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COMMON FIXED POINT RESULTS IN PARTIAL SYMMETRIC SPACES

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Abstract. In this paper, we prove some common fixed point results for weakly compatible mappings employing (E.A)-property and (CLR_g)-property in partial symmetric spaces. Our results extend and improve several existing results in literature. We also give some examples which exhibit the utility of our results.

Keywords: Partial symmetric; common fixed point; (E.A.)-property; (CLR_g)-property.

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1. Introduction

The classical Banach contraction principle due to S. Banach [1] continues to be inspiring results for the researchers of metric fixed point theory. This principle ensures the existence and uniqueness of fixed point of a contraction map in the setting of complete metric spaces. In the last several decades, Banach contraction principle has been extended and generalized in many different directions by several authors. One direction of proving such generalized results is to vary spaces (see [2–8]) and several others.

Often researchers of the metric fixed point theory reduce the force of the metric conditions and observed that in the absence of some conditions, several metric fixed point results can still survive. Inspired by this fact, several researchers established fixed point and common fixed point results in semi-metric space [9] (or symmetric space). A symmetric d on a non-empty set X is a function $d : X \times X \to \mathbb{R}_+$ which satisfies $d(\varsigma, \sigma) = d(\sigma, \varsigma)$ and $d(\varsigma, \sigma) = 0$ if and only if $\varsigma = \sigma$ for all $\varsigma, \sigma \in X$. On the other hand, Matthews [2], introduced the notion of partial metric spaces as a part of the study of denotational semantics of dataflow networks and proved an analogue of Banach contraction theorem besides proving Kannan-Ćirić and Ćirić quasi type fixed point results. By combining these two ideas, Asim et al. [10] introduced yet another notion and called it partial symmetric spaces and proved some related fixed point results for single valued and multivalued mappings.

In 1986, Jungck [11] generalized the idea commuting pair of mappings by introducing the notion of compatible mappings. In 1998, Jungck and Rhoades [12] introduced the

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concept of weakly compatibility maps. In the recent past, Imdad et al. [13, 14] almost simultaneously gave some fixed point results by using weakly compatible maps.

Inspired by foregoing observations, we prove some existence and uniqueness results on common fixed point in partial symmetric space. Finally, we also adopt some examples which exhibit the utility of our results.

2. Preliminaries

In what follows, we collect some relevant definitions and examples which are needed in our subsequent discussions.

First, we recall the definition of partial metric spaces

Definition 2.1. *[15] Let X be a non-empty set. A mapping $p : X \times X \to \mathbb{R}^+$ is said to be a partial metric on X if*

(1) $\varsigma = \sigma \Leftrightarrow p(\varsigma, \varsigma) = p(\varsigma, \sigma) = p(\sigma, \sigma)$;
(2) $p(\varsigma, \varsigma) \leq p(\varsigma, \sigma)$;
(3) $p(\varsigma, \sigma) = p(\sigma, \varsigma)$;
(4) $p(\varsigma, \rho) \leq p(\varsigma, \sigma) + p(\sigma, \rho) - p(\sigma, \sigma)$.

The pair (X, p) is said to be a partial metric space.

Obviously, if $p(\varsigma, \varsigma) = 0$ for all $\varsigma \in X$, then (X, p) is a metric space.

Next, recall the definition of partial symmetric space as follows:

Definition 2.2. *[10] Let X be a non-empty set. A mapping $\mathcal{P} : X \times X \to \mathbb{R}_+$ is said to be a partial symmetric if*

(1) $\varsigma = \sigma \Leftrightarrow p(\varsigma, \varsigma) = p(\varsigma, \sigma) = p(\sigma, \sigma)$;
(2) $p(\varsigma, \varsigma) \leq p(\varsigma, \sigma)$;
(3) $p(\varsigma, \sigma) = p(\sigma, \varsigma)$.

Then the pair $(X, \mathcal{P})$ is said to be a partial symmetric space.

In partial symmetric space $(X, \mathcal{P})$ if for all $\varsigma \in X$, $\mathcal{P}(\varsigma, \varsigma) = 0$, then $(X, \mathcal{P})$ is symmetric space. It is clear that every symmetric space is a partial symmetric space. However, the converse of this fact is not true in general.

Example 2.1. *[10] Suppose $X = [0, \pi)$ and a mapping $\mathcal{P} : X \times X \to \mathbb{R}_+$ is define by:*

$$\mathcal{P}(\varsigma, \sigma) = \sin|\varsigma - \sigma| + \alpha \quad \text{for any } \alpha > 0.$$

Then the pair $(X, \mathcal{P})$ is a partial symmetric space.

Example 2.2. *[10] Suppose $X = \mathbb{R}_+$ and a mapping $\mathcal{P} : X \times X \to \mathbb{R}_+$ is define by:*

$$\mathcal{P}(\varsigma, \sigma) = (\max\{\varsigma, \sigma\})^p + (\max\{\varsigma, \sigma\})^q \quad \text{for any } p, q > 1.$$

Then the pair $(X, \mathcal{P})$ is a partial symmetric space.

Example 2.3. *Suppose $X = \mathbb{R}_+$ and a mapping $\mathcal{P} : X \times X \to \mathbb{R}_+$ is define by:*

$$\mathcal{P}(\varsigma, \sigma) = (\max\{\varsigma, \sigma\})^p + |\varsigma - \sigma|^q \quad \text{for any } p, q > 1.$$

Then the pair $(X, \mathcal{P})$ is a partial symmetric space.

Example 2.4. *Suppose $X = [0, \pi)$ and a mapping $\mathcal{P} : X \times X \to \mathbb{R}_+$ is define by:*

$$\mathcal{P}(\varsigma, \sigma) = (\max\{\varsigma, \sigma\})^p + e^{|\varsigma - \sigma|} \quad \text{for any } p > 1$$

Then the pair $(X, \mathcal{P})$ is a partial symmetric space.

Observe that, in above examples (that is, Examples 2.1-2.4) $\mathcal{P}$ is neither partial metric nor symmetric but partial symmetric and $(X, \mathcal{P})$ is partial symmetric space.

