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FUZZY BIPOLAR SOFT TOPOLOGICAL SPACES

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Abstract. Topological structures on soft, fuzzy soft, bipolar soft sets and etc. were studied by several researchers. In this paper, we define topological space on fuzzy bipolar soft sets and investigate the main results. We define fuzzy bipolar soft point and then we give the definition of fuzzy bipolar soft interior and closure points. Also we search properties of fuzzy bipolar soft interior and closure of fuzzy bipolar soft set.

Keywords: fuzzy bipolar soft set,fuzzy bipolar soft topological spaces, fuzzy bipolar soft point, fuzzy bipolar soft interior, fuzzy bipolar soft closure.

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1. Introduction

Analyzing the characteristics of data might give an insight, if one aims to model the real life problems by using mathematical tools. For instance, vagueness and uncertainty are important characteristics of the data which are mostly involved in the real life problems with which we encounter in different areas such as engineering, economics, medicine, social sciences etc. Since the vagueness violates the certainty of information, classical mathematical methods fail to solve those problems accurately. To surpass this insufficiency many theories are proposed such as theory of fuzzy sets [17], theory of rough sets [11], theory of intuitionistic fuzzy sets [3]. Among these theories, Zadeh's fuzzy set theory [17] is the most popular one. This theory has been used by many researchers and thus it spread to different fields. Although these theories have brought several novelties into the classical theories, they all have some kind of difficulties. In 1999, Molodtsov [8] defined soft set theory and he stated that this new theory is exempt from the difficulties seen in other theories since it has sufficient parametrization tools. Molodtsov also connected his theory successfully with different fields as Riemann integration, smoothness of functions, probability, game theory etc. Soft set theory is studied by many researchers and is applied to several branches, see for example [1, 2, 5, 10, 13, 18].

As it is known, in Boolean logic a property is either present or absent, i.e. it takes values in the set $\{0, 1\}$. In the advanced theories mentioned above, the membership

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function determines up to which degree a property exists or an element belongs to a set. In other words, the existence of a property is determined by the parameterized function which takes values in the interval $[0, 1]$. However, the theory of rough sets introduced by Pawlak emphasize the insufficiency of parametrization [11]. In order to deal with this problem, Maji et al. [7] defined the fuzzy soft sets in 2001 as a generalization of soft sets and in this way the fuzziness of data is incorporated. This hybrid model was also studied by several researchers, see [7, 12, 14, 16, 6].

As one can realize that the Boolean and fuzzy approaches focus only on the existence of a property. Hence, there is no chance to model the coexistence of a property with these approaches. Coexistence is in general associated with bipolarity of an information which roughly presents the existence or counter-existence of a property. Thus, bipolarity is also an important characteristic of the data which should be taken into account. As a further extension, fuzzy bipolar sets are introduced by Zhang [19]. Fuzzy bipolar sets whose membership function takes values within the range $[1, 1]$ can be seen as an extension of fuzzy sets. In a fuzzy bipolar set, the membership function of an element can be interpreted as follows:

- If the membership degree of an element is equal to 0, then the element is irrelevant to the property.
- If the membership function takes values in $(0, 1]$, then the element is positively related to the property, i.e. it up to some extent satisfies the property.
- If the membership function takes values in $[-1, 0)$, then the element is negatively related to the property, i.e. it up to some extent satisfies the counter-property.

Moreover, fuzzy bipolar soft set is a new type of hybrid models for vagueness and is first introduced by Naz and Shabir in 2013 [9]. In their work, they considered the algebraic structures of fuzzy bipolar soft sets and they illustrated an application of the fuzzy bipolar soft sets to a decision making problem.

In the present study, we first define the fuzzy bipolar soft topological spaces and present the main properties of this new topological structure. We further examine the relations among fuzzy bipolar soft topology, fuzzy soft topology and soft topology. Finally, we define the fuzzy bipolar soft interior and closure points and subsequently give the relevant main results.

2. Preliminaries

In this section, we briefly present several definitions which are necessary to build fuzzy bipolar soft topological spaces. Hence, throughout the paper U denotes the universe, E denotes the parameter set and $FP(U)$ denotes the collection off all fuzzy subsets of U. Let A, B and C be nonempty sets of E.

