TWMS J. App. and Eng. Math. V.11, N.2, 2021, pp. 319-328

EXISTENCE OF FIXED POINT AND BEST POINT OF PROXIMITY FOR MULTIFUNCTIONAL NON SELF MAPPINGS IN A PARTIAL METRIC SPACE

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Abstract. In this paper we give some theorems of existence of best points of proximity for a multifunctional non-self-mapping in a partial metric space and some approximations on the sets of the best points of proximity. Other results are also given.

Keywords: Partial metric space; Fixed point; best point; proximity.

AMS Subject Classification: 47H09; 54E50; 37C25; 47H10.

1. Introduction

Let A, B be two nonempty subsets of a metric space (X, d). The aim of this paper is to establish the existence theorems of a best proximity point $\overline{x} \in A$, which satisfies $inf\{p(\overline{x}, y) : y \in F(\overline{x})\} = dist(A, B)$ for a non-self-mapping multifunction $F : A \to 2^B$. In this article, we prove that the results obtained in [3] can be enhanced and managed on partial metric spaces. We define

$$d(x, B) = \inf_{y \in B} d(x, y), \; e(A, B) = \sup_{x \in A} d(x, B)$$

and

$$D(A, B) = \max(e(A, B), e(B, A)).$$

Here $e(A, B)$ is the excess of A over B and $D(A, B)$ is the Pompeiu-Hausdorff distance between A and B. And let

$$A_0 = \{x \in A : d(x, y) = d(A, B), for\, some\, y \in B\},$$

$$B_0 = \{y \in B : d(x, y) = d(A, B), for\, some\, x \in A\}.$$

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§ Manuscript received: February 27, 2019; accepted: February 14, 2021.
TWMS Journal of Applied and Engineering Mathematics, Vol.11, No.4
The authors are partially supported by operator laboratory, Eloued, Algeria.

Definition 1.1. [3] *Let (X,d) be a partial metric space and (A,B) a pair of nonempty subsets of X, and $A \neq \emptyset$.*
We say that the pair (A,B) satisfies the P-property if and only if

$$\begin{cases} d(x_1,y_1) = d(A,B) \\ d(x_2,y_2) = d(A,B) \end{cases} \quad \Rightarrow \quad d(x_1,y_1) = d(x_2,y_2).$$

Where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

Theorem 1.1. [3] *Let (X,d) be a complete metric space and A, B two closed and nonempty subsets of X such that $A_0 \neq \emptyset$ and the pair (A,B) satisfies the P-property. We assume that $F : A \to 2^B$ a multifunction with bounded and closed value. If there exists a constant $\theta \in (0,1)$ such that $\Big[D(F(x),F(y)) \leq \theta d(x,y), \qquad \forall x,y \in A\Big]$ and $F(x) \subseteq B_0$ for all $x \in A_0$. Then F has a best proximity point $\overline{x}$ in A.*

Definition 1.2. *We say that the function $p : X \times X \to [0,+\infty[$ is a partial metric on X if the following conditions are satisfied:*

(a) *$p(x,x) = p(y,y) = p(x,y)$ if and only if $x = y$ for all $x,y \in X$,*
(b) *$p(x,x) \leq p(x,y)$ for all $x,y \in X$,*
(c) *$p(x,y) = p(y,x)$ for all $x,y \in X$,*
(d) *$p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$ for all $x,y,z \in X$.*

Then (X,p) is called a partial metric space.

For a partial metric p on X, the function $p^s : X \times X \to \mathbb{R}^+$ given by

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y) \tag{1}$$

and

$$p^w(x,y) = p(x,y) - min\{p(x,x), p(y,y)\} \tag{2}$$

are metrics on X. Each partial metric p on X Generates a T_0-topology τ_p with a basis the families of open p-balls $\{B_p(x,\varepsilon) : x \in X, \ \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Definition 1.3. [17]

(i) *A sequence $\{x_n\}$ in a partial metric space (X,p) converges to $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$.*
(ii) *A sequence $\{x_n\}$ in a partial metric space (X,p) is called Cauchy sequence if and only if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists and finite.*
(iii) *A partial metric space (X,p) est is complete if any Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.*