Definition 2.3. *[10] Let $(X, \mathcal{P})$ be a partial symmetric space. A sequence $\{\varsigma_n\}$ in $(X, \mathcal{P})$ $\mathcal{P}$-converges to $\varsigma \in X$ if $\lim_{n\to\infty} \mathcal{P}(\varsigma_n, \varsigma) = \mathcal{P}(\varsigma, \varsigma)$.*

Definition 2.4. *[10] Let $(X, \mathcal{P})$ be a partial symmetric space. We say that*

(A1) $\lim_{n\to\infty} \mathcal{P}(\varsigma_n, \varsigma) = \mathcal{P}(\varsigma, \varsigma)$ and $\lim_{n\to\infty} \mathcal{P}(\varsigma_n, \sigma) = \mathcal{P}(\varsigma, \sigma)$ imply $\varsigma = \sigma$, for sequence $\{\varsigma_n\}$, ς and σ in X,

(A2) $\lim_{n\to\infty} \mathcal{P}(\varsigma_n, \varsigma) = \mathcal{P}(\varsigma, \varsigma)$ and $\lim_{n\to\infty} \mathcal{P}(\varsigma_n, \sigma_n) = \mathcal{P}(\varsigma, \varsigma)$ imply $\lim_{n\to\infty} \mathcal{P}(\sigma_n, \varsigma) = \mathcal{P}(\varsigma, \varsigma)$, for sequences $\{\varsigma_n\}$, $\{\sigma_n\}$ and ς in X.

Definition 2.5. *[16] Let f_1 and f_2 be two self-mappings defined on a partial symmetric space $(X, \mathcal{P})$. The maps f_1 and f_2 are said to satisfy (E.A)-property if there exists sequence $\{\varsigma_n\}$ in X such that $\{f_1\varsigma_n\}$ and $\{f_2\varsigma_n\}$ are convergent to some $\tau \in X$ and $\mathcal{P}(\tau, \tau) = 0$.*

Definition 2.6. *Let f_1, f_2, g_1 and g_2 be two self-mappings defined on a partial symmetric space $(X, \mathcal{P})$. The maps f_1, f_2, g_1 and g_2 are said to satisfy (E.A) common property if there exist two sequences $\{\varsigma_n\}$ and $\{\sigma_n\}$ in X such that*

$$\lim_{n\to\infty} f_1\varsigma_n = \lim_{n\to\infty} g_1\varsigma_n = \lim_{n\to\infty} f_2\sigma_n = \lim_{n\to\infty} g_2\sigma_n = \tau, \text{ for some } \tau \in X$$

whenever $\lim_{n\to\infty} f_1\varsigma_n = \lim_{n\to\infty} g_1\varsigma_n = \tau$ and $\lim_{n\to\infty} f_2\sigma_n = \lim_{n\to\infty} g_2\sigma_n = \tau$, for some $\tau \in X$.

Now, we recall the following definition inspired by the definitions introduced in [11,12, 17,18]

Definition 2.7. *Let f_1, f_2, g_1 and g_2 be two self-mappings of a partial symmetric space $(X, \mathcal{P})$. We say that*

- *An element $t \in X$ is said a coincidence point of f_1 and f_2 if*
$$f_1t = f_2t.$$
- *An element $t \in X$ is said a coincidence point of f_1, f_2, g_1 and g_2 if*
$$f_1t = f_2t = g_1t = g_2t.$$
- *An element $\bar{t} \in X$ is said a point of coincidence point of f_1 and f_2 if $f_1t = f_2t = \bar{t}$, where $t \in X$ is coincidence point of f_1 and f_2.*
- *An element $\bar{t} \in X$ is said a point of coincidence point of f_1, f_2, g_1 and g_2 if $f_1t = f_2t = g_1t = g_2t = \bar{t}$, where $t \in X$ is coincidence point of f_1 and f_2.*
- *A coincidence point $t \in X$ of f_1 and f_2 is said a common fixed point of f_1 and f_2 if $f_1t = f_2t = t$.*
- *A coincidence point $t \in X$ of f_1, f_2, g_1 and g_2 is said a common fixed point of f_1, f_2, g_1 and g_2 if $f_1t = f_2t = g_1t = g_2t = t$.*
- *The maps f_1 and f_2 are said to be commuting if $f_1f_2t = f_2f_1t$.*
- *The maps f_1 and f_2 are said to be compatible if*
$$\lim_{n\to\infty} \mathcal{P}(f_1f_2\varsigma_n, f_2f_1\varsigma_n) = \mathcal{P}(t, t),$$
whenever $\{\varsigma_n\}$ is sequence in X such that $\lim_{n\to\infty} \mathcal{P}(f_1\varsigma_n, t) = \lim_{n\to\infty} \mathcal{P}(f_2\varsigma_n, t) = \mathcal{P}(t, t)$, for some $t \in X$.
- *The maps f_1 and f_2 are said to be weakly compatible if they commute at their coincidence points, that is, if $f_1t = f_2t$ for some $t \in X$, then $f_1f_2t = f_2f_1t$.*

Recently, Asim et al. [10] proved the following:

Theorem 2.1. *[10] Let $(X,\mathcal{P})$ be complete partial symmetric space and $f : X \to X$. Suppose the following conditions are satisfied:*

(i) for some $\kappa \in [0,1)$,

$$\mathcal{P}(f\varsigma, f\sigma) \leq \kappa \mathcal{P}(\varsigma,\sigma), \ \forall\ \varsigma,\sigma \in X,$$

(ii) there exists $\varsigma_0 \in X$ such that

$$\sup\{\mathcal{P}(f^i\varsigma_0, f^j\varsigma_0) : \ \forall\ i,j \in \mathbb{N}\} < \infty,$$

(iii) either

(a) f is continuous or

(b) $(X,\mathcal{P})$ enjoys the $(A1)$ property.

Then f has a unique fixed point $\varsigma \in X$ such that $\mathcal{P}(\varsigma,\varsigma) = 0$.

Theorem 2.2. *[10] Let $(X,\mathcal{P})$ be complete partial symmetric space and $f : X \to X$. Suppose the following conditions are satisfied:*

(i) for some $\kappa \in [0,1)$,

$$\mathcal{P}(f\varsigma, f\sigma) \leq \kappa \max\{\mathcal{P}(\varsigma, f\varsigma), \mathcal{P}(\sigma, f\sigma)\},$$

(ii) f is continuous.

Then f has a unique fixed point $\varsigma \in X$ such that $\mathcal{P}(\varsigma,\varsigma) = 0$.

Theorem 2.3. *[10] Let $(X,\mathcal{P})$ be complete partial symmetric space and $f : X \to X$. Suppose the following conditions are satisfied:*

(i) for some $\kappa \in [0,1)$,

$$\mathcal{P}(f\varsigma, f\sigma) \leq \kappa \max\big\{\mathcal{P}(\varsigma,\sigma), \mathcal{P}(\varsigma, f\varsigma), \mathcal{P}(\sigma, f\sigma), \mathcal{P}(\varsigma, f\sigma), \mathcal{P}(\sigma, f\varsigma)\big\},$$

(ii) there exists $\varsigma_0 \in X$ such that

$$\sup\{\mathcal{P}(f^i\varsigma_0, f^j\varsigma_0) : \ \forall\ i,j \in \mathbb{N}\} < \infty,$$

(iii) f is continuous.