Definition 2.1. [9] *A triplet (F, G, A) is called a fuzzy bipolar soft set (hereafter, fbs set) over (U, E), if $F : A \mapsto FP(U)$ and $G : \neg A \mapsto FP(U)$*

$$0 \leq F(e)(x) + G(\neg e)(x) \leq 1$$

for all $e \in A$.

Definition 2.2. [9] *Let (F_1, G_1, A_1) and (F_2, G_2, A_2) be two fbs set over (U, E). (F_1, G_1, A_1) is called fbs subset of (F_2, G_2, A_2) if*

(1) $A_1 \subseteq A_2$,
(2) $F_1(e) \subseteq F_2(e)$ *and* $G_2(\neg e) \subseteq G_1(\neg e)$ *for all* $e \in A_1$

and it is shown that $(F_1, G_1, A_1) \subseteq^{\sim} (F_2, G_2, A_2)$.
Two fbs sets (F_1, G_1, A_1) *and* (F_2, G_2, A_2) *are said to be equal if* $(F_1, G_1, A_1) \subseteq^{\sim} (F_2, G_2, A_2)$ *and* $(F_2, G_2, A_2) \subseteq^{\sim} (F_1, G_1, A_1)$.

Definition 2.3. [9] *The complement of a fbs set*(F, G, A) *over* (U, E) *is shown by* $(F, G, A)^c = (F^c, G^c, A)$ *where* F^c, G^c *are mappings defined by* $F^c(e) = G(\neg e)$ *and* $G^c(\neg e) = F(e)$ *for all* $e \in A$.

Definition 2.4. [9] *The fbs set* (F, G, A) *over* (U, E) *is called relative null fuzzy bipolar soft set, denoted by* (Φ, U, A)*, if for all* $e \in A$, $\Phi(e) = \emptyset$ *and* $U(\neg e) = U$.

Definition 2.5. [9] *The fbs set* (F, G, A) *over* (U, E) *is called relative absolute fbs set, denoted by* (U, Φ, A)*, if for all* $e \in A$, $U(e) = U$ *and* $\Phi(\neg e) = \emptyset$.

Definition 2.6. [9] *Let* (F_1, G_1, A_1) *and* (F_2, G_2, A_2) *be two fbs sets over over* (U, E)*. The union of* (F_1, G_1, A_1) *and* (F_2, G_2, A_2) *denoted by* $(F_1, G_1, A_1) \cup^{\sim} (F_2, G_2, A_2) = (H, I, C)$ *is a fbs set* (H, I, C) *over* (U, E) *where* $C = A \cup B$ *and is defined by*

$$H(e) = \begin{cases} F_1(e), & \text{if } e \in A - B, \\ F_2(e), & \text{if } e \in B - A, \\ max\{F_1(e), F_2(e)\}, & \text{if } e \in A \cap B \end{cases}$$

$$I(\neg e) = \begin{cases} G_1(\neg e), & \text{if } e \in \neg A - \neg B, \\ G_2(\neg e), & \text{if } e \in \neg B - \neg A, \\ min\{G_1(\neg e), G_2(\neg e)\}, & \text{if } e \in \neg A \cap \neg B \end{cases}$$

Definition 2.7. [9] *Let* (F_1, G_1, A_1) *and* (F_2, G_2, A_2) *be two fbs sets over over* (U, E)*. The intersection of* (F_1, G_1, A_1) *and* (F_2, G_2, A_2) *denoted by* $(F_1, G_1, A_1) \cap^{\sim} (F_2, G_2, A_2) = (H, I, C)$ *is a fbs set* (H, I, C) *over* (U, E) *where* $C = A \cup B$ *and is defined by*

$$H(e) = \begin{cases} F_1(e), & \text{if } e \in A - B, \\ F_2(e), & \text{if } e \in B - A, \\ min\{F_1(e), F_2(e)\}, & \text{if } e \in A \cap B \end{cases}$$

$$I(\neg e) = \begin{cases} G_1(\neg e), & \text{if } e \in \neg A - \neg B, \\ G_2(\neg e), & \text{if } e \in \neg B - \neg A, \\ max\{G_1(\neg e), G_2(\neg e)\}, & \text{if } e \in \neg A \cap \neg B \end{cases}$$

Theorem 2.1. [9] *Let* $(F_1, G_1, A_1), (F_2, G_2, A_2)$ *be two bfs sets over (U,E).*

(1) $[(F_1, G_1, A_1) \cup^{\sim} (F_2, G_2, A_2)]^c = (F_1, G_1, A_1)^c \cap^{\sim} (F_2, G_2, A_2)^c$.
(2) $[(F_1, G_1, A_1) \cap^{\sim} (F_2, G_2, A_2)]^c = (F_1, G_1, A_1)^c \cup^{\sim} (F_2, G_2, A_2)^c$.

