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CERTAIN SUBCLASS OF PASCU-TYPE BI-STARLIKE FUNCTIONS IN PARABOLIC DOMAIN

K. VIJAYA[1], §

Abstract. Estimates on the coefficients $|a_2|$ and $|a_3|$ are obtained for normalized analytic function f in the open disk with f and its inverse $g = f^{-1}$ satisfy the condition that $\dfrac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)}$ and $\dfrac{zg'(z) + \lambda z^2 g''(z)}{(1-\lambda)g(z) + \lambda z g'(z)}$ $(0 \le \lambda \le 1)$ are both subordinate to an analytic function in parabolic region. Furthermore, we estimate the Fekete-Szegö functional for $f \in \mathcal{P}_{\Sigma,P}(\lambda, \varphi_\alpha)$.

Keywords: Analytic functions, univalent functions, bi-univalent functions, bi-starlike functions, bi-convex functions, and subordination.

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1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

normailzed by the conditions $f(0) = 0 = f'(0) - 1$ defined in the open unit disk $\triangle = \{z \in \mathbb{C} : |z| < 1\}$. A function $f \in \mathcal{A}$ is said to be bi-univalent in $\triangle$ if both f and f^{-1} are univalent in $\triangle$. Let Σ denote the class of bi-univalent functions defined in the unit disk $\triangle$. Since $f \in \sum$ has the Maclaurian series given by (1), a computation shows that its inverse $g = f^{-1}$ has the expansion

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 + \cdots \tag{2}$$

An analytic function f is subordinate to an analytic function g, written $f(z) \prec g(z)$, provided there is an analytic function w defined on $\triangle$ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$. Ma and Minda [8] unified various subclasses of starlike and convex functions for which either of the quantity $\frac{z\,f'(z)}{f(z)}$ or $1 + \frac{z\,f''(z)}{f'(z)}$ is subordinate to a more general superordinate function. For this purpose, they considered an analytic function φ with positive real part in the unit disk $\triangle, \varphi(0) = 1, \varphi'(0) > 0$, and φ maps $\triangle$ onto a

[1] School of Advanced Sciences, Vellore Institute of Technology, Vellore-632 014, India.
e-mail: kvijaya@vit.ac.in; ORCID: http://orcid.org/0000-0002-3216-7038.

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region starlike with respect to 1 and symmetric with respect to the real axis. The class of Ma-Minda starlike functions consists of functions $f \in \mathcal{A}$ satisfying the subordination $\frac{z\ f'(z)}{f(z)} \prec \varphi(z)$. Similarly, the class of Ma-Minda convex functions of functions $f \in \mathcal{A}$ satisfying the subordination $1 + \frac{z\ f''(z)}{f'(z)} \prec \varphi(z)$. A function f is bi-starlike of Ma-Minda type or bi-convex of Ma-Minda type if both f and f^{-1} are respectively Ma-Minda starlike or convex. These classes are denoted respectively by $\mathcal{S}^*_\Sigma(\varphi)$ and $\mathcal{K}_\Sigma(\varphi)$.In the sequel, it is assumed that φ is an analytic function with positive real part in the unit disk $\triangle$, satisfying $\varphi(0) = 1, \varphi'(0) > 0$, and $\varphi(\triangle)$ is symmetric with respect to the real axis. Such a function has a series expansion of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, \quad (B_1 > 0). \tag{3}$$

Ali and Singh [2] introduced a new class of parabolic starlike functions denoted by $\mathcal{S}_p(\alpha)$ of order $\alpha(0 \leq \alpha < 1)$ salifies the following:

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < (1 - 2\alpha) + \Re\left(\frac{zf'(z)}{f(z)}\right). \tag{4}$$

Equivalently,

$$f \in \mathcal{S}_p(\alpha) \Longleftrightarrow \left(\frac{zf'(z)}{f(z)}\right) \in \Omega_\alpha,$$

where Ω_α denotes the parabolic region in the right half-plane

$$\Omega_\alpha = \{w = u + iv : v^2 < 4(1-\alpha)(u-\alpha)\} = \{w : |w - 1| < (1 - 2\alpha) + \Re(w)\}. \tag{5}$$

