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SOLVING TRI-LEVEL LINEAR PROGRAMMING PROBLEM BY A NOVEL HYBRID ALGORITHM

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Abstract. This paper presents a revised hybrid algorithm to solve a tri-level linear programming problem, a generalization of a bi-level one, involving three decision makers at the upper, middle, and lower levels. The decision-making priority is from top to bottom and the decision of each decision maker affects the decision space of others. A hybrid algorithm has been already proposed to solve this problem, but it does not ensure to converge whereas the proposed novel revised algorithm lacks this drawback and ensures convergence.

Keywords: Tri-Level Linear Programming, Convergence, Sequential Linear Complementary Problem, Decentralized Decision Making.

AMS Subject Classification: 90-XX, 91A65

1. Introduction

Tri-level programming that lies under a family of multi-level programming [13] derived from Stackelberg game theory [12], is a model developed for the modeling of decentralized decision-making problems that involve decision makers distributed throughout a tri-level hierarchy. The tri-level decision making has been developed in a tri-level hierarchy to achieve an interaction and compromise among the decision makers who are called the leader in the upper level and the followers in the middle and lower levels. To optimize their objectives, decision-makers start making their personal decisions in the upper level and continue in the middle and then in the lower levels [16]. The existing tri-level programming studies have focused mainly on the results of linear models. For instance, Bard and Falk [2] stated the necessary optimal conditions for tri-level linear programming (TLP) problems based on the Stackelberg game theory, generalized the rational reaction sets for each decision maker, and presented the cutting plane algorithm to solve TLP problems. White [15] introduced the penalty function method for TLP problems. Other methods including that of Kuhn-Tucker [1, 8], branch and bound algorithm [8, 3], and Kth-best

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algorithm [16] have been developed to solve bi-level linear programming (BLP) and TLP problems. Wen et al. [14] presented a hybrid method to solve TLP problems. Lai et al. [10] give a tri-level model to mitigate coordinated attacks on electric power systems, Gu et al. [6] offer a tri-level model for a private road competition problem. Moreover, some meta-heuristic algorithms and fuzzy approaches have been proposed for solving multi-level programming problems (See [7, 11] for example). This paper has been so organized as to present the mathematical form of the TLP problem and the required definitions in Section 2, provide the solution methods for the sequential and revised sequential linear complementary problems in Section 3, explain the novel TLP problem algorithm (and a numerical example to show how it runs) in Section 4, and, finally, conclude the findings in Section 5.

2. Tri-level linear programming

In a tri-level hierarchical decision making problem, each decision maker optimizes its objective in each level, and variables in each level are controlled by the decision-makers in the same level. Since selecting values for variables of each level can affect decisions made at other levels, the objective function can change in each level. Effort is now made to introduce the TLP problem which is a generalization of the BLP problem.
Consider the following TLP problem:

$$\begin{array}{ll} level3 & \max\ (f_3 = c_{31}x_1 + c_{32}x_2 + c_{33}x_3 : (x_3)) \\ level2 & s.t.: \max\ (f_2 = c_{21}x_1 + c_{22}x_2 : (x_2|x_3)) \\ level1 & s.t.: \max\ (f_1 = c_{11}x_1 : (x_1|x_2, x_3)) \\ s.t.: & Ax_1 + Bx_2 + Cx_3 \leq b \\ & x_1, x_2, x_3 \geq 0, \end{array} \tag{1}$$

