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CONVOLUTION OF SOME SLANTED HALF-PLANE MAPPINGS WITH HARMONIC STRIP MAPPINGS

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Abstract. In this paper, we show that the convolution of generalized half-plane mapping and harmonic vertical strip mapping with dilatation $e^{i\theta}\ z^n$ $(n \in \mathbb{N}, \theta \in \mathbb{R})$ is convex in a particular direction and also solve the problem proposed by Z. Liu et al. [Convolutions of harmonic half-plane mappings with harmonic vertical strip mappings, Filomat, 31 (2017), no. 7, 1843–1856].

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1. Introduction

Let $\mathcal{H}$ denotes the class of complex-valued functions $f = u + iv$ which are harmonic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, where u and v are real-valued harmonic functions in $\mathbb{D}$. Functions $f \in \mathcal{H}$ can also be expressed as $f = h + \overline{g}$, where h and g are analytic in $\mathbb{D}$, called the analytic and co-analytic parts of f, respectively. The Jacobian of $f = h + \overline{g}$ is given by $J_f(z) = |h'(z)|^2 - |\, g'(z)|^2$.

According to the Lewy's Theorem, every harmonic function $f = h + \overline{g} \in \mathcal{H}$ is locally univalent and sense preserving in $\mathbb{D}$ if and only if $J_f(z) > 0$ in $\mathbb{D}$ which is equivalent to the existence of an analytic function $\omega(z) = g'(z)/h'(z)$ in $\mathbb{D}$ such that

$$|\omega(z)| < 1 \quad \text{for all} \quad z \in \mathbb{D}.$$

The function ω is called the dilatation of f. By requiring harmonic functions to be sense-preserving, we retain some basic properties exhibited by analytic functions, such as the open mapping property, the argument principal, and zeros being isolated (see for detail [6]). The class of all univalent, sense preserving harmonic functions $f = h + \overline{g} \in \mathcal{H}$, with the normalized conditions $h(0) = 0 = g(0)$ and $h'(0) = 1$ is denoted by $S_{\mathcal{H}}$. If the function $f = h + \overline{g} \in S_{\mathcal{H}}$, then h and g are of the form

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n \quad (|b_1| < 1; z \in \mathbb{D})\,. \tag{1}$$

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A subclass of functions $f = h + \overline{g} \in S_{\mathcal{H}}$ with the condition $g'(0) = 0$ (or $\omega_f(0) = 0$) is denoted by $S^0_{\mathcal{H}}$. Further, the subclasses of functions f in $S_{\mathcal{H}}$ $\left(S^0_{\mathcal{H}}\right)$, denoted by $K_{\mathcal{H}}$ $\left(K^0_{\mathcal{H}}\right)$, consists of functions f that map the unit disk $\mathbb{D}$ onto a convex region.

We define the harmonic convolution (or Hadamard product) "$*$"of $f = h + \overline{g} \in \mathcal{H}$, where h and g as above in (1) and $F = H + \overline{G} \in \mathcal{H}$, where

$$H(z) = z + \sum_{n=2}^{\infty} A_n z^n \quad \text{and} \quad G(z) = \sum_{n=1}^{\infty} B_n z^n,$$

by

$$(f * F)(z) = z + \sum_{n=2}^{\infty} a_n A_n z^n + \overline{\sum_{n=1}^{\infty} b_n B_n z^n}.$$

In the present paper, authors study a generalized class of slanted half-plane mappings. It is proved that the convolution of generalized half-plane mapping and harmonic vertical strip mapping with dilatation $e^{i\theta}\ z^n$ $(n \in \mathbb{N}, \theta \in \mathbb{R})$ is convex in a particular direction.