Ali and Singh [2]showed that the normalized Riemann mapping function $\varphi_\alpha(z)$ from the open unit disk $\triangle$ onto Ω_α is given by

$$\begin{aligned} \varphi_\alpha(z) &= 1 + \frac{4(1-\alpha)}{\pi^2}\left[\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right]^2 \\ &= 1 + \frac{16}{\pi^2}(1-\alpha)z + \frac{32}{3\pi^2}(1-\alpha)z^2 + \frac{368}{45\pi^2}(1-\alpha)z^3 + \cdots \\ &= 1 + \sum_{k=1}^{\infty} B_k z^k, \end{aligned} \tag{6}$$

where

$$B_k = \frac{16(1-\alpha)}{k\pi^2}\sum_{j=0}^{k-1}\frac{1}{2j+1} \quad (k \in \mathbb{N}). \tag{7}$$

Due to Ma and Minda [8], we state the following Lemma.

Lemma 1.1. *If a function $f \in \mathcal{S}_p(\alpha)$, then*

$$\left(\frac{zf'(z)}{f(z)}\right) \in \varphi_\alpha(z),$$

where φ_α is given by (6).

Since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk $\triangle$. In fact, the Koebe one-quarter theorem [6] ensures that the image of $\triangle$. under every univalent function $f \in \mathcal{S}$ of the form (1), contains a disk of radius $\frac{1}{4}$. Thus every univalent function $f \in \mathcal{S}$ has an inverse f^{-1} which is defined by

$$f^{-1}(f(z)) = z \quad (z \in \triangle)$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right).$$

In fact, the inverse function f^{-1} is given by

$$f^{-1}(w) = w - a_2w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right) w^4 + \cdots . \tag{8}$$

Several authors have introduced and investigated subclasses of bi-univalent functions Σ and obtained bounds for the initial coefficients (see [4, 3, 10, 12]). Motivated by the work of Ali et al. [2, 7],in this paper, we introduce a new subclass $\mathcal{P}_{\Sigma,P}(\lambda, \varphi_\alpha)$ of bi-univalent functions and obtain the estimates on the coefficients $|a_2|$ and $|a_3|$ by subordination.Furthermore, we estimate the Fekete-Szegö functional for $f \in \mathcal{P}_{\Sigma,P}(\lambda, \varphi_\alpha)$.

Definition 1.1. *A function $f \in \Sigma$ is said to be in the class $\mathcal{P}_{\Sigma,p}(\lambda, \varphi_\alpha)$ if the following subordination hold:*

$$\left|\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda zf'(z)} - 1\right| < (1-2\alpha) + \Re\left(\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda zf'(z)}\right) \quad (z \in \triangle) \tag{9}$$

and

$$\left|\frac{wg'(w) + \lambda w^2 g''(w)}{(1-\lambda)g(w) + \lambda wg'(w)} - 1\right| < (1-2\alpha) + \Re\left(\frac{wg'(w) + \lambda w^2 g''(w)}{(1-\lambda)g(w) + \lambda wg'(w)}\right) \quad (w \in \triangle). \tag{10}$$

Due to Lemma1.1 and by the above the definition we can state

$$\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda zf'(z)} \prec \varphi_\alpha(z) \quad (z \in \triangle) \tag{11}$$

and

$$\frac{wg'(w) + \lambda w^2 g''(w)}{(1-\lambda)g(w) + \lambda wg'(w)} \prec \varphi_\alpha(w) \quad (w \in \triangle), \tag{12}$$

where φ_α is given by (6).