where, $\max(f_3 = c_{31}x_1 + c_{32}x_2 + c_{33}x_3 : (x_3))$ shows the maximization of f_3 on x_1, x_2 , and x_3 , but in the third level problem only x_3 is controlled, $\max(f_2 = c_{21}x_1 + c_{22}x_2 : (x_2|x_3))$ shows the maximization of f_2 on x_1 and x_2 for the fixed value of x_3, but in the second level problem only x_2 is controlled, $\max(f_1 = c_{11}x_1 : (x_1|x_2, x_3))$ shows the maximization of f_1 on x_1 for the fixed values of x_2 and x_3 , x_1 is the vector of the decision variables for the first level problem with dimensions $n_1 \times 1$, x_2 is the vector of the decision variables for the second level problem with dimensions $n_2 \times 1$, x_3 is the vector of the decision variables for the third level problem with dimensions $n_3 \times 1$, A is the technological coefficients matrix for the first level variables with dimensions $m \times n_1$,B is the technological coefficients matrix for the second level variables with dimensions $m \times n_2$,C is the technological coefficients matrix for the third level variables with dimensions $m \times n_3$, c_{11} , c_{21} , and c_{31} are the cost coefficients vectors for the first level variables in respectively the first, second, and third level problems with dimensions $1 \times n_1$, c_{22} and c_{32} are the cost coefficients vectors for the second level variables in respectively the second and third level problems with dimensions $1 \times n_2$, c_{33} is the cost coefficients vector for the third level variables in the third level problem with dimensions $1 \times n_3$, and b is the system resources capacity vector with dimensions $m \times 1$.
It is worth noting that TLP is a tri-level linear resource control problem that controls the third level of x_3 which in turn changes the resource space of the first and second level by the $Ax_1 + Bx_2 \leq b - Cx_3$ constraint and controls the second level of x_2 that changes the resource space of the first level by fixing x_2 and the $Ax_1 \leq b - Bx_2 - Cx_3$ constraint.

2.1. **Points and definitions.** First, we define S_1 , S_2 , and S_3 as follows:
$S_1 = \{x : Ax_1 + Bx_2 + Cx_3 \leq b\}$ is a convex set of R^n where $x = (x_1, x_2, x_3)$ and

$n = n_1 + n_2 + n_3$.
$S_2 = \{x : x \in S_1 \cap \{x \in R^n | c_{11}\widehat{x}_1 = \max(f_1 = c_{11}x_1 : (x_1|\widehat{x}_2, \widehat{x}_3))\}\}$ is a set of rational reactions of f_1 on S_1 that does not need to be convex.
$S_3 = \{x : x \in S_2 \cap \{x \in R^n | c_{21}\widehat{x}_1 + c_{22}\widehat{x}_2 = \max(f_2 = c_{21}x_1 + c_{22}x_2 : (x_2|\widehat{x}_3))\}\}$ is a set of rational reactions of f_2 on S_3 that does not need to be convex.
Therefore, the TLP can be shown as:

$$\begin{aligned} &\max \ (c_{31}x_1 + c_{32}x_2 + c_{33}x_3 : (x_3)) \\ &s.t. : \max \ (c_{21}x_1 + c_{22}x_2 : (x_2|x_3)) \\ &\qquad s.t. : x \in S_2 \end{aligned}$$

or

$$\begin{aligned} &\max \ (c_{31}x_1 + c_{32}x_2 + c_{33}x_3 : (x_3)) \\ &\qquad s.t. : x \in S_3 \end{aligned}$$

which is, in fact, the optimization of the linear function on a non-convex region.

Theorem 2.1. *: if x is the extreme point of S_3 , then it will be the extreme point of S_2 as well as S_1.*

Proof: refer to [14].

3. Sequential linear complementary problem (SLCP) method

To solve BLP problems, Bialas and Karwan [5] presented the generally non-convergent SLCP method; then, Judice and Faustino [9] modified it for convergence. Next, both are reviewed.
Consider the following BLP problem:

$$\begin{aligned} &\min_{y \in R^m_+ = \{y \in R^m : y \geq 0\}} \quad cx + dy \\ s.t. \quad &\min_{x \in R^n_+ = \{x \in R^n : x \geq 0\}} \quad ax \\ s.t. \quad &A_1x + A_2y \geq b \end{aligned} \tag{2}$$

and

$$\begin{aligned} &\min_{y \in R^n_+ = \{x \in R^n : x \geq 0\}} \quad ax \\ s.t. \quad &A_1x + A_2x \geq b, \end{aligned} \tag{3}$$

where A_1 and A_2 are $r \times n$and $r \times m$dimensional matrices, respectively.
If $S = \{(x, y) \in R^{n \times m} : A_1x + A_2y \geq b, x, y \geq 0\}$ set is bounded, the dual model of (3) will be:

$$\begin{aligned} &\max \ (b - A_2y)^t u \\ s.t. \quad &A_1^t u \leq a \\ &u \geq 0. \end{aligned} \tag{4}$$