Recall [3] that a function $f = h + \overline{g} \in S^0_H$ is called a slanted half-plane mapping if f maps $\mathbb{D}$ onto $H_\alpha := \{w : \Re e(e^{i\alpha} w) > -1/2\}$, where $0 \leq \alpha < 2\pi$. The class of all slanted half-plane mappings is denoted by $S^0(H_\alpha)$. A function $f = h + \overline{g} \in S_H$ is called a generalized slanted half-plane mapping if it maps $\mathbb{D}$ onto the region $H_{c,\alpha} := \{w : \Re e(e^{i\alpha} w) > -1/(1+c)\}$, where $0 \leq \alpha < 2\pi, c > 0$. Using the shearing method due to Clunie and Sheil Small [2] and the Riemann mapping theorem, such a mapping has the form

$$h(z) + e^{-2i\alpha} g(z) = \frac{2z}{(1+c)(1 - e^{i\alpha} z)}.$$

The class of all generalized slanted half-plane mappings is denoted by $S^c(H_\alpha)$. In the harmonic case, one can easily see that there are infinitely many slanted half-plane mappings for each fixed α. For $\alpha = 0$ and $c = 1$, the class $S^c(H_\alpha)$ reduces to the class of right half-plane mappings $f \in S^0(H_0)$ that map $\mathbb{D}$ onto $f(\mathbb{D}) = H_0 = \{w : \Re e(w) > -1/2\}$, and such a mapping clearly assumes the form

$$h(z) + g(z) = \frac{z}{1-z}.$$

We now recall fundamental result, called shearing theorem, due to Clunie and Sheil-Small as follows:

Theorem 1.1. [2]*A locally univalent harmonic function $f = h + \overline{g}$ in $\mathbb{D}$ is a univalent mapping of $\mathbb{D}$ onto a domain convex in the direction of $e^{i\gamma}$ if and only if $h - e^{2i\gamma} g$ is a analytic univalent mapping of $\mathbb{D}$ onto a domain convex in the direction of $e^{i\gamma}$.*

The class of slanted strip mappings $S^0(\Omega_{\gamma,\beta})$, defined in [1] consists of functions f that map $\mathbb{D}$ onto slanted strip domains

$$\Omega_{\gamma,\beta} = \left\{ w : \frac{\beta - \pi}{2\sin\beta} < \Re e(e^{i\gamma} w) < \frac{\beta}{2\sin\beta} \right\},$$

where $0 < \beta < \pi$ and $0 \leq \gamma < 2\pi$. Each $f = h + \overline{g} \in S^0(\Omega_{\gamma,\beta})$ has the form

$$h(z) + e^{-2i\gamma} g(z) = \frac{e^{-i\gamma}}{2i\sin\beta} \log \frac{1 + z e^{i(\gamma+\beta)}}{1 + z e^{i(\gamma-\beta)}}.$$

Let $L_{c,\alpha} = h_{c,\alpha} + \overline{g_{c,\alpha}} \in S^c(H_\alpha)$ be given by

$$h_{c,\alpha} + e^{-2i\alpha} g_{c,\alpha} = \frac{2z}{(1+c)(1-e^{i\alpha}z)},$$

where

$$h_{c,\alpha} = \frac{z - \frac{z^2}{1+c}e^{i\alpha}}{(1-e^{i\alpha}z)^2} \qquad \text{and} \qquad g_{c,\alpha} = \frac{\frac{1-c}{1+c}ze^{2i\alpha} - \frac{z^2}{1+c}e^{3i\alpha}}{(1-e^{i\alpha}z)^2}. \tag{2}$$

The mapping $L_{c,\alpha}$ maps $\mathbb{D}$ onto $H_{c,\alpha} := \{w : \Re e(e^{i\alpha}w) > -1/(1+c)\}$, where $c > 0$. If $\alpha = 0$, then we get generalized half-plane harmonic univalent mappings introduced by Muir [14].

Convolution of two harmonic convex mappings need not be convex, and it may even fail to be univalent. Therefore, it is interesting to study the convolution properties of two harmonic convex mappings. For some results on convolution of harmonic mappings see [5, 7, 8, 9, 11, 12, 15].

A domain $w \subset \mathbb{C}$ is said to be convex in the direction γ, $\gamma \in \mathbb{R}$, if and only if for every $a \in \mathbb{C}$, the set $\Omega \cap \{a + te^{i\gamma} : t \in \mathbb{R}\}$ is either connected or empty.