Then f has a unique fixed point $\varsigma \in X$ such that $\mathcal{P}(\varsigma,\varsigma) = 0$.

3. Results

In this section, we present some fixed point results for Ciric quasi contraction in the setting of partial symmetric spaces. To accomplish this we present some relevant definition and auxiliary results:

Definition 3.1. *Let $(X,\mathcal{P})$ partial symmetric space and $f_1, f_2, g_1, g_2 : X \to X$. Then f_1, f_2, g_1, g_2 are said to be Quasi-weak contraction if for all $\varsigma,\sigma \in X$ and $\kappa \in (0,1)$*

$$\mathcal{P}(f_1\varsigma, f_2\sigma) \leq \kappa \max\{\mathcal{P}(g_1\varsigma, g_2\sigma), \mathcal{P}(g_1\varsigma, f_1\varsigma), \mathcal{P}(g_2\sigma, f_2\sigma), \mathcal{P}(g_1\varsigma, f_2\sigma), \mathcal{P}(g_2\sigma, f_1\varsigma)\}. \quad (1)$$

Now, we state and prove the following lemma inspired by Lemma 3.1 by Imdad and Javid [19], which is needed in our main results:

Lemma 3.1. *Let $(X,\mathcal{P})$ be a partial symmetric space and $f_1, f_2, g_1, g_2 : X \to X$ such that:*

(Li) $\{f_1, g_1\}$ (or $\{f_2, g_2\}$) satisfies (E.A.)-property,

(Lii) $f_1(X) \subset g_2(X)$ (or $f_2(X) \subset g_1(X)$),

(Liii) If for every sequence $\{\sigma_n\}$ in X $g_1\sigma_n$ converges then $f_1\sigma_n$ converges (or if for every sequence $\{\sigma_n\}$ in X $g_2\sigma_n$ converges then $f_2\sigma_n$ converges),

(Liv) f_1, f_2, g_1, g_2 satisfy condition (1).

Then $\{f_1, g_1\}$ and $\{f_2, g_2\}$ satisfy (E.A.) common property.

Proof. Assume that, $\{f_1, g_1\}$ enjoys (E.A.)-property, then there exists $\{\varsigma_n\}$ in X such that

$$\lim_{n\to\infty} f_1\varsigma_n = \lim_{n\to\infty} g_1\varsigma_n = \tau, \text{ for some } \tau \in X.$$

By condition (Lii), we have $f_1(X) \subset g_2(X)$, then for each $\{\varsigma_n\}$ in X there exists $\{\sigma_n\}$ in X such that $f_1\varsigma_n = g_2\sigma_n$ which implies $\lim_{n\to\infty} f_1\varsigma_n = \lim_{n\to\infty} g_2\sigma_n = \tau$. Therefore, $f_1\varsigma_n \to \tau$, $g_1\varsigma_n \to \tau$ and $g_2\sigma_n \to \tau$. Now, we show that $f_2\sigma_n \to \tau$. Let on contrary that $f_2\sigma_n \nrightarrow \tau$. Then by using (1), we have

$$\begin{aligned}\mathcal{P}(f_1\varsigma_n, f_2\sigma_n) &\le \kappa \max\{\mathcal{P}(g_1\varsigma_n, g_2\sigma_n), \mathcal{P}(g_1\varsigma_n, f_1\varsigma_n), \mathcal{P}(g_2\sigma_n, f_2\sigma_n),\\ &\quad \mathcal{P}(g_1\varsigma_n, f_2\sigma_n), \mathcal{P}(g_2\sigma_n, f_1\varsigma_n)\}\end{aligned}$$

on making $n \to \infty$, we have

$$\begin{aligned}\mathcal{P}(\tau, f_2\sigma_n) &\le \kappa \max\{\mathcal{P}(\tau,\tau), \mathcal{P}(\tau,\tau), \mathcal{P}(\tau, f_2\sigma_n), \mathcal{P}(\tau, f_2\sigma_n), \mathcal{P}(\tau,\tau)\},\\ &\le \kappa\mathcal{P}(\tau, f_2\sigma_n)\end{aligned}$$

a contradiction. Hence, $f_2\sigma_n \to \tau$ which shows that $\{f_1, g_1\}$ and $\{f_2, g_2\}$ satisfies (E.A.) common property. ☐

Now, we state and prove our main result as follows:

Theorem 3.1. *Let $(X, \mathcal{P})$ be a partial symmetric space and $f_1, f_2, g_1, g_2 : X \to X$ are such that*

(i) the pairs $\{f_1, g_1\}$ and $\{f_2, g_2\}$ satisfy (E.A.) common property,
(ii) f_1, f_2, g_1 and g_2 satisfy the condition (1),
(iii) $g_1(X)$ and $g_2(X)$ are closed subsets of X.

Then, the pairs (f_1, g_1) and (f_2, g_2) has a coincidence point. Moreover, if the pairs $\{f_1, g_1\}$ and $\{f_2, g_2\}$ are weakly compatible, then f_1, f_2, g_1, g_2 have a unique common fixed point in X.

Proof. In view of (i), there exist two sequences $\{\varsigma_n\}$ and $\{\sigma_n\}$ in X, such that

$$\lim_{n\to\infty} f_1\varsigma_n = \lim_{n\to\infty} g_1\varsigma_n = \lim_{n\to\infty} f_2\sigma_n = \lim_{n\to\infty} g_2\sigma_n = \tau, \text{ for same } \tau \in X.$$