If α and β are the slack variables vectors corresponding to the primal and dual constraints, respectively, writing KKT conditions for (3) will result in the following BLP-equivalent nonconvex optimization problem:

$$\begin{aligned} & \min \quad cx + dy \\ s.t. \quad & \alpha = -b + A_1 x + A_2 y \\ & \beta = a - A_1^t u \\ & x^t \beta = u^t \alpha = 0 \\ & x, y, u, \alpha, \beta \geq 0. \end{aligned} \tag{5}$$

Therefore, the $(\overline{x}, \overline{y})$ absolute minimum of the BLP can be found by solving the non-convex program (5). In the SLCP method, the following parametric LCP is obtained by defining parameter λ and replacing the objective function of (5) with the $cx + dy \leq \lambda$ constraint:

$$\begin{aligned} & \text{LCP}(\lambda) \\ & \begin{bmatrix} \alpha \\ \beta \\ \nu_0 \end{bmatrix} = \begin{bmatrix} -b \\ a \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \lambda + \begin{bmatrix} 0 & A_1 & A_2 \\ -A_1^t & 0 & 0 \\ 0 & -c & -d \end{bmatrix} \begin{bmatrix} u \\ x \\ y \end{bmatrix} \\ & x, y, u, \alpha, \beta, \nu_0 \geq 0, x^t \beta = u^t \alpha = 0. \end{aligned} \tag{6}$$

The SLCP algorithm involves finding the LCP$(\overline{\lambda})$ solution ($\overline{\lambda}$ is the smallest value of λ) so that the LCP(λ) may have a solution for which the SLCP method solves a sequence of the LCP(λ_k) where $\{\lambda_k\}$ is a decent sequence defined as follows:

$$\text{Upper bound of } cx + dy \text{ on } S = \lambda_0$$

$$\lambda_k = cx^{k-1} + dy^{k-1} - |\gamma_k(cx^{k-1} + dy^{k-1})|, \tag{7}$$

where (x^{k-1}, y^{k-1}) is the solution to LCP(λ_{k-1}) and γ_k is a small positive number. The process will end in iteration k if LCP(λ_k) does not yield a solution. In such a case, the solution of LCP(λ_k) ,(x^{k-1}, y^{k-1}), will confirm the following inequality:

$$0 \leq cx^{k-1} + dy^{k-1} - VAL \leq |\gamma(cx^{k-1} + dy^{k-1})|, \tag{8}$$

where VAL is the value of the objective function in the optimal solution. Hence, if the defined set S is bounded and nonempty, the algorithm will find the optimal $-\epsilon$ solution of the BLP:

$$\epsilon = |\gamma_k(cx^{k-1} + dy^{k-1})|. \tag{9}$$

If γ_k is small, the (x^{k-1}, y^{k-1}) solution from LCP(λ_{k-1}) is usually the BLP optimal solution.

3.1. **The SLCP algorithm.** The SLCP algorithm is as follows:
Step 1- let $k = 0$ and λ_0 be the upper bound of $cx + dy$ on S.
Step 2- Solve LCP(λ_k). If it yields a solution, go to Step 3. Otherwise, let (x^k, y^k) be its solution:

$$\lambda_{k+1} = cx^k + dy^k - |\gamma_{k+1}(cx^k + dy^k)|,$$

where γ_{k+1} is a constant value. Let $k = k + 1$ and repeat Step 2.
Step 3- If $k = 0$, the BLP is infeasible. Otherwise, (x^{k-1}, y^{k-1}) is the optimal $-\epsilon$ solution of the BLP where ϵ is obtained from (9).
The efficiency of the SLCP algorithm depends mainly on how LCP(λ_k) is solved. We will now describe the BRES algorithm using an artificial variable. Consider LCP(λ_k) as follows:

$$\begin{array}{ll} (a). & w = q + Mz + Ny \\ (b). & w, z, y \geq 0 \\ (c). & z_i w_i = 0, i = 1, ..., r + n, \end{array} \tag{10}$$

where

$$w = \begin{bmatrix} \alpha \\ \beta \\ \nu_0 \end{bmatrix} \in R^{r+n+1}, z = \begin{bmatrix} u \\ x \\ \end{bmatrix} \in R^{r+n}, q = \begin{bmatrix} -b \\ a \\ \lambda_k \end{bmatrix} \in R^{r+n+1}, M = \begin{bmatrix} 0 & A_1 \\ -A_1^t & 0 \\ 0 & -c \end{bmatrix}_{(r+n+1)\times(r+n)},$$

$$N = \begin{bmatrix} A_2 \\ 0 \\ -d \end{bmatrix}_{(r+n+1)\times m}.$$

Let z_0 be an artificial variable andpbe a nonnegative vector on the condition that $p_i > 0$ for each i where $q_i < 0$. Consider the following:

$$\begin{array}{ll} & \min \ z_0 \\ s.t. & w = q + z_0 p + Mz + Ny \\ & z, w, y, z_0 \geq 0 \\ & z_i w_i = 0, i = 1, ..., r + n. \end{array} \tag{11}$$