Lemma 1.1. [17]

($a1$) *A sequence $\{x_n\}$ is Cauchy in a partial metric space (X,p) if and only if $\{x_n\}$ is Cauchy in the space (X,p^s).*
($a2$) *A partial metric space (X,p) is complete if and only if the space (X,p^s) Is complete. Morewer*

$$\lim_{n\to\infty} p^s(x,x_n) = 0 \Leftrightarrow p(x,x) = \lim_{n\to\infty} p(x,x_n) = \lim_{n,m\to\infty} p(x_n,x_m). \tag{3}$$

Let A, B be two nonempty subsets of a partial metric space (X, p), the partial surplus $e_p(A, B)$ from A to B and $D_p(A, B)$ are the partial distance of Pompeiu-Hausdorff between A and B defined as follows:

$$p(x, B) = \inf_{y \in B} p(x, y),$$

$$p(A, B) = \sup_{x \in A} p(x, B)$$

and

$$p(A, B) = \max(e_p(A, B), e_p(B, A)).$$

Remark 1.1. [6] *Let (X, p) be a partial metric space and A a nonempty subset of X, then $a \in \overline{A} \Leftrightarrow p(a, A) = p(a, a)$.*

Lemma 1.2. [7] *Let (X, p) be a partial metric space and A, B two closed, nonempty and bounded subsets of X, and $h > 1$. Then for each $a \in A$, there exists $b \in B$, such that $p(a, b) \leq hD_p(A, B)$.*

We next apply the notations

$$A_0 = \{x \in A : p(x, y) = p(A, B), for\, some\, y \in B\}$$

and

$$B_0 = \{y \in B : p(x, y) = p(A, B), for\, some\, x \in A\}.$$

Definition 1.4. *Let (X, p) be a partial metric space and (A, B) a pair of nonempty subsets of X, and $A \neq \emptyset$.*
We say that the pair (A, B) satisfies the P-property if and only if

$$\begin{cases} p(x_1, y_1) = p(A, B) \\ p(x_2, y_2) = p(A, B) \end{cases} \Rightarrow \quad p(x_1, y_1) = p(x_2, y_2).$$

Where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

2. Main results

2.1. **Existence of best proximity point for non-self mapping multifunction.**

Theorem 2.1. *Let (X, p) be a complete partial metric space and A, B two closed and nonempty subsets of X such that $A_0 \neq \emptyset$ and the pair (A, B) satisfies the P-property . We assume that $F : A \to 2^B$ a multifunction with bounded and closed values. If there exists a constant $\theta \in (0, 1)$ such that*

$$D_p(F(x), F(y)) \leq \theta p(x, y), \quad \forall x, y \in A.$$

And $F(x) \subseteq B_0$ for each $x \in A_0$. Then F has a best proximity point $\overline{x}$ in A.

Proof. Let $x_0 \in A_0$, and $y_0 \in F(x_0) \subseteq B_0$. Then there exists $x_1 \in A_0$, such that $p(x_1, y_0) = p(A, B)$. On the other hand, by lemma 1.2 we obtain $y_1 \in F(x_1) \subseteq B_0$ such that

$$p(y_0, y_1) \leq D_p(F(x_0), F(x_1)) + \theta.$$

Futher more, there exists $x_2 \in A_0$, such that $p(x_1, y_1) = p(A, B)$, and $y_2 \in F(x_2) \subseteq B_0$ such that

$$p(y_1, y_2) \leq D_p(F(x_1), F(x_2)) + \theta^2.$$

One continues for getting $x_n \in A_0$, such that $p(x_n, y_{n-1}) = p(A, B)$, and $y_n \in F(x_n) \subseteq B_0$ such that

$$p(y_{n-1}, y_n) \leq D_p(F(x_{n-1}), F(x_n)) + \theta^n.$$

Since $p(x_{n+1}, y_n) = p(A, B)$ and $p(x_n, y_{n-1}) = p(A, B)$, by the P-property, we obtain $p(x_n, x_{n+1}) = p(y_{n-1}, y_n)$. Which gives