We now state some results which were proved earlier and problem proposed by Liu et al. [12]:

Theorem 1.2. [3] *If $f_k \in S^0(H_{\gamma_k})$, $k = 1, 2$, and $f_1 * f_2$ is locally univalent in $\mathbb{D}$, then $f_1 * f_2$ is convex in the direction $-(\gamma_1 + \gamma_2)$.*

Theorem 1.3. [3] *Let $f = h + \overline{g} \in S^0(H_0)$ with the dilatation $\omega(z) = e^{i\theta}\, z^n (n = 1, 2)$, $\theta \in \mathbb{R}$. Then $f_0 * f \in S_H^0$ and is convex in the direction of real axis.*

Theorem 1.4. [10] *Let $f = h + \overline{g} \in S^0(H_\gamma)$ with the dilatation $\omega(z) = e^{i\theta}\, z^n (n = 1, 2)$, $\theta \in \mathbb{R}$. Then $f_0 * f \in S_H^0$ and is convex in the direction of $-\gamma$.*

Theorem 1.5. [12] *Let $L_{c,0} = h_{c,0} + \overline{g_{c,0}} \in S^c(H_0)$ be defined by (2) and $f = h + \overline{g} \in S^0(\Omega_\beta)$ with $\omega(z) = e^{i\theta}z$, where $\pi/2 \le \beta < \pi$ $(\theta \in \mathbb{R})$, then $L_{c,0} * f \in S_H^0$ and is convex in the horizontal direction for $0 < c \le 2$.*

Theorem 1.6. [12] *Let $L_{c,0} = h_{c,0} + \overline{g_{c,0}} \in S^c(H_0)$ be defined by (2) and $f = h + \overline{g} \in S^0(\Omega_\beta)$ with $\omega(z) = e^{i\theta}z^n$, where $\beta = \pi/2$ $(n \in \mathbb{N}, \theta \in \mathbb{R})$, then $L_{c,0} * f \in S_H^0$ and is convex in the horizontal direction for $0 < c \le \frac{2}{n}$.*

Problem (4.4) [12] Let $L_{c,0} = h_{c,0} + \overline{g_{c,0}} \in S^c(H_0)$ be defined by (2) and $f = h + \overline{g} \in S^0(\Omega_\beta)$ with $\omega(z) = e^{i\theta}z^n$, where $\pi/2 \le \beta < \pi$ $(n \in \mathbb{N}, \theta \in \mathbb{R})$, then $L_{c,0} * f \in S_H^0$ and is convex in the horizontal direction for $0 < c \le \frac{2}{n}$.

2. MAIN RESULTS

Lemma 2.1. *If a function $f = h + \overline{g} \in S^c(H_\alpha)$, then*

$$h(z) + e^{-2i\alpha}g(z) = \frac{2z}{(1+c)(1-e^{i\alpha}z)} \qquad (c > 0).$$

Proof. If $f = h + \overline{g} \in S^c(H_\alpha)$, then $\Re e\ \{e^{i\alpha}(h + \overline{g})\} > -1/(1+c)$ which means that $\Re e\{e^{i\alpha}h(z) + e^{-i\alpha}g(z)\} > -1/(1+c)$. In other words, $\Re e\ \{e^{i\alpha}(h(z) + e^{-2i\alpha}g(z))\} > -1/(1+c)$. Since f is a convex function, it follows from a convexity theorem by Clunie and Sheil-Small [2] that the function $h(z) + e^{-2i\alpha}g(z)$ is the convex in the direction $\pi/2 - \alpha$, and so f is univalent. It is also clear that $z \to h(z) + e^{-2i\alpha}g(z)$ maps $\mathbb{D}$ onto the region $H_{c,\alpha}$ which implies the result. $\square$

Lemma 2.2. *Let $L_{c,\alpha} = h_{c,\alpha} + \overline{g_{c,\alpha}}$ be defined by (2) and $f = h + \overline{g} \in S^0(\Omega_{\gamma,\beta})$ be the vertical strip mapping, with dilatation $\omega(z) = g'(z)/h'(z)$. then the dilatation of $\tilde{\omega}$ of $L_{c,\alpha} * f$ is given by*

$$\tilde{\omega}(z) = e^{2i\alpha}\left[\frac{(1-c)g'\left(ze^{i\alpha}\right) - cze^{i\alpha}g''\left(ze^{i\alpha}\right)}{(1+c)h'\left(ze^{i\alpha}\right) + cze^{i\alpha}h''\left(ze^{i\alpha}\right)}\right]. \tag{3}$$