The BRES algorithm is a revision of phase 1 of a two-phase simplex with an artificial variable [4]. The optimal solution of (11) can be obtained only by using the solutions that satisfy the constraints in (11). To ensure this condition, the non-basic variable z_i or w_i with positive decrease cost coefficient can be nominated as the entering variable if its complementary z_i or w_i is also non-basic or becomes so in this iteration. Accordingly, the BRES algorithm may have three endings:
$TERM = 1$ - Optimal solution $(\overline{w}, \overline{z}, \overline{y})$ for (11) with $\overline{z}_0 = 0$.
$TERM = 2$ - Optimal solution $(\overline{w}, \overline{z}, \overline{y})$ for (11) with $\overline{z}_0 > 0$.
$TERM = 3$ - A non-optimal basic solution for (11) while there is no candidate for the entering variable.

In case 1 ($TERM = 1$), $(\overline{w}, \overline{z}, \overline{y})$ is the solution to LCP (10). In case 2 ($TERM = 2$), this LCP does not yield a solution. In case 3 ($TERM = 3$), there is no conclusion on whether there is a solution; the revised BRES method has been designed for both the first two as well as case 3.
After some introductory points, effort will be made to address the revised BRES method. In model (10), Like linear programming, solutions (z, w, y) that satisfy linear constraints (a) and (b) are called feasible. The solution will be complementary if variables z_i and w_i satisfy constraint (c). The enumeration method tries to find the complementary solution using only the system's basic feasible solutions (a). For this purpose, the following tree (Figure 1) is examined, where $i_1, i_2, ...$ are integer numbers from $\{1, ..., r+n\}$ set. In node 1, the initial feasible solution is found through Phase 1 of a two-Phase simplex algorithm with an artificial single variable. Each of the other nodes is generated by solving a sub-problem that involves the minimization of z_i or w_i under the LCP linear constraints and some $z_i = 0$ or $w_i = 0$ constraints. For instance, to generate node 4 of tree (Figure 1), it is necessary to solve the following linear program (12):

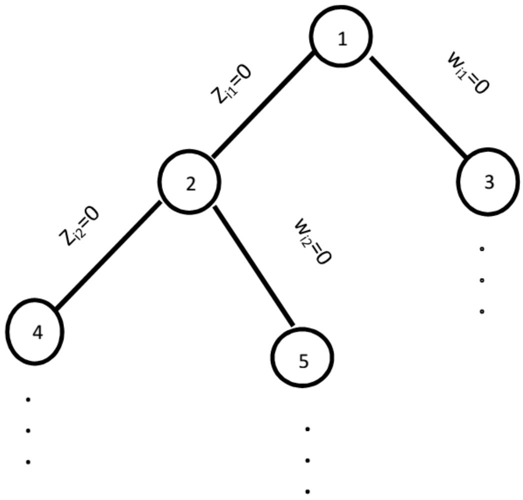

FIGURE 1. Tree of the BRES algorithm.

$$\begin{aligned} & \min \quad z_{i2} \\ s.t. \quad & w = q + Mz + Ny \\ & z, w, y \geq 0 \\ & z_{i1} = 0. \end{aligned} \tag{12}$$

Such a linear program is solved by a revision of Phase 2 of a two-Phase simplex method and two cases may occur:
1. If the minimized variable equals zero, it will be taken zero in all the paths down the tree.
2. If the minimized variable value is positive, the node has been fathomed and the branching is accomplished.
The enumeration method solves the LCP through generating consecutive tree nodes according to the process described above. The algorithm either finds the LCP solution (which will be the first feasible complementary solution) or proves that it does not have a solution (all the tree nodes have been fathomed).