Since $g_1(X)$ is a closed subset of X, hence $\lim_{n\to\infty} g_1\varsigma_n = \tau \in X$. Therefore, there exists $\varsigma \in X$ such that $g_1\varsigma = \tau$. Now, we show that $f_1\varsigma = g_1\varsigma$. Let on contrary that $f_1\varsigma \neq g_1\varsigma$. Then by using (1), we have

$$\begin{aligned}\mathcal{P}(f_1\varsigma, f_2\sigma_n) &\le \kappa \max\{\mathcal{P}(g_1\varsigma, g_2\sigma_n), \mathcal{P}(g_1\varsigma, f_1\varsigma), \mathcal{P}(g_2\sigma_n, f_2\sigma_n),\\ &\quad \mathcal{P}(g_1\varsigma, f_2\sigma_n), \mathcal{P}(g_2\sigma_n, f_1\varsigma)\}\end{aligned}$$

by taking limit as $n \to \infty$, we have

$$\begin{aligned}\mathcal{P}(f_1\varsigma, \tau) &\le \kappa \max\{\mathcal{P}(g_1\varsigma, \tau), \mathcal{P}(g_1\varsigma, f_1\varsigma), \mathcal{P}(\tau,\tau), \mathcal{P}(g_1\varsigma, \tau), \mathcal{P}(\tau, f_1\varsigma)\}\\ &\le \kappa\mathcal{P}(f_1\varsigma, \tau),\end{aligned}$$

a contradiction. Hence $f_1\varsigma = \tau = g_1\varsigma$, so that ς is a coincidence point of the pair (f_1, g_1). Moreover, $g_2(X)$ is closed subset of X, so that $\lim_{n\to\infty} g_2\sigma_n = \tau \in g_2(X)$. Hence, $g_2\sigma = \tau$ for some $\sigma \in X$. Now, we show that $g_2\varsigma = f_2\varsigma$. Let on contrary that, $g_2\varsigma \neq f_2\varsigma$. Then by using (1), we have

$$\begin{aligned}\mathcal{P}(f_1\varsigma_n, f_2\sigma) &\le \kappa \max\{\mathcal{P}(g_1\varsigma_n, g_2\sigma), \mathcal{P}(f_1\varsigma_n, g_1\varsigma_n), \mathcal{P}(f_2\sigma, g_2\sigma),\\ &\quad \mathcal{P}(g_1\varsigma_n, f_2\sigma), \mathcal{P}(g_2\sigma, f_1\varsigma_n)\}\end{aligned}$$

by taking $n \to \infty$, we have

$$\begin{aligned}\mathcal{P}(\tau, f_2\sigma) &\leq \kappa \max\{\mathcal{P}(\tau, g_2\sigma), \mathcal{P}(\tau,\tau), \mathcal{P}(f_2\sigma, g_2\sigma), \mathcal{P}(\tau, f_2\sigma), \mathcal{P}(g_2\sigma, \tau)\}\\ &\leq \kappa\mathcal{P}(g_2\sigma, f_2\sigma),\end{aligned}$$

a contradiction. Hence, $f_2\sigma = g_2\sigma$ implies that the pair (f_2, g_2) has a coincidence point (namely: σ). Since the pairs (f_1, g_1) and (f_2, g_2) are weakly compatible, therefore

$$f_1\tau = f_1g_1\varsigma = g_1f_1\varsigma = g_1\tau \text{ and } f_2\tau = f_2g_2\sigma = g_2f_2\sigma = g_2\tau.$$

Suppose that, $\mathcal{P}(f_1\tau, \tau) > 0$, then by using (1), we have

$$\begin{aligned}\mathcal{P}(f_1\tau, \tau) &\leq \kappa \max\{\mathcal{P}(g_1\tau, \tau), \mathcal{P}(f_1t, g_1\tau), \mathcal{P}(\tau,\tau), \mathcal{P}(g_1\tau, \tau), \mathcal{P}(\tau, f_1\tau)\}\\ &\leq \kappa\mathcal{P}(f_1\tau, \tau),\end{aligned}$$

a contradiction. Hence, $f_1\tau = \tau = g_1\tau$ implies that the pair (f_1, g_1) has a common fixed point (namely: τ). Similarly, one can easily show that τ is a common fixed point of the pair (f_2, g_2). For that reason, τ is a common fixed point of the mapping f_1, g_1, f_2 and g_2.

For uniqueness, we consider that w ($w \neq \tau$) be another common fixed point of the mapping f_1, g_1, f_2 and g_2. Then condition (1) gives arise

$$\begin{aligned}\mathcal{P}(\tau, w) &= \mathcal{P}(f_1\tau, f_2w)\\ &\leq \kappa \max\{\mathcal{P}(g_1\tau, g_2w), \mathcal{P}(g_1\tau, f_1\tau), \mathcal{P}(g_2w, f_2w), \mathcal{P}(g_1\tau, f_2w), \mathcal{P}(g_2w, f_1\tau)\}\\ &= \kappa \max\{\mathcal{P}(t, w), \mathcal{P}(t,t), \mathcal{P}(w,w), \mathcal{P}(\tau, w), \mathcal{P}(w, \tau)\}\\ &= \kappa\mathcal{P}(\tau, w),\end{aligned}$$

which deals a contradiction. Thus, $\tau = w$ and the uniqueness follows. This completes the proof. □

Theorem 3.2. *Theorem 3.1 remains true if condition* (ii) *is replaced by the following:*
$(ii)'$ $\overline{f_1(X)} \subset g_2(X)$ *and* $\overline{f_2(X)} \subset g_1(X)$.

Proof. Since, the pairs $\{f_1, g_1\}$ and $\{f_2, g_2\}$ satisfy (E.A.) common property, so there exist two sequences $\{\varsigma_n\}$ and $\{\sigma_n\}$ in X, such that

$$\lim_{n\to\infty} f_1\varsigma_n = \lim_{n\to\infty} g_1\varsigma_n = \lim_{n\to\infty} f_2\sigma_n = \lim_{n\to\infty} g_2\sigma_n = \tau, \text{ for same } \tau \in X.$$

Since, $\tau \in \overline{f_1(X)}$ and $\overline{f_1(X)} \subset g_2(X)$, so there exists $u \in X$ such that $g_2u = \tau$. By the proof of Theorem 3.1, one can show that the pair (f_2, g_2) has a coincidence point, namely u, i.e. $f_2u = g_2u$. Since, $\tau \in \overline{f_2(X)}$ and $\overline{f_2(X)} \subset g_1(X)$, then there exists $v \in X$ such that $g_1v = \tau$. Similarly, one can show that the pair (f_1, g_1) has a coincidence point, namely v, i.e. $f_1v = g_1u$. The rest of the proof runs along the lines of the proof of Theorem 3.1, hence it is omitted. □

We can have the following corollary of Theorem 3.2, which is also a variant of Theorem 3.1.

Corollary 3.1. *Theorems 3.1 and 3.2 remain true if conditions* (ii) *and* $(ii)'$ *are replaced by the following:*
$(ii)''$ $f_1(X)$ *and* $f_2(X)$ *are closed subsets of* X *provided* $f_1(X) \subset g_2(X)$ *and* $f_2(X) \subset g_1(X)$.