We note that $\mathcal{P}_{\Sigma,P}(0, \varphi_\alpha) = \mathcal{S}^*_{\Sigma,P}(\varphi_\alpha)$ [5] and $\mathcal{P}_{\Sigma,P}(1, \varphi_\alpha) = \mathcal{K}_{\Sigma,P}(\varphi_\alpha)$ as illustrated below:

Example 1.1. [5] *A function $f \in \Sigma$ is said to be in the class $\mathcal{S}_{\Sigma,P}(\varphi_\alpha)$ if the following subordination hold:*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < (1-2\alpha) + \Re\left(\frac{zf'(z)}{f(z)}\right) \quad (z \in \triangle)$$

and

$$\left|\frac{wg'(w)}{g(w)} - 1\right| < (1-2\alpha) + \Re\left(\frac{wg'(w)}{g(w)}\right) \quad (w \in \triangle).$$

Due to Lemma1.1 and by the above the definition we can state

$$\frac{zf'(z)}{f(z)} \prec \varphi_\alpha(z) \quad \textit{and} \quad \frac{wg'(w)}{g(w)} \prec \varphi_\alpha(w)$$

where $\varphi_\alpha(z)$ is given by (6) and $z, w \in \Delta$.

Example 1.2. *A function $f \in \Sigma$ is said to be in the class $\mathcal{K}_{\Sigma,P}(\varphi_\alpha)$ if the following subordination hold:*

$$\left|\frac{zf''(z)}{f'(z)}\right| < (1-2\alpha) + \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) \quad (z \in \triangle)$$

and

$$\left|\frac{wg''(w)}{g'(w)}\right| < (1-2\alpha) + \Re\left(1 + \frac{wg''(w)}{g'(w)}\right) \quad (w \in \triangle).$$

Due to Lemma1.1 and by the above the definition we can state

$$1+\frac{zf''(z)}{f'(z)} \prec \varphi_\alpha(z) \quad \textit{and} \quad 1+\frac{wg''(w)}{g'(w)} \prec \varphi_\alpha(w)$$

where $\varphi_\alpha(z)$ is given by (6) and $z, w \in \Delta$. In order to prove our main results, we require the following Lemma due to [11].

Lemma 1.2. *If $h \in \mathcal{P}$, then $|c_k| \leq 2$ for each k, where $\mathcal{P}$ is the family of all functions h analytic in $\triangle$ for which $\Re\{h(z)\} > 0$, where $h(z) = 1 + c_1z + c_2z^2 + \cdots$ for $z \in \triangle$.*

2. Section

Coefficient estimates for the function class $\mathcal{P}_{\Sigma,P}(\lambda, \varphi_\alpha)$

Theorem 2.1. *Let f given by (1) be in the class $\mathcal{P}_{\Sigma,P}(\lambda, \varphi_\alpha)$. Then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{|(1+2\lambda-\lambda^2)B_1^2+(1+\lambda)^2(B_1-B_2)|}} \tag{13}$$

and

$$|a_3| \leq B_1\left(\frac{B_1}{(1+\lambda)^2}+\frac{1}{2(1+2\lambda)}\right). \tag{14}$$

where $B_1 = \frac{16}{\pi^2}(1-\alpha)$ and $B_2 = \frac{32}{3\pi^2}(1-\alpha)$ from (7).

Proof. Let $f \in \mathcal{P}_{\Sigma,P}(\lambda, \varphi_\alpha)$ and $g = f^{-1}$. Then there are analytic functions $u, v : \triangle \longrightarrow \triangle$, with $u(0) = 0 = v(0)$, satisfying

$$\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)} = \varphi_\alpha(u(z)) \tag{15}$$

and

$$\frac{wg'(w)+\lambda w^2 g''(w)}{(1-\lambda)g(w)+\lambda wg'(w)} = \varphi_\alpha(v(w)). \tag{16}$$

Define the functions $p(z)$ and $q(z)$ by

$$p(z) := \frac{1+u(z)}{1-u(z)} = 1+p_1z+p_2z^2+\cdots$$

and

$$q(z) := \frac{1+v(z)}{1-v(z)} = 1+q_1z+q_2z^2+\cdots$$

or, equivalently,

$$u(z) := \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left[p_1z+\left(p_2-\frac{p_1^2}{2}\right)z^2+\cdots\right] \tag{17}$$

and

$$v(z) := \frac{q(z)-1}{q(z)+1} = \frac{1}{2}\left[q_1z+\left(q_2-\frac{q_1^2}{2}\right)z^2+\cdots\right]. \tag{18}$$