3.2. **Revised BRES method.** Consider node 1 generation from tree (Figure 1) (i.e. finding the first feasible LCP solution discussed earlier). This is done by solving the following linear program:

$$\begin{aligned} & \min \quad z_0 \\ s.t. \quad & w = q + z_0 p + Mz + Ny \\ & z, w, y, z_0 \geq 0, \end{aligned} \tag{13}$$

where p is a non-negative vector on the condition that for each i where $q_i < 0$, $p_i > 0$ and z_0 is an artificial variable. Since the objective is to find a complementary feasible solution, the BRES algorithm can be used to solve such a linear program. If this algorithm ends with $TERM = 1$, the solution to LCP (10) has been found, and if it ends with $TERM = 2$, the LCP does not have a solution and the enumeration process will stop. If

$TERM = 3$ occurs, the z_0 value can still reduce if one step of the simplex pivot (belonging to the column with positive decrease cost coefficient) is executed. When $TERM = 3$, the revised BRES algorithm involves the execution of such a pivot step and reuse of the BRES method and then the process is repeated; this algorithm can also be used to generate any other k node. Since, as discussed earlier, generating a node involves minimizing variable z_i , or w_i , the revised BRES algorithm for generating node k is as follows:
Step 1- Let NCP be the number of the (z_i, w_i) pairs of complementary variables where both w_i and z_i are basic in the solution (if $k = 0$, $NCP = 0$).
Step 2- Use the BRES algorithm; if the feasible complementary solution has been found, let $TERM = 1$ and $NCP = 0$, go to Step 4. If the BRES algorithm ends with $TERM = 1$ or $TERM = 2$,go to Step 4; otherwise, ($TERM = 3$), go to Step 3.
Step 3- Let $NCP = NCP + 1$ and then run one step of the simplex pivot (belonging to the column with positive decrease cost coefficient). Go to Step 2.
Step 4- If $TERM = 1$ and $NCP = 0$, the LCP solution has been found and the enumeration process will stop, if $TERM = 1$ and $NCP > 0$, node k will be generated, and if $TERM = 2$, node k cannot be generated and is fathomed.
The revised BRES method is implemented in the following example (for more details, refer to [9]).

Example 3.1.

$$\begin{aligned}
&\min_{y_1} \quad -y_1\\
&s.t.\\
&\min_{x_1} \quad x_1\\
&s.t.: 2x_1 + y_1 \geq 10\\
&2x_1 - y_1 \geq -6\\
&x_1 - 2y_1 \geq -21\\
&-2x_1 - y_1 \geq -38\\
&-2x_1 + y_1 \geq -18\\
&x_1, y_1 \geq 0.
\end{aligned}$$

4. Revised hybrid algorithm

Now, a novel hybrid method is proposed that makes use of the Kth-best algorithm and the revised BRES instead of the BRES method, and ensures the method convergence because, as noted in Section 3, the BRES method is not always convergent.
Step 1- Let $k = 1$ and $\alpha = 0$. To obtain the optimal solution of $\widehat{x}_{[1]}$, solve problem (T_1) by the simplex method. Let $T = \emptyset$ and $W = \{\widehat{x}_{[1]}\}$. Go to Step 2.

$$\begin{aligned}
(T_1): \quad & \max \ (c_{31}x_1 + c_{32}x_2 + c_{33}x_3 : (x_3))\\
s.t.: \quad & x \in S_1.
\end{aligned}$$

Step 2- Solve problem (T_2) through the bounded simplex method.

$$\begin{aligned}
(T_2): \quad & \max \ c_{11}x_1\\
s.t.: \quad & x \in S_1 \cap \{x \in R^n | x_2 = \widehat{x}_2, x_3 = \widehat{x}_3\}.
\end{aligned}$$

Suppose this problem has the optimal solution $(\overline{x}_1, \widehat{x}_2, \widehat{x}_3)$. If $\overline{x}_1 \neq \widehat{x}_1$, then $\widehat{x} \notin S_2$; therefore, $\widehat{x}$ is not in S_3. Go to Step 3. If $\overline{x}_1 = \widehat{x}_1$, then $\widehat{x} \in S_2$. Go to Step 5.
Step 3- Suppose $W_{[k]}$ represents a set of extreme points. x adjacent to $\widehat{x}_{[k]}$ so that $c_3x \leq c_3\widehat{x}_{[k]}$. Update T and W as follows:

$$T = T \cup \{\widehat{x}_{[k]}\}, W = (W \cup W_{[k]}) \setminus T.$$

Step 4- Let $k = k + 1$ and select $\widehat{x}_{[k]}$ so that $c_3\widehat{x}_{[k]} = \max\{c_3x|x \in W\}$. Go to step 2.
Step 5- (First-best check) - Find the optimal solution to the following problem:

$$\begin{aligned} (T_3): \quad & \max \quad c_{21}x_1 + c_{22}x_2 \\ s.t.: \quad & x \in S_1 \cap \{x \in R^n | x_3 = \widehat{x}_3\}. \end{aligned}$$