Proof. Let $f = h + \overline{g} \in S^c(H_\alpha)$ with $\omega(z) = g'(z)/h'(z)$. Let $L_{c,\alpha} * f = h_{c,\alpha} * h + \overline{g_{c,\alpha} * g} = h_1 + \overline{g_1}$, where

$$\begin{aligned} h_1(z) &= \frac{1}{1+c}\left[\frac{z}{1-ze^{i\alpha}} + \frac{cz}{(1-ze^{i\alpha})^2}\right] * h \\ &= \frac{e^{-i\alpha}}{1+c}\left[h(ze^{i\alpha}) + cze^{i\alpha}h'\left(ze^{i\alpha}\right)\right], \end{aligned}$$

and

$$\begin{aligned} g_1(z) &= \frac{1}{1+c}\left[\frac{z}{1-ze^{i\alpha}} - \frac{cz}{(1-ze^{i\alpha})^2}\right] * g \\ &= \frac{e^{i\alpha}}{1+c}\left[g(ze^{i\alpha}) - cze^{i\alpha}g'\left(ze^{i\alpha}\right)\right]. \end{aligned}$$

The dilatation is given by

$$\tilde{\omega}(z) = \frac{g_1'(z)}{h_1'(z)} = e^{2i\alpha}\left[\frac{(1-c)g'\left(ze^{i\alpha}\right) - cze^{i\alpha}g''\left(ze^{i\alpha}\right)}{(1+c)h'\left(ze^{i\alpha}\right) + cze^{i\alpha}h''\left(ze^{i\alpha}\right)}\right].$$

□

Lemma 2.3. [1] *If the mapping $f = h + \overline{g} \in S^0(\Omega_{\gamma,\beta})$, then*

$$h(z) + e^{-2i\gamma}g(z) = \frac{e^{-i\gamma}}{2i\sin\beta}\log\frac{1+ze^{i(\gamma+\beta)}}{1+ze^{i(\gamma-\beta)}}.$$

Lemma 2.4. *Let a mapping $L_{c,\alpha} = h_{c,\alpha} + \overline{g_{c,\alpha}}$ be defined by (2) and $f = h + \overline{g} \in S^0(\Omega_{\gamma,\beta})$. If the mapping $L_{c,\alpha} * f$ is sense preserving, then $L_{c,\alpha} * f$ is convex in the direction $-(\alpha+\gamma)$.*

Proof. By similar method used in [9], we can prove this lemma. So we omit the details. □

Note: If we take $c = 1, \gamma = 0, \beta = \frac{\pi}{2}$ in above lemma, we get the result proved in [9, Theorem 3.2].

Theorem 2.1. *Let $L_{c,\alpha} = h_{c,\alpha} + \overline{g_{c,\alpha}} \in S^c(H_\alpha)$ be defined by (2) and $f = h + \overline{g} \in S^0(\Omega_{\gamma,\beta})$ with $\omega(z) = e^{i\theta}z^n$, where $0 < \beta < \pi$ $(n \in \mathbb{N}, \theta \in \mathbb{R})$, then $L_{c,\alpha} * f \in S_H^0$ and is convex in the direction $-(\alpha+\gamma)$ for $0 < c \leq \frac{2}{n}$.*

Proof. To prove this theorem, it suffices to show by Lemma 2.4 that the dilatation of $\tilde{\omega}$ of $L_{c,\alpha} * f$ given by (3) satisfies $|\tilde{\omega}(z)| < 1$ for all $z \in \mathbb{D}$. By (3), we see that $\tilde{\omega}(z) = W(ze^{i\alpha})$, where

$$W(z) = e^{2i\alpha}\left[\frac{(1-c)g'(z) - czg''(z)}{(1+c)h'(z) + czh''(z)}\right].$$