$$\begin{aligned} p(x_n, x_{n+1}) = p(y_{n-1}, y_n) &\leq D_p(F(x_{n-1}), F(x_n)) + \theta^n \\ &\leq \theta p(x_{n-1}, x_n) + \theta^n \\ &\leq \theta p(y_{n-2}, y_{n-1}) + \theta^n) \\ &\leq \theta(D_p(F(x_{n-2}), F(x_{n-1})) + \theta^{n-1}) + \theta^n \\ &\leq \theta^2 p(x_{n-2}, x_{n-1}) + 2\theta^n \\ &\vdots \\ &\leq \theta^n p(x_0, x_1) + n\theta^n \to 0, \ when\, n \to +\infty. \end{aligned}$$

Next by definition of p^s, we get $p^s(x_n, x_{n+1}) \leq 2p(x_n, x_{n+p}) \to 0$, when $n \to +\infty$. We now prove that (x_n) is a Cauchy sequence in (X, p^s). Suppose that there exist an $\epsilon > 0$ and $k \in \mathbb{N}$, there exist $m_k, n_k \in \mathbb{N}$ such that $m_k > n_k > k$ and $p(x_{m_k}, x_{n_k}) > \epsilon$. We have

$$\begin{aligned} p(A, B) &\leq p(y_{n_k-1}, x_{m_k}) \\ &\leq p(y_{n_k-1}, x_{n_k}) + p(x_{n_k}, x_{m_k}) - p(x_{n_k}, x_{n_k}) \\ &\leq p(y_{n_k-1}, x_{n_k}) + p(x_{n_k}, x_{m_k}) \\ &\leq p(y_{n_k-1}, x_{n_k}) + p(x_{n_k}, x_{n_k+1}) + ... + p(x_{m_k-1}, x_{m_k}). \end{aligned}$$

Taking the limits on k we get

$$\lim_{k\to\infty} p(y_{n_k-1}, x_{n_k}) + \lim_{k\to\infty} p(x_{n_k}, x_{m_k}) = p(A, B).$$

Which gives $\lim_{k\to\infty} p(x_{n_k}, x_{m_k}) = 0$. Since $p^s(x_{n_k}, x_{m_k}) < 2p(x_{n_k}, x_{m_k}) \to 0$, then (x_n) is a Cauchy sequence in (X, p^s), and since (X, p) is complete then by lemma 1.1, (X, p^s) is a complete metric space and so the sequence (x_n) converges in X. Let $\overline{x} = \lim_n x_n$. Since A is closed, we have $\overline{x} \in A$.
Now, since $p(x_n, x_{n+1}) = p(y_{n-1}, y_n)$, the sequence (y_n) is convergent in X. Let $\overline{y} = \lim_n y_n$.

Sine B is closed, we have $\overline{y} \in B$. On the other hand, by lemma 1.1, we have

$$p(\overline{x},\overline{x}) = \lim_{n\to+\infty} p(x_n,\overline{x}) = \lim_{n,m\to+\infty} p(x_n,x_m) = 0$$

and

$$p(\overline{y},\overline{y}) = \lim_{n\to+\infty} p(y_n,\overline{y}) = \lim_{n,m\to+\infty} p(y_n,y_m) = 0$$

and $p(\overline{x},\overline{y}) = p(A,B)$. Hence,

$$\begin{aligned} 0 &\leq p(y_n, F(\overline{x})) \\ &\leq D_p(F(x_n), F(\overline{x})) \\ &\leq \theta p(x_n,\overline{x}). \end{aligned}$$

We take the limit $n \to +\infty$, we obtain $p(\overline{y}, F(\overline{x})) = 0$ which gives $p(\overline{y}, F(\overline{x})) = p(\overline{y},\overline{y})$, using remark 1.1, to get $\overline{y} \in \overline{F(\overline{x})} = F(\overline{x})$. One obtains that $\overline{x}$ is a best proximity point in A which satisfies $p(\overline{x}, F(\overline{x})) = p(A,B)$. □

As a direct result of Theorem 2.1, we have the following result.