3. Fuzzy Bipolar Soft Topological Spaces

In this section, we present our main results. For this purpose, we first define the fuzzy bipolar soft topological space.

Definition 3.1. *A fuzzy bipolar soft topological space is a triplet* (U, τ, E) *(hereafter, FBST space) where* τ *is a family of fbs sets over* (U, E) *satisfying the following properties:*

(1) $(\Phi, U, A), (U, \Phi, A) \in \tau$.
(2) *If* $(F_1, G_1, A_1), (F_2, G_2, A_2) \in \tau$ *then* $(F_1, G_1, A_1) \cap^{\sim} (F_2, G_2, A_2) \in \tau$.
(3) *If* $(F_i, G_i, A_i) \in \tau$ *for all* $i \in I$ *then* $\bigcup_{i \in I}^{\sim} (F_i, G_i, A_i) \in \tau$.

τ is called a fuzzy bipolar soft topology of fbs sets over (U, E). The members of τ are called fuzzy bipolar soft open sets (hereafter fbs open set).

Definition 3.2. *Let (U, τ, E) be a FBST space. A fbs set set (F, G, A) over (U, E) is called fbs closed set if the complement of this set is a fbs set.*

Theorem 3.1. *Let (U, τ, E) be a FBST space*

1. *$(\Phi, U, A), (U, \Phi, A)$ are fbs closed sets.*
2. *The arbitrary intersection of fbs closed sets is fbs closed set.*
3. *The intersection of two fbs closed set is fbs closed set.*

Proof. The proof can be seen easily by the complement operation. □

Example 3.1. *Let $U = \{x, y, z\}, E = \{e_1, e_2, e_3\}$. Then*

$$\tau = \{(\Phi, U, E), (U, \Phi, E), (F_1, G_1, A_1), (F_2, G_2, A_2), (F_3, G_3, A_3), (F_4, G_4, A_4)\}$$

is a FBST space where:

$$\begin{aligned}
F_1(e_1) &= \{x_{0.4}, y_{0.5}, z_{0.2}\}, \quad & G_1(\neg e_1) &= \{x_{0.5}, y_{0.3}, z_{0.4}\}\\
F_1(e_2) &= \{x_{0.6}, y_{0.3}, z_{0.1}\}, \quad & G_1(\neg e_2) &= \{x_{0.3}, y_{0.6}, z_{0.8}\}\\
F_2(e_1) &= \{x_{0.3}, y_{0.4}, z_{0.6}\}, \quad & G_2(\neg e_1) &= \{x_{0.4}, y_{0.5}, z_{0.3}\}\\
F_2(e_3) &= \{x_{0.2}, y_{0.3}, z_{0.4}\}, \quad & G_2(\neg e_3) &= \{x_{0.6}, y_{0.5}, z_{0.2}\}\\
F_3(e_1) &= \{x_{0.4}, y_{0.5}, z_{0.6}\}, \quad & G_3(\neg e_1) &= \{x_{0.4}, y_{0.3}, z_{0.3}\}\\
F_3(e_2) &= \{x_{0.6}, y_{0.3}, z_{0.1}\}, \quad & G_3(\neg e_2) &= \{x_{0.3}, y_{0.6}, z_{0.8}\}\\
F_3(e_3) &= \{x_{0.2}, y_{0.3}, z_{0.4}\}, \quad & G_3(\neg e_3) &= \{x_{0.6}, y_{0.5}, z_{0.2}\}\\
F_4(e_1) &= \{x_{0.3}, y_{0.4}, z_{0.2}\}, \quad & G_4(\neg e_1) &= \{x_{0.5}, y_{0.5}, z_{0.4}\}\\
F_4(e_2) &= \{x_{0.6}, y_{0.3}, z_{0.1}\}, \quad & G_4(\neg e_2) &= \{x_{0.3}, y_{0.6}, z_{0.8}\}\\
F_4(e_3) &= \{x_{0.2}, y_{0.3}, z_{0.4}\}, \quad & G_4(\neg e_3) &= \{x_{0.6}, y_{0.5}, z_{0.2}\}
\end{aligned}$$