Suppose the problem has an optimal solution $(\widehat{x}_1^*, \widehat{x}_2^*, \widehat{x}_3)$. Check if point $(\widehat{x}_1^*, \widehat{x}_2^*, \widehat{x}_3)$ belongs in S_2 by solving the following problem:

$$\begin{aligned} (T_4): \quad & \max \quad c_{11}x_1 \\ s.t.: \quad & x \in S_1 \cap \{x \in R^n | x_2 = \widehat{x}_2, x_3 = \widehat{x}_3. \end{aligned}$$

The optimal solution $(\overline{x}_1, \widehat{x}_2^*, \widehat{x}_3)$ is obtained. If $\overline{x}_1 = \widehat{x}_1^*$, then solution (T_3) is in S_2. For S_3 − Check, go to Step 6. If $\overline{x}_1 \neq \widehat{x}_1^*$, then, solution (T_3) is not in S_2; put $\alpha = c_{21}\widehat{x}_1 + c_{22}\widehat{x}_2$ and go to Step 7.
Step 6- If $(\widehat{x}_1^*, \widehat{x}_2^*) = (\widehat{x}_1, \widehat{x}_2)$, then $\widehat{x} \in S_3$ and, hence, $\widehat{x}$ is the solution to the tri-level problem and stop. Otherwise $\widehat{x} \notin S_3$, go to Step 3.
Step 7- (CP-check) Solve the following system using the revised BRES method presented in the SLCP algorithm:

$$\begin{aligned} (T_5): & Ax_1 + Bx_2 + y = b - Cx_3 \\ & (A)^t u - \nu = (c_{11})' \\ & c_{21}x_1 + c_{22}x_2 \geq \alpha + \delta \\ & u^t y = 0, (x_1)^t \nu = 0 \\ & x_1, x_2, y, u, \nu \geq 0, \end{aligned}$$

where δ is a sufficiently small positive number. If finding a feasible solution for (T_5) is not possible, $(\widehat{x}_1^*, \widehat{x}_2^*)$ is the solution to the bi-level problem (level 2 and level 1) where x_3 has a value equal to $\widehat{x}_3$, then $\widehat{x} \in S_3$ and $\widehat{x}$ is the optimal solution to the tri-level problem and stop. Otherwise $\widehat{x} \notin S_3$, go to Step 3.

4.1. **Convergence of the revised hybrid algorithm.** The hybrid algorithm starts with the Kth-best algorithm [16] to search for the Kth-best extreme point over the entire feasible solution region of S_1 which is bounded and has finite extreme point (note that $S_3 \subset S_2 \subset S_1$). Therefore, since the $c_3\widehat{x}$ objective function is non-increasing after the next optimal solution is found in different iterations, and there are only a finite number of bases, the algorithm will be fathomed after a finite number of iterations (since the points being studied are generated by the Kth-best algorithm, there will be no rotation). Note that in Step 7, the CP-check is also convergent because it has made use of the revised BRES method (Section 3). Additionally, if the hybrid algorithm stops in Step 7 with the absolute optimal solution, the algorithm will be fathomed with a small deviation (δ) from the absolute optimal solution.

Example 4.1. [14]. *Let us solve the following tri-level problem using the proposed revised hybrid algorithm.*

$$\begin{aligned}
&\max \quad -4x_1 + 3x_2 + 7x_3 : (x_1)\\
&s.t. : \max \quad x_2 : (x_2|x_3)\\
&s.t. : \max \quad x_1 : (x_1|x_2, x_3)\\
&s.t. : x_1 + x_1 + x_3 \leq 3\\
&\qquad x_1 + x_2 - x_3 \geq 1\\
&\qquad x_1 - x_2 + x_3 \leq 1\\
&\qquad -x_1 + x_2 + x_3 \leq 1\\
&\qquad x_3 \leq 0.5\\
&\qquad x_1, x_2, x_3 \geq 0.
\end{aligned} \tag{14}$$

Variable x_i is controlled by level i ($i = 1, 2, 3$).