Theorem 3.3. *Let* $(X, \mathcal{P})$ *be a partial symmetric space and* $f_1, f_2, g_1, g_2 : X \to X$ *are such that*

(i) the pair $\{f_1, g_1\}$ *(or* $\{f_2, g_2\}$*) satisfy (E.A.)-property,*
(ii) f_1, f_2, g_1 *and* g_2 *satisfy the condition* (1),

(iii) $f_1(X) \subset g_2(X)$ *(or* $f_2(X) \subset g_1(X)$*)*
(iv) $g_1(X)$ *(or* $g_2(X)$*) is a closed subset of* X*.*

Then, the pairs (f_1, g_1) *and* (f_2, g_2) *have a coincidence point. Moreover, if the pairs* $\{f_1, g_1\}$ *and* $\{f_2, g_2\}$ *are weakly compatible, then* f_1, f_2, g_1, g_2 *have a unique common fixed point in* X*.*

Proof. In view of Lemma 3.1, the pairs $\{f_1, g_1\}$ and $\{f_2, g_2\}$ share the (E.A.) common property, i.e., there exist two sequences $\{\varsigma_n\}$ and $\{\sigma_n\}$ in X, such that

$$\lim_{n\to\infty} f_1\varsigma_n = \lim_{n\to\infty} g_1\varsigma_n = \lim_{n\to\infty} f_2\sigma_n = \lim_{n\to\infty} g_2\sigma_n = \tau, \text{ for same } \tau \in X.$$

If g_1 is closed subset of X, then by the proof of Theorem 3.1, the pair (f_1, g_1) has a coincidence point, namely u, i.e., $f_1 u = g_1 u$. Since, $f_1 u \in f_1(X)$ and $f_1(X) \subset g_2(X)$, then there exists $v \in X$ such that $f_1 u = g_2 v$. Now, we assert that $f_2 v = g_2 v$. Let on contrary that $f_2 v \neq g_2 v$, then by (1), we have

$$\begin{aligned}\mathcal{P}(f_1\varsigma_n, f_2 v) &\leq \kappa \max\{\mathcal{P}(g_1\varsigma_n, g_2 v), \mathcal{P}(g_1\varsigma_n, f_1\varsigma_n), \mathcal{P}(g_2 v, f_2 v),\\ &\quad \mathcal{P}(g_1\varsigma_n, f_2 v), \mathcal{P}(g_2 v, f_1\varsigma_n)\}\end{aligned}$$

on making limit as $n \to \infty$, we have

$$\begin{aligned}\mathcal{P}(\tau, f_2 v) &\leq \kappa \max\{\mathcal{P}(\tau, g_2 v), \mathcal{P}(\tau, \tau), \mathcal{P}(g_2 v, f_2 v), \mathcal{P}(\tau, f_2 v), \mathcal{P}(g_2 v, \tau)\}\\ \text{or } \mathcal{P}(g_2 v, f_2 v) &\leq \kappa \max\{\mathcal{P}(g_2 v, g_2 v), \mathcal{P}(g_2 v, g_2 v), \mathcal{P}(g_2 v, f_2 v),\\ &\quad \mathcal{P}(g_2 v, f_2 v), \mathcal{P}(g_2 v, g_2 v)\}\\ &\leq \kappa \mathcal{P}(g_2 v, f_2 v)\end{aligned}$$

a contradiction. Then $f_2 v = g_2 v$, hence v is a coincidence point of the pair (f_2, g_2). Rest of the proof is on the lines of the proof of Theorem 3.1, therefore details are omitted. ☐

Remark 3.1. *Theorem 3.3 is also a partial improvement of Theorem 3.1 besides relaxing the closedness of one of the subspaces.*

Now, we furnish the following example which illustrates Theorem 3.1

Example 3.1. *Consider* $X = [5, 25)$ *and partial symmetric space* $\mathcal{P} : X \times X \to [0, \infty]$ *defined by,* $\mathcal{P}(\varsigma, \sigma) = (\max\{\varsigma, \sigma\})^2$ *for all* $\varsigma, \sigma \in X$*. Define self-mappings* f_1, f_2, g_1 *and* g_2 *on* X *by:*

$$f_1\varsigma = \begin{cases} 5, & \varsigma \in \{5\} \cup (6, 25);\\ 6, & \varsigma \in (5, 6], \end{cases} \qquad f_2\varsigma = \begin{cases} 5, & \varsigma \in \{5\} \cup (6, 25);\\ 5.5, & \varsigma \in (5, 6], \end{cases}$$

$$g_1\varsigma = \begin{cases} 5, & \varsigma = 5;\\ 13, & \varsigma \in (5, 6];\\ \frac{\varsigma+94}{20}, & \varsigma \in (6, 25), \end{cases} \qquad g_2\varsigma = \begin{cases} 5, & \varsigma = 5;\\ 16, & \varsigma \in (5, 6];\\ \frac{\varsigma+119}{25}, & \varsigma \in (6, 25). \end{cases}$$

By choosing sequences, $\{\varsigma_n\} = \{6 + \frac{1}{2n}\}$, $\{\sigma_n\} = \{5\}$ *and* $\{\varsigma_n\} = \{5\}$ $\{\sigma_n\} = \{6 + \frac{1}{2n}\}$ *in* X*, we have*

$$\lim_{n\to\infty} \mathcal{P}(f_1\varsigma_n, 5) = \lim_{n\to\infty} \mathcal{P}(g_1\varsigma_n, 5) = \lim_{n\to\infty} \mathcal{P}(f_2\sigma_n, 5) = \lim_{n\to\infty} \mathcal{P}(g_2\sigma_n, 5) = \mathcal{P}(5, 5).$$

Therefore, both the pair (f_1, g_1) *and* (f_2, g_2) *enjoy the (E.A) common property. Also,* g_1 *and* g_2 *are closed subset of* X*. Thus, all the conditions of then Theorem 3.1 are satisfied and* $\varsigma = 5$ *is a unique common fixed point os the pairs* (f_1, g_1) *and* (f_2, g_2) *which also remains coincidence point as well. Also, all the involved mappings are even discontinuous at their unique common fixed point (namely* $\varsigma = 5$*). Moreover,*

$$f_1(X) = \{5, 6\} \not\subseteq [5, 5.95] \cup \{16\} = g_2(X)$$

and

$$f_2(X) = \{5, 6\} \not\subseteq [5, 5.76] \cup \{13\} = g_1(X).$$

The following example illustrates Theorems 3.2 and 3.3.