Definition 3.3. *$(F, G, \{e\})$ is called a fuzzy bipolar soft point (hereafter bfs point) over (U, E) if there exists $x \in U$ such that $F(e)(x) = p, (0 \le p \le 1)$, $G(\neg e)(x) = q, (0 \le q \le 1)$ and $0 \le p + q \le 1$. We denote this fuzzy soft point by $(e_x^p, \neg e_x^q)$.*

Definition 3.4. *Let $(e_x^p, \neg e_x^q)$ be a bfs point and (F, G, A) be bfs set over (U, E) .$(e_x^p, \neg e_x^q)$ is called an element of (F, G, A), denoted by $(e_x^p, \neg e_x^q) \in^\sim (F, G, A)$ iff $p \le F(e)(x), G(\neg e)(x) \le q$.*

Example 3.2. *Let $U = \{x, y, z\}$, $E = \{e_1, e_2, e_3\}$ and (F, G, A) be a fbs set over (U, E) where,*

$$\begin{aligned}
F(e_1) &= \{x_{0.3}, y_{0.4}, z_{0.5}\}, \quad & G(\neg e_1) &= \{x_{0.4}, y_{0.5}, z_{0.2}\}\\
F(e_2) &= \{x_{0.2}, y_{0.5}, z_{0.9}\}, \quad & G(\neg e_2) &= \{x_{0.3}, y_{0.4}, z_{0.1}\}
\end{aligned}$$

Then $(e_{1_y}^{0.3}, \neg e_{1_y}^{0.7}) \in^\sim (F, G, A)$.

Lemma 3.1. *Let (H, I, B) be a fbs set and $(F, G, \{e\}) = (e_x^p, \neg e_x^q)$ be a fbs point over (U, E). Then the followings hold:*

1. *$(F, G, \{e\}) \in^\sim (H, I, B)$ iff $(F, G, \{e\}) \subseteq^\sim (H, I, B)$.*
2. *If $(F, G, \{e\}) \cap^\sim (H, I, B) = (\Phi, U, E)$ then $(F, G, \{e\}) \notin^\sim (H, I, B)$.*

Proof. The proof is clear. □

Theorem 3.2. *Let (U, τ, E) be a FBST space. Then the collection $\tau_e = \{F_E : (F, G, E) \in \tau\}$ defines a fuzzy soft topology on (U,E).*

Proof. (1) Since $\Phi(e) = 0$ and $U(e) = 1$, $\forall e \in E$ the first conditıon comes true.

(2) Since $(F_1, G_1, E), (F_2, G_2, E) \in \tau$ then $(F_1, G_1, E) \cap^{\sim} (F_2, G_2, E) \in \tau$ and so $F_{1_E} \wedge F_{2_E} \in \tau_e$.

(3) Since $(F_i, G_i, E) \in \tau$ then $\bigcup^{\sim}_{i\in I} \in \tau$ and so $\vee_{i\in I} F_{i_E} \in \tau_e$.

□

Theorem 3.3. *Let (U, τ, E) be a FBST space. Then the family $\tau = \{(F, G, E) : (F, E) \in \tau, G(\neg e) = F'(e) = U - F(e), \forall \neg e \in E\}$ defines a FBST space over (U,E).*

Proof. The proof is clear. □

Definition 3.5. *Let $\tau = \{(\Phi, U, E), (U, \Phi, E)\}$. Then τ is called indiscrete FBST space . Let τ be the collection of all fbs sets defined. Then τ is called discrete FBST space.*

Definition 3.6. *Let (U, τ_1, E) and (U, τ_2, E) be two FBST spaces . τ_1 is said to be coarser than τ_2 (or τ_2 is finer than τ_1) if $\tau_1 \subseteq \tau_2$ ($\tau_2 \subseteq \tau_1$).*

Theorem 3.4. *Let (U, τ_1, E) and (U, τ_2, E) be two FBST spaces . Then $(U, \tau_1 \cap \tau_2, E)$ is a FBST space .*

Proof. (1) Since $(\Phi, U, A), (U, \Phi, A) \in \tau_1, \tau_2$ then $(\Phi, U, A), (U, \Phi, A) \in \tau_1 \cap \tau_2$.