Then $p(z)$ and $q(z)$ are analytic in $\triangle$ with $p(0) = 1 = q(0)$. Since $u, v : \triangle \to \triangle$, the functions $p(z)$ and $q(z)$ have a positive real part in $\triangle$, and $|p_i| \leq 2$ and $|q_i| \leq 2$. Using (17) and (18) in (15) and (16) respectively, we have

$$\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)} = \varphi\left(\frac{1}{2}\left[p_1z+\left(p_2-\frac{p_1^2}{2}\right)z^2+\cdots\right]\right) \tag{19}$$

and

$$\frac{wg'(w)+\lambda w^2 g''(w)}{(1-\lambda)g(w)+\lambda wg'(w)} = \varphi\left(\frac{1}{2}\left[q_1 w + \left(q_2 - \frac{q_1^2}{2}\right)w^2 + \cdots\right]\right). \tag{20}$$

In light of (1) - (3), from (19) and (20), it is evident that

$$1+(1+\lambda)a_2 z + [2(1+2\lambda)a_3 - (1+\lambda)^2 a_2^2]z^2 + \cdots$$
$$= 1 + \frac{1}{2}B_1 p_1 z + [\frac{1}{2}B_1(p_2 - \frac{p_1^2}{2}) + \frac{1}{4}B_2 p_1^2]z^2 + \cdots$$

and

$$1-(1+\lambda)a_2 w - [2(1+2\lambda)a_3 + (\lambda^2 - 6\lambda - 3)a_2^2]w^2 + \cdots$$
$$= 1 + \frac{1}{2}B_1 q_1 w + [\frac{1}{2}B_1(q_2 - \frac{q_1^2}{2}) + \frac{1}{4}B_2 q_1^2]w^2 + \cdots$$

which yields the following relations.

$$(1+\lambda)a_2 = \frac{1}{2}B_1 p_1 \tag{21}$$

$$-(1+\lambda)^2 a_2^2 + 2(1+2\lambda)a_3 = \frac{1}{2}B_1(p_2 - \frac{p_1^2}{2}) + \frac{1}{4}B_2 p_1^2 \tag{22}$$

$$-(1+\lambda)a_2 = \frac{1}{2}B_1 q_1 \tag{23}$$

and

$$-(\lambda^2 - 6\lambda - 3)a_2^2 - 2(1+2\lambda)a_3 = \frac{1}{2}B_1(q_2 - \frac{q_1^2}{2}) + \frac{1}{4}B_2 q_1^2. \tag{24}$$

From (21) and (23), it follows that

$$p_1 = -q_1 \tag{25}$$

and

$$8(1+\lambda)^2 a_2^2 = B_1^2(p_1^2 + q_1^2). \tag{26}$$

From (22), (24) and (26), we obtain

$$a_2^2 = \frac{B_1^3(p_2+q_2)}{4[(1+2\lambda-\lambda^2)B_1^2 + (1+\lambda)^2(B_1 - B_2)]}. \tag{27}$$

Applying Lemma 1.2, for the coefficients p_2 and q_2, we immediately got the desired estimate on $|a_2|$ as asserted in (2.2).

By subtracting (24) from (22) and using (25) and (26), we get

$$a_3 = a_2^2 + \frac{B_1(p_2 - q_2)}{8(1+2\lambda)} = \frac{B_1^2(p_1^2+q_1^2)}{8(1+\lambda)^2} + \frac{B_1(p_2 - q_2)}{8(1+2\lambda)}. \tag{28}$$

Applying Lemma 1.2 once again for the coefficients p_1, p_2, q_1 and q_2, we get the desired estimate on $|a_3|$ as asserted in (2.2) $\square$

Remark 2.1. *For $\lambda = 0$ and $B_1 = \frac{16}{\pi^2}(1-\alpha)$ and $B_2 = \frac{32}{3\pi^2}(1-\alpha)$ the inequality (2.2) reduces to the estimate of $|a_2|$ and $|a_3|$.* [5].

By taking $\lambda = 1$ we get the following result for $f \in \mathcal{K}_{\Sigma,P}(\varphi_\alpha)$

Theorem 2.2. *Let f given by (1) be in the class $\mathcal{K}_{\Sigma,P}(\varphi_\alpha)$. Then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{2B_1^2 + |4(B_1 - B_2)|}}$$

and

$$|a_3| \leq \frac{B_1^2}{4} + \frac{B_1}{6}.$$

where $B_1 = \frac{16}{\pi^2}(1-\alpha)$ *and* $B_2 = \frac{32}{3\pi^2}(1-\alpha)$ *from* (7).