Example 3.2. *In setting of Example 3.1, replace the self-mappings f_1, f_2, g_1 and g_2 by:*

$$f_1\varsigma = \begin{cases} 5, & \varsigma \in \{5\} \cup (6, 25); \\ 6, & \varsigma \in (5, 6], \end{cases} \quad f_2\varsigma = \begin{cases} 5, & \varsigma \in \{5\} \cup (6, 25); \\ 7, & \varsigma \in (5, 6], \end{cases}$$

$$g_1\varsigma = \begin{cases} 5, & \varsigma = 5; \\ 17, & \varsigma \in (5, 6]; \\ \frac{\varsigma+4}{2}, & \varsigma \in (6, 25), \end{cases} \quad g_2\varsigma = \begin{cases} 5, & \varsigma = 5; \\ 2\varsigma, & \varsigma \in (5, 6]; \\ \frac{\varsigma+14}{4}, & \varsigma \in (6, 25). \end{cases}$$

It is easy to check that, $f_1(X) = \{5, 6\} \subset [5, 9.75] \cup (10, 12] = g_2(X)$ and $f_2(X) = \{5, 6\} \subset [7, 14.5] \cup \{17\} = g_1(X)$. Thus, all the conditions of then Theorems 3.2 and 3.3 are satisfied and $\varsigma = 5$ is a unique common fixed point os the pairs (f_1, g_1) and (f_2, g_2) which also remains coincidence point as well. Also, all the involved mappings are even discontinuous at their unique common fixed point (namely $\varsigma = 5$). As g_2 is not a closed subset of X so that Theorem 3.1 is not applicable in the context of the present example.

We further need the following definition in our next fixed point results via (CLR_g)-property in partial symmetric spaces:

Definition 3.2. *[20] Let f and g be two self-mappings defined on a partial symmetric space $(X, \mathcal{P})$. The maps f and g are said to satisfy (CLR_g)-property if there exists sequence $\{\varsigma_n\}$ in X such that*

$$\lim_{n\to\infty} f\varsigma_n = \lim_{n\to\infty} g\varsigma_n = g\tau, \text{ for some } \tau \in X.$$

Definition 3.3. *Let f_1, f_2, g_1 and g_2 be two self-mappings defined on a partial symmetric space $(X, \mathcal{P})$. The maps f_1, f_2, g_1 and g_2 are said to satisfy $(CLR_{g_1 g_2})$ common property if there exist two sequences $\{\varsigma_n\}$ and $\{\sigma_n\}$ in X such that*

$$\lim_{n\to\infty} f_1\varsigma_n = \lim_{n\to\infty} g_1\varsigma_n = \lim_{n\to\infty} f_2\varsigma_n = \lim_{n\to\infty} g_2\varsigma_n = \zeta, \text{ for same } \zeta \in g_1(X) \cap g_2(X)$$

whenever $\lim_{n\to\infty} f_1\varsigma_n = \lim_{n\to\infty} g_1\varsigma_n = g_1\tau = \zeta$, for some $\zeta \in g_1(X)$ and $\lim_{n\to\infty} f_2\sigma_n = \lim_{n\to\infty} g_2\sigma_n = \zeta$, for some $\zeta \in g_2(X)$.

Firstly, we give the following lemma which is needed in our subsequent discussions:

Lemma 3.2. *Let $(X, \mathcal{P})$ be a partial symmetric space and $f_1, f_2, g_1, g_2 : X \to X$ such that:*

(Li) $\{f_1, g_1\}$ *satisfies* (CLR_{g_1})*-property (or* $\{f_2, g_2\}$ *satisfies* (CLR_{g_2})*-property),*
(Lii) $f_1(X) \subset g_2(X)$ *(or* $f_2(X) \subset g_1(X)$*),*
(Liii) $g_1(X)$ *and* $g_2(X)$ *are closed subsets of* X*,*
(Liv) If for every sequence $\{\sigma_n\}$ *in* X $g_1\sigma_n$ *converges then* $f_1\sigma_n$ *converges (or if for every sequence* $\{\sigma_n\}$ *in* X $g_2\sigma_n$ *converges then* $f_2\sigma_n$ *converges),*
(Lv) f_1, f_2, g_1 *and* g_2 *satisfy the condition* (1)

Then $\{f_1, g_1\}$ *and* $\{f_2, g_2\}$ *satisfy* $(CLR_{g_1 g_2})$ *common property.*

Proof. Assume that, $\{f_1, g_1\}$ enjoys (CLR_{g_1})-property, then there exists $\{\varsigma_n\}$ in X such that

$$\lim_{n\to\infty} f_1\varsigma_n = \lim_{n\to\infty} g_1\varsigma_n = \zeta, \text{ for some } \zeta \in g_1(X).$$

By condition (Lii), we have $f_1(X) \subset g_2(X)$, then for each $\{\varsigma_n\}$ in X there exists $\{\sigma_n\}$ in X such that $f_1\varsigma_n = g_2\sigma_n$. Since, $g_1(X)$ is closed then $\lim_{n\to\infty} f_1\varsigma_n = \lim_{n\to\infty} g_2\sigma_n = \zeta$. Therefore, $\zeta \in g_2(X)$ so that $\zeta \in g_1(X) \cap g_2(X)$. Hence, we get $f_1\varsigma_n \to \zeta$, $g_1\varsigma_n \to \zeta$ and $g_2\sigma_n \to \zeta$. Now, we will show that $f_2\sigma_n \to \zeta$. Let on contrary that $f_2\sigma_n \to w \neq \zeta$. Then by using (1), we have

$$\begin{aligned} \mathcal{P}(f_1\varsigma_n, f_2\sigma_n) &\leq \kappa \max\{\mathcal{P}(g_1\varsigma_n, g_2\sigma_n), \mathcal{P}(g_1\varsigma_n, f_1\varsigma_n), \mathcal{P}(g_2\sigma_n, f_2\sigma_n), \\ &\quad \mathcal{P}(g_1\varsigma_n, f_2\sigma_n), \mathcal{P}(g_2\sigma_n, f_1\varsigma_n)\} \end{aligned}$$

on making $n \to \infty$, we have

$$\begin{aligned} \mathcal{P}(\zeta, w) &\leq \kappa \max\{\mathcal{P}(\zeta,\zeta), \mathcal{P}(\zeta,\zeta), \mathcal{P}(\zeta, w), \mathcal{P}(\zeta, w), \mathcal{P}(\zeta,\zeta)\}, \\ &\leq \kappa \mathcal{P}(\zeta, w) \end{aligned}$$

which deals a contradiction. Hence, $\zeta = w$, that is, $f_2\sigma_n \to \zeta$ which shows that $\{f_1, g_1\}$ and $\{f_2, g_2\}$ satisfy $(CLR_{g_1g_2})$ common property. □

By using Lemma 3.2, we show the existence of unique common fixed point in the following Theorem:

Theorem 3.4. *Let $(X, \mathcal{P})$ partial symmetric space and $f_1, f_2, g_1, g_2 : X \to X$ are such that*

(i) the pairs $\{f_1, g_1\}$ and $\{f_2, g_2\}$ satisfy $(CLR_{g_1g_2})$ common property,
(ii) f_1, f_2, g_1 and g_2 satisfy the condition (1).