(2) Since $(F_1, G_1, A_1), (F_2, G_2, A_2) \in \tau_1, \tau_2$ then $(F_1, G_1, A_1) \cap^{\sim} (F_2, G_2, A_2) \in \tau_1 \cap \tau_2$.

(3) Since $(F_i, G_i, A_i) \in \tau_1, \tau_2$ for all $i \in I$ then $\bigcup^{\sim}_{i\in I}(F_i, G_i, A_i) \in \tau_1 \cup \tau_2$.

□

Remark 3.1. *The union of two FBST space is not a FBST space generally as seen the following example.*

Example 3.3. *Let $U = \{x, y, z\}, E = \{e_1, e_2, e_3\}$, τ_1 be the BFST space given in Example 3.1. and*

$$\tau_2 = \{(\Phi, U, E), (U, \Phi, E), (F_1, G_1, A_1), (F_2, G_2, A_2), (F_3, G_3, A_3), (F_4, G_4, A_4)\}$$

is as follows:

$$\begin{aligned}
F_1(e_1) &= \{x_{0.3}, y_{0.5}, z_{0.3}\}, & G_1(\neg e_1) &= \{x_{0.4}, y_{0.3}, z_{0.4}\}\\
F_1(e_2) &= \{x_{0.2}, y_{0.3}, z_{0.9}\}, & G_1(\neg e_2) &= \{x_{0.4}, y_{0.6}, z_{0.1}\}\\
F_2(e_2) &= \{x_{0.4}, y_{0.5}, z_{0.6}\}, & G_2(\neg e_2) &= \{x_{0.6}, y_{0.3}, z_{0.3}\}\\
F_3(e_1) &= \{x_{0.3}, y_{0.5}, z_{0.3}\}, & G_3(\neg e_1) &= \{x_{0.4}, y_{0.3}, z_{0.4}\}\\
F_3(e_2) &= \{x_{0.4}, y_{0.5}, z_{0.9}\}, & G_3(\neg e_2) &= \{x_{0.4}, y_{0.3}, z_{0.1}\}\\
F_4(e_1) &= \{x_{0.3}, y_{0.5}, z_{0.3}\}, & G_4(\neg e_1) &= \{x_{0.4}, y_{0.3}, z_{0.4}\}\\
F_4(e_2) &= \{x_{0.2}, y_{0.3}, z_{0.6}\}, & G_4(\neg e_2) &= \{x_{0.6}, y_{0.6}, z_{0.1}\}
\end{aligned}$$

It can be seen that although τ_1, τ_2 are FBST spaces, $\tau_1 \cup \tau_2$ is not a FBST space.

Definition 3.7. *Let (U, τ, E) be a FBST space, (F, G, A) be a bfs set and $(e^p_x, \neg e^q_x)$ be a bfs point over (U, E). (F, G, A) is called a fuzzy bipolar soft neighborhood of $(e^p_x, \neg e^q_x)$ if there exists a bfs open set (F_1, G_1, A_1) such that $(e^p_x, \neg e^q_x) \in^{\sim} (F_1, G_1, A_1) \subseteq^{\sim} (F, G, A)$.*

Definition 3.8. *Let (U, τ, E) be a BFST space, (F, G, A), (F_1, G_1, A_1) be two bfs sets over (U, E). (F_1, G_1, A_1) is called a fuzzy bipolar soft neighborhood of (F, G, A) if there exists a bfs open set (F_2, G_2, A_2) such that $(F, G, A) \subseteq^{\sim} (F_2, G_2, A_2) \subseteq^{\sim} (F_1, G_1, A_1)$.*

Theorem 3.5. *Let (U,τ,E) be FBST space. Then the following hold:*

1. *There exists a bfs open neighborhood (F,G,A) for each fbs point $(e_x^p,\neg e_x^q)$.*
2. *If $(F_1,G_1,A_1),(F_2,G_2,A_2)$ are bfs neighborhoods of $(e_x^p,\neg e_x^q)$ then $(F_1,G_1,A_1)\cap^{\sim}(F_2,G_2,A_2)$ is a bfs neighborhoods of $(e_x^p,\neg e_x^q)$.*
3. *If (F_1,G_1,A_1) is a bfs neighborhoods of $(e_x^p,\neg e_x^q)$ and $(F_1,G_1,A_1)\subseteq^{\sim}(F_2,G_2,A_2)$ then (F_2,G_2,A_2) is a bfs neighborhoods of $(e_x^p,\neg e_x^q)$.*

Proof. (1) The proof is clear.