Corollary 2.1. [7] *Let (X,p) be a complete partial metric space and $F : X \to 2^X$ be a multifunction with bounded and closed values. If there exists a constant $\theta \in (0,1)$ such that*

$$D_p(F(x),F(y)) \leq \theta p(x,y), \;\; \forall x \in X \text{ and } \forall y \in X.$$

Then F has a fixed point $\overline{x}$ in X.

In the following, let PF_i be the set of best proximity points for a multifunction F_i.

Theorem 2.2. *Let (X,d) be a complete metric space and A, B two closed nonempty subsets of X such that $A_0 \neq \emptyset$ and the pair (A,B) satisfies the P-property. Let $F_i : A \to 2^B, i = 1,2$ be two multifunctions with compact non empty values. If there exist two constants $\theta_1, \theta_2 \in (0,1)$ such that*

$$D(F_1(x),F_1(y)) \leq \theta_1 d(x,y) \;\forall x \in A \text{ and } \forall y \in A,$$

$$D(F_2(x),F_2(y)) \leq \theta_2 d(x,y) \;\forall x \in A \text{ and } \forall y \in A$$

and $F_i(x) \subseteq B_0, i = 1,2$, for each $x \in A_0$. Then

$$D(PF_1,PF_2) \leq \frac{1}{1-max\{\theta_1,\theta_2\}}[\sup_{x\in A} D(F_1(x),F_2(x))].$$

Proof. Let $\varepsilon > 0$, we choose $\beta > 0$ such that $\beta \sum n\theta_2^n < 1$ and $\alpha = \dfrac{\beta\varepsilon}{(1-\theta_2)}$. Let $x_{0,1} = x_0 \in PF_1$, there exists $y_{0,1} \in F_1(x_0) \subseteq B_0$ such that $d(x_0,y_{0,1}) = d(A,B)$. On the other hand, there exist $y_0 \in F_2(x_0) \subseteq B_0$ and $x_1 \in A_0$ such that $d(x_1,y_0) = d(A,B)$. By the P-property, we get $d(x_0,x_1) = d(y_{0,1},y_0)$. Which gives

$$d(x_0,x_1) \leq D(F_1(x_0),F_2(x_0)) + \varepsilon.$$

We take $y_1 \in F(x_1) \subseteq B_0$ such that

$$d(y_0,y_1) \leq D(F_2(x_0),F_2(x_1)) + \theta_2\alpha.$$

Furthermore, there exists $x_1 \in A_0$, such that $d(x_1, y_1) = d(A, B)$, and there exists $y_2 \in F_2(x_2) \subseteq B_0$ such that

$$d(y_1, y_2) \leq D(F_2(x_1), F_2(x_2)) + \theta_2^2 \alpha.$$

We continue to find $x_n \in A_0$, such that $d(x_n, y_{n-1}) = d(A, B)$, and there exists $y_n \in F_2(x_n) \subseteq B_0$ such that

$$d(y_{n-1}, y_n) \leq D(F_2(x_{n-1}), F_2(x_n)) + \theta_2^n \alpha.$$

Since $d(x_{n+1}, y_n) = d(A, B)$ and $d(x_n, y_{n-1}) = d(A, B)$, by the P-property, we get $d(x_n, x_{n+1}) = d(y_{n-1}, y_n)$. Which gives

$$\begin{aligned} d(x_n, x_{n+1}) = d(y_{n-1}, y_n) &\leq D(F_2(x_{n-1}), F_2(x_n)) + \theta_2^n \alpha \\ &\leq \theta_2 d(x_{n-1}, x_n) + \theta_2^n \alpha \\ &\leq \theta_2 d(y_{n-2}, y_{n-1}) + \theta_2^n \alpha \\ &\leq \theta_2 (D(F_2(x_{n-2}), F_2(x_{n-1})) + \theta_2^{n-1}) + \theta_2^n \alpha \\ &\leq \theta_2^2 d(x_{n-2}, x_{n-1}) + 2\theta_2^n \alpha \\ &\vdots \\ &\leq \theta_2^n d(x_0, x_1) + n\theta_2^n \alpha \to 0, \ when\, n \to +\infty. \end{aligned} \tag{4}$$

