frames have been discussed in [5, 11] . In this section, we introduce perturbation of c-K-g-fusion frames.

Definition 4.1. *Let $\Lambda = \{\Lambda_\omega \in B(F(\omega), H_\omega) : \omega \in \Omega\}$ and $\widetilde{\Lambda} = \{\widetilde{\Lambda}_\omega \in B(\widetilde{F}(\omega), H_\omega) : \omega \in \Omega\}$ where $F : \Omega \to \mathbb{H}$ and $\widetilde{F} : \Omega \to \mathbb{H}$ are weakly measurable and $\widetilde{v} : \Omega \to \mathbb{R}^+$ be measurable function. Let $0 < \lambda_1, \lambda_2 < 1$ and $\varepsilon > 0$. We say that $(\widetilde{\Lambda}, \widetilde{F}, \widetilde{v})$ is a $(\lambda_1, \lambda_2, \varepsilon)$-Perturbation of (Λ, F, v) if for each $h \in H$ and , $\omega \in \Omega$*

$$\|v(\omega)\Lambda_\omega \pi_{F(\omega)}(h) - \widetilde{v}(\omega)\widetilde{\Lambda}_\omega \pi_{\widetilde{F}(\omega)}(h)\| \le \lambda_1 \|v(\omega)\Lambda_\omega \pi_{F(\omega)}(h)\| + \lambda_2 \|\widetilde{v}(\omega)\widetilde{\Lambda}_\omega \pi_{\widetilde{F}(\omega)}(h)\| + \varepsilon v(\omega)\|K^* h\|.$$

Theorem 4.1. *Let (Λ, F, v) be a c-K-g-fusion frame for H with respect to $v \in L^2(\Omega)$ with bounds A and B. Choose $0 \le \lambda_1 < 1$ and $\varepsilon > 0$ such that*

$$0 < (1 - \lambda_1)\sqrt{A} - \varepsilon \|K\| \Big(\int_\Omega v^2(\omega)\, d\mu \Big)^{\frac{1}{2}}. \tag{11}$$

Furthermore, if $(\widetilde{\Lambda}, \widetilde{F}, \widetilde{v})$ is a $(\lambda_1, \lambda_2, \varepsilon)$-Perturbation of (Λ, F, v), then $(\widetilde{\Lambda}, \widetilde{F}, \widetilde{v})$ is a c-K-g-fusion frame for H with respect to $\widetilde{v}$ with bounds

$$\Big(\frac{(1 + \lambda_1)\sqrt{B} + (1 - \lambda_1)\sqrt{A}}{1 - \lambda_2} \Big)^2$$

and

$$\Big(\frac{\sqrt{A}(1 - \lambda_1)(\|K\| - 1)}{\|K\|(1 + \lambda_2)} \Big)^2.$$

Proof. We first verify the upper frame bound condition. For each $h \in H$ and $\omega \in \Omega$, we get

$$\begin{aligned}
&\Big(\int_\Omega \|\widetilde{v}^2(\omega)\widetilde{\Lambda}_\omega \pi_{\widetilde{F}(\omega)}(h)\|^2 \, d\mu \Big)^{\frac{1}{2}} \\
&= \Big(\int_\Omega \|\widetilde{v}(\omega)\widetilde{\Lambda}_\omega \pi_{\widetilde{F}(\omega)}(h) - v(\omega)\Lambda_\omega \pi_{F(\omega)}(h) + v(\omega)\Lambda_\omega \pi_{F(\omega)}(h)\|^2 \, d\mu \Big)^{\frac{1}{2}} \\
&\le \Big(\int_\Omega \Big\{ (1 + \lambda_1)\|v(\omega)\Lambda_\omega \pi_{F(\omega)}(h)\| + \lambda_2 \|\widetilde{v}(\omega)\widetilde{\Lambda}_\omega \pi_{\widetilde{F}(\omega)}(h)\| + \varepsilon v(\omega)\|K^* h\| \Big\}^2 \, d\mu \Big)^{\frac{1}{2}} \\
&\le (1 + \lambda_1)\Big(\int_\Omega \|v(\omega)\Lambda_\omega \pi_{F(\omega)}(h)\|^2 \, d\mu \Big)^{\frac{1}{2}} + \lambda_2 \Big(\int_\Omega \|\widetilde{v}(\omega)\widetilde{\Lambda}_\omega \pi_{\widetilde{F}(\omega)}(h)\|^2 \, d\mu \Big)^{\frac{1}{2}} \\
&+ \varepsilon \|K^* h\| \Big(\int_\Omega v^2(\omega)\, d\mu \Big)^{\frac{1}{2}}.
\end{aligned}$$

By (11), we have

$$\int_\Omega \widetilde{v}^2(\omega)\|\widetilde{\Lambda}_\omega \pi_{\widetilde{F}(\omega)}(h)\|^2 \, d\mu \le \Big(\frac{(1 + \lambda_1)\sqrt{B} + (1 - \lambda_1)\sqrt{A}}{1 - \lambda_2} \Big)^2 \|h\|^2.$$

Therefore, $(\widetilde{\Lambda}, \widetilde{F}, \widetilde{v})$ is a c-g-fusion Bessel for H with respect to $\widetilde{v}$. Now, we show that $(\widetilde{\Lambda}, \widetilde{F}, \widetilde{v})$ has the lower c-K-g-fusion frame condition. For each $h \in H$, we have

$$
\begin{aligned}
&\Big(\int_\Omega \|\widetilde{v}(\omega)\widetilde{\Lambda}_\omega \pi_{\widetilde{F}(\omega)}(h)\|^2\,d\mu\Big)^{\frac{1}{2}} \\
&= \Big(\int_\Omega \|\widetilde{v}(\omega)\widetilde{\Lambda}_\omega \pi_{\widetilde{F}(\omega)}(h) - v(\omega)\Lambda_\omega\pi_{F(\omega)}(h) + v(\omega)\Lambda_\omega\pi_{F(\omega)}(h)\|^2\,d\mu\Big)^{\frac{1}{2}} \\
&\geq \Big(\int_\Omega \Big\{(1-\lambda_1)\|v(\omega)\Lambda_\omega\pi_{F(\omega)}(h)\| - \lambda_2\|\widetilde{v}(\omega)\widetilde{\Lambda}_\omega\pi_{\widetilde{F}(\omega)}(h)\| - \varepsilon v(\omega)\|K^*h\|\Big\}^2\,d\mu\Big)^{\frac{1}{2}} \\
&\geq (1-\lambda_1)\Big(\int_\Omega \|v(\omega)\Lambda_\omega\pi_{F(\omega)}(h)\|^2\,d\mu\Big)^{\frac{1}{2}} - \lambda_2\Big(\int_\Omega \|\widetilde{v}(\omega)\widetilde{\Lambda}_\omega\pi_{\widetilde{F}(\omega)}(h)\|^2\,d\mu\Big)^{\frac{1}{2}} \\
&- \varepsilon\|K^*h\|\Big(\int_\Omega v^2(\omega)\,d\mu\Big)^{\frac{1}{2}}.
\end{aligned}
$$

Thus

$$
\int_\Omega \widetilde{v}^2(\omega)\|\widetilde{\Lambda}_\omega\pi_{\widetilde{F}(\omega)}(h)\|^2\,d\mu \geq \Big(\frac{\sqrt{A}(1-\lambda_1)(\|K\|-1)}{\|K\|(1+\lambda_2)}\Big)^2\|K^*h\|^2,
$$

and the proof is complete. □

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