(2) Let $(F_1,G_1,A_1),(F_2,G_2,A_2)$ are bfs neighborhoods of $(e_x^p,\neg e_x^q)$. Then there exist fbs open sets $(H_1,I_1,B_1),(H_2,I_2,B_2)$ such that $(e_x^p,\neg e_x^q)\in^{\sim}(H_1,I_1,B_1)\subseteq^{\sim}(F_1,G_1,A_1)$ and $(e_x^p,\neg e_x^q)\in^{\sim}(H_2,I_2,B_2)\subseteq^{\sim}(F_2,G,A_2)$. Hence $(e_x^p,\neg e_x^q)\in^{\sim}(H_1,I_1,B_1)\cap^{\sim}(H_2,I_2,B_2)\subseteq^{\sim}(F_1,G_1,A_1)\cap^{\sim}(F_2,G_2,A_2)$. Thus $(F_1,G_1,A_1)\cap^{\sim}(F_2,G_2,A_2)$ is a bfs neighborhoods of $(e_x^p,\neg e_x^q)$.

(3) Let (F_1,G_1,A_1) is a bfs neighborhoods of $(e_x^p,\neg e_x^q)$ and $(F_1,G_1,A_1)\subseteq^{\sim}(F_2,G_2,A_2)$. Then there exists a fbs open set (H_1,I_1,B_1) such that $(e_x^p,\neg e_x^q)\in^{\sim}(H_1,I_1,B_1)\subseteq^{\sim}(F_1,G_1,A_1)$. Hence$(e_x^p,\neg e_x^q)\in^{\sim}(H_1,I_1,B_1)\subseteq^{\sim}(F_2,G_2,A_2)$. This shows that (F_2,G_2,A_2) is a bfs neighborhoods of $(e_x^p,\neg e_x^q)$.

□

Definition 3.9. *Let (U,τ,E) be FBST space, $(e_x^p,\neg e_x^q)$ be a fbs point and (F,G,A) be a fbs set over (U,E). $(e_x^p,\neg e_x^q)$ is called a bfs interior point of (F,G,A) if there exists a bfs open set (H,I,B) such that $(e_x^p,\neg e_x^q)\in^{\sim}(H,I,B)\subseteq^{\sim}(F,G,A)$.*
The set of all fbs interior points of (F,G,A) is called the fuzzy bipolar soft interior of (F,G,A) and is shown by $fbsint(F,G,A)$

Theorem 3.6. *Let (U,τ,E) be FBST space and (F,G,A) be a fbs set over (U,E).Then the followings hold:*

1. *$fbsint(F,G,A)$ contains the all fbs open sets contained in (F,G,A).*
2. *$fbsint(F,G,A)\subseteq^{\sim}(F,G,A)$.*
3. *$fbsint(F,G,A)$ is a fbs open set.*
4. *$fbsint(F,G,A)$ is the biggest fbs open set contained in (F,G,A).*
5. *(F,G,A) is a fbs open set iff $fbsint(F,G,A)=(F,G,A)$.*

Proof. (1) Let $(e_x^p,\neg e_x^q)$ be a fbs interior point of (F,G,A). Then $(e_x^p,\neg e_x^q)\in^{\sim}(H_i,I_i,B_i)\subseteq^{\sim}(F,G,A)$ where $(H_i,I_i,B_i)\in\tau$. Thus $(e_x^p,\neg e_x^q)\in^{\sim}\bigcup^{\sim}(H_i,I_i,B_i)$.
Conversely let $(e_x^p,\neg e_x^q)\in^{\sim}(H_i,I_i,B_i)\subseteq^{\sim}(F,G,A)$ where $(H_i,I_i,B_i)\in\tau$. Hence $(e_x^p,\neg e_x^q)\in^{\sim}fbsint(F,G,A)$.

(2) The proof is clear from Definition 3.9.

(3) The proof is clear since $fbsint(F,G,A)$ is the union of fbs open sets.

(4) The proof is clear by (1).

(5) Let (F,G,A) be a fbs open set. As proved in (1), the fbs interior of (F,G,A) is the biggest fbs open set contained in (F,G,A) . Hence $fbsint(F,G,A)=(F,G,A)$.
Conversely let $fbsint(F,G,A)=(F,G,A)$. Since the fbs interior of (F,G,A) is a fbs open set (F,G,A) is fbs open.