Step 1- *Let $k = 1$ and $\alpha = 0$, solve problem (15) by the simplex method:*

$$x \equiv (x_1, x_2, x_3)^t$$

$$\begin{aligned}
&\max \quad -4x_1 + 3x_2 + 7x_3\\
s.t. : \quad &x_1 + x_1 + x_3 \leq 3\\
&x_1 + x_2 - x_3 \geq 1\\
&x_1 - x_2 + x_3 \leq 1\\
&-x_1 + x_2 + x_3 \leq 1\\
&x_3 \leq 0.5\\
&x_1, x_2, x_3 \geq 0.
\end{aligned} \tag{15}$$

The solution to problem (15) is $\widehat{x}_{[1]} = (0, 1, 0)^t$. Put:

$$T = \emptyset, W = \{\widehat{x}_{[1]}\}$$

Step 2- *Solve problem (16) by the bounded simplex method:*

$$\begin{aligned}
&\max \quad x_1\\
s.t. : \quad &x_1 + x_1 + x_3 \leq 3\\
&x_1 + x_2 - x_3 \geq 1\\
&x_1 - x_2 + x_3 \leq 1\\
&-x_1 + x_2 + x_3 \leq 1\\
&x_2 = 1\\
&x_3 = 0\\
&x_1 \geq 0.
\end{aligned} \tag{16}$$

The solution to problem (16) is $\overline{x} = (2, 0, 1)^t$ where $\overline{x} \neq \widehat{x}_{[1]} = (0, 1, 0)^t$ hence $\widehat{x}_{[1]} \notin S_2$ and $\widehat{x}_{[1]} \notin S_3$.

Step 3- *The extreme points adjacent to $\widehat{x}_{[1]}$ are $\{(1, 0, 0)^t, (1, 2, 0)^t, (0.75, 0.75, 0.5^t)$:*
Now we will obtain $W_{[1]}$ which shows a set of extreme points x adjacent to $\widehat{x}_{[1]}$ so that $c_3x \leq c\widehat{x}_{[1]}$.
$c_3 = (-4, 3, 4), c_3\widehat{x}_{[1]} = 3, c_3(1, 0, 0)^t = -4$

$, c_3(1,2,0)^t = 2, c_3(0.75,0.75,0.5)^t = 2.75 \Longrightarrow W_{[1]} = \{(1,0,0)^t,(1,2,0)^t,(0.75,0.75,0.5)^t\}$
$T = T \cup \{\widehat{x}_{[1]}\} = \{(0,1,0)^t\}, W = (W \cup W_{[1]}) \setminus T = W_{[1]}$.
Step 4- *Let* $k = k+1 = 2$ *, find* $\widehat{x}_{[2]}$ *as follows:*

$c_3\widehat{x}_{[2]} = \max\{c_3 x\} = \max\{-4, 2, 2.75\} = 2.75 \Longrightarrow \widehat{x}_{[2]} = (0.75, 0.75, 0.5)$
Step 2- *Solve problem (17):*

$$\begin{aligned} & \max \quad x_1 \\ s.t.: \quad & x_1 + x_1 + x_3 \le 3 \\ & x_1 + x_2 - x_3 \ge 1 \\ & x_1 - x_2 + x_3 \le 1 \\ & -x_1 + x_2 + x_3 \le 1 \\ & x_2 = 0.75 \\ & x_3 = 0.5 \\ & x_1 \ge 0. \end{aligned} \tag{17}$$

The solution to problem (17) is $\overline{x} = (1.75, 0.75, 0.5)^t$ *where* $\overline{x} \neq \widehat{x}_{[2]} = (0.75, 0.75, 0.5)^t$*; hence* $\widehat{x}_{[2]} \notin S_2$.
Step 3- *The set of extreme points adjacent to* $\widehat{x}_{[2]}$ *are:*