On the other hand, we prove that (x_n) is a Cauchy sequence in A. One suppose that there exists $\epsilon > 0$ and for all $k \in \mathbb{N}$, there exist $m_k, n_k \in \mathbb{N}$ such that $m_k > n_k > k$ and $d(x_{m_k}, x_{n_k}) > \epsilon$. We have

$$\begin{aligned} d(A, B) &\leq d(y_{n_k-1}, x_{m_k}) \\ &\leq d(y_{n_k-1}, x_{n_k}) + d(x_{n_k}, x_{m_k}) \\ &\leq d(y_{n_k-1}, x_{n_k}) + d(x_{n_k}, x_{m_k}) \\ &\leq d(y_{n_k-1}, x_{n_k}) + d(x_{n_k}, x_{n_k+1}) + ... + d(x_{m_k-1}, x_{m_k}). \end{aligned} \tag{5}$$

Taking the limit, we get

$$\lim_{k \to \infty} d(y_{n_k-1}, x_{n_k}) + \lim_{k \to \infty} d(x_{n_k}, x_{m_k}) = d(A, B).$$

Which gives $\lim\limits_{k \to \infty} d(x_{n_k}, x_{m_k}) = 0$. We get (x_n) a Cauchy sequence in a complete metric space and so the sequence (x_n) is convergent in X. Let $\overline{x_2} = \lim\limits_n x_n$. Since A is closed, we get $\overline{x_2} \in A$.

Since $d(x_n, x_{n+1}) = d(y_{n-1}, y_n)$, the sequence (y_n) converges in X. Let $\overline{y} = \lim\limits_n y_n$. Since B is closed, we get $\overline{x_2} \in B$ and

$$\begin{aligned} 0 &\leq d(y_n, F_2(\overline{x_2})) \\ &\leq D(F_2(x_n), F_2(\overline{x_2})) \\ &\leq \theta_2 d(x_n, \overline{x_2}). \end{aligned}$$

Taking the limit, $n \to +\infty$, we get $d(\overline{y}, F_2(\overline{x_2})) = 0$. We obtain $\overline{x_2}$ as the best proximity point in A which satisfies $d(\overline{x_2}, F_2(\overline{x_2})) = d(A, B)$. Then we get

$$\begin{aligned} d(x_0, \overline{x_2}) &\leq \sum_{n=0}^{\infty} d(x_n, x_{n+1}) \\ &\leq \sum_{n=0}^{\infty} (\theta_2^n d(x_0, x_1) + n\theta_2^n \alpha) \\ &\leq \frac{1}{1-\theta_2} d(x_0, x_1) + \sum_{n=1}^{\infty} n\theta_2^n \alpha) \\ &\leq \frac{1}{1-\theta_2}(d(x_0, x_1) + \varepsilon) \\ &\leq \frac{1}{1-\theta_2}(D(F_1(x_0), F_2(x_0)) + 2\varepsilon) \end{aligned} \tag{6}$$

We obtain for all $\varepsilon > 0$,

$$d(x_{0,1}, \overline{x_2}) \leq \frac{1}{1-\theta_2}(D(F_1(x_{0,1}), F_2(x_{0,1})) + \varepsilon).$$

As previously, let $x_{0,2} \in PF_2$, there exists $\overline{x_1} \in PF_1$ such that

$$d(x_{0,2}, \overline{x_1}) \leq \frac{1}{1-\theta_1}(D(F_1(x_{0,1}), F_2(x_{0,1})) + \varepsilon), \ \forall \varepsilon > 0.$$

□

Theorem 2.3. *Let (X, p) a complete partial metric space and A, B two closed nonempty subsets of X such that $A_0 \neq \emptyset$ and the pair (A, B) satisfies the P-property. Let $F_i : A \to 2^B, i = 1, 2$ two multifunctions with compact values. If there exist two constants $\theta_1, \theta_2 \in (0, 1)$ such that*

$$D_p(F_1(x), F_1(y)) \leq \theta_1 p(x, y) \ \forall x \in A\,,\, \forall y \in A,$$
$$D_p(F_2(x), F_2(y)) \leq \theta_2 p(x, y) \ \forall x \in A\,,\, \forall y \in A,$$

and $F_i(x) \subseteq B_0, i = 1, 2,$ for each $x \in A_0$. Then

$$D_p(PF_1, PF_2) \leq \frac{1}{1 - max\{\theta_1, \theta_2\}}[\sup_{x \in A} D_p(F_1(x), F_2(y))].$$