□

Theorem 3.7. *Let (U,τ,E) be FBST space, (F,G,A), (H,I,B) be fbs sets over (U,E). Then the followings hold:*

1. *$fbsint(\Phi,U,A)=(\Phi,U,A)$ and $fbsint(U,\Phi,A)=(U,\Phi,A)$.*
2. *$fbsint(fbsint(F,G,A))=fbsint(F,G,A)$.*

(3) *If* $(F,G,A) \subseteq^{\sim} (H,I,B)$ *then* $fbsint(F,G,A) \subseteq^{\sim} fbsint(H,I,B)$.
(4) $fbsint(F,G,A) \cap^{\sim} fbsint(H,I,B) = fbsint[(F,G,A) \cap^{\sim} (H,I,B)]$.
(5) $fbsint(F,G,A) \cup^{\sim} fbsint(H,I,B) \subseteq^{\sim} fbsint[(F,G,A) \cup^{\sim} (H,I,B)]$.

Proof. (1) Obvious.
(2) Let $fbsint(F,G,A) = (H,I,B)$. Since (H,I,B) is a fbs open set $fbsint(H,I,B) = (H,I,B)$. Hence the proof is completed.
(3) Let $(F,G,A) \subseteq^{\sim} (H,I,B)$. It is known that $fbsint(F,G,A) \subseteq^{\sim} (F,G,A)$ and so, $fbsint(F,G,A) \subseteq^{\sim} (H,I,B)$. Also $fbsint(H,I,B) \subseteq^{\sim} (H,I,B)$ and $fbsint(H,I,B)$ is the biggest fbs open set contained in (H,I,B). Thus, $fbsint(F,G,A) \subseteq^{\sim} fbsint(H,I,B)$.
(4) It can be easily seen that $fbsint(F,G,A) \subseteq^{\sim} (F,G,A)$ and $fbsint(H,I,B) \subseteq^{\sim} (H,I,B)$. Hence $fbsint(F,G,A) \cap^{\sim} fbsint(H,I,B)$ is a fbs open set contained in $(F,G,A) \cap^{\sim} (H,I,B)$. It is also known that $fbsint[(F,G,A) \cap^{\sim} (H,I,B)]$ is the biggest fbs open set contained in $(F,G,A) \cap^{\sim} (H,I,B)$. Thus, $fbsint(F,G,A) \cap^{\sim} fbsint(H,I,B) \subseteq^{\sim} fbsint[(F,G,A) \cap^{\sim} (H,I,B)]$.
Conversely, it is clear that $fbsint[(F,G,A) \cap^{\sim} (H,I,B)] \subseteq^{\sim} fbsint(F,G,A)$ and $fbsint[(F,G,A) \cap^{\sim} (H,I,B)] \subseteq^{\sim} fbsint(H,I,B)$.
Thus, $fbsint[(F,G,A) \cap^{\sim} (H,I,B)] \subseteq^{\sim} fbsint(F,G,A) \cap^{\sim} fbsint(H,I,B)$. Hence $fbsint(F,G,A) \cap^{\sim} fbsint(H,I,B) = fbsint[(F,G,A) \cap^{\sim} (H,I,B)]$.
(5) It is known that $fbsint(F,G,A) \subseteq^{\sim} (F,G,A)$ and $fbsint(H,I,B) \subseteq^{\sim} (H,I,B)$. Hence $fbsint(F,G,A) \cup^{\sim} fbsint(H,I,B) \subseteq^{\sim} (F,G,A) \cup^{\sim} (H,I,B)$. Also the biggest fbs open set contained in $(F,G,A) \cup^{\sim} (H,I,B)$ is $fbsint[(F,G,A) \cup^{\sim} (H,I,B)]$. Thus $fbsint(F,G,A) \cup^{\sim} fbsint(H,I,B) \subseteq^{\sim} fbsint[(F,G,A) \cup^{\sim} (H,I,B)]$. □

Definition 3.10. *Let* (U,τ,E) *be a FBST space and* (F,G,A) *be a fbs set over* (U,E) *. The intersection of all fbs closed sets containing in* (F,G,A) *is called the fbs closure of* (F,G,A) *and is denoted by* $fbscl(F,G,A)$.