Then, the pairs (f_1, g_1) and (f_2, g_2) has a coincidence point. Moreover, if the pairs $\{f_1, g_1\}$ and $\{f_2, g_2\}$ are weakly compatible, then f_1, f_2, g_1, g_2 have a unique common fixed point in X.

Proof. Since the pairs $\{f_1, g_1\}$ and $\{f_2, g_2\}$ satisfy $(CLR_{g_1g_2})$ common property, then there exist two sequences $\{\varsigma_n\}$ and $\{\sigma_n\}$ in X, such that

$$\lim_{n\to\infty} f_1\varsigma_n = \lim_{n\to\infty} g_1\varsigma_n = \lim_{n\to\infty} f_2\sigma_n = \lim_{n\to\infty} g_2\sigma_n = \zeta, \text{ for same } \zeta \in g_1(X) \cap g_2(X).$$

Since $f_1(X) \subset g_2(X)$, so there exists a point $\varsigma \in X$, such that $f_1\varsigma = \zeta$. Now, we show that $f_1\varsigma = g_1\varsigma = \zeta$ Let on contrary that $f_1\varsigma \neq g_1\varsigma$. Then by using (1), we have

$$\begin{aligned} \mathcal{P}(f_1\varsigma, f_2\sigma_n) &\leq \kappa \max\{\mathcal{P}(g_1\varsigma, g_2\sigma_n), \mathcal{P}(g_1\varsigma, f_1\varsigma), \mathcal{P}(g_2\sigma_n, f_2\sigma_n), \\ &\quad \mathcal{P}(g_1\varsigma, f_2\sigma_n), \mathcal{P}(g_2\sigma_n, f_1\varsigma)\} \end{aligned}$$

by taking limit as $n \to \infty$, we have

$$\begin{aligned} \mathcal{P}(f_1\varsigma, \zeta) &\leq \kappa \max\{\mathcal{P}(g_1\varsigma, \zeta), \mathcal{P}(g_1\varsigma, f_1\varsigma), \mathcal{P}(\zeta,\zeta), \mathcal{P}(g_1\varsigma, z), \mathcal{P}(\zeta, f_1\varsigma)\} \\ &\leq \kappa \mathcal{P}(f_1\varsigma, \zeta), \end{aligned}$$

a contradiction. Hence $f_1\varsigma = \zeta = g_1\varsigma$, so that ς is a point of a coincidence of the pair (f_1, g_1). As $\zeta \in g_2(X)$, then there exists a point $\sigma \in X$ such that $g_2\sigma = \zeta$. We claim that $f_2\varsigma = g_2\varsigma = \zeta$ Let on contrary that $f_2\varsigma \neq g_2\varsigma$. Then by using (1), we have

$$\begin{aligned} \mathcal{P}(f_1\varsigma_n, f_2\sigma) &\leq \kappa \max\{\mathcal{P}(g_1\varsigma_n, g_2\sigma), \mathcal{P}(f_1\varsigma_n, g_1\varsigma_n), \mathcal{P}(f_2\sigma, g_2\sigma), \\ &\quad \mathcal{P}(g_1\varsigma_n, f_2\sigma), \mathcal{P}(g_2\sigma, f_1\varsigma_n)\} \end{aligned}$$

by taking $n \to \infty$, we have

$$\begin{aligned} \mathcal{P}(\zeta, f_2\sigma) &\leq \kappa \max\{\mathcal{P}(\zeta, g_2\sigma), \mathcal{P}(\zeta,\zeta), \mathcal{P}(f_2\sigma, g_2\sigma), \mathcal{P}(\zeta, f_2\sigma), \mathcal{P}(g_2\sigma, \zeta)\} \\ &\leq \kappa \mathcal{P}(\zeta, f_2\sigma), \end{aligned}$$

a contradiction. Hence, $f_2\sigma = \zeta = g_2\sigma$ implies that σ is a coincidence point of the pair (f_2, g_2).

Since the pair (f_1, g_1) is weakly compatible. Therefore $f_1\zeta = f_1g_1\varsigma = g_1f_1\varsigma = g_1\zeta$. Suppose that, $f_1\zeta \neq \zeta$, then by using (1), we have

$$\begin{aligned}\mathcal{P}(f_1\zeta, \zeta) &\leq \kappa \max\{\mathcal{P}(g_1\zeta, \zeta), \mathcal{P}(f_1\zeta, g_1\zeta), \mathcal{P}(\zeta, \zeta), \mathcal{P}(g_1\zeta, \zeta), \mathcal{P}(\zeta, f_1\zeta)\}\\ &\leq \kappa\mathcal{P}(f_1\zeta, \zeta),\end{aligned}$$

which deals a contradiction. Hence, $f_1\zeta = \zeta = g_1\zeta$ implies that ζ is a common fixed point of the pair (f_1, g_1).

Similarly, since the pair (f_2, g_2) is weakly compatible. Therefore $f_2\zeta = f_2g_2\sigma = g_2f_2\sigma = g_2\zeta$. Suppose that, $f_2\zeta \neq \zeta$, then by using (1), we have

$$\begin{aligned}\mathcal{P}(\zeta, f_2\zeta) &\leq \kappa \max\{\mathcal{P}(\zeta, g_2\zeta), \mathcal{P}(\zeta, \zeta), \mathcal{P}(f_2\zeta, g_2\zeta), \mathcal{P}(\zeta, f_2\zeta), \mathcal{P}(g_2\zeta, \zeta)\}\\ &\leq \kappa\mathcal{P}(\zeta, f_2\zeta),\end{aligned}$$

which deals a contradiction. Hence, $f_2\zeta = \zeta = g_2\zeta$ implies that ζ is a common fixed point of the pair (f_2, g_2). For uniqueness, we consider that w $(w \neq \zeta)$ be another common fixed point of the mapping f_1, g_1, f_2 and g_2. Then condition (1) gives arise

$$\begin{aligned}\mathcal{P}(\zeta, w) &= \mathcal{P}(f_1\zeta, f_2w)\\ &\leq \kappa \max\{\mathcal{P}(g_1\zeta, g_2w), \mathcal{P}(g_1\zeta, f_1\zeta), \mathcal{P}(g_2w, f_2w), \mathcal{P}(g_1\zeta, f_2w), \mathcal{P}(g_2w, f_1\zeta)\}\\ &= \kappa \max\{\mathcal{P}(\zeta, w), \mathcal{P}(\zeta, \zeta), \mathcal{P}(w, w), \mathcal{P}(\zeta, w), \mathcal{P}(w, \zeta)\}\\ &= \kappa\mathcal{P}(\zeta, w),\end{aligned}$$

which deals a contradiction. Thus, $\zeta = w$ and ζ is unique common fixed point of the mapping f_1, g_1, f_2 and g_2. This completes the proof. □