$\{(0,1,0)^t, (1.25, 1.25, 0.5)^t, (1, 0.5, 0.5)^t\}$
$c_3\widehat{x}_{[1]} = 2.75, c_3(0,1,0)^t = 3, c_3(1.25,1.25,0.5)^t = 2.25, c_3(1,0.5,0.5)^t = 1$
$\Longrightarrow W_{[2]} = \{(1.25,1.25,0.5)^t,(1,0.5,0.5)^t\}, T = T \cup \{\widehat{x}_{[2]}\} = \{(0,1,0)^t,(0.75,0.75,0.5)^t\}$

$W = (W \cup W_{[2]}) \setminus T \Longrightarrow W = \{(1,0,0)^t,(1,2,0)^t,(1.25,1.25,0.5)^t,(1,0.5,0.5)^t\}$
Step 4- *Let* $k = k+1 = 3$ *:*
$c_3\widehat{x}_{[3]} = \max_{x \in W}\{c_3 x\} = \max\{-4, 2, 2.25, 1\} = 2.25 \Rightarrow \widehat{x}_{[3]} = (1.25, 1.25, 0.5)^t$
Step 2- *Solve problem (18):*

$$\begin{aligned} & \max \quad x_1 \\ s.t.: \quad & x_1 + x_1 + x_3 \le 3 \\ & x_1 + x_2 - x_3 \ge 1 \\ & x_1 - x_2 + x_3 \le 1 \\ & -x_1 + x_2 + x_3 \le 1 \\ & x_2 = 1.25 \\ & x_3 = 0.5 \\ & x_1 \ge 0. \end{aligned} \tag{18}$$

The solution to problem (18) is $\overline{x} = (1.25, 1.25, 0.5)^t$ *where* $\overline{x} = \widehat{x}_{[3]}$ *; hence* $\widehat{x}_{[3]} \in S_2$.
Step 5- *Solve problem (19):*

$$
\begin{aligned}
& \max \quad x_1 \\
s.t.: \quad & x_1 + x_1 + x_3 \leq 3 \\
& x_1 + x_2 - x_3 \geq 1 \\
& x_1 - x_2 + x_3 \leq 1 \\
& -x_1 + x_2 + x_3 \leq 1 \\
& x_3 = 0.5 \\
& x_1, x_2 \geq 0.
\end{aligned} \tag{19}
$$

The solution to problem (19) is $(\widehat{x}_1^*, \widehat{x}_2^*, \widehat{x}_3)^t = (1.25, 1.25, 0.5)^t$ *. The following problem is solved to check if* $(\widehat{x}_1^*, \widehat{x}_2^*, \widehat{x}_3)$ *belongs in* S_2*:*

$$
\begin{aligned}
& \max \quad x_1 \\
s.t.: \quad & x_1 + x_1 + x_3 \leq 3 \\
& x_1 + x_2 - x_3 \geq 1 \\
& x_1 - x_2 + x_3 \leq 1 \\
& -x_1 + x_2 + x_3 \leq 1 \\
& x_2 = 1.25 \\
& x_3 = 0.5 \\
& x_1 \geq 0.
\end{aligned} \tag{20}
$$

The solution to problem (20) is $(\overline{x}_1, \widehat{x}_2^*, \widehat{x}_3)^t = (1.25, 1.25, 0.5)^t$*. Since* $\overline{x}_1 = \widehat{x}_1^*, \widehat{x}^* \in S_2$*, for* S_3*-Check go to Step 6.*

Step 6- *Since* $(\widehat{x}_1^*, \widehat{x}_2^*) = (\widehat{x}_1, \widehat{x}_2)$ *, in fact* $\widehat{x}_{[3]} = \widehat{x}^*$ *; hence,* $\widehat{x}_{[3]} \in S_3$ *, and* $\widehat{x}_{[3]}$ *is the optimal solution to the tri-level problem. Therefore,* $x^\star = (1.25, 1.25, 0.5)$ *is the absolute optimal solution, stop.*

5. Conclusions

The tri-level linear programming problem has more complexity compared to the bi-level one because it is necessary, when designing its solution algorithm, to consider the effects of the decisions of three decision makers on one another. To solve TLP problems, this paper combines the Kth-best and the revised BRES methods and proposes a novel hybrid algorithm a feature of which is to converge on the tri-level optimal solution; the algorithm's implementation capability has been illustrated through a numerical example.

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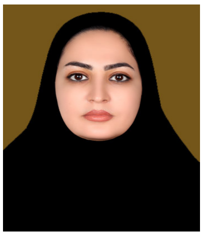

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