Proof. Let $\varepsilon > 0$, we choose $\beta > 0$ such that $\beta \sum n\theta_2^n < 1$ and $\alpha = \dfrac{\beta\varepsilon}{(1-\theta_2)}$. Let $x_{0,1} = x_0 \in PF_1$, there exist $y_{0,1} \in F_1(x_0) \subseteq B_0$ such that $d(x_0, y_{0,1}) = d(A, B)$. Otherwise, there exist $y_0 \in F_2(x_0) \subseteq B_0$ and $x_1 \in A_0$ such that $p(x_1, y_0) = p(A, B)$. By the P-property, we get $p(x_0, x_1) = p(y_{0,1}, y_0)$, which gives

$$p(x_0, x_1) \leq D_p(F_1(x_0), F_2(x_0)) + \varepsilon.$$

We take $y_1 \in F(x_1) \subseteq B_0$ such that

$$p(y_0, y_1) \leq D_p(F_2(x_0), F_2(x_1)) + \theta_2.$$

Else, there exists $x_1 \in A_0$, such that $p(x_1, y_1) = p(A, B)$, and there exists $y_2 \in F_2(x_2) \subseteq B_0$

such that

$$p(y_1, y_2) \leq D_p(F_2(x_1), F_2(x_2)) + \theta_2^2.$$

We continue to find $x_n \in A_0$, such that $p(x_n, y_{n-1}) = p(A, B)$, and $y_n \in F_2(x_n) \subseteq B_0$ such that

$$p(y_{n-1}, y_n) \leq D_p(F_2(x_{n-1}), F_2(x_n)) + \theta_2^n.$$

Since $p(x_{n+1}, y_n) = p(A, B)$ and $p(x_n, y_{n-1}) = p(A, B)$, by the P-property, we get $p(x_n, x_{n+1}) = p(y_{n-1}, y_n)$. Which gives

$$\begin{aligned} p(x_n, x_{n+1}) = p(y_{n-1}, y_n) &\leq D_p(F_2(x_{n-1}), F_2(x_n)) + \theta_2^n \\ &\leq \theta_2 p(x_{n-1}, x_n) + \theta_2^n \\ &\leq \theta_2(D_p(F_2(y_{n-2}), F_2(y_{n-1})) + \theta_2^{n-1}) \\ &\leq \theta_2(D_p(F_2(x_{n-2}), F_2(x_{n-1})) + \theta_2^{n-1}) + \theta_2^n \\ &\leq \theta_2^2 p(x_{n-2}, x_{n-1}) + 2\theta_2^n \\ &\vdots \\ &\leq \theta_2^n p(x_0, x_1) + n\theta_2^n \to 0, \; when\, n \to +\infty. \end{aligned}$$

By definition of p^s, we get $p^s(x_n, x_{n+1}) \leq 2p(x_n, x_{n+p}) \to 0$, when $n \to +\infty$. On the other hand, we prove that (x_n) is a Cauchy in (X, p^s). One suppose that there exist $\epsilon > 0$ and for all $k \in \mathbb{N}$, there exists $m_k, n_k \in \mathbb{N}$ such that $m_k > n_k > k$ and $p(x_{m_k}, x_{n_k}) > \epsilon$. We have

$$\begin{aligned} p(A, B) &\leq p(y_{n_k-1}, x_{m_k}) \\ &\leq p(y_{n_k-1}, x_{n_k}) + p(x_{n_k}, x_{m_k}) - p(x_{n_k}, x_{n_k}) \\ &\leq p(y_{n_k-1}, x_{n_k}) + p(x_{n_k}, x_{m_k}) \\ &\leq p(y_{n_k-1}, x_{n_k}) + p(x_{n_k}, x_{n_k+1}) + ... + p(x_{m_k-1}, x_{m_k}). \end{aligned}$$