For $|\tilde{\omega}(z)| < 1$, it is enough to prove that $|W(z)| < 1$. Since $f = h + \overline{g} \in S^0(\Omega_{\gamma,\beta})$, we have

$$h(z) + e^{-2i\gamma}g(z) = \frac{e^{-i\gamma}}{2i\sin\beta}\log\frac{1+ze^{i(\gamma+\beta)}}{1+ze^{i(\gamma-\beta)}}. \tag{4}$$

Since the dilatation $\omega(z) = e^{i\theta}z^n$, we have

$$g'(z) = e^{i\theta}z^n h'(z) \qquad \text{and} \qquad g''(z) = ne^{i\theta}z^{n-1}h'(z) + e^{i\theta}z^n h''(z)$$

so that $W(z)$ defined above takes the form

$$W(z) = e^{i(2\alpha+\theta)}z^n\left(\frac{1-c-nc-cu[h(z)]}{1+c+cu[h(z)]}\right), \tag{5}$$

where

$$u[h(z)] = z\frac{h''(z)}{h'(z)}.$$

From (4), with the dilatation $\omega(z) = e^{i\theta}z^n$, we find that

$$h'(z) = \frac{1}{(1+e^{-2i\gamma}\omega(z))(1+ze^{i(\gamma+\beta)})(1+ze^{i(\gamma-\beta)})}$$

and

$$h''(z) = -\frac{2(ze^{2i\gamma}+e^{i\gamma}\cos\beta)(1+e^{-2i\gamma}\omega(z))+e^{-2i\gamma}\omega'(z)(1+ze^{i(\gamma+\beta)})(1+ze^{i(\gamma-\beta)})}{(1+e^{-2i\gamma}\omega(z))^2(1+ze^{i(\gamma+\beta)})^2(1+ze^{i(\gamma-\beta)})^2},$$

Therefore,

$$\begin{aligned} u[h(z)] &= -\frac{2z(ze^{2i\gamma}+e^{i\gamma}\cos\beta)}{(1+ze^{i(\gamma-\beta)})(1+ze^{i(\gamma+\beta)})} - \frac{ze^{-2i\gamma}\omega'(z)}{1+e^{-2i\gamma}\omega(z)} \\ &= -\frac{2z(ze^{2i\gamma}+e^{i\gamma}\cos\beta)}{(1+ze^{i(\gamma-\beta)})(1+ze^{i(\gamma+\beta)})} - n\frac{e^{i(\theta-2\gamma)}z^n}{1+e^{i(\theta-2\gamma)}z^n} \\ &= -\frac{ze^{i(\gamma-\beta)}}{1+ze^{i(\gamma-\beta)}} - \frac{ze^{i(\gamma+\beta)}}{1+ze^{i(\gamma+\beta)}} - n\frac{e^{i(\theta-2\gamma)}z^n}{1+e^{i(\theta-2\gamma)}z^n}. \end{aligned}$$

which proves that $X := \Re e\ u[h(z)] > -1 - \frac{n}{2}$ for all $z \in \mathbb{D}$ and thus, $2 + n + 2X > 0$. From (5), to show $\left|\frac{1-c-nc-cu[h(z)]}{1+c+cu[h(z)]}\right| \leq 1$, it suffices to show that

$$T := (1+c+cX)^2 - (1-c-nc-cX)^2 \geq 0, \text{ for all } z \in \mathbb{D} \tag{6}$$

or

$$T = c\,(2-cn)\,(2+n+2X) \geq 0. \tag{7}$$

Since, $c > 0$ and $2 + n + 2X > 0$, we conclude that $T \geq 0$ if and only if $0 < c \leq \frac{2}{n}$. This completes the proof of Theorem 2.1. □

Substituting $\alpha = 0, \gamma = 0$, we get the following result:

Corollary 2.1. [12] *Let $L_{c,0} = h_{c,0} + \overline{g_{c,0}} \in S^c(H_0)$ be defined by (2) and $f = h + \overline{g} \in S^0(\Omega_{0,\beta})$ with $\omega(z) = e^{i\theta}z^n$, where $0 < \beta < \pi$ $(n \in \mathbb{N}, \theta \in \mathbb{R})$, then $L_{c,0} * f \in S_H^0$ and is convex in the horizontal direction for $0 < c \leq \frac{2}{n}$.*

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