2.1. **Subsection.** Fekete-Szegö inequalities for the Function Class $\mathcal{P}_{\Sigma,P}(\lambda, \varphi_\alpha)$ Making use of the values of a_2^2 and a_3, and motivated by the recent work of Zaprawa [13], we prove the following Fekete-Szegö result for the function class $f \in \mathcal{P}_{\Sigma,P}(\lambda, \varphi_\alpha)$.

Theorem 2.3. *Let the function $f(z)$ be in the class $\mathcal{P}_{\Sigma,P}(\lambda, \varphi_\alpha)$ and $\mu \in \mathbb{C}$, then*

$$|a_3 - \mu a_2^2| \leq 2B_1 \left| \left(\Theta(\mu) + \frac{1}{8(1+2\lambda)}\right) + \left(\Theta(\mu) - \frac{1}{8(1+2\lambda)}\right) \right|, \tag{29}$$

where

$$\Theta(\mu) = \frac{B_1^2(1-\mu)}{4[(1+2\lambda-\lambda^2)B_1^2 + (1+\lambda)^2(B_1 - B_2)]}, B_1 > 0.$$

where $B_1 = \frac{16}{\pi^2}(1-\alpha)$ *and* $B_2 = \frac{32}{3\pi^2}(1-\alpha)$ *from* (7)

Proof. From (28), we have

$$a_3 = a_2^2 + \frac{B_1(p_2 - q_2)}{8(1+2\lambda)}.$$

Using (27), by simple calculation we get

$$a_3 - \mu a_2^2 = B_1 \left[\left(\Theta(\mu) + \frac{1}{8(1+2\lambda)}\right) p_2 + \left(\Theta(\mu) - \frac{1}{8(1+2\lambda)}\right) q_2 \right],$$

where $\Theta(\mu) = \frac{B_1^2(1-\mu)}{4[(1+2\lambda-\lambda^2)B_1^2+(1+\lambda)^2(B_1-B_2)]}$. Since all B_j are real and $B_1 > 0$, we have

$$|a_3 - \mu a_2^2| \leq 2B_1 \left| \left(\Theta(\mu) + \frac{1}{8(1+2\lambda)}\right) + \left(\Theta(\mu) - \frac{1}{8(1+2\lambda)}\right) \right|,$$

which completes the proof. □

Remark 2.2. *Specializing $\lambda = 0$ we can obtain the Fekete-Szegö inequality for the function class $\mathcal{S}_{\Sigma,P}(\varphi_\alpha)$ as in* [5] *.*

Specializing $\lambda = 1$ we can obtain the Fekete-Szegö inequality for the function class $\mathcal{K}_{\Sigma,P}(\varphi_\alpha)$ as given below.

Corollary 2.1. *Let the function $f(z)$ be in the class $\mathcal{K}_{\Sigma,P}(\varphi_\alpha)$ and $\mu \in \mathbb{C}$, then*

$$|a_3 - \mu a_2^2| \leq 2B_1 \left| \left(\Theta(\mu) + \frac{1}{24}\right) + \left(\Theta(\mu) - \frac{1}{24}\right) \right|,$$

where

$$\Theta(\mu) = \frac{B_1^2(1-\mu)}{4[2B_1^2 + 4(B_1 - B_2)]}, B_1 > 0.$$

where $B_1 = \frac{16}{\pi^2}(1-\alpha)$ *and* $B_2 = \frac{32}{3\pi^2}(1-\alpha)$ *from* (7)

3. Conclusions

By taking $B_1 = \frac{16}{\pi^2}(1-\alpha)$ and $B_2 = \frac{32}{3\pi^2}(1-\alpha)$ and specializing the parameter $\lambda = 1$ we state the results for the class of bi convex functions in parabolic domain which has not been studied. Further by specializing $\lambda = 0$ we can obtain the results for bi-starlike functions in parabolic domain as in [5] .

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