Theorem 3.8. *Let* (U,τ,E) *be a FBST space and* (F,G,A) *and* (H,I,B) *be two fbs sets over (U,E). Then the followings hold:*
(1) $fbscl(F,G,A)$ *is a fbs closed set.*
(2) $(F,G,A) \subseteq^{\sim} fbscl(F,G,A)$.
(3) $fbscl(F,G,A)$ *is the smallest fbs closed set containing* (F,G,A).
(4) *If* $(F,G,A) \subseteq^{\sim} (H,I,B)$ *then* $fbscl(F,G,A) \subseteq^{\sim} fbscl(H,I,B)$.
(5) (F,G,A) *is fbs closed set iff* $fbscl(F,G,A) = (F,G,A)$.
(6) $fbscl(fbscl(F,G,A)) = fbscl(F,G,A)$.
(7) $fbscl[(F,G,A) \cup^{\sim} (H,I,B)] = fbscl(F,G,A) \cup^{\sim} fbscl(H,I,B)$.
(8) $fbscl[(F,G,A) \cap^{\sim} (H,I,B)] \subseteq^{\sim} fbscl(F,G,A) \cap^{\sim} fbscl(H,I,B)$.

Proof. (1) The proof is clear from Definition 3.10 since the intersection of fbs closed sets is fbs closed.
(2) The proof is clear from Definition 3.10.
(3) The proof is clear from Definition 3.10.
(4) Let $(F,G,A) \subseteq^{\sim} (H,I,B)$. It can be easily seen that $(F,G,A) \subseteq^{\sim} fbscl(F,G,A)$ and $(H,I,B) \subseteq^{\sim} fbscl(H,I,B)$ so, $(F,G,A) \subseteq^{\sim} fbscl(H,I,B)$. Since the smallest fbs closed set containing (F,G,A) is $fbscl(F,G,A)$ we obtain that $fbscl(F,G,A) \subseteq^{\sim} fbscl(H,I,B)$.
(5) The proof is clear.
(6) Suppose that $fbscl(F,G,A) = (H,I,B)$. Since (H,I,B) is a fbs closed set $fbscl(H,I,B) = (H,I,B)$. Hence the proof is completed.

(7) It is known that $(F,G,A) \subseteq^{\sim} fbscl(F,G,A)$ and $(H,I,B) \subseteq^{\sim} fbscl(H,I,B)$ and hence $(F,G,A) \cup^{\sim} (H,I,B) \subseteq^{\sim} fbscl(F,G,A) \cup^{\sim} fbscl(H,I,B)$. Also it can be easily seen that $(F,G,A) \cup^{\sim} (H,I,B) \subseteq^{\sim} fbscl[(F,G,A) \cup^{\sim} (H,I,B)]$. Since the smallest fbs closed set containing $(F,G,A) \cup^{\sim} (H,I,B)$ is $fbscl[(F,G,A) \cup^{\sim} (H,I,B)]$ we obtain that $fbscl[(F,G,A) \cup^{\sim} (H,I,B)] \subseteq^{\sim} fbscl(F,G,A) \cup^{\sim} fbscl(H,I,B)$. Conversely, it is known that $fbscl(F,G,A) \subseteq^{\sim} fbscl[(F,G,A) \cup^{\sim} (H,I,B)]$ and $fbscl(H,I,B) \subseteq^{\sim} fbscl[(F,G,A) \cup^{\sim} (H,I,B)]$. Hence we get $fbscl(F,G,A) \cup^{\sim} fbscl(H,I,B) \subseteq^{\sim} fbscl[(F,G,A) \cup^{\sim} (H,I,B)]$.

(8) It can be easily seen that $fbscl[(F,G,A) \cap^{\sim} (H,I,B)] \subseteq^{\sim} fbscl(F,G,A)$ and $fbscl[(F,G,A) \cap^{\sim} (H,I,B)] \subseteq^{\sim} fbscl(H,I,B)$. Thus, $fbscl[(F,G,A) \cap^{\sim} (H,I,B)] \subseteq^{\sim} fbscl(F,G,A) \cap^{\sim} fbscl(H,I,B)$.

$\square$

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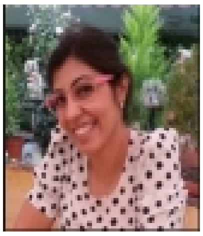

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