By restricting the underlying contraction one can derived the following corollary. Which can be viewed as variants of several previously known results specially due to [19]:

Corollary 3.2. *The conclusions of Theorems 3.1, 3.2, 3.3 and 3.4 remain true if the contractive condition* (1) *is replaced by any one of the following:*

(i) $\mathcal{P}(f_1\varsigma, f_2\sigma) \leq \kappa \max\big\{\mathcal{P}(g_1\varsigma, g_2\sigma), \mathcal{P}(g_1\varsigma, f_1\varsigma)\big\}$;

(ii) $\mathcal{P}(f_1\varsigma, f_2\sigma) \leq \kappa \max\big\{\mathcal{P}(g_1\varsigma, g_2\sigma), \mathcal{P}(g_1\varsigma, f_1\varsigma), \mathcal{P}(g_2\sigma, f_2\sigma)\big\}$;

(iii) $\mathcal{P}(f_1\varsigma, f_2\sigma) \leq \kappa \max\Big\{\mathcal{P}(g_1\varsigma, g_2\sigma), \mathcal{P}(g_1\varsigma, f_1\varsigma), \mathcal{P}(g_2\sigma, f_2\sigma), \frac{\mathcal{P}(g_1\varsigma, f_2\sigma)+\mathcal{P}(g_2\sigma, f_1\varsigma)}{2}\Big\}$;

(iv) $\mathcal{P}(f_1\varsigma, f_2\sigma) \leq \kappa \max\Big\{\mathcal{P}(g_1\varsigma, g_2\sigma), \frac{\mathcal{P}(g_1\varsigma, f_1\varsigma)+\mathcal{P}(g_2\sigma, f_2\sigma)}{2}, \mathcal{P}(g_1\varsigma, f_2\sigma), \mathcal{P}(g_2\sigma, f_1\varsigma)\Big\}$;

(v) $\mathcal{P}(f_1\varsigma, f_2\sigma) \leq \kappa \max\Big\{\mathcal{P}(g_1\varsigma, g_2\sigma), \frac{\mathcal{P}(g_1\varsigma, f_1\varsigma)+\mathcal{P}(g_2\sigma, f_2\sigma)}{2}, \frac{\mathcal{P}(g_1\varsigma, f_2\sigma)+\mathcal{P}(g_2\sigma, f_1\varsigma)}{2}\Big\}$.

4. Conclusion

In this paper, we adopt the definitions of (E.A)-property, (E.A) common property, (CLR_g)-property and (CLR_g) common property for four self maps in our newly introduced partial symmetric spaces and utilize the same to prove some common fixed point results for pairs of weakly compatible mappings. Moreover, we adopt some examples to established the utility of our newly proved results.

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References

[1] Banach, S., (1922), Sur les operations dans les ensembles abstraits et leur application aux equations integrals, Fund. Math. 3, pp. 133-181.

[2] S. G. Matthews, S.G., (1994), Partial metric topology, Annals of the New York Academy of Sciences, 728, pp. 183-197.

[3] A. Amini-Harandi, Metric-like spaces, partial metric spaces and fixed points. *Fixed Point Theory Appl.*, 2012, Article ID 204, (2012).

[4] Czerwik, S., (1993), Contraction mappings in b-metric spaces, Acta Mathematica et Informatica Universitatis Ostraviensis, 1 (1), pp. 5-11.

[5] Branciari, A., (2000) A fixed point theorem of Banach–Caccioppoli type on a class of generalized metric spaces, Publ. Math., 57, pp. 31-37.

[6] Long-Guang, H. and Xian, Z., (2007), Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl., 332, pp. 1468–1476.

[7] Asim, M. Khan, A.R. and Imdad, M., (2019), Rectangular M_b-metric spaces and fixed point results, Journal of mathematical analysis, 10 (1), pp. 10–18.

[8] Asim, M. Imdad, M. and Radenovic, S., (2019), Fixed point results in extended rectangular b-metric spaces with an application, U.P.B. Sci. Bull., Series A, 81 (2), pp. 43–50.

[9] Wilson, W. A., (1931), On semi-metric spaces, American Journal of Mathematics, 53 (2), pp. 361–373.

[10] Asim, M. Khan, A.R. and Imdad, M., (2019), Fixed point results in partial symmetric spaces with an application, Axioms, 8 (13), doi:10.3390.

[11] Jungck, G., (1986), Compatible mappings and common fxed points. International Journal of Mathematics and Mathematical Sciences, 9 (4), pp. 771–779.

[12] Jungck, G. and Rhoades, B.E., (1998), Fixed points for set valued functions without continuity, Indian J. Pure. Appl. Math., 29 (3), pp. 227–238.

[13] Imdad, M. Asim, M and Gubran, R., (2017), Order-theoretic fixed point results $(\psi, \phi, \eta)_g$-generalized weak contractive mappings, Journal of Mathematics Analysis, 8(6), pp. 169-179.

[14] Imdad, M. Asim, M. and Gubran, R., (2018), Common fixed point theorems for g-Generalized contractive mappings in b-metric spaces, Indian Journal of Mathematics, 60(1), pp. 85-105.

[15] Matthews, S.G., (1994), Partial metric topology. Proc. 8th Summer Conference on General Topology and Application, Ann. New York Acad. Sci. 728, pp. 183-197.

[16] Aamri, M. and Moutawakil. D., (2002), Some new common fixed point theorems under strict contractive conditions, J. Math. Anal. Appl., 270 (1), pp. 181-188.

[17] Jungck, G., (1976), Commuting maps and fixed points, Am. Math. Mon, 83 (4), pp. 261–263.

[18] Jungck, G., (1996), Common fixed points for noncontinuous nonself maps on non-metric spaces, Far East J. Math. Sci., 4, pp. 199–215.

[19] Imdad, M. and Ali, J., (2009), Common fixed point theorems in symmetric spaces employing a new implicit function and common property (E.A), Bull. Belg. Math. Soc. Simon Stevin, 16, pp. 421–433.

[20] Sintunavarat W. and Kumam, P., (2011), Common fixed point theorems for a pair of weakly compartible mappings in fuzzy metric spaces, J. Appl. Math., 2011, pp. 1-14.

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