Taking the limit we get

$$\lim_{k\to\infty} p(y_{n_k-1}, x_{n_k}) + \lim_{k\to\infty} p(x_{n_k}, x_{m_k}) = p(A, B).$$

Which gives $\lim\limits_{k\to\infty} p(x_{n_k}, x_{m_k}) = 0$. Since $p^s(x_{n_k}, x_{m_k}) < 2p(x_{n_k}, x_{m_k}) \to 0$, we get (x_n) a Cauchy sequence in (X, p^s), and since (X, p) is complete then by lemma 1.1, (X, p^s) is a complete metric space and the sequence (x_n) converges in X. Let $\overline{x_2} = \lim\limits_n x_n$. Since A is closed we have $\overline{x_2} \in A$. Next, since $p(x_n, x_{n+1}) = p(y_{n-1}, y_n)$, the sequence (y_n) converges in X. Let $\overline{y} = \lim\limits_n y_n$. Since B is closed we have $\overline{y} \in B$.
Otherwise, by lemma 1.1, we have

$$p(\overline{x_2}, \overline{x_2}) = \lim_{n\to+\infty} p(x_n, \overline{x_2}) = \lim_{n,m\to+\infty} p(x_n, x_m) = 0,$$

$$p(\overline{y}, \overline{y}) = \lim_{n \to +\infty} p(y_n, \overline{y}) = \lim_{n,m \to +\infty} p(y_n, y_m) = 0,$$

and $p(\overline{x_2}, \overline{y}) = p(A, B)$. As result,

$$\begin{aligned} 0 &\leq p(y_n, F_2(\overline{x_2})) \\ &\leq D_p(F_2(x_n), F_2(\overline{x_2})) \\ &\leq \theta_2 p(x_n, \overline{x_2}). \end{aligned}$$

Taking the limit $n \to +\infty$, we get $p(\overline{y}, F_2(\overline{x_2})) = 0$ which gives $p(\overline{y}, F_2(\overline{x_2})) = p(\overline{y}, \overline{y})$, using remark 1.1, to get $\overline{y} \in \overline{F_2(\overline{x_2})} = F_2(\overline{x_2})$. Then we obtain $\overline{x_2}$ as a best proximity point in A which satisfies $p(\overline{x_2}, F_2(\overline{x_2})) = p(A, B)$. Then we get

$$\begin{aligned} p(x_0, \overline{x_2}) &\leq \sum_{n=0}^{\infty} p(x_n, x_{n+1}) - \sum_{n=1}^{\infty} p(x_n, x_n) \\ &\leq \sum_{n=0}^{\infty} p(x_n, x_{n+1}) \\ &\leq \sum_{n=0}^{\infty} (\theta_2^n p(x_0, x_1) + n\theta_2^n \alpha) \\ &\leq \frac{1}{1-\theta_2} p(x_0, x_1) + \sum_{n=1}^{\infty} n\theta_2^n \alpha) \\ &\leq \frac{1}{1-\theta_2} (p(x_0, x_1) + \varepsilon) \\ &\leq \frac{1}{1-\theta_2} (D_p(F_1(x_0), F_2(x_0)) + 2\varepsilon). \end{aligned} \tag{7}$$

We obtain for all $\varepsilon > 0$,

$$p(x_{0,1}, \overline{x_2}) \leq \frac{1}{1-\theta_2} (D_p(F_1(x_{0,1}), F_2(x_{0,1})) + \varepsilon).$$

As previously, let $x_{0,2} \in PF_2$, there exists $\overline{x_1} \in PF_1$ such that

$$p(x_{0,2}, \overline{x_1}) \leq \frac{1}{1-\theta_1} (D_p(F_1(x_{0,1}), F_2(x_{0,1})) + \varepsilon), \ \forall \varepsilon > 0.$$

□

3. Conclusions

In this paper, we have proved some results of best points of proximity for a multifunctional non-self-mapping in a partial metric space and some approximations on the sets of the best points of proximity.

Acknowledgement. The authors would like to express their sincere apreciation to the referees for their very helpful suggestions and many kind comments.

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